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# PRIME POISSON SUSPENSIONS

FRANÇOIS PARREAU AND EMMANUEL ROY

ABSTRACT. We establish a necessary and sufficient condition for a Poisson suspension to be prime. The proof is based on the Fock space structure of the  $L^2$ -space of the Poisson suspension. We give examples of explicit infinite measure preserving systems that give way to prime Poisson suspensions, in particular non-singular compact group rotations. We also compare those prime Poisson suspensions to the existing prime transformations known so far, showing our examples are new.

## 1. INTRODUCTION

A measure-preserving transformation  $(X, \mathcal{A}, \mu, T)$  is said to be *prime* if  $\mathcal{A}$  and  $\{X, \emptyset\}$  are the only factors of the system. The first examples were Ornstein's mixing rank one constructions [10], proved to be prime by Polit in [12]. Indeed those systems are part of the larger class of simple systems which possess their own theory: they are those systems  $(X, \mathcal{A}, \mu, T)$  whose ergodic selfjoinings are either the product joining or graph joinings  $\Delta_S$  with  $S \in C(T)$ , the centralizer of  $T$ . In particular factors of simple systems correspond to compact groups of  $K \subset C(T)$ :

$$\mathcal{K} := \{A \in \mathcal{A}, SA = A, A \in K\}$$

The factor system  $T/\mathcal{K}$  is simple if and only if  $K$  is normal in  $C(T)$ . Therefore, if  $K$  is a maximal compact subgroup of the centralizer of a simple system  $T$ , then it induces a prime system  $T/\mathcal{K}$ . The most drastic situation occurs when the centralizer of the simple system is reduced to the powers of the transformation, it is then said to have minimal self-joinings (MSJ(2)) (mixing rank one transformations are such, also Chacon transformation and many others).

There also exists example of rigid (and therefore not MSJ(2)), simple, prime transformations (see [2]) and examples of simple systems with a centralizer possessing a non normal maximal compact subgroup  $K$ , giving way to prime, non simple systems are given in [4]. Observe that some non-zero time map of a horocycle flow are prime. As they can always be seen as factors of a simple system [16], they are part of the above theory. Let us

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mention also that if  $T$  is MSJ(2), then the symmetric factor of  $T \times T$  is prime and so is the map  $(x, y) \mapsto (y, Tx)$ .

Many of the examples of prime maps are also rank one. Indeed, mildly mixing rank one maps are prime as King showed that a strict factor of a rank one map is rigid. It is yet unknown if prime rank one maps are not factors of simple systems.

We believe that we've listed all the examples of finite measure preserving prime transformations known so far.

The aim of this paper is to introduce new examples of prime transformation as special cases of Poisson suspensions. The main structural result is given in Section 3 and examples in Section 5.

## 2. TECHNOLOGY

**2.1. Poisson measure.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $(X^*, \mathcal{A}^*, \mu^*)$  be the corresponding Poisson measure space.

It is frequent to define the identity on  $X^*$  by  $N$ , that is  $N(\nu) = \nu$  where  $\nu$  is a counting measure on  $X$ . Under the distribution  $\mu^*$ ,  $N$  is therefore a random measure distributed as  $\mu^*$  and  $\mathcal{A}^* = \sigma\{N(A), A \in \mathcal{A}\}$ .

**2.1.1. Fock space.** We recall that  $L^2(\mu^*)$  has a Fock-space structure based on  $L^2(\mu)$ . Namely:

$$L^2(\mu^*) \simeq \mathbb{C} \oplus L^2(\mu) \oplus L^2_{\text{sym}}(\mu^{\otimes 2}) \oplus \cdots \oplus L^2_{\text{sym}}(\mu^{\otimes n}) \oplus \cdots$$

Through this identification,  $L^2(\mu^*)$  decomposes into an orthogonal sum of chaos  $H^n$ , where  $H^n$  is linearly spanned by multiple integrals, for  $f$  in  $L^1(\mu) \cap L^2(\mu)$ , defined by:

$$J^{(n)}(f^{\otimes n}) :=$$

$$\int \cdots \int_{\Delta_n^c} f(x_1) \cdots f(x_n) (N(dx_1) - \mu(dx_1)) \cdots (N(dx_n) - \mu(dx_n))$$

In the identification,  $J^{(n)}(f^{\otimes n})$  corresponds to  $\sqrt{n!}f^{\otimes n} \in L^2_{\text{sym}}(\mu^{\otimes n})$  and we have the isometry formula:

$$\left\langle J^{(n)}(f), J^{(p)}(g) \right\rangle_{L^2(\mu^*)} = n! \langle f^{\otimes n}, g^{\otimes n} \rangle_{L^2_{\text{sym}}(\mu^{\otimes n})} \mathbf{1}_{n=p}$$

We can therefore consider that  $J^{(n)}(h)$  is defined for all  $h \in L^2_{\text{sym}}(\mu^{\otimes n})$ .

