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# Edge Decompositions of Hypercubes by Paths and by Cycles

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## Abstract

If  $H$  is isomorphic to a subgraph of  $G$ , we say that  $H$  *divides*  $G$  if there exist embeddings  $\theta_1, \theta_2, \dots, \theta_k$  of  $H$  such that

$$\{\{E(\theta_1(H)), E(\theta_2(H)), \dots, E(\theta_k(H))\}$$

is a partition of  $E(G)$ . For purposes of simplification we will often omit the embeddings, saying that we have an edge decomposition by copies of  $E(H)$ .

Many authors ([1], [2], [3], [8], [9], [12]) have studied this notion for various subgraphs of hypercubes. We continue such a study in this paper.

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# 1 Introduction and Preliminary Results

**Definition 1** *If  $H$  is isomorphic to a subgraph of  $G$ , we say that  $H$  divides  $G$  if there exist embeddings  $\theta_1, \theta_2, \dots, \theta_k$  of  $H$  such that*

$$\{\{E(\theta_1(H)), E(\theta_2(H)), \dots, E(\theta_k(H))\}$$

*is a partition of  $E(G)$ .*

Ramras [8] has defined a more restrictive concept.

**Definition 2** *A fundamental set of edges of a graph  $G$  is a subset of  $E(G)$  whose translates under some subgroup of the automorphism group of  $G$  partition  $E(G)$ .*

Edge decompositions of graphs by subgraphs has a long history. For example, there is a Steiner triple system of order  $n$  if and only if the complete graph  $K_n$  has an edge-decomposition by  $K_3$ . In 1847 Kirkman [5] proved that for a Steiner triple system to exist it is necessary that  $n \equiv 1 \pmod{6}$  or  $n \equiv 3 \pmod{6}$ . In 1850 he proved the converse hold also [6].

**Theorem 1** *A Steiner system of order  $n \geq 3$  exists if and only if  $n \equiv 1 \pmod{6}$  or  $n \equiv 3 \pmod{6}$ .*

In more modern times (1964) G. Ringel [11] stated the following conjecture, which is still open.

**Conjecture 1** *If  $T$  is a fixed tree with  $m$  edges then  $K_{2m+1}$  is edge-decomposable into  $2m + 1$  copies of  $T$ .*

By  $Q_n$  we mean the  $n$ -dimensional hypercube. We regard its vertex set,  $V(Q_n)$ , as  $\mathcal{P}(\{1, 2, \dots, n\})$ , the set of subsets of  $\{1, 2, \dots, n\}$ . Two vertices  $x$  and  $y$  are considered adjacent (so  $\langle x, y \rangle \in E(Q_n)$ ) if  $|x \Delta y| = 1$ , where  $\Delta$  denotes the symmetric difference of the two subsets  $x$  and  $y$ .  $(V(Q_n), \Delta)$  is isomorphic as a group to  $(\mathbb{Z}_2^n, +)$ . Occasionally, when convenient, we shall use the vector notation for vertices; thus  $\vec{x}$  and  $\vec{y}$  are adjacent precisely when they differ in exactly one component. Note that for  $k < n$ ,  $\mathcal{P}(\{1, 2, \dots, k\}) \subset \mathcal{P}(\{1, 2, \dots, n\})$  so that  $V(Q_k) \subset V(Q_n)$ . In fact, from the definition of adjacency, it follows that  $Q_k$  is an induced subgraph of  $Q_n$ .

Beginning in the early 1980's, interest in hypercubes (and similar hypercube-like networks such as "cube-connected cycles" and "butterfly" networks) increased dramatically with the construction of massively parallel-processing

computers, such as the “Connection Machine” whose architecture is that of the 16-dimensional hypercube, with  $2^{16} = 65,536$  processors as the vertices. Problems of routing message packets simultaneously along paths from one processor to another led to an interest in questions of edge decompositions of  $E(Q_n)$  by paths. An encyclopedic discussion of this and much more can be found in [7].

In [8] we have shown that if  $\mathcal{G}$  is a subgroup of  $\text{Aut}(Q_n)$  and for all  $g \in \mathcal{G}$ , with  $g \neq id$  (where  $id$  denotes the identity element),  $g(E(H)) \cap E(H) = \emptyset$ , then there is a packing of these translates of  $E(H)$  in  $Q_n$ , *i.e.* they are pairwise disjoint. If, in addition,  $|E(H)| \cdot |\mathcal{G}| = n \cdot 2^{n-1} = |E(Q_n)|$ , then the translates of  $E(H)$  by the elements of  $\mathcal{G}$  yield an edge decomposition of  $Q_n$ . In [8] it is shown that every tree on  $n$  edges can be embedded in  $Q_n$  as a fundamental set. (This result for edge decompositions was obtained independently by Fink [3]). In [9] this is extended to certain trees and certain cycles on  $2n$  edges. Decompositions of  $Q_n$  by  $k$ -stars are proved for all  $k \leq n$  in [2]. Recently, Wagner and Wild [12] have constructed, for each value of  $n$ , a tree on  $2^{n-1}$  edges that is a fundamental set for  $Q_n$ . The structure of  $\text{Aut}(Q_n)$  is discussed in [8]. For each subset  $A$  of  $\{1, 2, \dots, n\}$ , the complementing automorphism  $\sigma_A$  is defined by  $\sigma_A(x) = A\Delta\{x\}$ . Another type of automorphism arises from the group of permutations  $\mathcal{S}_n$  of  $\{1, 2, \dots, n\}$ . For  $x = \{x_1, x_2, \dots, x_m\} \subseteq \{1, 2, \dots, n\}$  and  $\theta \in \mathcal{S}_n$  we denote by  $\rho_{\theta(x)}$  the vertex  $\{\theta(x_1), \theta(x_2), \dots, \theta(x_m)\}$ . The mapping  $\rho_\theta : V(Q_n) \rightarrow V(Q_n)$  defined in this way is easily seen to belong to  $\text{Aut}(Q_n)$ . Every automorphism in  $\text{Aut}(Q_n)$  can be expressed uniquely in the form  $\sigma_A \circ \rho_\theta$ , where this notation means that we first apply  $\rho_\theta$ . Note:  $\rho_\theta \circ \sigma_A = \sigma_{\theta(A)} \circ \rho_\theta$ .

To avoid ambiguity in what follows we make this definition:

**Definition 3** *By  $P_k$ , the “ $k$ -path”, we mean the path with  $k$  edges.*

### Questions

- (1) For which  $k$  dividing  $n \cdot 2^{n-1}$  does  $P_k$  divide  $Q_n$ ?
- (2) For which  $k$  dividing  $n \cdot 2^{n-1}$  does  $C_k$ , the cycle on  $k$  edges, divide  $Q_n$ ?
- (3) For those  $k$  for which the answer to either (1) or (2) is “yes”, is the edge set used in the decomposition a fundamental set for  $Q_n$ ?

We begin this introductory section with some examples. In later sections we prove a variety of results relating to these questions, and in the final section we summarize our findings.

