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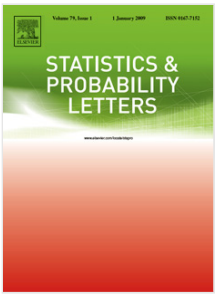

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Recursive equations for the predictive distributions of some determinantal processes

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Abstract

The paper provides recursive equations for the predictive distributions of one-dependent and two-dependent determinantal processes. Fixed order recursive equations can be applied both to efficiently simulate trajectories and to explore properties of the process.

Keywords: Determinantal processes; predictive distributions; m-dependence; negative association
2000 MSC: 60G10; 60G25

1. Introduction

Determinantal processes have been studied by many authors, mainly in the framework of fermion point processes and fields, and applied in different contexts, including physics, random matrix theory, representation theory, and ergodic theory (see Lyons, 2003, Lyons and Steif, 2003, Shirai and Takahashi, 2003 and the references therein). In words, a determinantal process is a $\{0,1\}$-valued process, indexed by a finite or countable set, whose cylinder probabilities are determinants.

A recent delightful paper on one-dependent determinantal processes is Borodin et al. (2009). Besides many examples from combinatorics to graph theory, it contains the important result that any one-dependent point process on the integers is determinantal.

Determinantal processes are interesting because they allow one to model conditional negative association (see Lyons, 2003, Lyons and Steif, 2003 and

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Shirai and Takahashi, 2003 for the precise definition and proofs). In practice, a "one" at a certain site decreases the probability of further "ones" in the process, whatever the values of the process at different sites. For this reason, determinantal processes are used in statistical physics to represent systems of repulsive particles. Stationary determinantal processes (SDP) have interesting properties, such as mixing, entropy positivity, Bernoulli shift, and Gibbs properties (see Lyons and Steif, 2003 and Shirai and Takahashi, 2003).

The present paper focuses on finitely dependent SDP's. In particular, the problem of computing the predictive distributions of one-dependent and two-dependent SDP's is treated.

Finitely dependent SDP's can be used to model local negative interactions between random variables that propagate through the stochastic sequence. In fact, if $(X_n)_{n\in\mathbb{Z}}$ is m-dependent, X_n and X_j are independent when $n < j-m$. Nevertheless, they are not conditionally independent given $X_{n+1}, \ldots, X_{j-1}$. This feature is in some sense the opposite of the Markov property.

Predictive distributions of SDP's are cumbersome. In fact, no simple sufficient statistics can be identified. On the other hand, it is proved in the present paper that recursive equations of fixed order exist for the predictive distributions of one-dependent and two-dependent SDP's.

This paper is organized as follows. In Section 2 we recall the precise definition of an SDP and discuss its parameters. In Section 3 we provide recursive equations for the predictive distributions of one-dependent SDP's and give a characterization of the process in terms of predictive covariances. Some of these results are contained in the technical report Cifarelli and Fortini (2005). The two-dependent case is treated in Section 4. There recursive equations are given, separately, for complex and real parameters. Equations of lower degree hold when the parameters are real. The results are applied to explore properties of the predictive probabilities and to suggest efficient simulation procedures.

2. Definitions and preliminary results

Let E be a finite or countable set and let K be an $E\times E$ matrix of complex numbers satisfying $K_{j,m} = \overline{K}_{m,j}$ for every $j, m \in E$ and $\sum_{j,m\in E} K_{j,m}u_j\overline{u}_m \in [0,1]$ for every u in the complex Hilbert space $l^2(E)$. For every finite $E_1 \subseteq E$ let $K|E_1$ denote the submatrix of K whose rows and columns are indexed by E_1. It is possible to define a probability measure P^K on the cylinder

sigma-algebra of $\{0,1\}^E$ such that

$$P^K\{x \in \{0,1\}^E : x_e = 1, e \in E_1\} = \det(K|E_1) \tag{1}$$

($E_1 \subseteq E$, E_1 finite) (see Lyons, 2003, Lyons and Steif, 2003 and Shirai and Takahashi, 2003).