**2.1.2. Difference operators.** In this section, we refer to the very useful paper of Last and Penrose [7]. We shall need the following “operators”, called *difference operator*:

Let  $F \in L^2(\mu^*)$ . Define  $D_y^1 F$  by:

$$D_y^1 F(\nu) := F(\nu + \delta_y) - F(\nu)$$

and by induction  $D_{y_1, \dots, y_n}^n F$

$$D_{y_1, \dots, y_n}^n F := D_{y_2, \dots, y_n}^{n-1} (D_{y_1}^1 F)$$

It can be observed that this operator is symmetric in  $y$ 's for any  $n \in \mathbb{N}$ .

Those operators are acting between  $L^2(\mu^*)$  and  $L^2(\mu^* \otimes \mu^{\otimes n})$ , that is  $F \in L^2(\mu^*)$  is mapped to

$$(\nu, y_1 \dots, y_n) \mapsto D_{y_1, \dots, y_n}^n F(\nu)$$

And by taking expectation we get an operator which maps  $F \in L^2(\mu^*)$  to  $P_n F \in L_{\text{sym}}^2(\mu^{\otimes n})$ :

$$P_n F(y_1 \dots, y_n) = \mathbb{E} [D_{y_1, \dots, y_n}^n F]$$

It is convenient to set  $P_0 F = \mathbb{E} [F]$  and we get the remarkable property that  $F$  decomposes in the Fock space as:

$$F \simeq P_0 F + \dots + P_n F + \dots$$

In particular  $P_n J^{(n)}(f^{\otimes n}) = f^{\otimes n}$  and in this case, we can even remove the expectation:

$$D_{y_1, \dots, y_n}^n J^{(n)}(f^{\otimes n})(\nu) = f^{\otimes n}(y_1 \dots, y_n)$$

for  $\mu^* \otimes \mu^{\otimes n}$ -almost all  $\nu, y_1 \dots, y_n \in X^* \times X^n$ .

**Lemma 1.** *If  $F \in H^n$ , then, for  $\mu$ -almost all  $a \in X$ ,  $D_a^1 F \in H^{n-1}$ .*

*Proof.* First, by Eqs (3.9), (3.11) and Theorem 3.3 in [7]:

$$\int_X \mathbb{E} [(D_a^1 F)^2] \mu(da) = n \mathbb{E} [F^2],$$

$D_a^1 F$  is thus in  $L^2(\mu^*)$  for  $\mu$ -almost all  $a \in X$ .

Let  $k \neq n-1$ , we get  $P_k(D_a^1 F)(y_1 \dots, y_k) = \mathbb{E} [D_{y_1, \dots, y_k}^k (D_a^1 F)] = \mathbb{E} [D_{y_1, \dots, y_k, a}^{k+1} F] = P_{k+1} F(y_1 \dots, y_k, a)$ .

But as  $F \in H^n$ ,  $P_{k+1} F(y_1 \dots, y_k, y_{k+1})$  is zero for  $\mu^{\otimes k+1}$ -almost all  $y_1 \dots, y_{k+1} \in X^{k+1}$ , therefore we deduce that for  $\mu$ -almost all  $a \in X$ ,  $P_k(D_a^1 F)(y_1 \dots, y_k)$  is zero for  $\mu^{\otimes k}$ -almost all  $y_1 \dots, y_k \in X^k$ . This proves our claim.  $\square$

We shall need this formula in the sequel:

$$(2.1) \quad \int_{X^*} \int_X h(\nu, x) \nu(dx) \mu^*(d\nu) = \int_{X^*} \int_X h(\nu + \delta_x, x) \mu(dx) \mu^*(d\nu)$$

valid for all positive measurable functions  $h$  defined on  $X^* \times X$ .

**2.2. Poisson suspension.** If  $T$  is a measure preserving automorphism of  $(X, \mathcal{A}, \mu)$ , then  $T_* : \nu \mapsto \nu(T^{-1}(\cdot))$  is a measure preserving automorphism of  $(X^*, \mathcal{A}^*, \mu^*)$ .  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is the *Poisson suspension* over the base  $(X, \mathcal{A}, \mu, T)$ .  $T$  acts unitarily on  $L^2(\mu)$  by  $U_T : f \mapsto f \circ T$  and similarly  $T_*$  on  $L^2(\mu^*)$  by  $U_{T_*} : F \mapsto F \circ T_*$ .  $U_{T_*}$  preserves every chaos and, through the above identification, it is easy to see that  $U_{T_*}$  corresponds to  $U_T$  on  $H^1 \simeq L^2(\mu)$ , and more generally to  $U_T^{\otimes n}$  on  $H^n \simeq L_{\text{sym}}^2(\mu^{\otimes n})$ .

