



HAL
open science

A quantum improvement to the gravitational collapse of radiating stars

F Fayos, R Torres

► **To cite this version:**

F Fayos, R Torres. A quantum improvement to the gravitational collapse of radiating stars. *Classical and Quantum Gravity*, 2011, 28 (10), pp.105004. <10.1088/0264-9381/28/10/105004>. <hal-00699369>

HAL Id: hal-00699369

<https://hal.science/hal-00699369v1>

Submitted on 21 May 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

A quantum improvement to the gravitational collapse of radiating stars.

F. Fayos[‡] and R. Torres

Department of Applied Physics, UPC, Barcelona, Spain.

Abstract. Based on previous works by Bonanno and Reuter [1][2] we postulate a renormalization group improved solution for the exterior of a collapsing object that emits radiation. We show that, contrary to Vaidya's radiating solution, backscattered radiation is bounded in the eikonal approximation, so that this solution provides us with a more consistent description of the last stages of collapse. We also show its limitations as the exterior of collapsing stars endowed with very high negative pressures. Finally, we are able to describe the profile of the luminosity of a collapsing object as seen by a far away observer in situations where Vaidya's solution can not deliver an admissible answer. We illustrate this with models for the collapse of scalar fields where such a profile had not been possible to obtain so far.

PACS numbers: 04.20.Cv, 04.40.-b, 04.90.+e, 98.10.+z

1. Introduction

The pioneering work on stellar models in the framework of General Relativity by Oppenheimer-Snyder [3] treated the collapse of a dust cloud surrounded by a void. Subsequent models tried to overcome the idealizations assumed in this work in order to get more realistic models. For example, a natural step forward was to replace Schwarzschild's void exterior by an exterior which took into account the outward flux of radiation coming out from the star. The first work in this regard was due to Vaidya [4], who found a solution modeling the outgoing incoherent radiation[§].

[§] See also [5] for more on the physical interpretation of this solution.

Following this line, many authors considered radiating stellar models using this *outgoing Vaidya solution* to model the radiating exterior together with a wide variety of interiors. See, for example, [6][7][8] and references therein.

It soon became clear that the use of radiating exteriors could give rise to the possibility that a model could radiate away its entire mass. Such behaviour was apparently first considered by Bondi [9] and was later studied in many works. In this way, it was found that the *evaporation* process could give rise, on the one hand, to models that avoid the formation of singularities (see, for example, [10][11][12]) and, on the contrary, to models which develop naked singularities at the final evaporating event (see, for example, [13][14][15]). These evaporating models had, however, a setback when Waugh and Lake [16] showed that, when approaching zero mass, the backscattered radiation in an outgoing Vaidya metric becomes blueshifted without bound. This means that the exterior of a collapsing and radiating star approaching its evaporation event can not be correctly modeled by the outgoing Vaidya solution.

On the other hand, it is expected that quantum effects will play a key role in the late stages of gravitational collapse. However, since the outgoing Vaidya solution is derived classically, it is obvious that these effects are not taken into account, what can be viewed as a second problem for Vaidya's solution in these extreme situations.

In this paper we consider an alternative to Vaidya's solution that could solve both problems up to a certain degree. The origin of this alternative can be traced back to a paper by Bonanno and Reuter [1] in which they introduced an effective quantum spacetime that took into account the quantum gravitational effects in spherically symmetric black holes. They did this by using the idea of the Wilsonian renormalization group [17] in order to study quantum effects in the Schwarzschild spacetime. Specifically, they obtained a *renormalization group improvement* of the Schwarzschild metric based upon a scale dependent Newton constant G obtained from the exact renormalization group equation for gravity [18] describing the scale dependence of the effective average action [19][20].

Later [2], in order to investigate the impact of the leading quantum gravity effects on the dynamics of the Hawking evaporation process of a black hole, the improved Schwarzschild solution was slightly modified. In this way, the obtained

solution now also included ingoing negative energy radiation so that the black hole could be evaporating, i.e., its total mass could decrease. We will call this solution the *improved ingoing Vaidya solution*.

The aim of this paper is to slightly modify the improved Schwarzschild solution, in the spirit of [2], in order to get a solution containing, however, *outgoing* positive energy radiation: The *improved outgoing Vaidya solution*. Provided such a solution has the correct properties, it could be used as a quantum replacement to the outgoing Vaidya's solution as the exterior of collapsing radiating models. In particular, if we are able to show that the backscattered radiation is bounded in this solution, the two problems stated above would be (at least, partially) solved.

The paper is divided as follows: in section 2 we deduce the improved outgoing Vaidya solution from the improved Schwarzschild solution and interpret its physical meaning in terms of contributions from the vacuum and from the radiation. This will allow us to ascertain the total luminosity as measured by an observer at rest at infinity. In this section we will also study other properties of the solution, such as the existence of singularities and the behaviour of its horizons. Section 3 is devoted to the study of the backscattered radiation in this solution. Some restrictions to its possible interiors are shown in section 4. Finally, in section 5 we use the new solution to get some stellar models that were not possible to obtain using the outgoing Vaidya solution for their exterior. In this way, for example, their total luminosity profile is obtained for the first time.

Throughout this article we will use *natural units* in which $c = \hbar = 1$.

2. Improved outgoing Vaidya solution

The *renormalization group improved* Schwarzschild solution found by Bonanno and Reuter [1] can be written as

$$ds^2 = - \left(1 - \frac{2G(R)M}{R} \right) dt^2 + \left(1 - \frac{2G(R)M}{R} \right)^{-1} dR^2 + R^2 d\Omega^2. \quad (1)$$

where

$$G(R) = \frac{G_0 R^3}{R^3 + \tilde{\omega} G_0 (R + \gamma G_0 M)}, \quad (2)$$

G_0 is Newton's universal gravitational constant, M is the mass measured by an observer at infinity and $\tilde{\omega}$ and γ are constants coming from the non-perturbative renormalization group theory and from an appropriate "cutoff identification", respectively. It is argued [1][2] that the qualitative properties of this solution are fairly insensitive to the precise value of γ . In this way, following [2], we can choose $\gamma = 0$ henceforth, what will allow us to perform many calculations analytically. On the other hand, $\tilde{\omega}$ can be found by comparison with the standard perturbative quantization of Einstein's gravity [21]. It can be deduced that its precise value is $\tilde{\omega} = 167/30\pi$, but again the properties of the solution do not rely on its precise value as long as it is strictly positive. (Note, for example, that the value $\tilde{\omega} = 0$ would turn off the quantum corrections).