Definition 1. *A probability measure P^K on on the cylinder sigma-algebra of $\{0,1\}^E$ satisfying (1) is called a determinantal probability measure. The coordinate process $(X_n)_{n\in\mathbb{Z}}$ of P^K is called a determinantal process.*

It follows from (1) that for every finite, disjoint $E_1, E_2 \subseteq E$,

$$P^K(\{x \in \{0,1\}^E : x_{e_1} = 0, e_1 \in E_1, x_{e_2} = 1, e_2 \in E_2\}) = \det(K^{E_1}|E_1 \cup E_2), \tag{2}$$

where, for every $E_1 \subseteq E$, K^{E_1} denotes the matrix that coincides with K except on the rows indexed by E_1, where it coincides with $I - K$ (see Lyons, 2003 or Shirai and Takahashi, 2003). It follows from (2) that $(1 - X_e)_{e\in E}$ has probability distribution P^{I-K} if $(X_e)_{e\in E}$ has probability distribution P^K. In the following we will refer to this property as "duality".

This paper deals with determinantal probability measures on the cylinder sigma-algebra of $\{0,1\}^{\mathbb{Z}}$. Furthermore attention is restricted to matrices K satisfying $K_{j,m} = k(j-m)$ for some function $k : \mathbb{Z} \to \mathbb{C}$. Under this condition P^K is stationary. In fact $K|(E_1 + n) = K|E_1$ for every $E_1 \subseteq \mathbb{Z}$ and every $n \in \mathbb{Z}$. It is proved in Lyons and Steif (2003) and Shirai and Takahashi (2003) that $k(j)$ ($j \in \mathbb{Z}$) is the j-th Fourier coefficient $\hat{f}_j$ of a $[0,1]$-valued Lebesgue-measurable function f on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

In the following we will write P^f instead of P^K to stress the dependence on f. Furthermore, we will assume that f is neither a.s. equal to one nor a.s. equal to zero. This last condition ensures that P^f has full support (see Lyons and Steif, 2003).

Lemma 1. *Let $f : \mathbb{T} \to [0,1]$ be a Lebesgue-measurable function. If $g(s) = f(s+t)$ for some $t \in \mathbb{T}$, then $P^g = P^f$.*

Proof. Let K_f and K_g be the infinite dimensional matrices whose (j,m)-th entry is $\hat{f}_{m-j}$ and $\hat{g}_{m-j}$, respectively ($j, m \in \mathbb{Z}$). Since $\hat{g}_j = \exp(i2\pi jt)\hat{f}_j$ for $j \in \mathbb{Z}$, then $K_g = (D^{-1}K_f D)$, where D is the infinite dimensional diagonal matrix whose (j,j)-th entry is $\exp(i2\pi jt)$. For every finite $A \subseteq \mathbb{Z}$, we have that $\det(K_g|A) = \det(D^{-1}K_f D|A) = \det(D^{-1}|A)\det(K_f|A)\det(D|A) = \det(K_f|A)$. □

According to the above lemma, there is some degree of arbitrariness in parameterizing an SDP. In particular, one of the k_j's can be taken to be real positive, without loss of generality.

Let us now introduce cylinder probabilities, predictive distributions and predictive probabilities. For every sequence $(x_n)_{n\in\mathbb{Z}}$, let $x_{(n)} = (x_1, x_2, \ldots, x_n)$ $(n \in \mathbb{N})$. Analogously, let $1_{(n)}$ and $0_{(n)}$ denote the vectors of n ones and zeroes, respectively $(n \geq 1)$. The probability measures P_n^f defined by

$$P_n^f(A) = P^f(\{x \in \{0,1\}^{\mathbb{Z}} : x_{(n)} \in A\})$$

$(n = 1, 2, \ldots, A \subseteq \{0,1\}^n)$ will be called the cylinder probabilities of P^f or of its coordinate process $(X_n)_{n\in\mathbb{Z}}$. Since P^f has full support, $P_n^f(A) > 0$ for every non-empty $A \subseteq \{0,1\}^n$ and every $n \in \mathbb{N}$. The predictive distributions of P^f or of its coordinate process $(X_n)_{n\in\mathbb{Z}}$ are defined by $P^f(X_n \in A | X_{(n-1)} = x_{(n-1)})$ $(A \subseteq \{0,1\}, n \geq 2, x_{(n-1)} \in \{0,1\}^{n-1})$. Since SDP's are $\{0,1\}$-valued, the predictive distributions are completely characterized by the functions

