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Can a λ -model have an r.e. theory?

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Types and Computations: A Colloquium in Honor of
Mario Coppo,
Mariangiola Dezani-Ciancaglini,
Simona Ronchi Della Rocca

Outline

- 1 **The problem**
 - λ -calculus models and theories
 - The conjecture(s)
- 2 **Methodology**
- 3 **Effective models of λ -calculus**
 - Effective and weakly λ -models
 - Can an effective λ -model have an r.e. theory?
 - Some results
- 4 **Graph models**
 - Definition
 - Effective graph models
 - Can a graph theory be r.e.?

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Syntax of the untyped λ -calculus

Terms of untyped λ -calculus

λ -terms: $M, N ::= x \mid MN \mid \lambda x.M$

Examples:

$\mathbf{I} \equiv \lambda x.x$, $\mathbf{T} \equiv \lambda xy.x$, $\mathbf{F} \equiv \lambda xy.y$, $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$.

β -conversion

$(\lambda x.M)N =_{\beta} M[x := N]$.

λ -theory

Any congruence on Λ containing β -conversion.

Examples of λ -theories

$\lambda\beta$ = the smallest λ -theory,

$\lambda\beta\eta$ = the smallest extensional λ -theory.

The lattice of λ -theories

- λ -theories form a complete lattice $(\lambda\mathcal{T}, \subseteq)$.

There is a continuum of λ -theories

$\text{card}(\lambda\mathcal{T}) = 2^{\aleph_0}$: the set of r.e. λ -theories is *dense* in $\lambda\mathcal{T}$.
 \mathcal{T} is r.e. if $\{(M, N) : M =_{\mathcal{T}} N\}$ is r.e.

General aim

Understanding the structure of $\lambda\mathcal{T}$.

Remark

Few λ -theories come from syntactical considerations ($\lambda\beta$, $\lambda\beta\eta$, BT , \mathcal{H}^*), most come as the equational theory of a model:

$$\text{Eq}(\mathcal{M}) = \{(M, N) : \mathcal{M} \models M = N\}$$

AIM: Investigate this conjecture

Conjecture

No (proper = non-syntactical) λ -calculus model belonging to the “main semantics”:

- Continuous semantics (Scott),
simplest subclass: **Graph models** (never extensional),
- Stable semantics (Berry and Girard),
- Strongly stable semantics (Bucciarelli and Ehrhard),

can have an r.e. equational (order) theory.

Recall

All these models are *partially ordered* with a \perp element.

Can a proper model have an r.e. theory?

Every (uniform) class of p.o. models:

- (Kerth 1995, Gouy, Bastonero 1996) represents 2^{\aleph_0} λ -theories,
- (Salibra 2001) omits 2^{\aleph_0} λ -theories.

Remark

Most models do not have an r.e. theory:

- there are only countably many r.e. theories,
- \mathcal{T} sensible $\Rightarrow \mathcal{T}$ non r.e.,
- the few known theories of models (BT, \mathcal{H}^* & co.) are non-r.e.

Is this question interesting?

A longstanding open problem...

Problem (Honsell \simeq 1984)

$\exists?$ a proper model \mathcal{M} such that $Eq(\mathcal{M}) = \lambda\beta, \lambda\beta\eta$ (r.e.)?

Theorem (Selinger 1995)

If $Eq(\mathcal{M}) = \lambda\beta, \lambda\beta\eta$ then any p.o. on (\mathcal{M}, \bullet) is trivial on (the interpretations of) closed λ -terms.

Known results

- **NO!** for graph models (corollary).
- (DiG-H-P,1995) \exists proper model \mathcal{M} such that $Eq(\mathcal{M}) = \lambda\beta\eta$ in a weakly continuous semantics and not strictly proper!!!
- (BBB 98-00) \exists strictly proper model \mathcal{M} of System F in the Scott-continuous semantics such that $Eq(\mathcal{M}) = \lambda\beta\eta^F$ Typed calculus!!!