### Example 1

Let  $T$  be the 2-star (= the 2-path) contained in  $Q_3$  with center 000, and leaves 100, 010. Then  $\mathcal{G} = \{id, \sigma_{123}, \sigma_1\rho_{(123)}, \sigma_{12}\rho_{(132)}, \sigma_3\rho_{(132)}, \sigma_{23}\rho_{(123)}\}$  is a (cyclic) subgroup of  $\text{Aut}(Q_3)$  of order 6, and the 6 translates of  $T$  under  $\mathcal{G}$  yield an edge decomposition of  $Q_3$ .  $\square$

Note, however, that  $\mathcal{G}$  does not work for the 2-star  $T'$ , whose center is 000 and whose leaves are 100 and 001. The subgroup which works for this 2-star is  $\mathcal{G}' = \{id, \sigma_{123}, \sigma_1\rho_{(132)}, \sigma_{13}\rho_{(123)}, \sigma_2\rho_{(123)}, \sigma_{23}\rho_{(132)}\}$ .

### Example 2

$P_6$  does not divide  $Q_3$ . For since  $Q_3$  has 12 edges, if  $P_6$  *did* divide  $Q_3$  then  $Q_3$  would have an edge-decomposition consisting of 2 copies of  $P_6$ . The degree sequence (in decreasing order) of each  $P_6$  is 2, 2, 2, 2, 2, 1, 1, 0, whereas  $Q_3$ , of course, is 3-regular. Thus the vertex of degree 0 in one  $P_6$  would require a degree of 3 in the other, which is impossible.  $\square$

### Example 3

$P_4$  does not divide  $Q_3$ . Since  $P_4$  has 4 edges, we would need 3 copies of  $P_4$  for an edge-decomposition of  $Q_3$ . The degree sequence of  $P_4$  is 2, 2, 2, 1, 1, 0, 0, 0. Call the three copies of  $P_4$   $P^{(1)}$ ,  $P^{(2)}$ , and  $P^{(3)}$ . At each vertex  $v$  of  $Q_3$ ,  $\sum_{1 \leq i \leq 3} \deg_{P^{(i)}}(v) = 3$ . For each  $v$  with  $\deg_{P^{(1)}}(v) = 0$  or 2, exactly one of  $\deg_{P^{(2)}}(v)$  or  $\deg_{P^{(3)}}(v)$  is 1. Thus a total of  $6 + 2 = 8$  1's are required. However, each of the 3 copies of  $P_4$  has exactly 2 vertices of degree 1, and so the total number of 1's in the  $3 \times 8$  array  $(a_{iv})$  formed by the 3 degree sequences is 6. Hence  $P_4$  does not divide  $Q_3$ .  $\square$

### Example 4

The *other* tree on 4 edges *does* divide  $Q_3$ . Let  $T$  = the 3-star centered at 000 union the edge  $\langle 001, 101 \rangle$ . Let  $\mathcal{G} = \langle \sigma_{23}\rho_{(123)} \rangle$ , which is a cyclic subgroup of  $\text{Aut}(Q_3)$  of order 3. A straight-forward calculation shows that the translates of  $T$  under  $\mathcal{G}$  form an edge decomposition of  $Q_3$ .  $\square$

**Proposition 1** For  $k \geq 3$ ,  $P_{2^k}$  does not divide  $Q_{2^{k+1}}$ .

*Proof.* Suppose that  $k \geq 3$ , and suppose that  $P_{2^k}$  divides  $Q_{2^{k+1}}$ . The matrix  $(a_{iv})$  formed by the degree sequences of copies of  $P_{2^k}$  has  $2^{2^{k+1}}$  columns, and

$$(2k + 1) \cdot 2^{2^k} / 2^k = (2k + 1)2^k$$

rows. Then since each row has exactly two 1's, the entire matrix has  $(2k + 1)2^{2^{k+1}}$  1's. But since each vertex of  $Q_{2^{k+1}}$  has degree  $2k + 1$ , each column sum is  $2k + 1$ , and thus each column has at least one 1. Thus there must be at least  $2^{2^{k+1}}$  1's in the matrix. Therefore,  $(2k + 1)2^{2^{k+1}} \geq 2^{2^{k+1}}$ . This is equivalent to  $2k + 1 \geq 2^k$ . But for  $k \geq 3$  this is clearly false. Thus for  $k \geq 3$ ,  $P_{2^k}$  does not divide  $Q_{2^{k+1}}$ .  $\square$

We will prove in Section 3 that for  $k = 2$ ,  $P_{2^k}$  does divide  $Q_{2^{k+1}}$ .

The next result is Proposition 8 of [9].

**Proposition 2** *Let  $n$  be odd, and suppose that  $P_k$  divides  $Q_n$ . Then  $k \leq n$ .*

**Lemma 1** *“Divisibility” is transitive, i.e. if  $G_1$  divides  $G_2$  and  $G_2$  divides  $G_3$ , then  $G_1$  divides  $G_3$ .*

*Proof.* This follows immediately from the definition of “divides”.  $\square$

**Corollary 1** *If  $k$  divides  $n$  then  $P_k$  divides  $Q_n$ .*

*Proof.* By [8], Theorem 2.3,  $T$  divides  $Q_n$  for every tree  $T$  on  $n$  edges. In particular, then,  $P_n$  divides  $Q_n$ . Clearly, if  $k$  divides  $n$  then  $P_k$  divides  $P_n$ . Hence, by Lemma 1,  $P_k$  divides  $Q_n$ .  $\square$

We have the following partial converse.

**Proposition 3** *If  $P_k$  divides  $Q_n$  and  $k$  is odd, then  $k$  divides  $n$ .*

*Proof.* Since  $P_k$  divides  $Q_n$ ,  $k$  divides  $n \cdot 2^{n-1}$ . But since  $k$  is odd, this means that  $k$  divides  $n$ .  $\square$

**Proposition 4** *If  $k$  divides  $n$  then  $E(Q_k)$  is a fundamental set for  $Q_n$ . In particular,  $Q_k$  divides  $Q_n$ .*

*Proof.* Let  $n = k \cdot m$ . Define the permutation

$$\theta : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$$

by

$$\theta(i) = i + k \pmod{n}.$$

Let  $\rho_\theta$  be the automorphism of  $Q_n$  defined by

$$\rho_\theta(x) = \theta(x).$$

The order of  $\theta$  is  $m$  and so the order of the group  $\langle \rho_\theta \rangle$  is also  $m$ . Let  $\mathcal{H} = \{\sigma_A \mid A \subset \{k+1, k+2, \dots, n\}\}$ .  $\mathcal{H}$  is a subgroup of the automorphism group of  $Q_{n-k}$ . Finally, let  $\mathcal{G}$  be the group generated by  $\mathcal{H}$  and  $\rho_\theta$ , *i.e.*

$$\mathcal{G} = \{\sigma_A \rho_{\theta^j} \mid A \subset \{k+1, k+2, \dots, n\}, 1 \leq j \leq m\}.$$

$\mathcal{G}$  is a subgroup of  $\text{Aut}(Q_n)$  of order  $m \cdot 2^{n-k}$ . Hence

$$|\mathcal{G}| \cdot |E(Q_k)| = m \cdot 2^{n-k} (k \cdot 2^{k-1}) = n \cdot 2^{n-1} = |E(Q_n)|.$$