If  $\sigma$  is the maximal spectral type of  $U_T$  on  $L^2(\mu)$ ,  $\sigma^{*n}$  is the maximal spectral type of  $U_T^{\otimes n}$  on  $L_{\text{sym}}^2(\mu^{\otimes n})$ .

**2.3. Poissonian factors.** There are two main ways to obtain natural factors of a Poisson suspension  $(X^*, \mathcal{A}^*, \mu^*, T_*)$ .

First assume you can break down  $X$  into two  $T$ -invariant measurable pieces  $A$  and  $B$ . Then the Poisson measure restricted to  $A$  is such a factor. Indeed, the map

$$\begin{aligned} X^* &\rightarrow A^* \\ \nu &\mapsto \nu|_A \end{aligned}$$

realizes such a factor between  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  and  $(A^*, (\mathcal{A}|_A)^*, (\mu|_A)^*, (T|_A)^*)$ .

In terms of  $\sigma$ -algebra, the above factor corresponds to  $\sigma\{N(C), C \in \mathcal{A}, C \subset A\} \subset \mathcal{A}^*$ .

The second way consists in considering  $\sigma$ -finite factors of the base. Namely, if  $\mathcal{B} \subset \mathcal{A}$  is such a  $\sigma$ -finite factor then we obtain the following situation

$$(X, \mathcal{A}, \mu, T) \xrightarrow{\psi} (X/\mathcal{B}, \mathcal{A}/\mathcal{B}, \mu/\mathcal{B}, T/\mathcal{B})$$

and if we define  $\psi_*$  by  $\nu \mapsto \nu(\psi^{-1}(\cdot))$  we obtain the following factor relationship at the level of the Poisson suspensions:

$$(X^*, \mathcal{A}^*, \mu^*, T_*) \xrightarrow{\psi_*} ((X/\mathcal{B})^*, (\mathcal{A}/\mathcal{B})^*, (\mu/\mathcal{B})^*, (T/\mathcal{B})^*)$$

In terms of  $\sigma$ -algebra, it corresponds to  $\mathcal{B}^* := \sigma\{N(C), C \in \mathcal{B}\} \subset \mathcal{A}^*$ .

A Poissonian factor is a combination of both situations which is obtained by first considering a  $T$ -invariant subset  $A \subset X$  and then considering a  $\sigma$ -finite factor  $\mathcal{B}$  of the restricted system  $(A, \mathcal{A}|_A, \mu|_A, T|_A)$ .

For example, if  $(X, \mathcal{A}, \mu, T)$  is ergodic, then the Poissonian factors of  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  are:

- the trivial factor  $\{\emptyset, X^*\}$  which corresponds to the first situation with  $A = \emptyset$ ;
- $\mathcal{B}^*$ , for a  $\sigma$ -finite factor  $\mathcal{B} \subset \mathcal{A}$ .

We recall a result from [13]:

**Proposition 2.** *Let  $\mathcal{C} \subset \mathcal{A}^*$  be a factor of  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  and  $\Phi$  the corresponding conditional expectation. Assume moreover that  $\Phi$  preserves the first chaos  $H^1$  and doesn't vanish on  $H^1$ . Then:*

- $\Phi$  induces on  $L^2(\mu)$  a sub-Markov operator  $\Psi$ .

- *There exists a  $T$ -invariant set  $A \subset X$  such that  $\Psi$  restricted to  $L^2(\mu|_A)$  is a conditional expectation on a  $\sigma$ -finite factor  $\mathcal{G} \subset \mathcal{A}|_A$  and vanishes on  $L^2(\mu|_{A^c})$ .*

### 3. THE MAIN RESULT

**Definition 3.** A Poisson suspension is said to have the property ‘‘CP’’ (for ‘‘chaos-preserving’’) if any conditional expectation with respect to a factor preserves each chaos  $H^n$ .

**Example 4.** If the maximal spectral type  $\sigma$  of  $(X, \mathcal{A}, \mu, T)$  satisfies  $\sigma^{*n} \perp \sigma^{*m}$  then  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  has property CP (see [8] for the proof in the Gaussian case, completely analogous to the Poissonian one).

