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# Edge Decompositions of Hypercubes by Paths and by Cycles

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## Abstract

If  $H$  is (or is isomorphic to) a subgraph of  $G$ , we say that  $H$  *divides*  $G$  if there is an edge-decomposition of  $G$  by copies of  $H$ . Many authors ([1], [2], [3], [5], [6], [8]) have studied this notion for various subgraphs of hypercubes. We continue such a study in this paper.

## 1 Introduction and Preliminary Results

**Definition 1** *If  $H$  is (or is isomorphic to) a subgraph of  $G$ , we say that  $H$  divides  $G$  if there is an edge-decomposition of  $G$  by copies of  $E(H)$ .*

Ramras [5] has defined a more restrictive concept.

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**Definition 2** *A fundamental set of edges of a graph  $G$  is a subset of  $E(G)$  whose translates under some subgroup of the automorphism group of  $G$  partition  $E(G)$ .*

In [5] we have shown that if  $\mathcal{G}$  is a subgroup of  $\text{Aut}(Q_n)$  and for all  $g \in \mathcal{G}$ , with  $g \neq id$  (where  $id$  denotes the identity element),  $g(E(H)) \cap E(H) = \emptyset$ , then there is a packing of these translates of  $E(H)$  in  $Q_n$ , *i.e.* they are pairwise disjoint. If, in addition,  $|E(H)| \cdot |\mathcal{G}| = n \cdot 2^{n-1} = |E(Q_n)|$ , then the translates of  $E(H)$  by the elements of  $\mathcal{G}$  yield an edge decomposition of  $Q_n$ . In [5] it is shown that every tree on  $n$  edges can be embedded in  $Q_n$  as a fundamental set. In [6] this is extended to certain trees and certain cycles on  $2n$  edges. Decompositions of  $Q_n$  by  $k$ -stars are proved for all  $k \leq n$  in [2]. Recently, Wagner and Wild [8] have constructed, for each value of  $n$ , a tree on  $2^{n-1}$  edges that is a fundamental set for  $Q_n$ .

The structure of  $\text{Aut}(Q_n)$  is discussed in [5]. For each subset  $A$  of  $\{1, 2, \dots, n\}$ , the complementing automorphism  $\sigma_A$  is defined by  $\sigma_A(\vec{x}) = \vec{x} + \sum_{i \in A} e_i$ , where  $e_i$  is the  $n$ -tuple with a 1 in the  $i$ th coordinate and 0 elsewhere. Another type of automorphism arises from the group of permutations  $\mathcal{S}_n$  of  $\{1, 2, \dots, n\}$ . For  $\theta \in \mathcal{S}_n$ , we define  $\rho_\theta$  by  $\rho_\theta(x_1, x_2, \dots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \dots, x_{\theta(n)})$ . Every automorphism in  $\text{Aut}(Q_n)$  can be expressed uniquely in the form  $\sigma_A \circ \rho_\theta$ .

To avoid ambiguity in what follows we make this definition:

**Definition 3** *By  $P_k$ , the “ $k$ -path”, we mean the path with  $k$  edges.*

### Questions

- (1) For which  $k$  dividing  $n \cdot 2^{n-1}$  does  $P_k$  divide  $Q_n$ ?
- (2) For which  $k$  dividing  $n \cdot 2^{n-1}$  does  $C_k$ , the cycle on  $k$  edges, divide  $Q_n$ ?
- (3) For those  $k$  for which the answer to either (1) or (2) is “yes”, is the edge set a fundamental set for  $Q_n$ ?

We begin this introductory section with some examples. In later sections we prove a variety of results relating to these questions, and in the final section we summarize our findings.

### Example 1

Let  $T$  be the 2-star (= the 2-path) contained in  $Q_3$  with center 000, and leaves 100, 010. Then  $\mathcal{G} = \{id, \sigma_{123}, \sigma_{1\rho(123)}, \sigma_{12\rho(132)}, \sigma_{3\rho(132)}, \sigma_{23\rho(123)}\}$  is a (cyclic) subgroup of  $\text{Aut}(Q_3)$  of order 6, and the 6 translates of  $T$  under  $\mathcal{G}$  yield an edge decomposition of  $Q_3$ .  $\square$

Note, however, that  $\mathcal{G}$  does not work for the 2-star  $T'$ , whose center is 000 and whose leaves are 100 and 001. The subgroup which works for this 2-star is  $\mathcal{G}' = \{id, \sigma_{123}, \sigma_1\rho_{(132)}, \sigma_{13}\rho_{(123)}, \sigma_2\rho_{(123)}, \sigma_{23}\rho_{(132)}\}$ .

### Example 2

$P_6$  does not divide  $Q_3$ . For since  $Q_3$  has 12 edges, if  $P_6$  *did* divide  $Q_3$  then  $Q_3$  would have an edge-decomposition consisting of 2 copies of  $P_6$ . The degree sequence of  $P_6$  is 2, 2, 2, 2, 2, 1, 1, whereas  $Q_3$ , of course, is 3-regular. So 5 vertices meet 2 edges of one copy of  $P_6$ , and another 5 vertices meet the other copy of  $P_6$ . Since  $Q_3$  has only 8 vertices, the 2 5-sets must have at least 2 vertices in common. But that would force these 2 vertices to have degree 4, a contradiction.  $\square$

### Example 3

$P_4$  does not divide  $Q_3$ . Since  $P_4$  has 4 edges, we would need 3 copies of  $P_4$  for an edge-decomposition of  $Q_3$ . The degree sequence of  $P_4$  is 2, 2, 2, 1, 1, 0, 0, 0. Call the three copies of  $P_4$   $P^{(1)}$ ,  $P^{(2)}$ , and  $P^{(3)}$ . At each vertex  $v$  of  $Q_3$ ,  $\sum_{1 \leq i \leq 3} \deg_{P^{(i)}}(v) = 3$ . For each  $v$  with  $\deg_{P^{(1)}}(v) = 0$  or 2, exactly one of  $\deg_{P^{(2)}}(v)$  or  $\deg_{P^{(3)}}(v)$  is 1. Thus a total of  $6 + 2 = 8$  1's are required. However, each of the 3 copies of  $P_4$  has exactly 2 vertices of degree 1, and so the total number of 1's in the  $3 \times 8$  array  $(a_{iv})$  formed by the 3 degree sequences is 6. Hence  $P_4$  does not divide  $Q_3$ .  $\square$