To show that the translates of  $E(Q_k)$  under  $\mathcal{G}$  yield an edge decomposition of  $Q_n$  it remains to show that for all  $A \subset \{k+1, k+2, \dots, n\}$ , and for all  $1 \leq j \leq m$ ,

$$\sigma_A \rho_{\theta^j}(E(Q_k)) \cap E(Q_k) = \emptyset \text{ or } E(Q_k).$$

Now if  $e = \langle x, y \rangle \in E(Q_k)$ , its direction (the unique element in which its endpoints differ) is some  $i$ , then  $1 \leq i \leq k$ . The direction of  $\rho_{\theta^j}(e)$  is  $\theta^j(i)$ . Automorphisms of the type  $\sigma_A$  leave edge directions fixed, so the direction of  $\sigma_A \rho_{\theta^j}(e) = \theta^j(i)$ . From the definition of  $\theta$ ,  $\theta(i) = k+i$ . One can then show by induction on  $j$  that  $\theta^j(i) = jk+i$  for all  $j$ . Hence for  $1 \leq j \leq m-1$ ,  $\theta^j(i) > k$ . Since all edges of  $Q_k$  have direction  $\leq k$ , for  $1 \leq j \leq m-1$ ,  $\sigma_A \rho_{\theta^j}(e) \notin E(Q_k)$ ; thus  $\sigma_A \rho_{\theta^j}(E(Q_k)) \cap E(Q_k) = \emptyset$ . Now if  $j = m$ ,  $\theta^m$  equals  $id$ , so  $\sigma_A \rho_{\theta^m} = \sigma_A$ . But if  $\sigma_A \neq id$  then  $A \neq \emptyset$ . Since  $x, y \subseteq \{1, \dots, k\}$ ,  $\sigma_A(x), \sigma_A(y) \notin \{1, \dots, k\}$  and so  $\sigma_A(e) \notin E(Q_k)$ . Hence  $E(Q_k) \cap \sigma_A \rho_{\theta^j}(E(Q_k)) = \emptyset$ .

Thus  $E(Q_k)$  is a fundamental set for  $E(Q_n)$ . □

The converse to Proposition 4 follows easily from the next lemma.

**Lemma 2** *Suppose that the subgraph  $H$  of  $G$  edge-divides  $G$ . If  $G$  is  $n$ -regular and  $H$  is  $k$ -regular, then  $k$  divides  $n$ .*

*Proof.* Since the copies of  $E(H)$  form an edge-partition of  $E(G)$ , each vertex  $v$  of  $H$  must belong to exactly  $n/k$  copies of  $H$  and so  $k$  divides  $n$ .  $\square$

**Corollary 2** *If  $Q_k$  divides  $Q_n$  then  $k$  divides  $n$ .*

*Proof.* Since  $Q_k$  is  $k$ -regular and  $Q_n$  is  $n$ -regular, this follows immediately from Lemma 2.  $\square$

Combining Proposition 4 and Corollary 2 we obtain

**Proposition 5**  *$Q_k$  divides  $Q_n$  if and only if  $k$  divides  $n$ .*

As an immediate consequence of Lemma 1 and Proposition 4 we have

**Corollary 3** *If  $k$  divides  $n$  and if  $P_j$  divides  $Q_k$  then  $P_j$  divides  $Q_n$ .*

We have a more general consequence.

**Corollary 4** *If  $k$  divides  $n$  and  $T$  is any tree on  $k$  edges, then there is an embedding of  $T$  which divides  $Q_n$ . In fact, that embedding of  $T$  is a fundamental set for  $Q_n$ .*

*Proof.* By [8], Theorem 2.3, by mapping any given vertex of  $T$  to  $\emptyset$  and assigning distinct labels  $1, 2, \dots, k$  to the edges of  $T$  we get a subtree of  $Q_k$  isomorphic to  $T$  that divides  $Q_k$ , in which no two different edges have the same direction. (We will abuse notation slightly and refer to this tree as  $T$ .) Hence by Lemma 1 and Proposition 4,  $T$  divides  $Q_n$ .

For the second assertion, let  $\mathcal{G}_1 = \{\sigma_A \rho_{\theta^j} \mid A \subseteq \{k+1, k+2, \dots, n\}\}$ , where  $\theta$  is defined as in Proposition 4. Note that the order of  $\theta$  is  $m$ , where  $mk = n$ . Let  $\mathcal{G}_2 = \{\sigma_B \mid B \subseteq \{1, 2, \dots, k\}, |B| \text{ even}\}$ . Finally, let  $\mathcal{G} \subseteq \text{Aut } Q_n$  be the subgroup generated by  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . If  $\sigma_B \in \mathcal{G}_2$  then  $\rho_{\theta} \sigma_B = \sigma_{\theta(B)} \rho_{\theta}$ . Note that  $\theta(B) \subseteq \{k+1, k+2, \dots, n\}$  and thus  $\sigma_{\theta(B)} \rho_{\theta} \in \mathcal{G}_1$ . So for all  $j$  with  $1 \leq j \leq m-1$ ,  $A \cap \theta^j(B) = \emptyset$ .

$$A \Delta \theta^j(B) \Delta \theta^{j-1}(B) \cdots \Delta \theta(B) = A \cup [\theta^j(B) \Delta \theta^{j-1}(B) \Delta \cdots \Delta \theta(B)].$$

Hence  $\sigma_{A \Delta \theta^j(B) \Delta \cdots \Delta \theta(B)} \rho_{\theta} \in (\Sigma_n, \rho_{\theta})$ , the subgroup of  $\text{Aut}(Q_n)$  generated by  $\rho_{\theta}$  and  $\Sigma_n = \{\sigma_C \mid C \subseteq \{1, 2, \dots, n\}\}$ , which is a group isomorphic to the power set of  $\{1, 2, \dots, n\}$  under the operation of symmetric difference. First we shall show that the translates of  $E = E(T)$  are edge-disjoint. So suppose



that  $\sigma_C \rho_{\theta^j}(e) = e'$ , where  $e, e' \in E(T)$  and  $|C|$  is even. Let the direction of  $e$  be  $i$ . Then the direction of  $\rho_{\theta}(e)$  is  $\theta(i)$ . Since translations  $\sigma_A$  preserve the directions of edges, the direction of  $e'$  is also  $i$ . Thus  $\theta^j(i) = i$ . Since  $\theta$  is a product of disjoint  $m$ -cycles, this implies that  $j \equiv 0 \pmod{m}$ , and hence  $\theta^j = \theta^m = id$ . Thus  $\sigma_C(e) = e', e, e' \in T$ . But no two distinct edges of  $T$  have the same direction in  $Q_k$ . So  $\sigma_C$  either fixes both ends of  $e$  and is therefore the identity, or else interchanges the ends of  $e$ . But then  $|C| = 1$ , contradicting the assumption that  $|C|$  is even. So the translates of  $E(T)$  are disjoint.