**Theorem 5.** *Let  $\Phi$  be a conditional expectation on a  $\sigma$ -algebra  $\mathcal{C} \subset \mathcal{A}^*$  that preserves  $H^n$  for every  $n \geq 1$ . If  $\Phi$  is zero on  $H^1$ , then  $\Phi$  is zero on every  $H^n$ , for every  $n \geq 1$ . In other words,  $\Phi$  is the conditional expectation on the trivial  $\sigma$ -algebra  $\{X^*, \emptyset\}$ .*

*Proof.* Let  $F$  be in  $H^n$ ,  $n \geq 2$  and  $a \in X$ . We will compute  $D_a^1 \Phi F$ .

$$\begin{aligned} \text{We have } \mathbb{E} \left[ (D_a^1 \Phi F)^2 \right] &= \mathbb{E} \left[ ((\Phi F)(\cdot + \delta_a) - \Phi F)^2 \right] = \mathbb{E} \left[ (\Phi F)^2(\cdot + \delta_a) \right] + \\ &\mathbb{E} \left[ (\Phi F)^2 \right] - 2\mathbb{E} [(\Phi F)(\cdot + \delta_a)(\Phi F)] \end{aligned}$$

We use Formula 2.1 with  $h(\nu, x) = (\Phi F)^2(\nu) f(x)$  where  $f \in L^1(\mu) \cap L^2(\mu)$  is a nonnegative function.

We get

$$\int_{X^*} \int_X (\Phi F)^2(\nu) f(x) \nu(dx) \mu^*(d\nu) = \int_{X^*} \int_X (\Phi F)^2(\nu + \delta_x) f(x) \mu(dx) \mu^*(d\nu)$$

which can be rewritten into

$$\mathbb{E} \left[ (\Phi F)^2 N(f) \right] = \int_X \mathbb{E} \left[ (\Phi F)^2(\cdot + \delta_x) \right] f(x) dx$$

$$\begin{aligned} \text{And } \mathbb{E} \left[ (\Phi F)^2 N(f) \right] &= \mathbb{E} \left[ (\Phi F)^2 (N(f) - \int_X f(x) dx) \right] + \mathbb{E} \left[ (\Phi F)^2 \int_X f(x) dx \right] = \\ &\mathbb{E} \left[ (\Phi F)^2 J^{(1)}(f) \right] + \int_X \mathbb{E} \left[ (\Phi F)^2 \right] f(x) dx \end{aligned}$$

But as  $\Phi$  is the conditional expectation on  $\mathcal{C}$ ,  $\Phi F$  is  $\mathcal{C}$ -measurable and so is  $(\Phi F)^2$ . Therefore:

$$\mathbb{E} \left[ (\Phi F)^2 J^{(1)}(f) \right] = \mathbb{E} \left[ \Phi \left[ (\Phi F)^2 J^{(1)}(f) \right] \right] = \mathbb{E} \left[ (\Phi F)^2 \Phi \left[ J^{(1)}(f) \right] \right]$$

But, by assumption,  $\Phi$  vanishes on  $H^1$ , which implies that  $\Phi \left[ J^{(1)}(f) \right] = 0$ .

Then we get:

$$\mathbb{E} \left[ (\Phi F)^2 N(f) \right] = \int_X \mathbb{E} \left[ (\Phi F)^2 \right] f(x) dx$$

and we deduce the equality

$$\int_X \mathbb{E} [(\Phi F)^2] f(x) dx = \int_X \mathbb{E} [(\Phi F)^2 (\cdot + \delta_x)] f(x) dx$$

valid for any nonnegative  $f \in L^1(\mu)$ . We obtain

$$\mathbb{E} [(\Phi F)^2] = \mathbb{E} [(\Phi F)^2 (\cdot + \delta_x)]$$

for  $\mu$ -almost all  $x \in X$ .

Now

$$\begin{aligned} \mathbb{E}[(\Phi F)(\cdot + \delta_a)(\Phi F)] &= \mathbb{E}[(\Phi F)(\cdot + \delta_a) - (\Phi F)](\Phi F) + \mathbb{E}[(\Phi F)^2] \\ &= \mathbb{E}[(D_a^1 \Phi F)(\Phi F)] + \mathbb{E}[(\Phi F)^2] \end{aligned}$$

But, as  $\Phi F$  is in  $H^n$ ,  $D_a^1 \Phi F$  is in  $H^{n-1}$  for  $\mu$ -almost all  $a \in X$ , thanks to Lemma 1. These two vectors are therefore orthogonal which means that  $\mathbb{E}[(\Phi F)(\cdot + \delta_a)(\Phi F)] = \mathbb{E}[(\Phi F)^2]$ .

If we sum up, we get, for  $\mu$ -almost all  $a \in X$ ,

$$\mathbb{E}[(D_a^1 \Phi F)^2] = 0$$

and this implies that  $D_a^1 \Phi F(\nu) = 0$  for  $\mu^* \otimes \mu$ -a.e.  $(\nu, a)$ , and it follows that  $D_{y_1, \dots, y_n}^n \Phi F = 0$  for  $\mu^* \otimes \mu^{\otimes n}$ -a.e.  $(\nu, y_1, \dots, y_n)$  (thanks to repeated use of Lemma 2.4 in [7]).