### Example 4

The *other* tree on 4 edges *does* divide  $Q_3$ . Let  $T =$  the 3-star centered at 000 union the edge  $\langle 001, 101 \rangle$ . Let  $\mathcal{G} = \langle \sigma_{23}\rho_{(123)} \rangle$ , which is a cyclic subgroup of  $\text{Aut}(Q_3)$  of order 3. A straight-forward calculation shows that the translates of  $T$  under  $\mathcal{G}$  form an edge decomposition of  $Q_3$ .  $\square$

**Proposition 1** For  $k \geq 3$ ,  $P_{2^k}$  does not divide  $Q_{2^{k+1}}$ .

*Proof.* Suppose that  $k \geq 3$ , and suppose that  $P_{2^k}$  divides  $Q_{2^{k+1}}$ . The matrix  $(a_{iv})$  has  $2^{2^{k+1}}$  columns, and

$$(2k + 1) \cdot 2^{2^k} / 2^k = (2k + 1)2^k$$

rows. Then since each row has exactly two 1's, the entire matrix has  $(2k + 1)2^{k+1}$  1's. But since each vertex of  $Q_{2k+1}$  has degree  $2k + 1$ , each column sum is  $2k + 1$ , and thus each column has at least one 1. Thus there must be at least  $2^{2k+1}$  1's in the matrix. Therefore,  $(2k + 1)2^{k+1} \geq 2^{2k+1}$ . This is equivalent to  $2k + 1 \geq 2^k$ . But for  $k \geq 3$  this is clearly false. Thus for  $k \geq 2$ ,  $P_{2^k}$  does not divide  $Q_{2k+1}$ .  $\square$

We can generalize this proposition.

**Lemma 1** *Given any positive integer  $j$  there exists an integer  $k_0 > 0$  such that for all  $k$  with  $k \geq k_0$ ,  $2^{k-j-1} > k$ .*

**Corollary 1** *For any positive integer  $j$  there exists an integer  $k_0$  such that for all integers  $k \geq k_0$ ,  $P_{2^{k-j}}$  does not divide  $Q_{2k+1}$ .*

*Proof.* Choose  $k_0$  by the lemma. We argue as in the proof of Proposition 1. Suppose that  $P_{2^{k-j}}$  divides  $Q_{2k+1}$ . Counting the number of 1's appearing in the matrix  $(a_{iv})$  we get the inequality  $(2k + 1)2^{k-j+1} \geq 2^{2k+1}$ . This is equivalent to  $2k + 1 \geq 2^{k-j}$ , i.e.  $2k \geq 2^{k-j}$ . This in turn is equivalent to  $k \geq 2^{k-j-1}$ . But by the lemma, this is false for  $k \geq k_0$ . So for  $k \geq k_0$ ,  $P_{2^{k-j}}$  does not divide  $Q_{2k+1}$ .  $\square$

The next result is Proposition 8 of [6].

**Proposition 2** *Let  $n$  be odd, and suppose that  $P_k$  divides  $Q_n$ . Then  $k \leq n$ .*

**Lemma 2** *"Divisibility" is transitive, i.e. if  $G_1$  divides  $G_2$  and  $G_2$  divides  $G_3$ , then  $G_1$  divides  $G_3$ .*

*Proof.* This follows immediately from the definition of "divides".  $\square$

**Corollary 2** *If  $k$  divides  $n$  then  $P_k$  divides  $Q_n$ .*

*Proof.* By [5], Theorem 2.3,  $T$  divides  $Q_n$  for every tree  $T$  on  $n$  edges. In particular, then,  $P_n$  divides  $Q_n$ . Clearly, if  $k$  divides  $n$  then  $P_k$  divides  $P_n$ . Hence, by Lemma 2,  $P_k$  divides  $Q_n$ .  $\square$

We have the following partial converse.

**Proposition 3** *If  $P_k$  divides  $Q_n$  and  $k$  is odd, then  $k$  divides  $n$ .*

*Proof.* Since  $P_k$  divides  $Q_n$ ,  $k$  divides  $n \cdot 2^{n-1}$ . But since  $k$  is odd, this means that  $k$  divides  $n$ .  $\square$

**Proposition 4** *If  $k$  divides  $n$  then  $E(Q_k)$  is a fundamental set for  $Q_n$ . In particular,  $Q_k$  divides  $Q_n$ .*

*Proof.* Let  $n = k \cdot m$ . Define the permutation

$$\theta : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$$

by

$$\theta = (1, k+1, \dots, (m-1)k+1)(2, k+2, \dots, (m-1)k+2) \cdots (k, 2k, \dots, mk).$$

Let  $\rho_\theta$  be the automorphism of  $Q_n$  defined by

$$\rho_\theta(x_1, \dots, x_n) = x_{\theta(1)}, \dots, x_{\theta(n)}.$$

The order of  $\theta$  is  $m$  and so the order of the group  $\langle \rho_\theta \rangle$  is also  $m$ . Let  $\mathcal{H} = \{\sigma_A \mid A \subset \{k+1, k+2, \dots, n\}, |A| \text{ is even}\}$ .  $\mathcal{H}$  is a subgroup of the automorphism group of  $Q_{n-k}$ . Finally, let  $\mathcal{G}$  be the group generated by  $\mathcal{H}$  and  $\rho_\theta$ , *i.e.*

$$\mathcal{G} = \{\sigma_A \rho_{\theta^j} \mid A \subset \{k+1, k+2, \dots, n\}, |A| \text{ is even}, 1 \leq j \leq m, \}.$$

$\mathcal{G}$  is a subgroup of  $\text{Aut}(Q_n)$  of order  $m \cdot 2^{n-k-1}$ . Hence

$$|\mathcal{G}| \cdot |E(Q_k)| = m \cdot 2^{n-k-1} (k \cdot 2^k) = n \cdot 2^{n-1} = |E(n)|.$$