$$p_n^f(x_{(n-1)}) = P^f(X_n = 1 | X_{(n-1)} = x_{(n-1)}) \tag{3}$$

$(n \geq 2,\ x_{(n-1)} \in \{0,1\}^{n-1})$. Furthermore, since P^f has full support,

$$p_n^f(x_{(n-1)}) = P_n^f(\{(x_1, \ldots, x_{n-1}, 1)\})/P_{n-1}^f(\{(x_1, \ldots, x_{n-1})\}.$$

In the following, we will call the p_n^f the predictive probabilities of P^f or of its coordinate process $(X_n)_{n\in\mathbb{Z}}$.

Finally, let us recall the definition of m-dependence.

Definition 2. *A random sequence $(X_n)_{n\in\mathbb{Z}}$ (or its probability distribution) is said to be m-dependent $(m \in \mathbb{N})$ if $(X_n)_{n<j}$ is stochastically independent of $(X_n)_{n\geq j+m}$ for every $j \in \mathbb{Z}$.*

Since for $\{0,1\}$-valued processes zero correlation implies stochastic independence and since, for SDP, $cov(X_n, X_{n+j}) = -|k_j|^2$ $(j = 1, 2, \ldots)$, $(X_n)_{n\in\mathbb{Z}}$ is m-dependent if and only if $k_j = 0$ for every $j > m$ $(m \in \mathbb{N})$. Hence an SDP is m-dependent if and only if $f(s)$ can be written as $\sum_{j=-m}^{m} k_j \exp(i2\pi js)$.

3. One-dependent stationary determinantal processes

Let P^f be a one-dependent SDP such that f is neither a.s. equal to zero nor a.s.equal to one. By the results in the previous section, we can suppose that

$$f(s) = k_0 + 2k_1 \cos(2\pi s) \qquad (s \in \mathbb{T}, k_0 \in (0,1), k_1 \in [0, min(k_0, 1-k_0)/2]). \tag{4}$$

Lemma 2. *Let P^f be a stationary determinantal probability measure with f as in (4). For every $x \in \{0,1\}^{\mathbb{Z}}$ and every $n \geq 2$, we have*

$$P_n^f(\{x_{(n)}\}) = k_0^{x_n}(1-k_0)^{1-x_n} P_{n-1}^f(\{x_{(n-1)}\}) - (-1)^{x_n+x_{n-1}} k_1^2 P_{n-2}^f(\{x_{(n-2)}\}), \tag{5}$$

where $P_0^f(\{x_{(0)}\}) := 1$.

Proof. Let $x_1, x_2, \ldots$ be a fixed $\{0,1\}$-valued sequence. Define the infinite dimensional matrix $[A_{jl}]_{j,l=1,2\ldots}$ by $A_{jl} = k_0^{x_j}(1-k_0)^{1-x_j}$ if $l = j$, $A_{jl} = (-1)^{x_j+1}k_1$ if $l = j-1, j+1$, and $A_{jl} = 0$ otherwise. Then $P_n^f(\{x_{(n)}\}) = \det A|\{1,\ldots,n\}$. Equation (5) is obtained by developing $\det A|\{1,\ldots,n\}$ along the last row and column. □

Theorem 1. *Let P^f be a stationary determinantal probability measure with f as in (4). For every $x \in \{0,1\}^{\mathbb{Z}}$ and every $n \geq 2$, we have*

$$p_n^f(x_{(n-1)}) = k_0 + k_1^2(1 - p_{n-1}^f(x_{(n-2)}))^{-1}\left(1 - 1/p_{n-1}^f(x_{(n-2)})\right)^{x_{n-1}}, \tag{6}$$

where $p_1^f(x_{(0)}) := P_1^f(\{1\})$.