At this point, Bonanno and Reuter [2] introduced an *advanced* time parameter replacing the time coordinate t in order to obtain their evaporating black hole solution. However, since we want an outward radiating solution, we will use a *retarded* time parameter u through the coordinate change defined by

$$du = dt - \left(1 - \frac{2G(R)M}{R}\right)^{-1} dR. \quad (3)$$

The next step is to allow the mass to change with this retarded time. This lead us to the *improved outgoing Vaidya solution* which can be written in radiative coordinates $\{u, R, \theta, \phi\}$ [22] as

$$ds^2 = - \left(1 - \frac{2G(R)M(u)}{R}\right) du^2 - 2dudR + R^2 d\Omega^2. \quad (4)$$

In order to interpret the physical meaning of this solution let us suppose that it has been generated by an effective matter fluid in such a way that the coupled gravity-matter system satisfies Einstein's equations $G_{\mu\nu} = 8\pi G_0 T_{\mu\nu}$. Consider now a radially moving observer with an arbitrary 4-velocity \mathbf{u} and an orthonormal basis $\{\mathbf{u}, \mathbf{n}, \boldsymbol{\omega}_\theta, \boldsymbol{\omega}_\varphi\}$ such that $\boldsymbol{\omega}_\theta \equiv R d\theta$, $\boldsymbol{\omega}_\varphi \equiv R \sin\theta d\varphi$ and \mathbf{n} is a space-like 1-form. Any radially moving observer can write the energy-momentum tensor as:

$$\mathbf{T} \stackrel{\Sigma}{=} \mathcal{R} + \mathbf{V}, \quad (5)$$

where \mathcal{R} is the radiative part and \mathbf{V} is the part of the energy-momentum tensor corresponding to the vacuum energy density and pressure:

$$\mathcal{R} \stackrel{\Sigma}{=} \Phi^2 \mathbf{1} \otimes \mathbf{1} \quad (6)$$

$$\mathbf{V} \stackrel{\Sigma}{=} \varrho_V \mathbf{u} \otimes \mathbf{u} + p_V \mathbf{n} \otimes \mathbf{n} + p_{\perp} (\boldsymbol{\omega}_{\theta} \otimes \boldsymbol{\omega}_{\theta} + \boldsymbol{\omega}_{\varphi} \otimes \boldsymbol{\omega}_{\varphi}), \quad (7)$$

here $\vec{l} = \partial/\partial R$ is a radial light-like 4-vector pointing in the direction of the future directed radiation, ϱ_V is the vacuum energy density, p_V is the vacuum normal pressure, p_{\perp} is the vacuum tangential pressure and Φ^2 is the flux of radiation travelling along the radial null geodesics $u = \text{constant}$. By using the field equations, one can obtain their explicit expressions:

$$\Phi^2 = -\frac{GM_{,u}}{4\pi G_0 R^2}, \quad (8)$$

$$\varrho_V = \frac{MG'}{4\pi G_0 R^2} = -p_V, \quad (9)$$

$$p_{\perp} = -\frac{MG''}{8\pi G_0 R}, \quad (10)$$

where G' and G'' are, respectively, the first and second derivatives of G with respect to R .

Note that if $M(u) = \text{constant}$ then only terms coming from the vacuum already found in the improved Schwarzschild solution appear [1]. On the other hand, if $M(u) \neq \text{constant}$ then a new contribution to the energy-momentum tensor in the form of a flux of outgoing radiation appears.

The radially moving observer would measure an energy density $q \equiv T_{\mu\nu} u^{\mu} u^{\nu}$ with contributions from the vacuum and from the radiation. However, (2) and (9) indicate that the contribution from the vacuum decays very fast with distance, so that for an observer at infinity there is only contribution from the radiation. In this way, if we calculate the total luminosity measured by an observer at rest at infinity

$$L_{\infty}(u) = \lim_{R \rightarrow \infty, dR/d\tau \rightarrow 0} 4\pi R^2 q = -M_{,u}(u), \quad (11)$$

where τ is the observer's proper time, the result is equivalent to Vaidya's case [23]. Therefore, in order to the observer at rest at infinity to observe a non-negative luminosity, the inequality

$$M_{,u} \leq 0 \quad (12)$$

or, equivalently, $\Phi^2 \geq 0$ should be satisfied||.

|| Hence the square we use in Φ^2 .

2.1. On the presence of $R = 0$ -singularities in the improved Vaidya solution

First, we would like to point out that, whether $R = 0$ belongs to the spacetime or not, it is easy to see [24] that it is a *timelike* curve or singularity, respectively. Now, as a consequence of its definition, a spacetime does not possess $R = 0$ scalar curvature singularities if the scalar invariants polynomial in the Riemann tensor remain finite at $R = 0$. A full independent set of invariants was found in [25]. By using this set one arrives to the following statement [26][27] for our case: All scalar invariants polynomial in the Riemann tensor will be finite, preventing the existence of scalar curvature singularities in the improved Vaidya solution with an unspecified γ if, and only if,

$$\lim_{R \rightarrow 0} \frac{G(R, u)}{R^3} < \infty. \quad (13)$$

If we test this on G given by (2) with $\gamma \neq 0$ we get

$$\lim_{R \rightarrow 0} \frac{G(R, u)}{R^3} = (\gamma \tilde{\omega} G_0 M(u))^{-1}. \quad (14)$$

Therefore, as long as $M(u) \neq 0$, the scalar curvature singularities seem to be absent. On the other hand, if we carry out the same calculation for $\gamma = 0$, there always seems to be a scalar curvature singularity since (13) is not satisfied and the divergence goes like $1/R$ (what, anyhow, is milder than in Vaidya's case where the divergence goes like $1/R^3$). One should be careful with the truthfulness of these results since the approximations used to deduce the improved Schwarzschild solution fail when R is too small [1]. The safest prediction could be that, in case there were a real $R = 0$ singularity, it would be much weaker than its classical counterpart.

2.2. The horizons

In order to find the apparent 3-horizons [24] of the solution we have to find the solution to $\chi = 0$, where

$$\chi \equiv g^{\mu\nu} \partial_\mu R \partial_\nu R = 1 - \frac{2GM}{R}. \quad (15)$$

If we define $M_{cr} \equiv \sqrt{\tilde{\omega}/G_0}$ then, in the case $\gamma = 0$ ¶, for any given $u = u_0$ there will be [2]:

¶ A similar structure is found for $\gamma \neq 0$ [1].