To show that the translates of  $E(Q_k)$  under  $\mathcal{G}$  yield an edge decomposition of  $Q_n$  it remains to show that for all  $1 \leq i \leq k$ ,  $A \subset \{1, 2, \dots, k\}$ ,  $|A|$  even and for all  $1 \leq j \leq m$ ,

$$\sigma_A \rho_{\theta^j}(E(Q_k)) \cap E(Q_k) = \emptyset.$$

Now if  $e \in E(Q_k)$ , its direction (the coordinate in which its endpoints differ) is some  $i$ ,  $1 \leq i \leq k$ . The direction of  $\rho_{\theta^j}(e)$  is  $\theta^j(i)$ . Automorphisms of the type  $\sigma_A$  leave edge directions fixed, so the direction of  $\sigma_A \rho_{\theta^j}(e) = \theta^j(i)$ . From the definition of  $\theta$ ,  $\theta(i) = k+i$ . One can then show by induction on  $j$  that  $\theta^j(i) = jk+i$  for  $1 \leq j \leq m-1$ . Hence  $\theta^j(i) > k$ . Since all edges of  $Q_k$  have direction  $\leq k$ ,  $\sigma_A \rho_{\theta^j}(e) \neq id$ . Now if  $j = m$ ,  $\theta^m$  equals  $id$ , so

$\sigma_A \rho_{\theta^m} = \sigma_A$ . But if  $\sigma_A(e) = e$ , and  $\sigma_A$  is not the  $id$ , then  $A \neq \emptyset$ , and so  $\sigma_A$  interchanges the ends of edge  $e$ . This is impossible since  $|A|$  is even, by hypothesis.  $\square$

As an immediate consequence of Lemma 2 and Proposition 4 we have

**Corollary 3** *If  $k$  divides  $n$  and if  $P_j$  divides  $Q_k$  then  $P_j$  divides  $Q_n$ .*

We have a more general consequence.

**Corollary 4** *If  $k$  divides  $n$  and  $T$  is any tree on  $k$  edges, then  $T$  divides  $Q_n$ . In fact,  $T$  is a fundamental set for  $Q_n$ .*

*Proof.* By [5], Theorem 2.3,  $T$  divides  $Q_k$ . Hence by Lemma 2 and Proposition 4,  $T$  divides  $Q_n$ .

For the second assertion, let  $\mathcal{G}_1 = \{\sigma_A \rho_{\theta^j} \mid A \subseteq \{k+1, k+2, \dots, n\}, |A| \text{ even}\}$ , where  $\theta$  is defined as in Proposition 4. Note that the order of  $\theta$  is  $m$ , where  $mk = n$ . Let  $\mathcal{G}_2 = \{\sigma_B \mid B \subseteq \{1, 2, \dots, k\}, |B| \text{ even}\}$ . Finally, let  $\mathcal{G} \subseteq \text{Aut } Q_n$  be the subgroup generated by  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . If  $\sigma_B \in \mathcal{G}_2$  then  $\rho_{\theta} \sigma_B = \sigma_{\theta(B)} \rho_{\theta}$ . Note that  $\theta(B) \subseteq \{k+1, k+2, \dots, n\}$  and thus  $\sigma_{\theta(B)} \rho_{\theta} \in \mathcal{G}_1$ . So for all  $j$  with  $1 \leq j \leq m$ ,

$$\sigma_A \rho_{\theta^j} \sigma_B = \sigma_A \rho_{\theta^{j-1}} \sigma_{\theta(B)} \rho_{\theta} = \dots = \sigma_{A \Delta \theta^j(B) \Delta \theta^{j-1}(B) \dots \Delta \theta(B)} \rho_{\theta^j}.$$

$|A \Delta \theta^j(B) \Delta \dots \Delta \theta(B)|$  is even, and furthermore,

$$A \Delta \theta^j(B) \Delta \dots \Delta \theta(B) = A \cup [\theta^j(B) \Delta \theta^{j-1}(B) \Delta \dots \Delta \theta(B)].$$

Hence  $\sigma_{A \Delta \theta^j(B) \Delta \dots \Delta \theta(B)} \rho_{\theta^j} \in (\Sigma_n, \rho_{\theta^j})$ , where  $\Sigma_n = \{\sigma_C \mid C \subseteq \{1, 2, \dots, n\}\}$ . First we shall show that the translates of  $E = E(T)$  are edge-disjoint. So suppose that  $\sigma_C \rho_{\theta^j}(e) = e'$ , where  $e, e' \in E(T)$  and  $|C|$  is even. Let the direction of  $e$  be  $i$ . Then the direction of  $\theta(e)$  is  $\theta(i)$ . Since translations  $\sigma_A$  preserve the directions of edges, the direction of  $e'$  is also  $i$ . Thus  $\theta^j(i) = i$ . Since  $\theta$  is a product of disjoint  $m$ -cycles, this implies that  $j = m$ , and hence  $\theta^j = \theta^m = id$ . Thus  $\sigma_C = id$ . Hence  $\sigma_C \rho_{\theta^j} = id$ , and so the translates of  $E(T)$  are disjoint.

Finally, we must show that every edge  $e \in E(Q_n)$  belongs to some translate of  $E(T)$ . Let  $e \in E(Q_n)$ . Then since  $E(T)$  is fundamental for  $E(Q_k)$ ,  $e = g'(e')$  for some  $e' \in E(T)$ , and for some  $g' \in \mathcal{G}_2$ . Since  $E(Q_k)$  is fundamental for  $Q_n$  relative to  $\mathcal{G}_1$ ,  $e' = g''(e'')$ , where  $e'' \in T$  and  $g'' \in \mathcal{G}_1$ . Thus  $e = g'g''(e'')$  for some  $e'' \in T$ . Since  $g'g'' \in \mathcal{G}$ , the translates of  $E(T)$  by elements of  $\mathcal{G}$  do cover  $E(Q_n)$ . So  $E(T)$  is fundamental for  $E(Q_n)$ .  $\square$

**Proposition 5** *If  $n$  is even, and  $j < n$  then  $P_{2^j}$  divides  $Q_n$ .*

*Proof.* It is proved in [1] that the cycle  $C_{2^n}$  divides  $Q_n$ . The Hamiltonian cycle  $C_{2^n}$  is divisible by any path  $P_q$ , as long as  $q$  divides  $2^n$  and  $q < 2^n$ . Thus  $C_{2^n}$  is divisible by  $P_{2^j}$  provided  $j < n$ . The result now follows from Lemma 2.  $\square$