This means that  $P_n(\Phi F) = 0$ . As we already know that  $P_k(\Phi F) = 0$  for all  $k \neq n$  (as  $\Phi F \in H^n$ ), we can conclude that  $\Phi F = 0$ .  $\square$

We are now in position to prove the main structural result of the paper:

**Theorem 6.** *Let  $(X, \mathcal{A}, \mu, T)$  be an ergodic measure preserving system such that  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  has property CP. If  $\mathcal{C} \subset \mathcal{A}^*$  is a non-trivial factor, then it contains a non-trivial Poissonian factor. In particular, if we assume moreover that  $(X, \mathcal{A}, \mu, T)$  is prime, then  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is prime.*

*Proof.* Let  $\mathcal{C}$  be a non-trivial  $T_*$ -invariant  $\sigma$ -algebra included in  $\mathcal{A}^*$  and  $\Phi$  the corresponding conditional expectation.  $\Phi$  preserves  $H^1$  and by means of Proposition 2,  $\Phi$  induces on  $L^2(\mu)$  a sub-Markov operator  $\Psi$ . Assume  $\Psi \neq 0$ , then  $\Psi$  is also an orthogonal projection and it exists a  $T$ -invariant set  $K \subset X$  such that  $\Psi f = 0$  for all  $f \in L^2(\mu)$  supported on  $K$  and  $\Psi$  restricted to  $L^2(K^c, \mathcal{A}|_{K^c}, \mu|_{K^c})$  is a Markov operator and thus a conditional expectation on a  $T$ -invariant  $\sigma$ -algebra  $\mathcal{T}$  included in  $\mathcal{A}|_{K^c}$ . But  $T$  is ergodic, therefore  $K = \emptyset$ . The image of  $\Psi$  contains all the indicator functions of finite measure sets contained in  $\mathcal{T}$ . But this means that the image of  $\Phi$  contains all the vector of the form  $N(A) - \mu(A)$ , for  $A \in \mathcal{T}$  of finite measure which are therefore  $\mathcal{C}$ -measurable. This proves that  $\mathcal{C}$  contains the Poissonian factor  $\mathcal{T}^*$ .

Now we assume that  $\Psi = 0$ , that is,  $\Phi$  vanishes on  $H^1$ . But we can now apply Theorem 5 to conclude that  $\mathcal{C} = \{X^*, \emptyset\}$  which is impossible as we have assumed  $\mathcal{C}$  to be a non-trivial.  $\square$

#### 4. Some consequences

It has been proved that when  $S$  has MSJ, then  $S \odot S$  (the symmetric factor of the direct product  $S \times S$ ) is prime and so is the map  $(x, y) \mapsto (y, Sx)$  with respect to the product measure. It is therefore natural to ask if this is the case in our context.

**Proposition 7.**  $T_* \odot T_*$  is never prime.

*Proof.* It is well known that  $(X^*, \mathcal{A}^*, (2\mu)^*, T_*)$  is a factor of the direct product  $(X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \times T_*)$  through the map  $\varphi : (\nu_1, \nu_2) \mapsto \nu_1 + \nu_2$  (the superposition of two independent Poisson measures of intensity  $\mu$  is a Poisson measure with intensity  $2\mu$ ). But we can remark that it is also a factor of  $T_* \odot T_*$ . We thus have the scheme

$$\begin{array}{c} (X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \times T_*) \\ \downarrow \\ (X^* \times X^*, \mathcal{A}^* \odot \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \odot T_*) \\ \downarrow \\ (X^*, \mathcal{A}^*, (2\mu)^*, T_*) \end{array}$$

However we know that

$$\begin{array}{c} (X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \times T_*) \\ \downarrow \\ (X^* \times X^*, \mathcal{A}^* \odot \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \odot T_*) \end{array}$$

is a compact extension. And

$$\begin{array}{c} (X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \times T_*) \\ \downarrow \\ (X^*, \mathcal{A}^*, (2\mu)^*, T_*) \end{array}$$

is a relatively weakly mixing extension. To see the latter, observe that the direct product  $(X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \times T_*)$  can be thought as the Poisson suspension

$$((X \times X)^*, (\mathcal{A} \otimes \mathcal{A})^*, (\mu \otimes \delta_\infty + \delta_\infty \otimes \mu)^*, (T \times T)_*)$$

where  $\infty$  is an artificially added point in  $X$  fixed by  $T$ . In this way,  $(X^*, \mathcal{A}^*, (2\mu)^*, T_*)$  appears as a Poissonian factor, and we know that the corresponding relatively independent joining is ergodic (see [13]).