Finally, we must show that every edge  $e \in E(Q_n)$  belongs to *some* translate of  $E(T)$ . Let  $e \in E(Q_n)$ . Then since  $E(T)$  is fundamental for  $E(Q_k)$ ,  $e = g'(e')$  for some  $e' \in E(T)$ , and for some  $g' \in \mathcal{G}_2$ . Since  $E(Q_k)$  is fundamental for  $Q_n$  relative to  $\mathcal{G}_1$ ,  $e' = g''(e'')$ , where  $e'' \in T$  and  $g'' \in \mathcal{G}_1$ . Thus  $e = g'g''(e'')$  for some  $e'' \in T$ . Since  $g'g'' \in \mathcal{G}$ , the translates of  $E(T)$  by elements of  $\mathcal{G}$  do cover  $E(Q_n)$ . So  $E(T)$  is fundamental for  $E(Q_n)$ .  $\square$

**Proposition 6** *If  $n$  is even, and  $j < n$  then  $P_{2^j}$  divides  $Q_n$ .*

*Proof.* It is proved in [1] that the cycle  $C_{2^n}$  divides  $Q_n$ . The Hamiltonian cycle  $C_{2^n}$  is divisible by any path  $P_q$ , as long as  $q$  divides  $2^n$  and  $q < 2^n$ . Thus  $C_{2^n}$  is divisible by  $P_{2^j}$  provided  $j < n$ . The result now follows from Lemma 1.  $\square$

**Proposition 7** *If  $n$  is even, and  $C$  is the  $2n$ -cycle with initial vertex  $\emptyset$ , and edge direction sequence  $(1, 2, \dots, n)^2 \stackrel{\text{def}}{=} (1, 2, \dots, n, 1, 2, \dots, n)$ , then  $Q_n$  is edge-decomposed by the copies of  $C$  under the action of  $\mathcal{G} = \{\sigma_A \mid A \subseteq \{1, 2, \dots, n-1\}, |A| \text{ even}\}$ . So  $E(C)$  is fundamental for  $Q_n$ .*

*Proof.*  $C$  consists of the path  $P$ , followed by  $\sigma_{\{1,2,\dots,n\}}(P)$ , where  $P$  is the path with initial vertex  $\emptyset$  and edge direction sequence  $1, 2, \dots, n$ . Note that for any  $B \subseteq \{1, 2, \dots, n\}$ , for any edge  $e$ ,  $\sigma_B(e) = e \implies B = \emptyset$  or  $|B| = 1$ . Now we shall show that for every subset  $A \subseteq \{1, 2, \dots, n-1\}$  with  $|A|$  even,  $\sigma_A(C) \cap C = \emptyset$ . It should be noted that these  $A$ 's form a subgroup of  $\text{Aut}(Q_n)$  of order  $2^{n-2}$ . So suppose that  $e = \langle x, y \rangle \in C \cap \sigma_A(C)$ . Let the direction of  $e$  be  $i$ . Then the direction of  $\sigma_A(e) = i$ . If  $A \neq \emptyset$ , then since  $|A|$  is even,  $\sigma_A(e) \neq e$ . The only other edge in  $C$  with direction  $i$  is  $\sigma_{\{1,2,\dots,n\}}(e)$ . So if  $\sigma_A(e) \in C$ , then  $\sigma_A(e) = \sigma_{\{1,2,\dots,n\}}(e)$ . Therefore  $\sigma_A \cdot \sigma_{\{1,2,\dots,n\}}(e) = e$ , *i.e.*  $\sigma_{A \Delta \{1,2,\dots,n\}}(e) = e$ . Since  $A$  and  $\{1, 2, \dots, n\}$  are

even, so is  $A\Delta\{1, 2, \dots, n\} = \overline{A}$ . Hence  $A\Delta\{1, 2, \dots, n\} = \emptyset$ , i.e.  $A = \{1, 2, \dots, n\}$ . But  $n \notin A$ , so we have a contradiction.

Thus we have a group  $\mathcal{G}$  of automorphisms of  $C$  of order  $2^{n-2}$ , such that for  $g \in \mathcal{G}, g \neq id$ ,  $g(E(C)) \cap E(C) = \emptyset$ . Furthermore, since  $|E(C)| = 2n$ , it follows that  $|\mathcal{G}| \cdot |E(C)| = |E(Q_n)|$ . Hence by [8], Lemma 1.1, the translates of  $E(C)$  via the elements of  $\mathcal{G}$  form an edge decomposition of  $Q_n$ .  $\square$

**Corollary 5** *If  $n$  is even,  $k < n$  and  $k$  divides  $n$ , then  $P_{2k}$  divides  $Q_n$ .*

*Proof.* Since  $k$  divides  $n$ ,  $2k$  divides  $2n$ , and thus since  $2k < 2n$ ,  $P_{2k}$  divides the  $2n$ -cycle  $C$  of Proposition 7. Hence by Proposition 7,  $P_{2k}$  divides  $Q_n$ .  $\square$

**Corollary 6** *If  $n$  and  $k$  are both even and  $k$  divides  $n$ , and  $C$  is the  $2k$ -cycle with initial vertex  $\emptyset$ , and edge direction sequence  $(1, 2, \dots, k)^2$ , then  $C$  divides  $Q_n$ .*

*Proof.* By the proposition,  $C$  divides  $Q_k$ , and by Proposition 4,  $Q_k$  divides  $Q_n$ . The result now follows from Lemma 1.  $\square$

## 2 $P_4$ divides $Q_5$

If  $k$  is odd then by Proposition 3 and Lemma 1  $P_k$  divides  $Q_n$  and only if  $k$  divides  $n$ . Thus the smallest value of  $k$  for which Question (1) remains open is  $k = 4$ . Corollary 5 settles the matter in the affirmative when  $n$  is even and thus we now only need to consider the case of  $n$  odd. Example 3 shows that  $P_4$  does *not* divide  $Q_3$ .

In the next two sections we show that for all odd  $n$  with  $n \geq 5$ ,  $P_4$  divides  $Q_n$ . We first, in this section, prove the result for  $n = 5$ . The strategy is to find a subgraph  $G$  of  $Q_5$ , show that  $G$  divides  $Q_5$ , and then show that  $P_4$  divides  $G$ . In the next section we deduce the general case.

We define  $G$  as follows (see figure 1). First, some notation. For  $b, c \in \{0, 1\}$ ,  $Q_5^{(**bc)}$  denotes the 3-cube induced by the vertices  $x_1x_2x_3x_4x_5$  with  $x_4 = b$  and  $x_5 = c$ . If  $a \in \{0, 1\}$   $Q_5^{(**abc)}$  is the 2-cube induced by the vertices with  $x_3 = a, x_4 = b$ , and  $x_5 = c$ . We take  $G$  to be the union of (1) :  $Q_5^{(**00)}$ , with the edges of  $Q_5^{(*0*00)}$  deleted; (2) :  $Q_5^{(**10)}$  with all edges deleted except for  $\langle 01010, 01110 \rangle$  and  $\langle 11010, 11110 \rangle$ ; (3) :  $Q_5^{(**01)}$  with all edges deleted except for  $\langle 01101, 11101 \rangle$  and  $\langle 01001, 11001 \rangle$ ; (4): the 4 matching