- Two zeros of $\chi(R, u = u_0) = 0$ ($R_+(u_0)$ and $R_-(u_0)$), if $M(u_0) > M_{cr}$.
- A double zero of $\chi(R, u = u_0) = 0$ ($R_+(u_0) = R_-(u_0)$), if $M(u_0) = M_{cr}$.
- No zero of $\chi(R, u = u_0) = 0$, if $M(u_0) < M_{cr}$,

where

$$R_{\pm}(u) = \frac{\tilde{\omega}}{M_{cr}} \left[\frac{M(u)}{M_{cr}} \pm \sqrt{\left(\frac{M(u)}{M_{cr}}\right)^2 - 1} \right]. \quad (16)$$

Clearly, the structure of the spacetime is similar to the Vaidya-Reissner-Nordström solution [28][26] which describes a spherically symmetric spacetime with radially directed radiation and a regular electromagnetic field, and where the role played by the charge Q is now played by M_{cr} . Therefore, it is easy to describe the improved Vaidya solution by using this analogy and the results in the references above. Here we would just like to point out a case that will be later used in which, from a certain $u = u_R$ (such that $M(u_R) > M_{cr}$), the mass is radiated until reaching $M = 0$. The corresponding Penrose diagram and its description can be found in figure 1.

3. Backscattered radiation in the eikonal approximation

While there is a flux of outgoing radiation travelling along the $u = \text{constant}$ null geodesics with tangent 4-vector \mathbf{l} , one expects that part of this radiation will be backscattered due to the curvature of the spacetime [30][31][32] and will travel inwards along the affinely parametrized radial null geodesics with tangent null 4-vector \mathbf{k} . In order to evaluate the relevance of the backscattered radiation in this spacetime let us consider a given point where the ingoing radiation is emitted and another point at its future where this radiation will be observed. The associated frequency shift will be

$$\frac{\nu_e}{\nu_o} = \frac{\varepsilon_e}{\varepsilon_o}, \quad (17)$$

where ν_e (ν_o) is the emitted (observed) frequency and $\varepsilon_{e(o)} \equiv v_{e(o)\alpha} k^\alpha$, being $\mathbf{v}_{e(o)}$ the timelike 4-velocity of the emitter (or the observer). Clearly, if $\nu_o \gg \nu_e$, as in the outgoing Vaidya's case [16], the backscattering radiation can not be neglected.

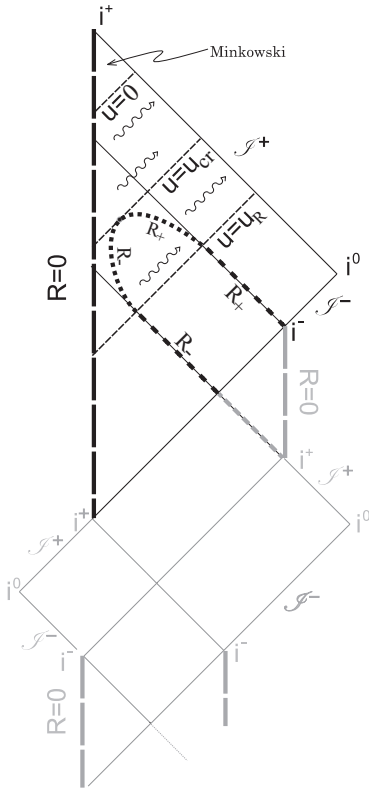


Figure 1. Here we show the Penrose diagram corresponding to an improved Vaidya solution radiating all its mass. A single chart in radiative coordinates covers only the upper region drawn in black, while the lower region drawn in grey corresponds to a possible extension to the past. (The reader could consult on such extensions [29][26]). In this example there are three important values for the retarded time: u_R that corresponds to the beginning of radiation emission (depicted here by wavy arrows), u_{cr} that marks when the mass reaches its critical value M_{cr} (and, therefore, where the outer apparent 3-horizon R_+ converges with the inner apparent 3-horizon R_-) and $u = 0$ that has been chosen as the time when all the mass is radiated away. To the future of the lightlike hypersurface $u = 0$ all is left is Minkowski's spacetime.

In order to evaluate (17) we will use that the radial null geodesics of the backscattered test field satisfy

$$\frac{dR}{du} = -\frac{1}{2} \left(1 - \frac{2G(R)M}{R} \right) \quad (18)$$

and that the geodesic equations with affine parameter λ are

$$\frac{d^2u}{d\lambda^2} + \frac{RMG' - GM}{R^2} \left(\frac{du}{d\lambda} \right)^2 = 0, \quad (19)$$

$$\frac{d^2 R}{d\lambda^2} + \frac{GM_{,u}}{R} \left(\frac{du}{d\lambda} \right)^2 = 0. \quad (20)$$

By using (18), one gets

$$\varepsilon = -\frac{1}{2} \frac{du/d\lambda}{du/d\tau}, \quad (21)$$

where τ is the proper time of the emitter (or the observer). The results of Waugh and Lake [16] showed that, in the case of the outgoing Vaidya solution, $du/d\lambda$ grows without bound as the geodesic approaches $R = 0$. Then also the frequency shift (17) must do so and, therefore, one cannot neglect the ingoing radiation around $R = 0$. We will now show that this is not the case for the improved outgoing Vaidya solution in which $du/d\lambda$ remains finite as $R = 0$ is approached providing a perfectly bound backscattered radiation.

Equation (19) can be integrated into the form

$$\frac{du}{d\lambda} = A \exp \left(\int \frac{GM - RMG'}{R^2} du \right), \quad (22)$$

where A is an integration constant. If we now use (18) and the explicit expression for $G(R)$ with $\gamma = 0$ in (2) we can rewrite this as

$$\frac{du}{d\lambda} = \tilde{A} \left(\frac{R^2 + G_0 \tilde{\omega}}{R} \right) \exp \left[\frac{1}{2} \int \left(\frac{1}{R} - \frac{G'}{G} \right) du \right], \quad (23)$$

where \tilde{A} is a constant. Every ingoing radial null geodesic approaching $R = 0$ must do so, according to (18), satisfying

$$\lim_{R \rightarrow 0} \frac{dR}{du} = -\frac{1}{2}. \quad (24)$$

So that if we arbitrarily choose that the geodesic reaches $R = 0$ at $u = 0$ then we can write

$$R = -\frac{1}{2}u + O(u^2). \quad (25)$$

If we now insert this into (23) we finally show that $du/d\lambda$ remains finite when approaching $u = R = 0$ since

$$\frac{du}{d\lambda} \simeq 2\tilde{A}G_0\tilde{\omega}. \quad (26)$$

4. A constraint on the interiors of the improved outgoing Vaidya solution

Consider that we want to model a collapsing object using the standard procedure of matching a spherically symmetric interior spacetime $\bar{\mathcal{V}}$ to a radiating exterior spacetime \mathcal{V} , that we choose to be the improved outgoing Vaidya solution. Not all possible interior solutions will be able to be matched, by the accurate use of the Darmois [33] junction conditions, through a time-like hypersurface Σ to the improved outgoing Vaidya solution and, even if they match, not all possible matched models will be meaningful. In this section we will show a restriction for the possible interior solutions in the usual case in which the interior has a *Type I* [34] energy-momentum tensor. With this purpose, let us consider the orthonormal basis in which the energy-momentum tensor diagonalizes $\{\tilde{\mathbf{u}}_d, \tilde{\mathbf{n}}_d, \boldsymbol{\omega}_\theta, \boldsymbol{\omega}_\varphi\}$. In this basis the energy-momentum tensor can be written as

$$\mathbf{T} = \rho_d \tilde{\mathbf{u}}_d \otimes \tilde{\mathbf{u}}_d + p_d \tilde{\mathbf{n}}_d \otimes \tilde{\mathbf{n}}_d + p_T (\boldsymbol{\omega}_\theta \otimes \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\varphi \otimes \boldsymbol{\omega}_\varphi). \quad (27)$$