This proves that  $(X^*, \mathcal{A}^*, (2\mu)^*, T_*)$  is a strict and non trivial factor of  $(X^* \times X^*, \mathcal{A}^* \odot \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \odot T_*)$ .  $\square$



**Proposition 8.** *If  $T_*$  is prime and  $T_* \times T_*$  has property CP, then  $(\nu_1, \nu_2) \mapsto (\nu_2, T_*\nu_1)$  is prime.*

*Proof.* We will use the representation of  $(X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu^* \otimes \mu^*, T_* \times T_*)$  introduced in the proof of the last proposition, that is

$$((X \times X)^*, (\mathcal{A} \otimes \mathcal{A})^*, (\mu \otimes \delta_\infty + \delta_\infty \otimes \mu)^*, (T \times T)_*)$$

In this representation,  $(\nu_1, \nu_2) \mapsto (\nu_2, T_*\nu_1)$  becomes  $(\nu_1 \otimes \delta_\infty + \delta_\infty \otimes \nu_2) \mapsto (\nu_2 \otimes \delta_\infty + \delta_\infty \otimes T_*\nu_1)$  and is indeed a Poisson suspension automorphism where the base automorphism is

$$(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes \delta_\infty + \delta_\infty \otimes \mu, R)$$

where  $R(x, y) \mapsto (y, Tx)$ .

Observe that  $R^2 = T \times T$  and that  $R$  is ergodic. Indeed, if  $A$  is an  $R$ -invariant set, then it is also a  $T \times T$ -invariant set (with respect to the measure  $\mu \otimes \delta_\infty + \delta_\infty \otimes \mu$ !). But it is easy to see that, as  $T$  is ergodic, the only  $T \times T$ -invariant sets are, modulo null sets,  $\emptyset$ ,  $X \times X$ ,  $X \times \{\infty\}$  and  $\{\infty\} \times X$ . But the last two are obviously not  $R$ -invariant, so  $R$  is ergodic.

In the same vein, a  $\sigma$ -finite factor of  $R$  is also a  $\sigma$ -finite factor of  $T \times T$ . The only factor of  $T \times T$  is the symmetric factor which is not a factor of  $R$ , so  $R$  is prime.

It remains to check that  $((X \times X)^*, (\mathcal{A} \otimes \mathcal{A})^*, (\mu \otimes \delta_\infty + \delta_\infty \otimes \mu)^*, R_*)$  has property CP. It follows easily from the fact that  $(T \times T)_*$  (under the measure  $(\mu \otimes \delta_\infty + \delta_\infty \otimes \mu)^*$ ) has property CP and that  $R_*^2 = (T \times T)_*$ .  $\square$

If the maximal spectral type  $\sigma$  of  $T$  satisfies  $\sigma^{*n} \perp \sigma^{*m}$  then  $\sigma$  is also the maximal spectral type of  $T \times T$  with respect to the measure  $\mu \otimes \delta_\infty + \delta_\infty \otimes \mu$  and therefore  $T_* \times T_*$  has property CP.

**4.1. Disjointness.** The following disjointness results come all from [9] where the notion of Joining Primness of order  $n$  ( $JP(n)$ ) was introduced. Simple maps and their factors are  $JP(1)$  and direct products of such maps are  $JP(2)$ .

**Theorem 9.** [9] *A Poisson suspension is disjoint from any  $JP(n)$  map, for all  $n \geq 1$ .*

Therefore our prime Poisson suspensions are disjoint from prime maps that are simple or factor of simple maps.

**Proposition 10.** *If  $S$  is distally simple. Then  $S \odot S$  and  $K := (x, y) \mapsto (y, Sx)$  are disjoint from Poisson suspensions.*

*Proof.* The first point follows from the fact that  $S \times S$  is  $JP(2)$  and so is  $S \odot S$ .

For the second point, a non-trivial joining between  $K$  and a Poisson suspension  $T_*$  would yield a non-trivial joining of  $K^2 = S \times S$  with  $(T_*)^2 = (T^2)_*$  which is impossible by the above arguments.  $\square$

## 5. EXAMPLES

**5.1. Non-singular compact group rotations.** We introduce a family of examples that has been studied (among other sources) in [5] and we use the same presentation.

Consider  $\Omega := \{0, 1\}^{\mathbb{N}}$  equipped with the uniform Bernoulli probability measure  $(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\otimes \mathbb{N}}$  and the “carrier to the right” transformation  $\omega \mapsto \omega + \bar{1}$  where addition is modulo 2 and  $\bar{1} = (1, 0, 0, \dots)$ . Let  $h$  be a measurable integer-valued positive function and define  $X \subset \Omega \times \mathbb{N}$  as the set of points  $(\omega, n)$  such that  $1 \leq n \leq h(\omega)$ .

Let  $T$  be the transformation

$$T(\omega, n) = \begin{cases} (\omega, n+1) & \text{if } 1 \leq n < h(\omega) \\ (\omega + \bar{1}, n) & \text{if } n = h(\omega) \end{cases}$$

We endow  $X$  with the measure, preserved by  $T$  and defined, for a measurable positive function  $f$ :

$$\int_X f(\omega, n) \mu(d(\omega, n)) = \int_{\Omega} \left[ \sum_{n=1}^{h(\omega)} f(\omega, n) \right] \nu(d\omega)$$

We have built a Kakutani tower over  $\Omega$  with height function  $h$ .