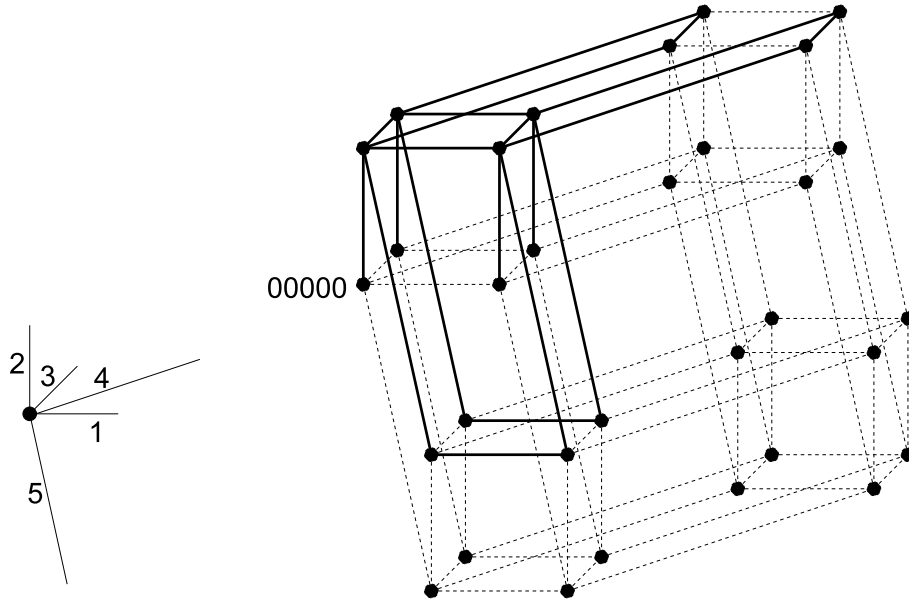


Figure 1:  $Q_5$  and the graph  $G$

edges between  $Q_5^{(*1*00)}$  and  $Q_5^{(*1*10)}$ ; and (5) the 4 matching edges between  $Q_5^{(*1*00)}$  and  $Q_5^{(*1*01)}$ . Thus  $|E(G)| = 20$ . Since  $|E(Q_5)| = 5 \cdot 2^4 = 80$ , we must exhibit  $80/20 = 4$  copies of  $E(G)$  that partition  $E(Q_5)$ .

**Lemma 3**  $G$  divides  $Q_5$ . In fact,  $E(G)$  is a fundamental set for  $Q_5$ .

*Proof.* By direct inspection of figure 2 the group of translations  $\mathcal{G} = \{id, \sigma_{24}, \sigma_{25}, \sigma_{45}\}$ , applied to  $E(G)$ , partitions  $E(Q_5)$ .  $\square$

**Lemma 4**  $P_4$  divides  $G$ .

*Proof.* It is easiest to describe the paths by their starting points and direction sequences (see figure 3).

Path	Starting Point	Direction Sequence
$A$	00000	2, 5, 1, 5
$B$	10100	2, 5, 1, 5
$C$	10000	2, 3, 1, 3
$D$	01000	1, 4, 3, 4
$E$	00100	2, 4, 3, 4

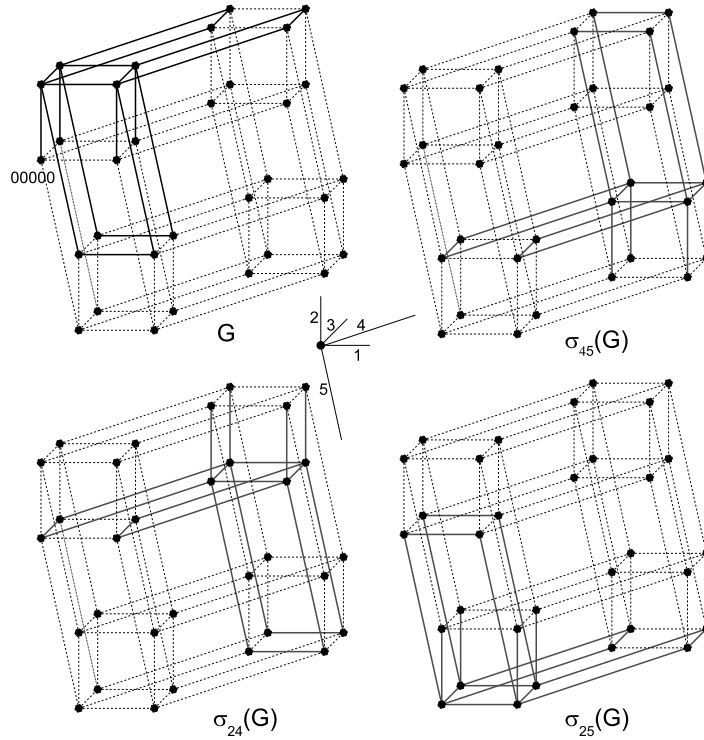


Figure 2:  $E(G)$  is a fundamental set for  $Q_5$

□

**Corollary 7**  $P_4$  divides  $Q_5$ .

*Proof.* This follows immediately from the previous two lemmas. □

### 3 $P_4$ divides $Q_n$ , for $n$ odd, $n \geq 5$

Let us write  $Q_5$  as  $Q_5 = Q_3 \square Q_2 = Q_3 \square C_4$ . Let  $G_0 = Q_5^{(***00)}$ ,  $G_1 = Q_5^{(***10)}$ ,  $G_2 = Q_5^{(***11)}$ ,  $G_3 = Q_5^{(***01)}$ . For  $i \in \{0, 1, 2, 3\}$  let  $\pi_i$  be the canonical mapping from  $G_i$  to  $Q_3$ .

\* From the decomposition of  $Q_5$  by  $P_4$  we have a coloring  $c : Q_5 \rightarrow$

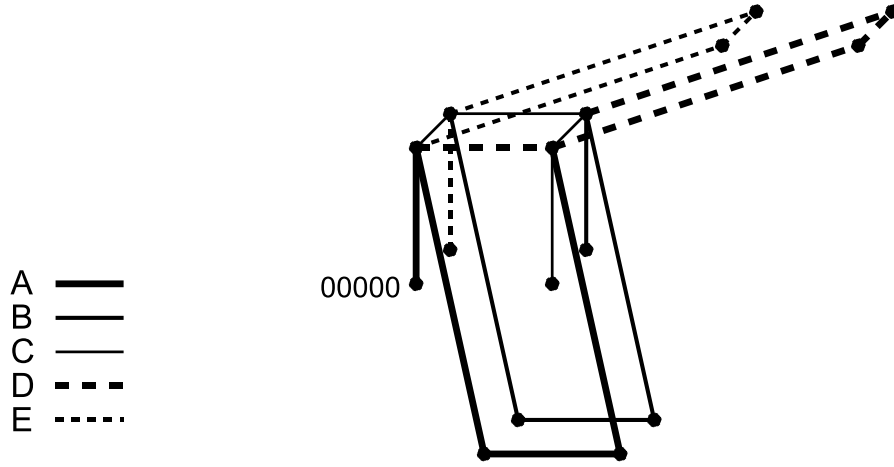


Figure 3:  $P_4$  divides  $G$

$\{1, 2, \dots, 20\}$  of the edges of  $Q_5$  such that for any  $i \in \{1, 2, \dots, 20\}$  the set of edges of  $Q_5$  colored  $i$  induces a  $P_4$ .

\* Consider now  $Q_3 \square C_{4k}$  for some  $k \geq 1$ . Let  $G'_0, \dots, G'_{4k-1} \simeq Q_3$ . Let  $\pi_{i'}$  be the canonical mapping from  $G'_{i'} \rightarrow Q_3$  for  $i' \in \{0, 1, \dots, 4k-1\}$ .

The edges of  $Q_3 \square C_{4k}$  are

Case A: the edges of  $G'_{i'}$ , for any  $i' \in \{0, 1, \dots, 4k-1\}$ .

Case B: for any  $i' \in \{0, 1, \dots, 4k-1\}$  the edges  $\langle x', y' \rangle$  for  $x' \in G'_{i'}$ ,  $y' \in G'_{j'}$ , where  $|j' - i'| \equiv 1 \pmod{4k}$  and  $\pi_{i'}(x') = \pi_{j'}(y')$ .