**Proposition 6** *If  $n$  is even, and  $C$  is the  $2n$ -cycle with initial vertex  $\emptyset$ , and edge direction sequence  $(1, 2, \dots, n)^2$ , then  $Q_n$  is edge-decomposed by the copies of  $C$  under the action of  $\mathcal{G} = \{\sigma_A \mid A \subset [n-1], |A| \text{ even}\}$ . So  $E(C)$  is fundamental for  $Q_n$ .*

*Proof.*  $C$  consists of the path  $P$ , followed by  $\sigma_{\{1,2,\dots,n\}}(P)$ , where  $P$  is the path with initial vertex  $\emptyset$  and edge direction sequence  $1, 2, \dots, n$ . Note that for any  $B \subseteq \{1, 2, \dots, n\}$ , for any edge  $e$ ,  $\sigma_B(e) = e \implies B = \emptyset$  or  $|B| = 1$ . Now we shall show that for every subset  $A \subset \{1, 2, \dots, n-1\}$  with  $|A|$  even,  $\sigma_A(C) \cap C = \emptyset$ . It should be noted that these  $A$ 's form a subgroup of  $\text{Aut}(Q_n)$  of order  $2^{n-2}$ . So suppose that  $e = \langle x, y \rangle \in C \cap \sigma_A(C)$ . Let the direction of  $e$  be  $i$ . Then the direction of  $\sigma_A(e) = i$ . If  $A \neq \emptyset$ , then since  $|A|$  is even,  $\sigma_A(e) \neq e$ . The only other edge in  $C$  with direction  $i$  is  $\sigma_{\{1,2,\dots,n\}}(e)$ . So if  $\sigma_A(e) \in C$ , then  $\sigma_A(e) = \sigma_{\{1,2,\dots,n\}}(e)$ . Therefore  $\sigma_A \cdot \sigma_{\{1,2,\dots,n\}}(e) = e$ , i.e.  $\sigma_{A\Delta\{1,2,\dots,n\}}(e) = e$ . Since  $A$  and  $\{1, 2, \dots, n\}$  are even, so is  $A\Delta\{1, 2, \dots, n\} = \overline{A}$ . Hence  $A\Delta\{1, 2, \dots, n\} = \emptyset$ , i.e.  $A = \{1, 2, \dots, n\}$ . But  $n \notin A$ , so we have a contradiction.

Thus we have a group  $\mathcal{G}$  of automorphisms of  $C$  of order  $2^{n-2}$ , such that for  $g \in \mathcal{G}, g \neq id$ ,  $g(E(C)) \cap E(C) = \emptyset$ . Furthermore, since  $|E(C)| = 2n$ , it follows that  $|\mathcal{G}| |E(C)| = |E(Q_n)|$ . Hence by [5], Lemma 1.1, the translates of  $E(C)$  via the elements of  $\mathcal{G}$  form an edge decomposition of  $Q_n$ .  $\square$

**Corollary 5** *If  $k < n$  and  $k$  divides  $n$  then  $P_{2^k}$  divides  $Q_n$ .*

*Proof.* Since  $k$  divides  $n$ ,  $2k$  divides  $2n$ , and thus since  $2k < 2n$ ,  $P_{2^k}$  divides the  $2n$ -cycle  $C$  of Proposition 6. Hence by Proposition 6,  $P_{2^k}$  divides  $Q_n$ .  $\square$

**Corollary 6** *If  $k$  is even and  $k$  divides  $n$ , and  $C$  is the  $2k$ -cycle with initial vertex  $\emptyset$ , and edge direction sequence  $(1, 2, \dots, k)^2$ , then  $C$  divides  $Q_n$ .*

*Proof.* By the proposition,  $C$  divides  $Q_k$ , and by Proposition 4,  $Q_k$  divides  $Q_n$ . The result now follows from Lemma 2.  $\square$



## 2 $P_4$ divides $Q_5$

If  $k$  is odd then by Proposition 3 and Lemma 2  $P_k$  divides  $Q_n$  and only if  $k$  divides  $n$ . Thus the smallest value of  $k$  for which Question (1) remains open is  $k = 4$ . Corollary 5 settles the matter in the affirmative when  $n$  is even and thus we now only need to consider the case of  $n$  odd. Example 3 shows that  $P_4$  does *not* divide  $Q_3$ .

In the next two sections we show that for all odd  $n$  with  $n \geq 5$ ,  $P_4$  divides  $Q_n$ . We first, in this section, prove the result for  $n = 5$ . The strategy is to find a subgraph  $G$  of  $Q_5$ , show that  $G$  divides  $Q_5$ , and then show that  $P_4$  divides  $G$ . In the next section we deduce the general case.