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Methodology

- We concentrate on the following two classes of p.o. models:
 - ① $\mathcal{E}ff$ = “Effective” λ -models (based on effective domains)
 - Reason 1: Effective models (in our sense) are omni-present in the literature.
 - Reason 2: The BBB model of System F, is effective in our sense.
 - ② \mathcal{G} = Graph models (Based on $(\mathcal{P}(D), \subseteq)$)
 - Reason: simplest class, deeply studied, generic (in some sense).
- We also look at order theories:
 - $Ord(\mathcal{M}) = \{(M, N) : \mathcal{M} \models M \sqsubseteq N\}$.
 - Note that: $Ord(\mathcal{M})$ r.e. $\Rightarrow Eq(\mathcal{M})$ r.e.
- (Visser 1980) If $U, V \subseteq \Lambda^0$ are β -closed, co-r.e. and non-empty, then $U \cap V \neq \emptyset$.

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Effective λ -models

Effective domains

An effective domain is a Scott-domain \mathcal{D} equipped with an adequate notion of computable elements:

$\mathcal{D}^{r.e.}$ = computable elements of \mathcal{D} ;

\mathcal{D}^{dec} = decidable elements of \mathcal{D} ; [BMS07]

(Scott-continuous) Effective λ -models

A λ -model $\mathcal{M} = (\mathcal{D}, A, \lambda)$ with $\lambda, A : [\mathcal{D} \rightarrow \mathcal{D}] \triangleleft \mathcal{D}$ is *weakly effective* if \mathcal{D} is an effective Scott domain, and A, λ are computable.

\mathcal{M} is *effective* modulo a further technical condition on A, λ .

All the models introduced individually in the literature can be proved effective in our sense...

More details

Effective domains

An effective domain is a Scott-domain $\mathcal{D} = (D, \sqsubseteq_{\mathcal{D}}, d)$ where $d : \mathbb{N} \rightarrow K(D)$ is surjective such that:

- the relation “ $d_n \sqcup d_m$ exists” is decidable in (m, n) ,
- the relation $d_k = d_n \sqcup d_m$ is decidable in (n, m, k) .

Computable and decidable elements

$x \in \mathcal{D}^{r.e.}$ iff $\{n : d_n \sqsubseteq x\}$ is r.e.

$x \in \mathcal{D}^{dec}$ iff $\{n : d_n \sqsubseteq x\}$ is decidable.

Computable functions

$f \in [\mathcal{D} \rightarrow \mathcal{D}']^{r.e.}$ iff the relation $d'_m \sqsubseteq f(d_n)$ is r.e. in (n, m) .

Properties

\mathcal{M} weakly effective

- (i) $|M| \in \mathcal{D}^{r.e.}$ for all $M \in \Lambda^o$,
- (ii) if $u \in \mathcal{D}^{dec}$ then $u^- = \{N \in \Lambda^o : |N| \sqsubseteq u\}$ is co-r.e.

Proof Sketch: Weak effectivity implies that the interpretation function is computable. Then (i) and (ii) follow.

\mathcal{M} effective

If $M, N \in \Lambda^o$ are β -normal, then:

- (i) $|M| \in \mathcal{D}^{dec}$, hence:
- (ii) $M^- = \{N \in \Lambda^o : |N| \sqsubseteq |M|\}$ is co-r.e., thus:
- (iii) $M^- \cap N^- \neq \emptyset$ by [Visser, 1980].

Can an effective λ -model have an r.e. theory?

Can an effective λ -model have an r.e. theory?

\mathcal{M} effective

- 1 $Ord(\mathcal{M})$ is non r.e.
- 2 $Eq(\mathcal{M}) \neq \lambda\beta, \lambda\beta\eta$,
- 3 If $\perp^- \neq \emptyset$ then $Eq(\mathcal{M})$ is non r.e.,
- 4 If \mathcal{M} is stable or strongly stable then $Eq(\mathcal{M})$ is non r.e.

Remark 1

$T^- \neq F^-$, where $T \equiv \lambda xy.x$, $F \equiv \lambda xy.y$.

Remark 2

If \mathcal{M} is effective, then:

- (i) \perp^- , T^- , F^- are co-r.e. β -closed subsets of Λ^0 .
- (ii) $T^- \cap F^- \neq \emptyset$.

Proof sketches

If \mathcal{M} is an effective λ -model...

Theorem 1.

$Ord(\mathcal{M})$ is non r.e.

Proof. If $Ord(\mathcal{M})$ is r.e., then M^- is r.e. β -closed and non-empty. Hence, Rem. 2 implies $F^- = T^- = \Lambda^o$. Contrad. Rem. 1!