Let us remark that this interior will be formed in general by contributions from a collection of different matter fields. Thus, an observer with 4-velocity $\tilde{\mathbf{u}}_d$ could measure an energy density ρ_d made up with contributions from, for example, a scalar field, an electromagnetic field, an anisotropic fluid non-comoving with $\tilde{\mathbf{u}}_d$ and so on.

Provided that we have obtained a matched model intended to represent an isolated spherically symmetric astrophysical body, the aforementioned constraint on the interior can be written as

Proposition 4.1 *The luminosity measured by an observer at rest at infinity will be non-negative if, and only if,*

$$p_d \stackrel{\Sigma}{\geq} p_V. \quad (28)$$

This result is easy to show if one considers a spherically symmetric matching hypersurface Σ provided with a tangent time-like 4-vector \mathbf{u} and a normal 4-vector \mathbf{n} that we can always relate to the previous basis through

$$\tilde{\mathbf{u}}_d = \cosh \xi \mathbf{u} + \sinh \xi \mathbf{n} \quad (29)$$

$$\tilde{\mathbf{n}}_d = \varsigma (\sinh \xi \mathbf{u} + \cosh \xi \mathbf{n}), \quad (30)$$

where $\zeta^2 = 1$. On the other hand, Israel's junction conditions [35], which are a consequence of Darmois' junction conditions, imply

$$[T_{\mu\nu}n^\mu n^\nu] \stackrel{\Sigma}{=} 0 \quad (31)$$

$$\Rightarrow \rho_d \sinh^2 \xi + p_d \cosh^2 \xi \stackrel{\Sigma}{=} N^2 \Phi^2 + p_V \quad (32)$$

and

$$[T_{\mu\nu}n^\mu a^\nu] \stackrel{\Sigma}{=} 0 \quad (33)$$

$$\Rightarrow (\rho_d + p_d) \sinh \xi \cosh \xi \stackrel{\Sigma}{=} N^2 \Phi^2, \quad (34)$$

where $[...] \stackrel{\Sigma}{=} 0$ means that the corresponding quantity, when measured from every side of Σ , must coincide. In addition, we have used for the exterior the energy-momentum tensor of the improved Vaidya solution (5,6,7) and we have defined the normalizing factor N such that $\mathbf{l} = \partial/\partial R = N(\mathbf{u} + \mathbf{n})$.

Finally, by combining these two equalities we arrive at

$$N^2 \Phi^2 \stackrel{\Sigma}{=} (p_d - p_V) e^\xi \cosh \xi. \quad (35)$$

Consequently, $p_d \stackrel{\Sigma}{\geq} p_V \Leftrightarrow \Phi^2 \stackrel{\Sigma}{\geq} 0$ what, taking into account (8) and (11), ends the proof.

5. Particular models of radiating collapsing stars

We will exemplify our results by choosing as our interior solution the well-known spatially homogeneous and anisotropic flat Robertson-Walker (RW) solution. We can write the line-element in its isotropic form as

$$ds_{RW}^2 = -dt^2 + a^2(t) [dr^2 + r^2 (d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}^2)]. \quad (36)$$

In order to have a complete stellar model we need to match the interior solution with the radiating exterior solution through the evolving star's surface. The matching of two spherically symmetric space-times through a time-like matching hypersurface Σ using Darmois' junction conditions can be found, for example, in [10]. First, we need to identify an embedding of Σ in both the improved-Vaidya and the RW space-times. To that end let us consider a general timelike hypersurface σ preserving the spherical symmetry of the exterior spacetime and with

intrinsic coordinates $\{\xi^a\} = \{\tau, \vartheta, \varphi\}$, where τ is a future-directed time coordinate defined only on the hypersurface. The general parametric equations of σ are: $u = u(\tau)$, $R = R(\tau)$, $\theta = \vartheta$, $\phi = \varphi$. The unit normal vector of σ is

$$\vec{n} = \frac{1}{\sqrt{\dot{u}(\chi\dot{u} + 2\dot{R})}} \left[-\dot{u} \frac{\partial}{\partial u} + (\chi\dot{u} + \dot{R}) \frac{\partial}{\partial R} \right]. \quad (37)$$

Similarly, let us take a general timelike hypersurface $\bar{\sigma}$ preserving the spherical symmetry of (36). This hypersurface is assumed to be diffeomorphic to σ , and thus the intrinsic coordinates are chosen to be the same $\{\xi^a\} = \{\tau, \vartheta, \varphi\}$. The general parametric equations for $\bar{\sigma}$ are: $t = t(\tau)$, $r = r(\tau)$, $\bar{\theta} = \vartheta$, $\bar{\phi} = \varphi$. This hypersurface is timelike whenever

$$\frac{\dot{r}^2}{\dot{t}^2} = \left(\frac{dr}{dt} \right)^2 < \frac{1}{a^2}. \quad (38)$$

The corresponding unit vector normal to $\bar{\sigma}$, whenever it is time-like, is chosen in agreement with (37) [10] to be

$$\vec{n} = \frac{1}{a \sqrt{\dot{t}^2 - a^2 \dot{r}^2}} \left(a^2 \dot{r} \frac{\partial}{\partial t} + \dot{t} \frac{\partial}{\partial r} \right).$$

At this stage, we impose the Darmois gravitational junction conditions [33][10], which are the best suited for our purposes, requiring that the first and second fundamental forms of Σ be identical when computed from either the interior or the exterior manifold. This provide us with a full set of matching conditions which in the case under consideration becomes after a little computation

$$R \stackrel{\Sigma}{=} ar, \quad (39)$$

$$MG \stackrel{\Sigma}{=} \frac{ar^3 a_{,t}^2}{2}, \quad (40)$$

$$\dot{u} \stackrel{\Sigma}{=} \frac{\dot{t} - ar}{1 + ra_{,t}}, \quad (41)$$

$$\dot{r} \stackrel{\Sigma}{=} - \frac{3\eta G + arG'}{a(3G - arG')} \dot{t}, \quad (42)$$

where $\stackrel{\Sigma}{=}$ means that both sides of the equality must be evaluated on Σ and we have defined:

$$\eta \equiv - \frac{2aa_{,tt} + a_{,t}^2}{3a_{,t}^2}. \quad (43)$$

It is interesting to note that $\eta(t)$ can be physically interpreted as the ratio between the total pressure and the total energy density as measured by an observer at $R = 0$ over the course of time⁺.