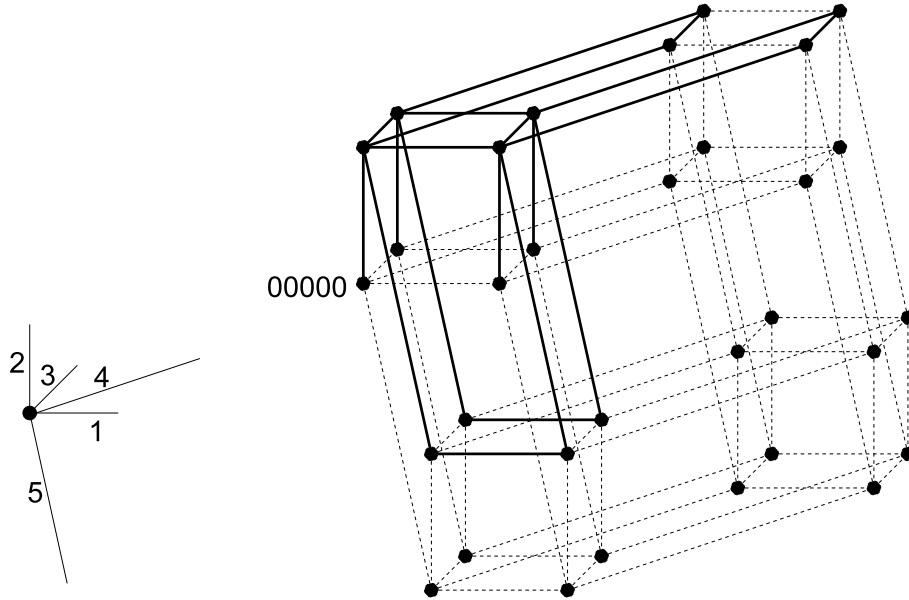


Figure 1:  $Q_5$  and the graph  $G$

We define  $G$  as follows (see figure 1). First, some notation. For  $b, c \in \{0, 1\}$ ,  $Q_5^{(**bc)}$  denotes the 3-cube induced by the vertices  $x_1x_2x_3x_4x_5$  with  $x_4 = b$  and  $x_5 = c$ . If  $a \in \{0, 1\}$   $Q_5^{(**abc)}$  is the 2-cube induced by the vertices with  $x_3 = a, x_4 = b$ , and  $x_5 = c$ . We take  $G$  to be the union of (1) :  $Q_5^{(**00)}$ , with the edges of  $Q_5^{(*0*00)}$  deleted; (2) :  $Q_5^{(**10)}$  with all edges deleted except for  $\langle 01010, 01110 \rangle$  and  $\langle 11010, 11110 \rangle$ ; (3) :  $Q_5^{(**01)}$  with all edges deleted except for  $\langle 01101, 11101 \rangle$  and  $\langle 01001, 11001 \rangle$ ; (4): the 4 matching

edges between  $Q_5^{(*1*00)}$  and  $Q_5^{(*1*10)}$ ; and (5) the 4 matching edges between  $Q_5^{(*1*00)}$  and  $Q_5^{(*1*01)}$ . Thus  $|E(G)| = 20$ . Since  $|E(Q_5)| = 5 \cdot 2^4 = 80$ , we must exhibit  $80/20 = 4$  copies of  $E(G)$  that partition  $E(Q_5)$ .

**Lemma 3**  $G$  divides  $Q_5$ . In fact,  $E(G)$  is a fundamental set for  $Q_5$ .

*Proof.* The group of translations  $\mathcal{G} = \{id, \sigma_{24}, \sigma_{25}, \sigma_{45}\}$ , applied to  $E(G)$ , partitions  $E(Q_5)$ . It is enough to check that for  $A = \{24\}, \{25\}$ , and  $\{45\}$ ,  $\sigma_A(E(G)) \cap E(G) = \emptyset$ . We will do this for the case  $A = \{24\}$ . The others are similar. Suppose  $e, e' \in E(G)$  and  $\sigma_{24}(e) = e'$ . If  $e$  lies in  $Q_5^{(***00)}$  then  $\sigma_{24}(e)$  is in  $Q_5^{(***10)}$ , which is (vertex) disjoint from  $Q_5^{(***00)}$ . Similarly, if  $e \in Q_5^{(***10)}$ ,  $\sigma_{24}(e) \in Q_5^{(***00)}$ . If  $e \in Q_5^{(***01)}$  then  $\sigma_{24}(e) \in Q_5^{(***11)}$ . If  $e$  is a matching edge  $\langle x, y \rangle$  between  $Q_5^{(***00)}$  and  $Q_5^{(***10)}$  then  $x_2 = y_2 = 1$ . Hence  $\sigma_{24}(e) = \langle x', y' \rangle$  where  $x'_2 = y'_2 = 0$ . Thus  $\sigma_{24}(e) \neq e'$ . Finally, if  $e$  is a matching edge  $\langle x, y \rangle$  between  $Q_5^{(***00)}$  and  $Q_5^{(***01)}$  then  $x_2 = y_2 = 1$ , while if  $\langle x', y' \rangle = \sigma_{24}(e)$ , then  $x'_2 = y'_2 = 0$  and so  $\sigma_{24}(e) \neq e'$ .  $\square$

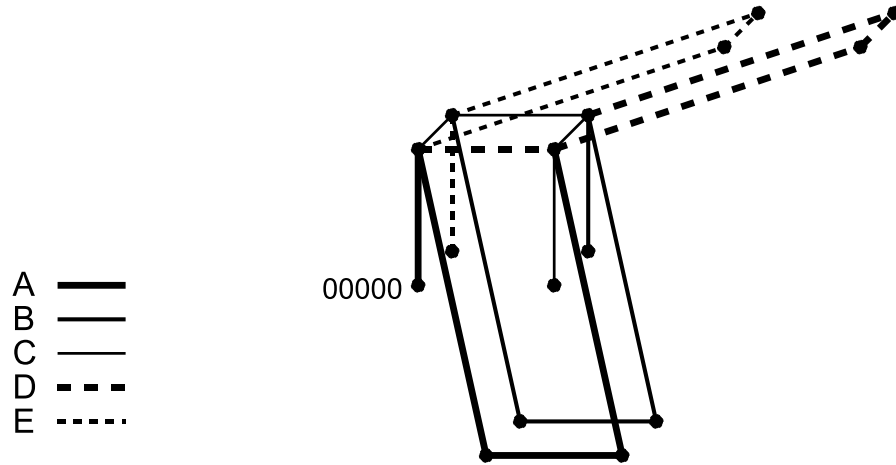


Figure 2:  $P_4$  divides  $G$

**Lemma 4**  $P_4$  divides  $G$ .

*Proof.* It is easiest to describe the paths by their starting points and direction sequences (see figure 2).

Path	Starting Point	Direction Sequence
A	00000	2, 5, 1, 5
B	10100	2, 5, 1, 5
C	10000	2, 3, 1, 3
D	01000	1, 4, 3, 4
E	00100	2, 4, 3, 4

□

**Corollary 7**  $P_4$  divides  $Q_5$ .

*Proof.* This follows immediately from the previous two lemmas. □

### 3 $P_4$ divides $Q_n$ , for $n$ odd, $n \geq 5$

Let us write  $Q_5$  as  $Q_5 = Q_3 \square Q_2 = Q_3 \square C_4$ . Let  $G_0 = Q_5^{(***00)}$ ,  $G_1 = Q_5^{(***10)}$ ,  $G_2 = Q_5^{(***11)}$ ,  $G_3 = Q_5^{(***01)}$ . For  $i \in \{0, 1, 2, 3\}$  let  $\pi_i$  be the canonical mapping from  $G_i$  to  $Q_3$ .