Proof. Equation (6) is obtained from Lemma 2 by computing

$$p_n^f(x_{(n-1)}) = P_n^f(\{(x_1,\ldots,x_{n-1},1)\})/P_{n-1}^f(\{(x_1,\ldots,x_{n-1})\})$$

and observing that

$$\frac{P_{n-1}^f(\{(x_1,\ldots,x_{n-1})\})}{P_{n-2}^f(\{(x_1,\ldots,x_{n-2})\})} = p_{n-1}^f(x_{(n-2)})^{x_{n-1}}(1-p_{n-1}^f(x_{(n-2)}))^{1-x_{n-1}}.$$

□

Example 1. Let $(X_n)_{n\in\mathbb{Z}}$ be a one-dependent SDP with parameter f as in (4). We have that $p := \lim_{n\to\infty} p_n^f(1_{(n-1)}) = \exp\int_{\mathbb{T}} \log f d\lambda$ (see Lyons and Steif, 2003). Theorem 1 can be used to evaluate p without computing the integral. In fact, by (6), p must satisfy the equation $p^2 - k_0 p + k_1^2 = 0$. Equation (6) together with $p_1^f = k_0$ implies that $p_n^f(1_{(n-1)}) \geq k_0/2$ for every n. Hence $p = k_0/2 + \sqrt{(k_0/2)^2 - k_1^2}$. By duality, the limit probability of a "one" after an n-long sequence of "zeroes" is $q = (1+k_0)/2 - \sqrt{((1-k_0)/2)^2 - k_1^2}$.

Depending on the values of k_0 and k_1, p can range between $k_0/2$ (when $k_0 \leq 1/2$ and $k_1 = k_0/2$) and k_0 (when $k_1 = 0$, i.e. in the i.i.d. case) and q can range between k_0 and $(1+k_0)/2$.

Equation (6), together with $P_1^f(\{1\}) = k_0$, characterizes one-dependent SDP's. The next result gives a characterization of SDP's within the class of one-dependent processes.

Corollary 1. *Let $(X_n)_{n\in\mathbf{Z}}$ be a $\{0,1\}$-valued, stationary, one-dependent process with probability distribution P. A necessary and sufficient condition for $P = P^f$ with f as in (4) is*

$$\begin{cases} E(X_1) = k_0 \\ Cov(X_{n+1}, X_{n+2}|X_1, \ldots, X_n) = -k_1^2 \quad P-a.s. \qquad (n \geq 0). \end{cases} \tag{7}$$

Proof. Let $(P_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ denote the sequences of the cylinder and predictive probabilities of P respectively. Let us first prove that (7) holds if $P = P^f$. We have that $E(X_1) = P^f(\{1\}) = k_0$. Furthermore, equation (6) yields

$$\begin{aligned} &Cov(X_{n+1}, X_{n+2}|X_1 = x_1, \ldots, X_n = x_n) \\ &= p_{n+1}^f(x_1, \ldots, x_n)p_{n+2}^f(x_1, \ldots, x_n, 1) - k_0 p_{n+1}^f(x_1, \ldots, x_n) = -k_1^2 \end{aligned}$$

for every fixed $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$.
To prove sufficiency, notice that the distribution of a $\{0,1\}$-valued, stationary, one-dependent process is completely characterized by the sequence $(P_n(\{1_{(n)}\}))_{n\geq 1}$. It follows from (7) that $P_1^f(\{1\}) = E(X_1) = k_0 = P_1^f(\{1\})$. Furthermore, by induction, (7) yields $P_n(\{1_{(n)}\}) \neq 0$ and $P_{n+2}(\{1_{(n+2)}\}) - k_0 P_{n+1}(\{1_{(n+1)}\}) = -k_1^2 P_n(\{1_{(n)}\})$. By Lemma 2, $P_n(\{1_{(n)}\}) = P_n^f(\{1_{(n)}\})$ for every $n \geq 1$. □