Assuming that we know the interior, i.e. $a(t)$, the general procedure to solve the matching conditions (39)-(42) is the following: (i) Equation (42), with the help of (39), can be considered a simple ordinary differential equation for the unknown $r(r)$, which will have solution if the right hand of (42) satisfies Lipschitz's conditions. The solution will depend on an arbitrary constant r_0 , and thus we obtain a one-parameter family of possible solutions for Σ . (ii) By substituting the solution $r(t, r_0)$ into Eq. (41), we obtain another differential equation for the unknown $u(t)$. Solving this equation, we obtain $u(t, r_0, u_0)$, where u_0 is a new arbitrary constant. (iii) From Eq. (39) we immediately find $R(t, r_0)$, which together with $u(t, r_0, u_0)$ provides the form of the hypersurface Σ as seen from the improved outgoing Vaidya spacetime. (iv) Finally, we get $M(t, r_0)$ from equation (40) and, combining this with $u(t, r_0, u_0)$, we find the mass $M(u, r_0, u_0)$ for the improved outgoing Vaidya solution explicitly.

It is worth noticing that, once a model satisfying the matching conditions (39)-(42) is obtained, by construction it will have for its exterior an improved outgoing Vaidya solution (4) which, at the same time, comes from a defined infrared cutoff identification [1]. Therefore, the cutoff originally chosen for a specific solution remains unaffected by the matching of this solution to another interior solution.

5.1. Measured luminosity

The luminosity measured by an observer at rest at infinity at the latest stages of the collapse is of the utmost importance since it gives away the signature for an evaporating collapse as seen from outside the collapsing object. Given an interior, i.e. $a(t)$, we can obtain L_∞ as a function of t through the matching conditions (39-42) and, after some algebraic manipulation, we can write it in the useful form

$$L_\infty = -M_{,u} \stackrel{\Sigma}{=} \frac{r^2(3G - RG')(3G\eta + RG')(1 + ra_{,t})^2 a_{,t}^2}{6G^3(1 + \eta)}, \quad (44)$$

where, according to (39), $R \stackrel{\Sigma}{=} ar$ and $G \stackrel{\Sigma}{=} G(R(t))$.

⁺ And, in fact, by any observer moving with 4-velocity $\partial/\partial t$.

5.2. Restrictions on the homogeneous-isotropic interiors

The temporality condition (38) on the matching hypersurface could be written now with the help of (39) and (42) as

$$-1 < \eta \stackrel{\Sigma}{<} \zeta(R) \equiv \frac{3G - 2RG'}{3G} \quad (45)$$

and using the explicit expression for G (2) with $\gamma = 0$

$$\zeta(R) = \frac{3R^2 - G_0\tilde{\omega}}{3(R^2 + G_0\tilde{\omega})}, \quad (46)$$

what imposes, for every value of the areal radius of the matching hypersurface at a precise inner time t , lower and upper bounds on η .

On the other hand, the application of the condition on the non-negative luminosity (28) to this case can be written using (39) and (40) as

$$\eta \stackrel{\Sigma}{\geq} \xi(R) \equiv -\frac{RG'}{3G}, \quad (47)$$

where ξ can be rewritten using (2) with $\gamma = 0$ as

$$\xi(R) = -\frac{2G_0\tilde{\omega}}{3(G_0\tilde{\omega} + R^2)}. \quad (48)$$

In this way, for every value of the (areal) radius of the collapsing object, this imposes the real minimum value for η . In particular, the absolute minimum of $\xi(R)$ is at $R = 0$ and its value is $-2/3$.

Another straightforward consequence of (47) when one takes into account (2) with $\gamma = 0$ is that the evolution equation (42) should satisfy

$$\frac{dr}{dt} \stackrel{\Sigma}{\leq} 0, \quad (49)$$

where the equal only applies in case the object does not radiate.

Restrictions (45) and (47) together imply

$$\xi \stackrel{\Sigma}{\leq} \eta \stackrel{\Sigma}{<} \zeta, \quad (50)$$

what can be conveniently summarized in a graphic, as shown in figure 2. The importance of the inequalities in (50) is that they provide us with some clues about the behaviour of the interior (η) as the star evolves. In this way, it can be seen that, if the model completely collapses, then the value of η at this moment should satisfy $-2/3 < \eta|_{R_{\Sigma}=0} < -1/3$. However, for an areal radius much greater than the

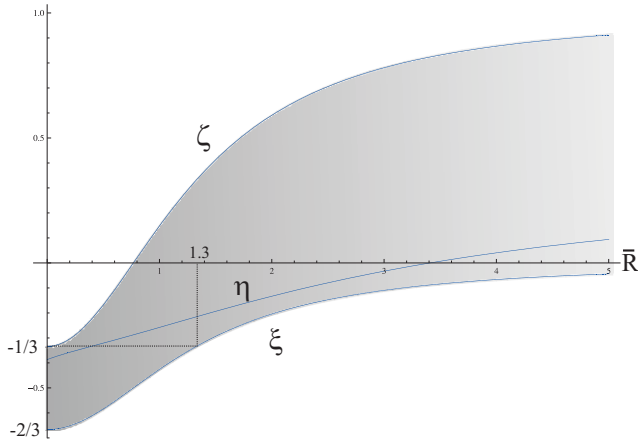


Figure 2. Given a homogeneous and isotropic interior matched to an improved outgoing Vaidya solution, the matching conditions provide us with the parametric equations $\{\bar{R}(t), \eta(t)\}$, where $\bar{R} \equiv R_{\Sigma}/\sqrt{G_0}$. These equations define a curve which is limited by the temporality condition (ζ) and by the non-negative-luminosity condition (ξ) and, therefore, should be inside the shadowed region of the graphic. We have depicted a possible curve (η) satisfying the conditions. On the other hand, in the *power-law* case (54) the curve for η is just a horizontal line (60) what, as shown, limits its applicability to a maximum $\bar{R} \simeq 1.3$, corresponding to $n \rightarrow 1$.