\* From the decomposition of  $Q_5$  by  $P_4$  we have a coloring  $c : Q_5 \rightarrow \{1, 2, \dots, 20\}$  of the edges of  $Q_5$  such that for any  $i \in \{1, 2, \dots, 20\}$  the set of edges of  $Q_5$  colored  $i$  induces a  $P_4$ .

\* Consider now  $Q_3 \square C_{4k}$  for some  $k \geq 1$ . Let  $G'_0, \dots, G'_{4k-1} \simeq Q_3$ . Let  $\pi'_{i'}$  be the canonical mapping from  $G'_{i'} \rightarrow Q_3$  for  $i' \in \{0, 1, \dots, 4k-1\}$ .

The edges of  $Q_3 \square C_{4k}$  are

Case A: the edges of  $G'_{i'}$ , for any  $i' \in \{0, 1, \dots, 4k-1\}$ .

Case B: for any  $i' \in \{0, 1, \dots, 4k-1\}$  the edges  $\langle x', y' \rangle$  for  $x' \in G'_{i'}$ ,  $y' \in G'_{j'}$ , where  $|j' - i'| \equiv 1 \pmod{4k}$  and  $\pi_{i'}(x') = \pi_{j'}(y')$ .

\* Let  $\theta$  be the mapping from  $Q_3 \square C_{4k} \rightarrow Q_5$  defined by: for any  $x' \in G'_{i'}$ ,  $\theta(x') = x$  where  $x$  is the element of  $G_i$ , with  $i \equiv i' \pmod{4}$  such that  $\pi_i(x) = \pi_{i'}(x')$ . (Note that  $\theta$  is not a one-to-one mapping.)

**Proposition 7** If  $\langle x', y' \rangle$  is an edge of  $Q_3 \square C_{4k}$  then  $\langle \theta(x'), \theta(y') \rangle$  is an edge of  $Q_5$ .

*Proof.*

Case A

$\langle x', y' \rangle \in G'_{i'}$ , for some  $i'$ . Then let  $i \equiv i' \pmod{4}$ . By the

definition of  $\theta$ ,  $\theta(x') \in G_i, \theta(y') \in G_i$ . This implies that  $\theta(x')$  and  $\theta(y')$  are adjacent.

Case B

Assume  $x' \in G_{i'}, y' \in G_{j'}$ , with  $|j' - i'| \equiv 1 \pmod{4k}$ . We have  $\pi_{i'}(x') = \pi_{j'}(y')$ . Then  $\theta(x') \in G_i$  and  $\theta(y') \in G_j$  where  $|j - i| \equiv 1 \pmod{4}$  since  $|j' - i'| \equiv 1 \pmod{4}$  implies that  $|j - i| \equiv 1 \pmod{4}$ . Furthermore

$$\pi_i(\theta(x')) \stackrel{\text{def of } \theta}{=} \pi(x') \stackrel{\text{edge}}{=} \pi(y') \stackrel{\text{def of } \theta}{=} \pi_j(\theta(y')).$$

Thus there exists an edge between  $\theta(x')$  and  $\theta(y')$  □

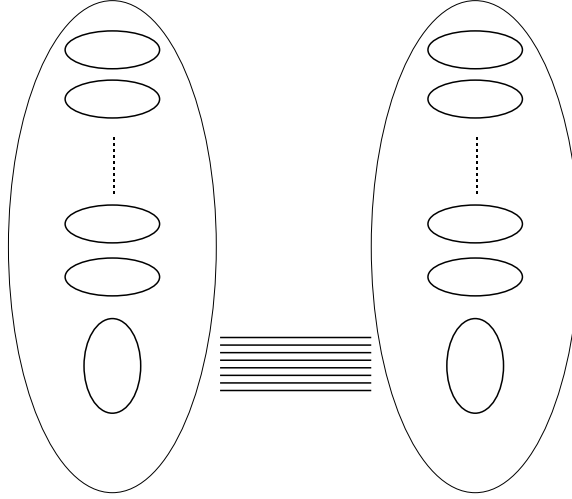


Figure 3: Decomposition of  $Q_{2m+1}$

**Definition 4** Consider the coloring  $E(Q_3 \square C_{4m}) \xrightarrow{c'} \{1, 2, \dots, 20\}$  of the edges of  $Q_3 \square C_{4m}$  defined by  $c'(\langle x', y' \rangle) = c(\langle \theta(x), \theta(y) \rangle)$ .

**Lemma 5** For any  $i \in \{1, 2, \dots, 20\}$  the set of edges of  $Q_3 \square C_{4k}$  such that  $c'(x', y') = i$  is a set of disjoint paths of length 4. Therefore  $P_4$  divides  $Q_3 \square C_{4m}$  for all  $m \geq 1$ .

*Proof.* By definition of  $c'$ , for any vertex  $x'$  of  $Q_3 \square C_{4k}$  the number of edges incident to  $x'$  colored  $i$  by  $c'$  is the number of edges incident to  $\theta(x')$  colored  $i$  by  $c$ . Therefore this number is  $\leq 2$ . Furthermore, there is no cycle colored  $i$  in  $Q_3 \square C_{4k}$  because the image by  $\theta$  of this cycle would be a cycle of  $Q_5$  colored  $i$  with  $c$ . Therefore the set of edges colored  $i$  by  $c'$  is a forest and more precisely, because of the degree, a set of disjoint paths. Notice that the image by  $\theta$  of a path colored  $i$  is a path of  $Q_5$  of the same length (because of the degree of the endpoints of the paths). Therefore all the paths are of length 4.  $\square$