\* Let  $\theta$  be the mapping from  $Q_3 \square C_{4k} \rightarrow Q_5$  defined by: for any  $x' \in G'_{i'}$ ,  $\theta(x') = x$  where  $x$  is the element of  $G_i$ , with  $i \equiv i' \pmod{4}$  such that  $\pi_i(x) = \pi_{i'}(x')$ . (Note that  $\theta$  is not a one-to-one mapping.)

**Proposition 8** *If  $\langle x', y' \rangle$  is an edge of  $Q_3 \square C_{4k}$  then  $\langle \theta(x'), \theta(y') \rangle$  is an edge of  $Q_5$ .*

*Proof.*

Case A

$\langle x', y' \rangle \in G'_{i'}$ , for some  $i'$ . Then let  $i \equiv i' \pmod{4}$ . By the definition of  $\theta$ ,  $\theta(x') \in G_i$ ,  $\theta(y') \in G_i$ . This implies that  $\theta(x')$  and  $\theta(y')$  are adjacent.

Case B

Assume  $x' \in G'_{i'}, y' \in G'_{j'}$ , with  $|j' - i'| \equiv 1 \pmod{4k}$ . We have  $\pi'_{i'}(x') = \pi'_{j'}(y')$ . Then  $\theta(x') \in G_i$  and  $\theta(y') \in G_j$  where  $|j - i| \equiv 1 \pmod{4}$  since  $|j' - i'| \equiv 1 \pmod{4}$  implies that  $|j - i| \equiv 1 \pmod{4}$ . Furthermore

$$\pi_i(\theta(x')) \stackrel{\text{def of } \theta}{=} \pi(x') \stackrel{\text{edge}}{=} \pi(y') \stackrel{\text{def of } \theta}{=} \pi_j(\theta(y')).$$

Thus there exists an edge between  $\theta(x')$  and  $\theta(y')$  □

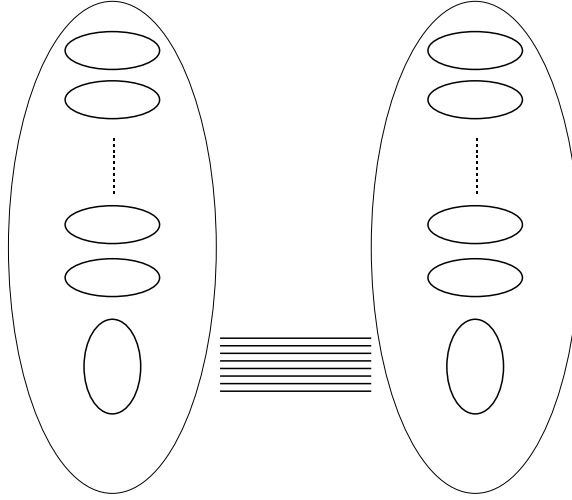


Figure 4: Decomposition of  $Q_{2m+1}$

**Definition 4** Consider the coloring  $E(Q_3 \square C_{4m}) \xrightarrow{c'} \{1, 2, \dots, 20\}$  of the edges of  $Q_3 \square C_{4m}$  defined by  $c'(\langle x', y' \rangle) = c(\langle \theta(x), \theta(y) \rangle)$ .

**Lemma 5** For any  $i \in \{1, 2, \dots, 20\}$  the set of edges of  $Q_3 \square C_{4k}$  such that  $c'(x', y') = i$  is a set of disjoint paths of length 4. Therefore  $P_4$  divides  $Q_3 \square C_{4m}$  for all  $m \geq 1$ .

*Proof.* By definition of  $c'$ , for any vertex  $x'$  of  $Q_3 \square C_{4k}$  the number of edges incident to  $x'$  colored  $i$  by  $c'$  is the number of edges incident to  $\theta(x')$  colored  $i$  by  $c$ . Therefore this number is  $\leq 2$ . Furthermore, there is no cycle colored

$i$  in  $Q_3 \square C_{4k}$  because the image by  $\theta$  of this cycle would be a cycle of  $Q_5$  colored  $i$  with  $c$ . Therefore the set of edges colored  $i$  by  $c'$  is a forest and more precisely, because of the degree, a set of disjoint paths. Notice that the image by  $\theta$  of a path colored  $i$  is a path of  $Q_5$  of the same length (because of the degree of the endpoints of the paths). Therefore all the paths are of length 4.  $\square$

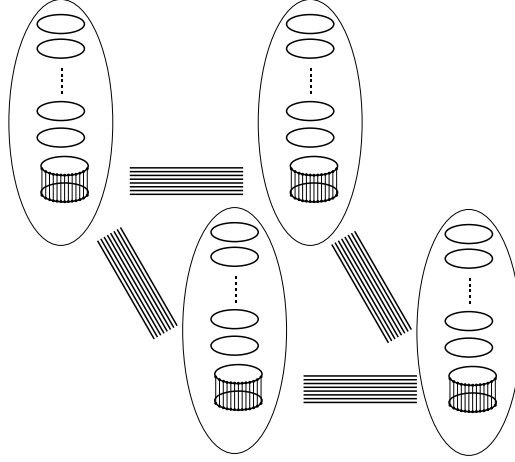


Figure 5: Decomposition of  $Q_{2m+3}$

**Theorem 2** For  $n \geq 4$ ,  $P_4$  divides  $Q_n$ .

*Proof.* If  $n$  is even, the result is true by Corollary 5. If  $n = 5$  then we are done by Corollary 7. Consider  $Q_{2m+3}$ , for  $m \geq 2$ .  $Q_{2m+3} = Q_{2m+1} \square Q_2$ .  $E(Q_{2m})$  can be decomposed into  $m$  cycles of length  $2^{2m}$  (Hamiltonian cycles) by Aubert and Schneider [1]. Let  $D$  be one of these cycles. The edges of  $Q_{2m+1}$  are the edges of the two copies of  $Q_{2m}$  and a matching. But every vertex of  $Q_{2m}$  appears exactly once in  $D$  so  $E(Q_{2m+1})$  can be decomposed into  $2(m-1)$  cycles of length  $2^{2m}$  and  $D \square Q_1 \simeq C_{2^{2m}} \square Q_1$  (see figure 4).

Every vertex of  $Q_{2m+1}$  appears once in  $D \square Q_1$ , thus, for the same reason,  $E(Q_{2m+3})$  can be decomposed into  $8(m-1)$  cycles of length  $2^{2k}$  and  $D \square Q_1 \square Q_2 \simeq C_{2^{2k}} \square Q_1 \square Q_2 \simeq C_{2^{2m}} \square Q_3$  (see figure 5).

Since  $k \geq 2$ ,  $\frac{2^{2m}}{4}$  is an integer strictly greater than 1 so the cycles of length  $2^{2m}$  are divisible by  $P_4$ . By Lemma 5,  $P_4$  divides  $C_{2^{2m}} \square Q_3$ , and  $P_4$  divides  $E(Q_n)$  for any odd  $n \geq 5$ .  $\square$

## 4 $Q_{2^k}$ has a fundamental Hamiltonian cycle.

**Remark.** We say that the direction of an edge  $\langle x, y \rangle$  of  $Q_n$  is  $i$  if  $x\Delta y = \{i\}$ . We shall describe walks in the hypercube by specifying the starting vertex (generally  $\emptyset$ ) and the sequence of edge directions.