Planck length, η should be *approximately* between 0 and 1 (being this range exact only when $R_{\Sigma} \rightarrow \infty$).

A final restriction that should be satisfied by a model is that, if we assume that τ and u should both grow towards de future, then the right hand of (41) should be positive. Combining this with the timelike condition for Σ (38) one gets the additional restriction

$$r_{\Sigma}(t)a_{,t} > -1. \quad (51)$$

5.3. Interiors dominated by a scalar field

Now we would like to construct a toy model for a collapsing star assuming an interior dominated by a scalar field, in addition to the simplifying assumption of its homogeneity and isotropy. This kind of models have been profusely used in the literature (see, for example, [36][37][38]). One expects that such an interior will also possess an inner mechanism responsible for the emission of radiation to the exterior,

in addition to an energy-momentum tensor with, at least, a term \mathcal{R} describing the inner radiation:

$$T_{\mu\nu} = \partial_\mu\phi \partial_\nu\phi - g_{\mu\nu} \left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi \partial_\beta\phi + V(\phi) \right] + \mathcal{R}_{\mu\nu} + \dots \quad (52)$$

or specifically for the homogeneous and isotropic case

$$T_{\mu\nu} = \dot{\phi}^2 \delta_\mu^t \delta_\nu^t + g_{\mu\nu} \left[\frac{1}{2}\dot{\phi}^2 - V(\phi) \right] + \mathcal{R}_{\mu\nu} + \dots \quad (53)$$

However, as the collapse proceeds the scalar field contribution dominates over the contribution from the radiation \mathcal{R} and the other terms. In this way, from now on we will assume, as usual [37][38][39], that their contributions can be neglected.

Let us consider in the first place the most usual scale factor studied in the literature: the *power-law case*, i.e.,

$$a(t) = A|t|^n, \quad (54)$$

where A and n are positive constants and the absolute value is taken since we choose the collapse to proceed in the range $-\infty < t \leq t_{final} (< 0)$.

The potential and the field are easily obtained since the field equations provide us with [39]

$$V(\phi(t)) = \frac{2a_{,t}^2 + a_{,tt} a}{8\pi G_0 a^2} \quad (55)$$

$$\phi_{,t}^2 = \frac{a_{,t}^2 - a_{,tt} a}{4\pi G_0 a^2} \quad (56)$$

what implies that in this power-law case

$$V = \frac{n(3n-1)}{8\pi G_0 t^2} \quad (57)$$

$$\phi = \phi_0 + \sigma \sqrt{\frac{n}{4\pi G_0}} \ln |t| \quad (58)$$

where $\sigma^2 = 1$. So that

$$V(\phi) = \frac{n(3n-1)}{8\pi G_0} \exp \left(-2\sigma \sqrt{\frac{4\pi G_0}{n}} (\phi - \phi_0) \right). \quad (59)$$

On the other hand, using (43) it is easy to check that in this case

$$\eta = \frac{2-3n}{3n}. \quad (60)$$

In this way, the condition on the non-negative luminosity (47) together with the temporality condition (45) can be satisfied around the collapsing event only if

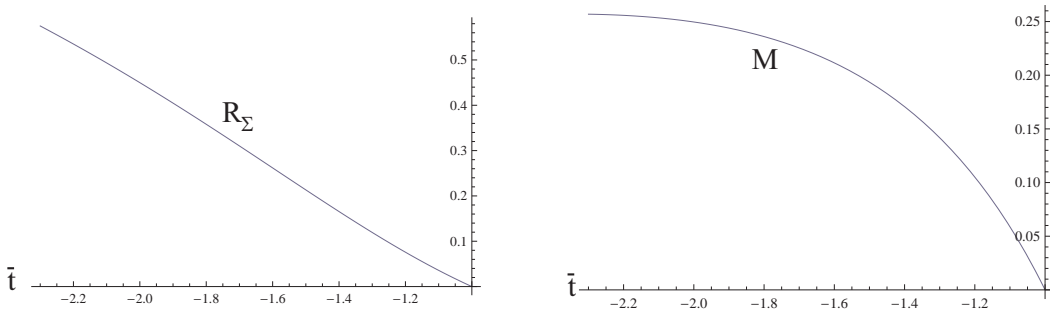


Figure 3. To the left we show the evolution of the areal radius of the star surface around the total collapse event as a function of time in Planck’s time units. The interior follows a power-law with $n = 3/2$, however the behaviour does not depend of this precise value, as long as it fulfills the required condition $1 < n \leq 2$. We have also chosen as the initial condition for the evolution equation $r(-1) = 0$. In other words, we choose that the energy density measured by an observer comoving with the center of the star to be extremely large at the collapse event: approximately one third of Planck’s density or $1.4 \cdot 10^{96} \text{kg/m}^3$. The evolution of the star’s mass during the same period is shown to the right. (Due to the mentioned limitations of the power-law behaviour it is meaningless to consider the evolution further to the past).

$1 < n \leq 2$. As a consequence of η being constant there will be a maximum areal radius for the star (see figure 2) that turns out to be too small to model any astrophysical object. Anyhow, we can consider the power-law behaviour as an approximation valid only around the collapsing event and obtain the evolution of the star’s surface around $R = 0$ from the *evolution equation* (42) and with the help of (39). This has to be done numerically and the result, which is qualitatively similar for all the allowed values for n , is shown in figure 3. We also show in the same figure the behaviour of the mass, which is radiated away until reaching the total evaporation. Despite extremely high energy densities are reached at the total evaporation event, no singularity appears.

As explained, the power-law case cannot model the complete evolution of an object of stellar size. In order to get a complete model we will use the following *ansatz*:

$$a(t) = (\alpha|t|^{-n} + \beta|t|^{-m})^{-1}, \quad (61)$$

where α , β , n and m are constants satisfying $1 < n \leq 2$ and $1/3 \leq m \leq 2/3$. The idea behind this *ansatz* is to take profit of the power-law behaviour for $|t| \sim t_P$ ($a \sim |t|^n$), that we have just analyzed, and also for $|t| \gg t_P$ ($a \sim |t|^m$) when we want the object to be of astrophysical size, provided that m is chosen to satisfy the restrictions on η for large R s (see figure 2).

Again, one can obtain the evolution of the star's surface from the *evolution equation* (42), and with the help of (39). The result, which is qualitatively similar for all the allowed values of n and m , is shown in figure 4. For example, the fact that the object should collapse is not surprising since this model satisfies $a_{,t} < 0$ what combined with (39) and (49) implies that $dR_\Sigma/dt < 0$ for $r_\Sigma > 0$. Since $a(0) = 0$, this implies that the evolving surface must reach $R_\Sigma = 0$ in the range $-\infty < t \leq 0^*$. We also show the big increase in the energy density measured from the center of the star in the last stages of the collapse. As expected, with the adequate values for α and β , there are no limitations to the past for the evolution's time. In this respect, let us remark that the values used for the figure provide exactly the allowed curve η that has been depicted in figure 2.