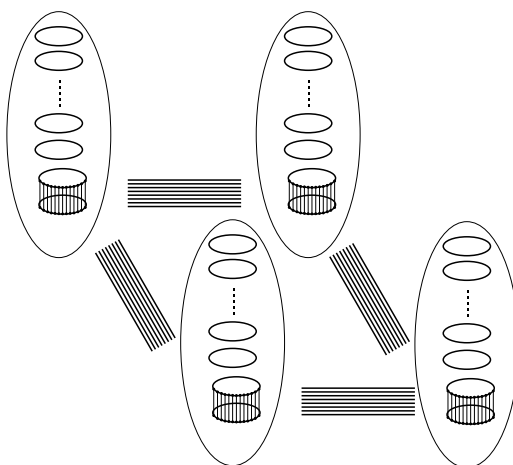


Figure 4: Decomposition of  $Q_{2m+3}$

**Theorem 1** For  $n \geq 4$ ,  $P_4$  divides  $Q_n$ .

*Proof.* If  $n$  is even, the result is true by Corollary 5. If  $n = 5$  then we are done by Corollary 7. Consider  $Q_{2m+3}$ , for  $m \geq 2$ .  $Q_{2m+3} = Q_{2m+1} \square Q_2$ .  $E(Q_{2m})$  can be decomposed into  $m$  cycles of length  $2^{2m}$  (Hamiltonian cycles) by Aubert and Schneider [1]. Let  $D$  be one of these cycles. The edges of  $Q_{2m+1}$  are the edges of the two copies of  $Q_{2m}$  and a matching. But every vertex of  $Q_{2m}$  appears exactly once in  $D$  so  $E(Q_{2m+1})$  can be decomposed into  $2(m-1)$  cycles of length  $2^{2m}$  and  $D \square Q_1 \simeq C_{2^{2m}} \square Q_1$  (see figure 3).

Every vertex of  $Q_{2m+1}$  appears once in  $D \square Q_1$ , thus, for the same reason,  $E(Q_{2m+3})$  can be decomposed into  $8(m-1)$  cycles of length  $2^{2k}$  and

$D \square Q_1 \square Q_2 \simeq C_{2^{2k}} \square Q_1 \square Q_2 \simeq C_{2^{2m}} \square Q_3$  (see figure 4).

Since  $k \geq 2$ ,  $\frac{2^{2m}}{4}$  is an integer strictly greater than 1 so the cycles of length  $2^{2m}$  are divisible by  $P_4$ . By Lemma 5,  $P_4$  divides  $C_{2^{2m}} \square Q_3$ , and  $P_4$  divides  $E(Q_n)$  for any odd  $n \geq 5$ .  $\square$

## 4 Products of fundamental sets.

We begin with an easy result about divisibility of the Cartesian product of two graphs. Its proof is straightforward and so we omit it.

**Proposition 8** *For  $i = 1, 2$ , let  $G_i$  be a graph that is divisible by the subgraph  $H_i$ . Then  $G_1 \square G_2$  is divisible by the subgraph  $H_1 \square H_2$ .*

**Lemma 6** *Let  $G_1$  and  $G_2$  be any two graphs, and for  $i = 1, 2$  let  $\phi_i \in \text{Aut}(G_i)$ . Define  $(\phi_1, \phi_2) : G_1 \square G_2 \rightarrow G_1 \square G_2$  by  $(\phi_1, \phi_2)((x, y)) = (\phi_1(x), \phi_2(y))$ . Then  $(\phi_1, \phi_2) \in \text{Aut}(G_1 \square G_2)$ .*

*Proof.* Let  $(x_1, x_2)$  and  $(y_1, y_2)$  be adjacent in  $G_1 \square G_2$ . Then either (1)  $x_1$  and  $y_1$  are adjacent in  $G_1$  and  $x_2 = y_2$  or (2)  $x_1 = y_1$  and  $x_2$  and  $y_2$  are adjacent in  $G_2$ . We must show that  $(\phi_1, \phi_2)(x_1, x_2)$  and  $(\phi_1, \phi_2)(y_1, y_2)$  are adjacent in  $G_1 \square G_2$ . By symmetry, it is sufficient to prove this for case (1). But then since  $\phi_1 \in \text{Aut}(G_1)$ ,  $\phi_1(x_1)$  and  $\phi_1(y_1)$  are adjacent in  $G_1$ , and since  $x_2 = y_2$ ,  $\phi_2(x_2) = \phi_2(y_2)$ . Therefore  $(\phi_1, \phi_2)(x_1, x_2)$  and  $(\phi_1, \phi_2)(y_1, y_2)$  are adjacent in  $G_1 \square G_2$ . Thus  $(\phi_1, \phi_2) \in \text{Aut}(G_1 \square G_2)$ .  $\square$

**Corollary 8** *Suppose that for  $i = 1, 2$ ,  $H_i \subseteq E(G_i)$  and  $H_i$  is a fundamental set for the graph  $G_i$  with subgroup  $\mathcal{H}_i$  of  $\text{Aut}(G_i)$ . Then  $H_1 \square H_2$  is a fundamental set for  $G_1 \square G_2$  with subgroup  $\mathcal{H}_1 \times \mathcal{H}_2$  of  $\text{Aut}(G_1 \square G_2)$ .*

*Proof.* By assumption, for  $i = 1, 2$  the translates of  $H_i$  by  $\mathcal{H}_i$  are edge-disjoint and their union is  $E(G_i)$ . Suppose that  $e = \langle (x_1, x_2), (y_1, y_2) \rangle \in E(G_1 \square G_2)$ . Then either Case (1):  $\langle x_1, y_1 \rangle \in E(G_1)$  and  $x_2 = y_2$  or Case (2):  $x_1 = y_1$  and  $\langle x_2, y_2 \rangle \in E(G_2)$ . By symmetry, we need only prove our assertions for Case (1). Since  $\langle x_1, y_1 \rangle \in E(G_1)$ , for some  $\phi_1 \in \mathcal{H}_1$ , there exists an edge  $\langle u_1, v_1 \rangle \in H_1$  such that  $\langle x_1, y_1 \rangle = \langle \phi_1(u_1), \phi_1(v_1) \rangle$ . Let  $id_i \in \mathcal{H}_i$  denote the identity automorphism of  $H_i$ . Then  $(\phi_1, id_2) \in \mathcal{H}_1 \times \mathcal{H}_2$  and

$\langle (x_1, x_2), (y_1, y_2) \rangle = (\phi_1, id_2)(\langle (u_1, x_2), (v_1, y_2) \rangle)$ . So the translates of  $H_1 \square H_2$  cover  $G_1 \square G_2$ .