It is well-known that the  $n$ -dimensional hypercube  $Q_n$  is Hamiltonian, and in fact has many Hamiltonian cycles. Aubert and Schneider [1] proved that for  $n$  even,  $Q_n$  has an edge decomposition into Hamiltonian cycles. However, their construction is technical. In contrast, in this last section we shall prove that for  $n = 2^k$ , there is a single Hamiltonian cycle  $C$  such that  $E(C)$  is a fundamental set for  $Q_n$ .

By  $G_1 \square G_2$  we denote the Cartesian product of the graphs  $G_1$  and  $G_2$ . We will start with two easy results about Cartesian product of graphs.

**Lemma 6** *Assume that  $\{C^1, C^2, \dots, C^p\}$  is an edge decomposition in Hamiltonian cycles of a graph  $G$ . Then  $\{C^1 \square C^1, C^2 \square C^2, \dots, C^p \square C^p\}$  is an edge decomposition of  $G \square G$ .*

*Proof.* Let  $(x_1, x_2)$  and  $(y_1, y_2)$  be adjacent in  $G \square G$ . Then either  $x_1$  and  $y_1$  are adjacent in  $G$  and  $x_2 = y_2$  or  $x_1 = y_1$  and  $x_2$  and  $y_2$  are adjacent in  $G$ . By symmetry, it is sufficient to consider the first case. Let  $i$  be such that  $\langle x_1, y_1 \rangle \in E(C^i)$ . Then since  $C^i$  is Hamiltonian  $x_2 = y_2 \in V(C^i)$ ; thus  $\langle (x_1, x_2), (y_1, y_2) \rangle \in E(C^i \square C^i)$ . Conversely  $\langle (x_1, x_2), (y_1, y_2) \rangle \in E(C^j \square C^j)$  implies  $\langle x_1, y_1 \rangle \in E(C^j)$  since  $x_2 = y_2$ ; thus  $j = i$ . Therefore the  $C^j \square C^j$ 's are disjoint and the conclusion follows.  $\square$

**Lemma 7** *Let  $G_1$  and  $G_2$  be any two graphs, and for  $i = 1, 2$  let  $\phi_i \in \text{Aut}(G_i)$ . Define  $(\phi_1, \phi_2) : G_1 \square G_2 \rightarrow G_1 \square G_2$  by  $(\phi_1, \phi_2)((x, y)) = (\phi_1(x), \phi_2(y))$ . Then  $(\phi_1, \phi_2) \in \text{Aut}(G_1 \square G_2)$ .*

*Proof.* Let  $(x_1, x_2)$  and  $(y_1, y_2)$  be adjacent in  $G_1 \square G_2$ . Then either (1)  $x_1$  and  $y_1$  are adjacent in  $G_1$  and  $x_2 = y_2$  or (2)  $x_1 = y_1$  and  $x_2$  and  $y_2$  are adjacent in  $G_2$ . We must show that  $(\phi_1, \phi_2)(x_1, x_2)$  and  $(\phi_1, \phi_2)(y_1, y_2)$  are adjacent in  $G_1 \square G_2$ . By symmetry, it is sufficient to prove this for case (1). But then since  $\phi_1 \in \text{Aut}(G_1)$ ,  $\phi_1(x_1)$  and  $\phi_1(y_1)$  are adjacent in  $G_1$ , and since  $x_2 = y_2$ ,  $\phi_2(x_2) = \phi_2(y_2)$ . Therefore  $(\phi_1, \phi_2)(x_1, x_2)$  and  $(\phi_1, \phi_2)(y_1, y_2)$  are adjacent in  $G_1 \square G_2$ . Conversely if  $(\phi_1, \phi_2)(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$  and  $(\phi_1, \phi_2)(y_1, y_2) = (\phi_1(y_1), \phi_2(y_2))$  are adjacent in  $G_1 \square G_2$  then  $\phi_1(x_1) = \phi_1(y_1)$  or  $\phi_2(x_2) = \phi_2(y_2)$ . We can assume the first case by symmetry then  $x_1 = y_1$



and  $x_2$  is adjacent to  $y_2$  in  $G_2$ . Thus  $(x_1, x_2)$  and  $(y_1, y_2)$  are adjacent in  $G_1 \square G_2$  and  $(\phi_1, \phi_2) \in \text{Aut}(G_1 \square G_2)$ .  $\square$

The starting point of the theorem of Aubert and Schneider is an earlier result of G. Ringel [10] who proved that for  $n = 2^k$ ,  $Q_n$  has an edge decomposition into Hamiltonian cycles. His proof is by induction on  $k$ . Let us recall the induction step. Let  $m = 2^n$ . Let  $\theta$  be the mapping from  $\{1, \dots, n\}$  to  $\{n+1, \dots, 2n\}$  defined by  $\theta(i) = i + n$ . Let  $C$  be a Hamiltonian cycle of  $Q_n$  then we can construct  $\Phi(C)$  and  $\Gamma(C)$  two disjoint Hamiltonian cycles of  $Q_{2n} = Q_n \square Q_n$  such that  $E(C \square C) = E(\Phi(C)) \cup E(\Gamma(C))$ . Indeed fix an arbitrary vertex (say 0) and represent  $C$  by the sequence of directions  $C = (c_1, \dots, c_m)$  then consider

$$\Phi(C) = ( \begin{array}{ll} c_1, \dots & \dots, c_{m-1}, c_{\theta(c_1)}, \\ c_m, c_1, \dots & \dots, c_{m-2}, c_{\theta(c_2)}, \\ c_{m-1}, c_m, c_1, \dots & \dots, c_{m-3}, c_{\theta(c_3)}, \\ \dots & \dots \\ c_2, \dots & \dots, c_m, c_{\theta(c_m)}, \end{array} )$$

and

$$\Gamma(C) = ( \begin{array}{ll} c_{\theta(1)}, \dots & \dots, c_{\theta(m-1)}, c_1, \\ c_{\theta(m)}, c_{\theta(1)}, \dots & \dots, c_{\theta(m-2)}, c_2, \\ c_{\theta(m-1)}, c_{\theta(m)}, c_{\theta(1)}, \dots & \dots, c_{\theta(m-3)}, c_3, \\ \dots & \dots \\ c_{\theta(2)}, \dots & \dots, c_{\theta(m)}, c_m, \end{array} )$$

It is immediate to check (see figure 6) that  $\Phi(C)$  and  $\Gamma(C)$  are disjoint and define a partition of the edges of  $C \square C$ . For  $n$  even let  $p = n/2$  and assume that  $\{C^1, C^2, \dots, C^p\}$  is an edge decomposition of  $Q_n$  in Hamiltonian cycles then as a consequence of Lemma 6,  $\{\Phi(C^1), \Phi(C^2), \dots, \Phi(C^p)\} \cup \{\Gamma(C^1), \Gamma(C^2), \dots, \Gamma(C^p)\}$  is an edge decomposition of  $Q_{2n}$  in Hamiltonian cycles.