Let's give the specifications that define  $h$ . Consider a sequence of integers  $\{m_i\}_{i \geq 0}$  where  $m_i \geq 3$  and set  $n_{j+1} = m_j n_j$  where  $n_0 = 1$ .

For  $\omega \in \Omega$ , let us denote  $k(\omega)$  the smallest integer  $k$  such that  $\omega_k = 0$  and

$$h(\omega) = n_{k(\omega)} - \sum_{j < k(\omega)} n_j.$$

It is easy to see that the measure  $\mu$  is infinite.

It can be noted (see [5]) that this system encodes an ergodic infinite measure preserving compact group rotation, the adding machine on  $\prod_{j \geq 0} \{0, \dots, m_j - 1\}$  with a measure singular with respect to the Haar measure on this group.

**5.2. Properties.** In [1], they obtained the following:

**Proposition 11.** *Joinings between  $(X, \mathcal{A}, c_1 \mu, T)$  and  $(X, \mathcal{A}, c_2 \mu, T)$  exist only for  $c_1 = c_2$  and are graph joinings  $\Delta_{T^n}$ ,  $n \in \mathbb{Z}$ . In particular, it is prime.*

In [5], they computed the maximal spectral type:

**Proposition 12.** *The maximal spectral type of  $T$  is the Riesz product*

$$\sigma := \prod_{j=0}^{+\infty} (1 + \cos 2\pi n_j t)$$

Those Riesz products are the most easy to deal with and we can obtain:

**Proposition 13.** *For all  $n \neq m$ ,  $\sigma^{*n} \perp \sigma^{*m}$ .*

*Proof.* First observe that for all  $n \geq 1$ ,  $\sigma^{*n} = \prod_{j=0}^{+\infty} \left(1 + \frac{1}{2^{n-1}} \cos 2\pi n_j t\right)$

We can therefore apply to those Riesz products the criterium of Peyrière [11] to verify their mutual singularity as  $\frac{n_{j+1}}{n_j} \geq 3$  and  $\sum_{k \in \mathbb{N}} \left(\frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}\right) = +\infty$ .  $\square$

**5.3. Poisson suspensions over  $(X, \mathcal{A}, \mu, T)$ .** As a direct application, we obtain our first examples of prime Poisson suspensions

**Proposition 14.** *The Poisson suspension  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is prime. Moreover, it is mildly mixing non mixing, has trivial centralizer and singular spectrum with infinite multiplicity.*

*Proof.* The requirements of Theorem 6 are satisfied as  $T$  is ergodic, prime, preserves an infinite measure, and has property CP as for all  $n \neq m$ ,  $\sigma^{*n} \perp \sigma^{*m}$ . This last property also implies (see [13]) that each transformation  $S$  that commutes with  $T_*$  is of the form  $R_*$  for a transformation  $R$  of the base, that commutes with  $T$ . Therefore, as the centralizer of  $T$  is trivial, so is the centralizer of  $T_*$ . A rigid transformation has an uncountable centralizer ([6]), therefore  $T_*$  is not rigid. As  $T_*$  is prime, the only rigid factor is the trivial one point system, consequently  $T_*$  is mildly mixing.

For such Riesz products  $\sigma$ , if we set  $U$ , the unitary operator acting on  $L^2(\sigma)$  by  $U : (z \mapsto f(z)) \mapsto (z \mapsto zf(z))$ , then the spectrum of  $U \odot U$  on  $L^2(\sigma) \odot L^2(\sigma)$  never has simple spectrum. This implies that  $U_{T_*}$  restricted to the second chaos which is unitarily isomorphic to  $U \odot U$  never has simple spectrum. As the multiplicity of a Poisson suspension is either 1 or infinity, we are in the latter case here.

To prove it is not mixing, it is sufficient to prove that  $\sigma$  is not Rajchman. This follows from the fact that  $\hat{\sigma}(n_j) = \frac{1}{2}$  for all  $j \geq 0$ .  $\square$

*Remark 15.* The fact that those systems possess a singular spectrum of infinite multiplicity makes them new examples of prime systems. Also they are not rank one (see Appendix) as they are mildly mixing.

With the above examples we obtain a Poisson suspensions with a continuum array of non-disjoint, non-isomorphic prime systems:

**Proposition 16.** *The Poisson suspension  $((X \times [0, 1])^*, (\mathcal{A} \otimes \mathcal{B})^*, (\mu \otimes \lambda_{[0,1]})^*, (T \times Id)_*)$  possesses the Poisson suspensions  $(X^*, \mathcal{A}^*, (c\mu)^*, T_*)$ ,  $0 < c \leq 1$  as factors. Those factors are prime, non-disjoint, unitarily isomorphic and non metrically isomorphic for different  $c$ 's.*

*Proof.* The factor relationship is implemented by the map  $\nu \mapsto \nu(\cdot \times [0, c])$ .