As explained when discussing on the restrictions for the interiors, in order to get a satisfactory model it is still necessary to check condition (51). We explicitly show the fulfillment of this condition for our specific model in figure 5. It is interesting to note that, as a consequence, none of the model's 2-spheres will be a closed trapped surface. This is easy to see since, the interior 2-spheres will be closed trapped surfaces if, and only if, $\bar{\chi} < 0$ [10], where

$$\bar{\chi} \equiv g^{\mu\nu} \partial_\mu R \partial_\nu R = (1 - a_{,t} r)(1 + a_{,t} r). \quad (62)$$

Now, since our models satisfy $a_{,t} < 0$ and (51), it is obvious that $\bar{\chi}_\Sigma > 0$. On the other hand, (62) tell us that the interior ($r \leq r_\Sigma$) satisfies $0 < \bar{\chi}_\Sigma(t_m) \leq \bar{\chi}(r, t_m) \leq 1$ for every t_m in the allowed range, i.e., the interior lacks of closed trapped surfaces. Taking into account that the matching conditions imply $\bar{\chi}(t_m) \stackrel{\Sigma}{=} \chi(u_m)$, where u_m is the advanced time parameter corresponding to t_m through the matching, then (2) and (15) tell us that the exterior satisfies $0 < \chi_\Sigma(u_m) \leq \chi(u_m, R) \leq 1$ for every u_m .

* Clearly, the same argument can be applied to the *power-law* case.

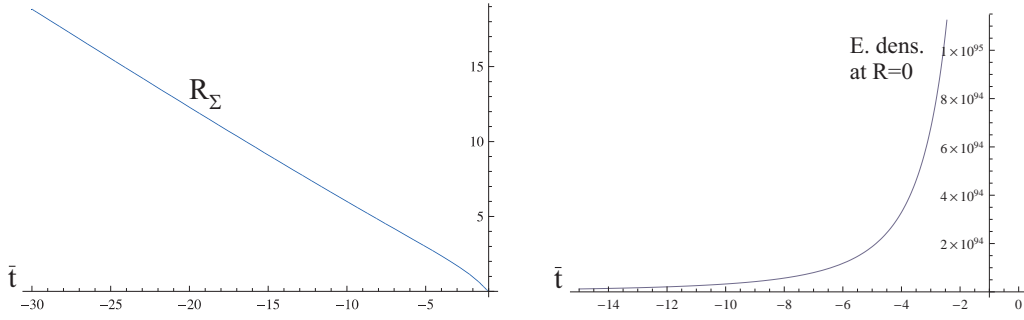


Figure 4. To the left we show the evolution of the areal radius of the star surface. Now the interior follows the *ansatz* (61) with $n = 3/2$ and $m = 1/2$, however the behaviour does not depend of these precise values, as long as they fulfill the required conditions. Our *ansatz* allows us to treat the collapse during any chosen period and from any initial radius. We have chosen the same initial condition that was chosen in the power-law case. This provides us with the same extremely high energy densities in the total evaporation event (see the caption of figure 3) which are reached in a short period during the last stages of collapse (graphic to the right).

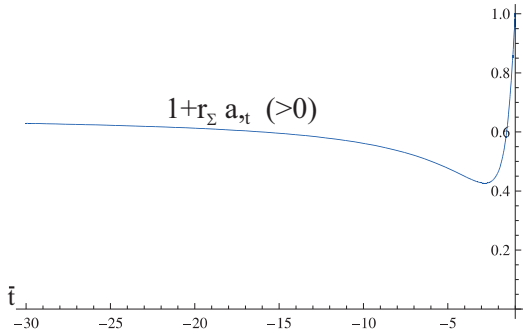


Figure 5. The graphic shows the fulfillment of condition (51) during the same period of time used in figure 4.

The absence of closed trapped surfaces both in the interior and the exterior shows that none of the model's 2-spheres will be a closed trapped surface.

Finally and of utmost importance, the luminosity can be also obtained numerically from (44) once the evolution equation has been integrated. The result for this case is shown in figure 6.

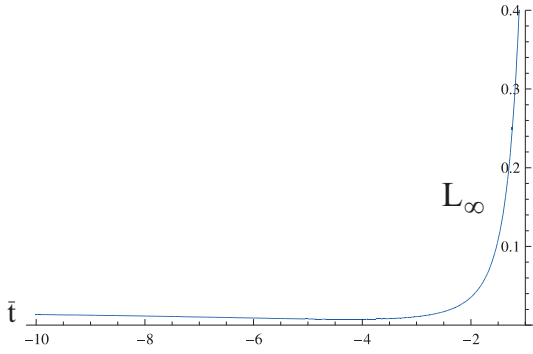


Figure 6. We show the evolution of the luminosity as measured by an observer at rest at infinity. The luminosity does not vary much during the lifetime of the star. Only in the last stages of the collapse the luminosity increases dramatically. This sudden final explosion signals the total evaporation of the star.

6. Conclusions

In this paper we have postulated the *improved outgoing Vaidya solution* as a replacement to Vaidya's solution in models in which the areal radius reaches small enough values. In this way, not only are quantum effects on the vacuum considered, but, as we have shown, also the problem of the unbound backscattered radiation, present in the original outgoing Vaidya's solution, disappears.

As can be deduced from proposition 4.1, a remarkable property of this solution is that it admits interiors possessing negative pressures. This is interesting since Vaidya's solution does not admit them [40][41]. Moreover, for the case of homogeneous and isotropic interiors we have shown that negative pressures are not only admitted, but required as the model approaches the final stages of the evolution (see figure 2 and compare with Vaidya's case where $0 \leq \eta < 1$ [10][40]). This is interesting since it has allowed us to model collapsing self-gravitating scalar fields which evolve from positive to negative pressures (what, incidentally, agrees with the behaviour expected in Loop Quantum Gravity interior models [42]). A Penrose diagram of the singularity-free model is shown in figure 7. Finally, we have been able to provide the profile of the luminosity of the collapsing scalar field as measured by an observer at rest at infinity, what had been impossible to obtain so far.