Next, we must show that these translates form a packing, *i.e.* that either  $(\phi_1, \phi_2)(H_1 \square H_2) = \emptyset$  or that  $\phi_1 = id_1$  and  $\phi_2 = id_2$ . Let  $(\langle (w_1, w_2), (z_1, z_2) \rangle) \in H_1 \square H_2$ . We shall show that  $(\phi_1, \phi_2) \in \mathcal{H}_1 \times \mathcal{H}_2$  is the identity automorphism of  $H_1 \square H_2$ . For from this hypothesis, we have  $\phi_1(\langle (w_1, z_1) \rangle) \in H_1$  and  $\phi_2(\langle (w_2, z_2) \rangle) \in H_2$ . Since  $H_1$  and  $H_2$  are fundamental sets for  $G_1$  and  $G_2$ , respectively,  $\phi_1 = id_1$  and  $\phi_2 = id_2$ . Thus  $H_1 \square H_2$  is a fundamental set for  $G_1 \square G_2$  with subgroup  $\mathcal{H}_1 \times \mathcal{H}_2$  of  $\text{Aut}(G_1 \square G_2)$ , as claimed.  $\square$

**Corollary 9** *If  $H$  is a fundamental set of edges for the graph  $G$ , with subgroup  $\mathcal{H}$  of  $\text{Aut}(G)$  then  $H \square H$  is a fundamental set for  $G \square G$  with subgroup  $\mathcal{H} \times \mathcal{H}$  of  $\text{Aut}(G \square G)$ .*

## 5 $Q_{2^k}$ has a fundamental Hamiltonian cycle.

**Remark.** We say that the direction of an edge of  $Q_n$  is  $i$  if the coordinate in which its two endpoints differ is the  $i^{\text{th}}$ . We shall describe walks in the hypercube by specifying the starting vertex (generally  $\vec{0}$ ) and the sequence of edge directions.

It is well-known that the  $n$ -dimensional hypercube  $Q_n$  is Hamiltonian, and in fact has many Hamiltonian cycles. Aubert and Schneider [1] proved that for  $n$  even,  $Q_n$  has an edge decomposition into Hamiltonian cycles. However, their construction is technical. In contrast, in this last section we shall show that for  $n = 2^k$ , there is a single Hamiltonian cycle  $C$  such that  $E(C)$  is a fundamental set for  $Q_n$ .

More generally, we shall show that if  $G$  is a graph with a Hamiltonian cycle whose edges form a fundamental set, relative to a subgroup  $\mathcal{G}$  of the automorphism group of  $G$ , then  $G^2 = G \square G$  has a Hamiltonian cycle whose edges form a fundamental set, relative to a subgroup of the automorphism group of  $G^2$ .

The next lemma is due to G. Ringel [7]. (The second statement is not stated explicitly there, but is a consequence of his proof.)

**Lemma 7** *From a Hamiltonian cycle  $C$  of a graph  $G$ , there are two edge-disjoint Hamiltonian cycles of  $G^2 = G \square G$ . Each of them is isomorphic to  $C \square C$ , via an automorphism  $(\phi_1, \phi_2)$ , where for  $i = 1, 2$ ,  $\phi_i \in \text{Aut}(G)$ .*

**Corollary 10** *If  $G$  has a Hamiltonian cycle  $C$  and  $C$  is a fundamental set for  $G$  under the subgroup  $\mathcal{H}$  of  $\text{Aut } G$ , then  $C \square C$  is a Hamiltonian cycle of  $G \square G$  under the subgroup  $\mathcal{H} \times \mathcal{H}$  of  $\text{Aut}(G \square G)$ .*

*Proof.* This follows from Corollary 9 and Lemma 7.  $\square$

**Proposition 9**  *$Q_4$  has a Hamiltonian cycle that is a fundamental set.*

*Proof.* The edge direction sequence  $(12132124)^2$ , together with any starting vertex, yields a Hamiltonian cycle  $C$ . Let  $\theta$  be the permutation  $(13)(24)$ . Then the mapping  $\rho_\theta : V(Q_n) \rightarrow V(Q_n)$ , defined by  $\rho_\theta(x_1, \dots, x_n) = x_{\theta(1)}, \dots, x_{\theta(n)}$  is an automorphism, which in this case generates a subgroup of  $\text{Aut}(Q_4)$  of order 2. It is easy to check that  $\rho_{\theta(C)}$  is edge-disjoint from  $C$ , and so we have the desired edge decomposition.  $\square$

**Theorem 2** *For any  $k \geq 1$ ,  $Q_{2^k}$  has a Hamiltonian cycle that is a fundamental set.*

*Proof.* This is trivial for  $k = 1$  since  $Q_2 = C_2$  and holds for  $k = 2$  by Proposition 9. The desired result now follows by induction, from Lemma 7 and Corollary 9.  $\square$

**Corollary 11** *For  $n$  and  $m$  each a power of 2, with  $m \leq n$ , there is an  $m$ -cycle that divides  $Q_n$ .*

*Proof.* Let  $m = 2^p$ . By Theorem 2  $Q_m$  has a fundamental  $2^p$ -cycle, which therefore divides  $Q_m = Q_{2^p}$ . Since  $m$  and  $n$  are each powers of two,  $m$  divides  $n$ . Hence by Proposition 4 and Lemma 2, this cycle divides  $Q_n$ .  $\square$

## 6 Summary of Results

1. For  $k$  odd, if  $P_k$  is a divides  $Q_n$ , then  $k$  divides  $n$ . (Proposition 3)
2. If  $k$  divides  $n$ , any tree on  $k$  edges is a fundamental set for  $Q_n$ . (Corollary 4)
3. If  $k$  divides  $n$  and  $k < n$  then  $P_{2k}$  divides  $Q_n$ . (Corollary 5)
4. If  $n$  is even and  $j < n$  then  $P_{2j}$  divides  $Q_n$ . (Proposition 5)
5. For  $k = 2n$  there is a  $k$ -cycle which is a fundamental set for  $Q_n$  when  $n$  is even. (Proposition 6)



6. For  $n =$  a power of 2, there is a Hamiltonian cycle which is a fundamental set for  $Q_n$ . (Theorem 2)
7. For  $n =$  a power of 2 and  $m =$  a power of 2, with  $m \leq n$ , there is an  $m$ -cycle that divides  $Q_n$ . (Corollary 11)
8. For  $n \geq 4$ ,  $P_4$  divides  $Q_n$ . (Theorem 1)
9. If  $k$  divides  $n$  then  $Q_k$  is a fundamental set for  $Q_n$ . (Proposition 4)
10. For  $k \geq 3$ ,  $P_{2^j}$  does *not* divide  $Q_{2^{k+1}}$ . (Proposition 1)

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