Theorem 2.

$Eq(\mathcal{M}) \neq \lambda\beta, \lambda\beta\eta$.

Proof. Follows from Rem 2(ii) and Selinger's result, since $N \in T^- \cap F^-$ implies $\mathcal{M} \models N \not\subseteq T$.

Proof sketches

If \mathcal{M} is an effective λ -model...

Theorem 3.

If $\perp^- \neq \emptyset$ then $Eq(\mathcal{M})$ is non r.e.,

Proof. $\perp^- \subsetneq \Lambda^0$ is a β -closed, co-r.e. set by Rem. 2(i).
Moreover \perp^- is just 1 equivalence class! Hence $Eq(\mathcal{M})$ cannot be r.e.

Theorem 4.

\mathcal{M} stable or strongly stable implies $Eq(\mathcal{M})$ non r.e.

Proof. Follows from Thm. 3 since in the stable and strongly stable semantics $T^- \cap F^- = \perp^- \neq \emptyset$ (nearly true).

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Graph models (Scott-continuous semantics)

- **Kernel of the definition.** A *graph model* is generated by a “web” (D, i) , where:
 - D is an infinite set,
 - $i : \mathcal{P}_f(D) \times D \rightarrow D$ is a *total* injection.

The underlying reflexive cpo is $(\mathcal{P}(D), \subseteq)$.

- **Furthermore:** there is a “*free completion*” process for generating a web (and hence a graph model) from a partial pair (A, j) . Ex: Engeler, P_ω, \dots

Löwenheim Skolem for graph models

For all graph models \mathcal{G} there exists \mathcal{G}' with countable web such that $Eq(\mathcal{G}) = Eq(\mathcal{G}')$ (and $Ord(\mathcal{G}) = Ord(\mathcal{G}')$).

\Rightarrow Wlog we can suppose D countable or even $D = \mathbb{N}$.

Effective graph models

- \mathcal{G} graph model with web (\mathbb{N}, i) .
- *Effective domain*: $\mathcal{D} = (\mathcal{P}(\mathbb{N}), \subseteq)$,
computable elements: r.e. subsets,
decidable elements: decidable subsets.
- $\mathcal{M} = ((P(D), \subseteq), A, \lambda)$,
 - $\lambda, A : [P(D) \rightarrow P(D)] \triangleleft P(D)$ are defined by:
 - $\lambda(f) \equiv \{i(d, \alpha) : d \in P_f(D) \text{ and } \alpha \in f(d)\}$,
 - $A(u)(v) \equiv \{\alpha \in D : (\exists d \subseteq_f v) i(d, \alpha) \in u\}$.

Effective graph models

Proposition

- (i) If i is computable and $dom(i)$ is decidable, then \mathcal{G} is weakly effective.
- (ii) If furthermore $range(i)$ is decidable, then \mathcal{G} is effective.

Examples: All free completions of finite partial pairs.

Similar for all classes of webbed models

K-models (Krivine) \supset Scott's and Park's models are effective.
F-ilter models (Coppo-Dezani-Honsell-Longo-Barendregt),
G-models (Girard's reflexive coherent spaces),
H-models (Ehrhard's reflexive hypercoherences),

Can a graph theory be r.e.?

Can a graph theory be r.e.?

Theorem 1

There is a minimal equational/order graph theory, which happens to be the theory of a (non unique) effective model \mathcal{G} .

Theorem 2

If \mathcal{G} is a graph model then $Ord(\mathcal{G})$ is not r.e.

Theorem 3

If \mathcal{G} is an effective graph model generated by a partial pair finite modulo its group of automorphisms, then $Eq(\mathcal{G})$ is not r.e.

Corollary

No graph model generated by a finite partial pair can have an r.e. theory.

Conclusions

- The weaker conjecture for order theories is:
 - 1 Proved for all graph models,
 - 2 Proved for all effective models.
- Concerning effective models the conjecture is:
 - 1 Proved for the *stable* and *strongly stable* semantics.
 - 2 Open for the continuous semantics,
 - 3 Even for the restricted case of graph models.
 - 4 But our results cover all the models individually introduced in the literature.
- Concerning graph models the conjecture is:

Proved for large families of effective graph models.

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