**Theorem 3** *For any  $k \geq 1$ ,  $Q_{2^k}$  has a Hamiltonian cycle that is a fundamental set.*

*Proof.* This is trivial for  $k = 1$  since  $Q_2 = C_4$ . The desired result follows by induction from Ringel's construction. Indeed let  $n = 2^k, k \geq 1$  and assume that there exists an edge decomposition  $\{C^1, C^2, \dots, C^p\}$  of  $Q_n$  obtained as the translate of an Hamiltonian cycle  $C^1$  under some subgroup  $\mathcal{E}$  of  $\text{Aut}(Q_n)$ . For any automorphism  $\phi \in \text{Aut}(Q_n)$ ,  $(\phi, \phi) \in \text{Aut}(Q_{2n})$  by Lemma 7. Furthermore if  $\phi(C^1) = C^i$  then  $(\phi, \phi)(\Phi(C^1)) = \Phi(C^i)$  and

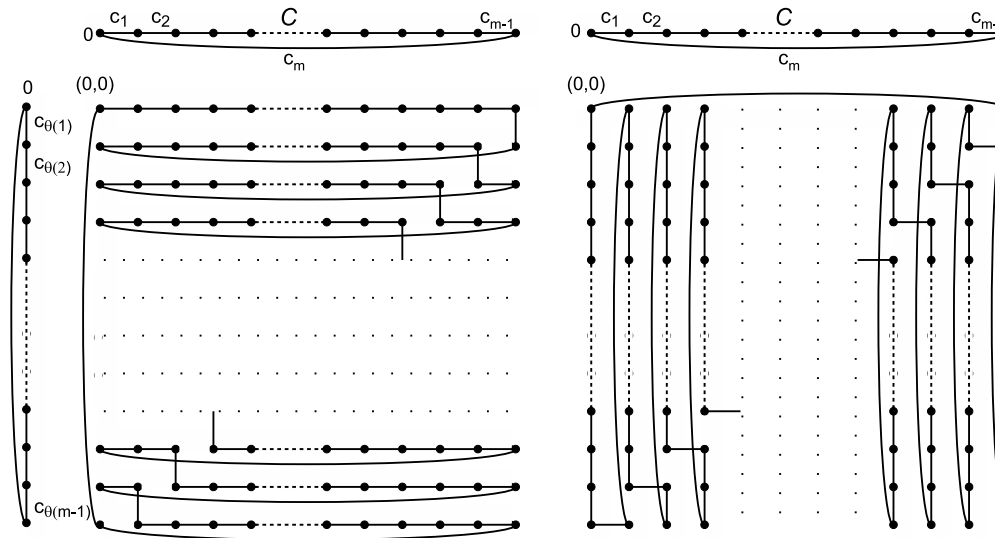


Figure 6: Construction of  $\Phi(C)$  and  $\Gamma(C)$  from  $C$

$(\phi, \phi)(\Gamma(C^1)) = \Gamma(C^i)$ . If we consider now the permutation  $\theta$  on  $\{1, \dots, 2n\}$  defined by  $\theta(i) = i + n \pmod{2n}$  then  $\rho_\theta(\Phi(C^i)) = \Gamma(C^i)$ . The conclusion follows since the subgroup of  $\text{Aut}(Q_{2n})$ , isomorph to  $\mathcal{E} \times S_2$ , defined by  $\mathcal{H} = \{(\phi, \phi); \phi \in \mathcal{E}\} \cup \{\rho_\theta \circ (\phi, \phi); \phi \in \mathcal{E}\}$  is such that  $\{\Phi(C^1), \Phi(C^2), \dots, \Phi(C^p)\} \cup \{\Gamma(C^1), \Gamma(C^2), \dots, \Gamma(C^p)\}$  are the translate of  $\Phi(C^1)$  under  $\mathcal{H}$ .  $\square$

**Corollary 8** *For  $n$  and  $m$  each a power of 2, with  $m \leq n$ , there is an  $m$ -cycle that divides  $Q_n$ .*

*Proof.* Let  $m = 2^p$ . By Theorem 3  $Q_m$  has a fundamental  $2^p$ -cycle, which therefore divides  $Q_m = Q_{2^p}$ . Since  $m$  and  $n$  are each powers of two,  $m$  divides  $n$ . Hence by Proposition 4 and Lemma 1, this cycle divides  $Q_n$ .  $\square$

## 5 Summary of Results

1. For  $k$  odd, if  $P_k$  is a path on  $k$  edges that divides  $Q_n$ , then  $k$  divides  $n$ . (Proposition 3)
2. If  $k$  divides  $n$ , any tree on  $k$  edges is a fundamental set for  $Q_n$ . (Corollary 4)
3. If  $k$  divides  $n$  and  $k < n$  then  $P_{2k}$  divides  $Q_n$ . (Corollary 5)
4. If  $n$  is even and  $j < n$  then  $P_{2j}$  divides  $Q_n$ . (Proposition 6)
5. For  $k = 2n$  there is a  $k$ -cycle which is a fundamental set for  $Q_n$  when  $n$  is even. (Proposition 7)
6. For  $n =$  a power of 2, there is a Hamiltonian cycle which is a fundamental set for  $Q_n$ . (Theorem 3)
7. For  $n =$  a power of 2 and  $m =$  a power of 2, with  $m \leq n$ , there is an  $m$ -cycle that divides  $Q_n$ . (Corollary 8)
8. For  $n \geq 4$ ,  $P_4$  divides  $Q_n$ . (Theorem 2)
9.  $Q_k$  is a fundamental set for  $Q_n$  if and only if  $k$  divides  $n$ . (Proposition 5)
10. For  $k \geq 3$ ,  $P_{2j}$  does *not* divide  $Q_{2k+1}$ . (Proposition 1)

## References

- [1] Aubert and Schneider, *Décomposition de la somme Cartésienne d'un cycle et de l'union de deux cycles Hamiltoniens en cycles Hamiltoniens*, Disc. Math. **38**, (1982), 7 – 16.
- [2] Darryn E. Bryant, Saad El-Zanati, Charles Vanden Eynden, and Dean G. Hoffman, *Star decompositions of cubes*, Graphs and Comb. **17**, (2001), no.1, 55–59.
- [3] J.F. Fink, *On the decomposition of  $n$ -cubes into isomorphic trees*, J. Graph Theory **14**, (1990), 405–411.
- [4] E. Gilbert, *Gray codes and paths on the  $n$ -cube*, Bell System Tech. J. **37**, (1958) 815–826.
- [5] T. Kirkman, *On a Problem in Combinatorics*, The Cambridge and Dublin Math. J. **2**, (1847), 191–204.

- [6] T. Kirkman, *Note on an unanswered prize question*”, The Cambridge and Dublin Math. J. **5**, (1850), 258–262.
- [7] F. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes*, M. Kaufmann Publishers, San Mateo, California, 1992.
- [8] M. Ramras, *Symmetric edge-decompositions of hypercubes*, Graphs and Comb. **7**, (1991), 65–87.
- [9] M. Ramras, *Fundamental Subsets of Edges of Hypercubes*, Ars Combinatoria **46** (1997), 3–24.
- [10] G. Ringel, *Über drei kombinatorische Probleme am  $n$ -dimensionalen Würfel und Würfelgitter*, Abh. Math. Sem. Univ. Hamburg **20** (1955), 10 – 15.
- [11] G. Ringel, *Problem 25*, Theory of Graphs and its Applications, Nakl. ČSĀN, Praha, (1964), p. 162.
- [12] S. Wagner; M. Wild, *Decomposing the hypercube  $Q_n$  into  $n$  isomorphic edge-disjoint trees*, Discrete Mathematics (2012), doi: 10.1016/j.disc.2012.01.033