The reader should be aware that we have not considered the leading *quantum*

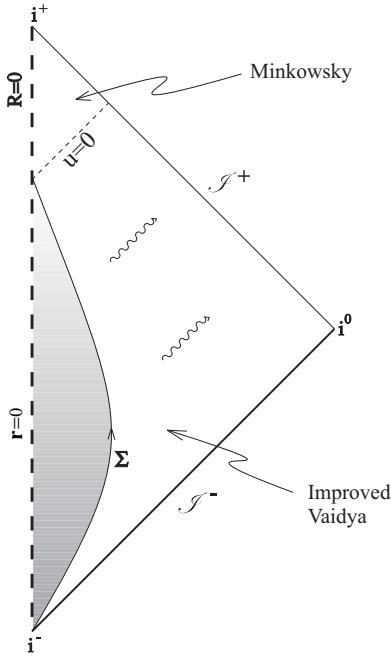


Figure 7. The complete model with a homogeneous and isotropic interior. Note the absence of singularities.

gravity corrections that would modify the interior while keeping it homogeneous and isotropic. Some work at this respect has already been done in a cosmological context (see, for example, [43] and references therein) in which the homogeneity and isotropy assumptions imply that the infrared cutoff can only be a function of the *cosmic time* t . In case a similar approach worked for a homogeneous and isotropic *stellar interior*, one could expect an inner running Newton's constant $G(t)$ and a much more complicated energy-momentum tensor which should now contain information not only about a possible scalar field but also, for example, about the vacuum polarization. In this way, the same scale factor could produce different physical interpretations depending on whether it is used in a RG-improved homogeneous and isotropic solution or not. However, if the matching with the improved outgoing Vaidya solution were carried out, we would not expect modifications in our main results for the matched models \ddagger since these results depend only on the homogeneity and isotropy of the interior and its specific scale factor. Despite this, we consider the

\ddagger Namely, the evolution of the model's surface, the behaviour of its luminosity or the restrictions on the scale factor.

search for renormalization-group improved interior solutions an interesting subject for future research.

With regard to the limitations of the improved Vaidya solution and, therefore, to the models presented in this paper, we expect this effective solution to be reliable only for values of the areal radius much greater than Planck's length. However, for R of the order of the Planck length the renormalization effects become strong and the quantum corrected geometry differs too significantly from the classical one. Probably only a full Quantum Gravity Theory could provide us with the accurate behaviour in the very final stages of collapse.

References

- [1] Bonanno A and Reuter M 2000 *Phys. Rev. D* **62** 043008
- [2] Bonanno A and Reuter M 2006 *Phys. Rev. D* **73** 083005
- [3] Oppenheimer J R and Snyder H 1939 *Phys. Rev.* **56** 455
- [4] Vaidya P C 1951 *Proc. Indian Acad. Sci.* **A33** 264
- [5] Griffiths J B 1974 *Gen. Rel. Grav.* **5** 458
- [6] Herrera L and Núñez L 1990 *Fundamental Cosmic Physics* **14** 235
- [7] Bonnor W B, Oliveira A K G and Santos N O 1989 *Phys. Rep.* **181** 269
- [8] Fayos F, Jaen X, LLanta E and Senovilla J M M 1992 *Phys. Rev. D* **45** 2732
- [9] Bondi H 1964 *Proc. R. Soc. London* **A281** 39
- [10] Fayos F, Senovilla J M M and Torres R 1996 *Phys. Rev. D* **54** 4862
- [11] Fayos F and Torres R 2008 *Class. Quantum Grav.* **25** 175009
- [12] Joshi P S and Goswami R 2007 *Class. Quantum Grav.* **24** 2917
- [13] Demianski L and Lasota J P 1968 *Astrop. Lett.* **1** 205
- [14] Lake K and Hellaby C 1981 *Phys. Rev. D* **24** 3019
- [15] Unruh W G 1985 *Phys. Rev. D* **31** 2693
- Lake K and Hellaby C 1985 *Phys. Rev. D* **31** 2695
- [16] Waugh B and Lake K 1986 *Phys. Lett. A* **116** 154
- [17] Wilson K and Kogut J 1974 *Phys. Rep.* **12** 75
- [18] Reuter M 1998 *Phys. Rev. D* **57** 971
- [19] Wetterich C 1993 *Phys. Lett. B* **301** 90
- [20] Reuter M and Wetterich C 1994 *Nucl. Phys. B* **427** 291
- [21] Bjerrum-Bohr N E J, Donoghue J F and Holstein B R 2003 *Phys. Rev. D* **68** 084005
- [22] Bondi H, van der Burg M G J and Metzner A W K 1962 *Proc. Roy. Soc. A* **269** 21
- [23] Lindquist R W, Schwartz R A and Misner C W 1965 *Phys. Rev. B* **137** 1364

- [24] Fayos F and Torres R 2010 *Class. Quantum Grav.* **27** 125011
- [25] Carminati J and McLenaghan R G 1991 *J. Math. Phys.* **32** 3135
- [26] Torres R 2001 *Construcción de modelos de estrellas y burbujas de vacío o radiativas, en el marco de la Relatividad General: Un planteamiento global*. Ph. D. Thesis, Barcelona: Universitat Politècnica de Catalunya
- [27] Fayos F, Senovilla J M M and Torres R 2003 *Class. Quantum Grav.* **20** 2579
- [28] Krori K D and Banerjee A J 1974 *J. Phys. A* **17** 2125
- [29] Fayos F, Martín-Prats M M and Senovilla J M M 1995 *Class. Quantum Grav.* **12** 2565
- [30] Misner C, Thorne K and Wheeler J A 1973 *Gravitation* W. H. Freeman and Company. New York.
- [31] Bardeen J M and Press W H 1973 *J. Math. Phys.* **14** 7
- [32] Karkowski J, Malec E and Swierczynski Z 2002 *Class. Quantum Grav.* **19** 953
- [33] Darmon G 1927 *Mémoires des Sciences Mathématiques*, Paris: Gautier-Villars, Fasc. **25**
- [34] Hawking S W and Ellis G F R 1973 *The large scale structure of space-time*. Cambridge: Cambridge University Press
- [35] Israel W 1966 *Nuovo Cimento* **44** 1
- [36] Choptuik M 1993 *Phys. Rev. Lett.* **70** 9
- [37] Christodoulou D 1994 *Ann. Math.* **140** 607
- [38] Giambò R 2005 *Class. Quantum Grav.* **22** 2295
- [39] Ellis G F R and Madsen M S 1991 *Class. Quantum Grav.* **8** 667
- [40] Fayos F & Torres R 2004 *Class. Quantum Grav.* **21** 1351
- [41] Torres R 2005 *Class. Quantum Grav.* **22** 4335
- [42] Goswami R, Joshi P S and Singh P 2006 *Phys. Rev. Lett.* **96** 031302
- [43] Bonanno A and Reuter N 2002 *Phys. Rev. D* **65** 043508