The systems  $(X, \mathcal{A}, c\mu, T)$  have the same properties as  $c$  spans  $\mathbb{R}_+^*$ . In particular  $(X^*, \mathcal{A}^*, (c\mu)^*, T_*)$  are prime and have the same spectrum, henceforth they all are unitarily isomorphic.

It is well known that adding two independent Poisson measures with intensity  $\nu_1$  and  $\nu_2$  leads to a Poisson measure with intensity  $\nu_1 + \nu_2$ . Therefore, if  $c_1 < c_2$  then  $(X^*, \mathcal{A}^*, (c_2\mu)^*, T_*)$  is a factor of the direct product of  $(X^*, \mathcal{A}^*, (c_1\mu)^*, T_*)$  with  $(X^*, \mathcal{A}^*, ((c_2 - c_1)\mu)^*, T_*)$ . This yields a joining between  $(X^*, \mathcal{A}^*, (c_2\mu)^*, T_*)$  and  $(X^*, \mathcal{A}^*, (c_1\mu)^*, T_*)$ ; it is not the independent one for obvious reasons.

Now assume there exists an isomorphism  $S$  between both systems. As  $\sigma \perp \sigma^{*n}$ ,  $n \geq 2$ , it implies, thanks to Proposition 5.2 in [13], that  $S = R^*$  for an isomorphism  $R$  between  $(X, \mathcal{A}, c_1\mu, T)$  and  $(X, \mathcal{A}, c_2\mu, T)$ , but this is not possible by Proposition 11.  $\square$

**5.4. A mixing example.** Another source of examples is furnished by recent Ryzhikov's infinite measure preserving "mixing" rank one transformations (see [15]).

He has proved, in particular, that all those systems have the *minimal self-joining* property in infinite measure (the only ergodic self-joinings are off-diagonal joinings) which implies that they are prime as in examples of the preceding section (see Proposition 11). Moreover he has proved that Poisson suspensions over such systems (with some extra assumptions) have simple (and singular) spectrum, which in turns implies that they have the property CP (indeed, a necessary condition for a Poisson suspension to have simple spectrum is that  $\sigma^{*n} \perp \sigma^{*m}$  for all  $n \neq m$ , where  $\sigma$  is the maximal spectral type of the base). If we sum up and apply Theorem 6, we get:

**Proposition 17.** *There exist prime Poisson suspensions which are mixing, with simple singular spectrum and trivial centralizer.*

Observe that, as any mixing rank one has MSJ, those prime mixing Poisson suspensions are disjoint from any previously known prime systems, thanks to Theorem 9 and Proposition 10.

## 6. Appendix

**Proposition 18.** *If a Poisson suspension is rank one, then it is rigid.*

*Proof.* We will use the following property of rank one systems, established by Ryzhikov in [14]. If  $\Phi$  is a Markov operator corresponding to an ergodic selfjoining of a rank one transformation  $T$ , then there exists  $a > 0$ , a Markov operator  $\Psi$  and a sequence  $n_k$  such that  $T^{n_k} \rightharpoonup a\Phi + (1 - a)\Psi$ .

Now we recall that a Poisson suspension  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  always has the so-called ELF property (see [3]), that is, limits of off diagonals joinings are ergodic. Therefore, in the above situation,  $a = 1$ . Observe that we cannot apply the above result with  $\Phi = Id$  as we have to rule out a sequence that would be identically zero after some time.

We recall also that we build a *Poissonian joining* (see [3] and [13]) of a Poisson suspension  $T_*$  by considering a sub-Markov operator  $\varphi$  that commutes with the base  $T$  and forming the exponential  $\tilde{\varphi}$  that acts on each

chaos  $H^n$  of  $L^2(\mu^*)$  as  $\varphi \otimes \cdots \otimes \varphi$  ( $n$  terms). Moreover, Poissonian joinings of an ergodic Poisson suspension are ergodic. Therefore we can apply Ryzhikov's result to the Markov operators  $(1 - \frac{1}{n}) \widetilde{Id}_{L^2(\mu)}$  for each  $n \in \mathbb{N}$ . As  $(1 - \frac{1}{n}) \widetilde{Id}_{L^2(\mu)}$  tends to  $Id_{L^2(\mu)}$ , then  $(1 - \frac{1}{n}) \widetilde{Id}_{L^2(\mu)}$  tends to  $\widetilde{Id}_{L^2(\mu)} = Id_{L^2(\mu^*)}$ . It is now easy to build a non-trivial sequence  $n_k$  such that  $T_*^{n_k} \rightharpoonup Id_{L^2(\mu^*)}$ , which is therefore a rigidity sequence for  $T_*$ .

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