Example 2. Equation (7) can be exploited to give a new proof that a one-dependent SDP $(X_n)_{n\in Z}$ is a two-block factor (see Broman, 2005). We recall that a process $(X_n)_{n\in Z}$ is an m-block factor if there exist a measurable function h of m variables and an i.i.d. process $(Y_n)_{n\in\mathbf{Z}}$ such that $(X_n)_{n\in Z} \stackrel{d}{=} (h(Y_n, Y_{n+1}, \ldots, Y_{n+m-1}))_{n\in Z}$, where $\stackrel{d}{=}$ denotes equality in distribution. Let $(X_n)_{n\in Z}$ be an SDP with f as in (4). We can suppose $0 < k_0 \leq 1/2$ without loss of generality. In fact, in the degenerate case $k_0 = 0$, the two-block factor condition is obvious; while for $k_0 > 1/2$ it can be proved by duality. To prove that $(X_n)_{n\in Z}$ is a two-block factor, consider a sequence $(Y_n)_{n\in\mathbf{Z}}$ of i.i.d. random variables with uniform distribution on $[0,1]$. We will show that $(1_A(Y_n, Y_{n-1}))_{n\in Z} \stackrel{d}{=} (X_n)_{n\in Z}$ for a suitable choice

of a Borel set $A \subseteq [0,1]^2$. Since, for every A, $(1_A(Y_n, Y_{n+1}))_{n\in Z}$ is a $\{0,1\}$-valued, stationary, one-dependent process, according to Corollary 1, it is sufficient to find A such that $E(1_A(Y_1, Y_2)) = k_0$ and

$$Cov(1_A(Y_{n+1}, Y_{n+2}), 1_A(Y_{n+2}, Y_{n+3})|1_A(Y_1, Y_2), \ldots, 1_A(Y_n, Y_{n+1})) = -k_1^2$$

for every $n \geq 1$. The first condition is true if $(\lambda \times \lambda)(A) = k_0$, where λ is the Lebesgue measure on $[0,1]$. A sufficient condition for the second equality is

$$\begin{aligned} &E(1_A(Y_{n+1}, Y_{n+2})1_A(Y_{n+2}, Y_{n+3})|Y_1, \ldots, Y_{n+1}) \\ &\quad = k_0 E(1_A(Y_{n+1}, Y_{n+2})|Y_1, \ldots, Y_{n+1}) - k_1^2 \end{aligned}$$

a.s. or, equivalently,

$$P((Y_{n+1}, Y_{n+2}) \in A, (Y_{n+2}, Y_{n+3}) \in A|Y_{n+1}) = k_0 P((Y_{n+1}, Y_{n+2}) \in A|Y_{n+1}) - k_1^2,$$

a.s. This last equation holds if A satisfies $\int_{A_y} \lambda(A_z)\lambda(dz) = k_0\lambda(A_y) - k_1^2$ a.s., where, for every x, $A_x = \{y \in [0,1] : (x,y) \in A\}$. An example is $A = A_1 \cup A_2 \cup A_3$ with $A_1 = [0, k_0/2 + \sqrt{k_0^2/4 - k_1^2}] \times [0, 1/2]$, $A_2 = [1 - k_0/2 - \sqrt{k_0^2/4 - k_1^2}, 1] \times [0, 1/2]$, and $A_3 = [0,1] \times [1/2, 1/2 + k_0/2 - \sqrt{k_0^2/4 - k_1^2}]$.

The next result can be used to sample sequentially from P^f. The advantage of this procedure, if compared with simulations using determinants, is that the computation complexity is not increasing with n.

Corollary 2. *Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, uniformly distributed on $[0,1]$. Furthermore, let $X_n = 1_{[0,W_n]}(Z_n)$ $(n \geq 1)$, $W_1 = k_0$ and*

$$W_n = k_0 + \frac{k_1^2}{1 - W_{n-1}} \left(1 - \frac{1}{W_{n-1}}\right)^{X_{n-1}} \qquad (n > 1),$$

with $k_0 \in (0,1)$ and $k_1 \in [0, min(k_0, 1-k_0)/2]$. Then, for every $n \geq 1$, $X_{(n)}$ has probability distribution P_n^f with $f(s) = k_0 + 2k_1 \cos(2\pi s)$ $(s \in \mathbb{T})$.

Proof. By induction, (6) implies $W_n = p_n^f(X_{(n-1)})$ for every $n \geq 1$. Hence for every $n \geq 1$, $P(X_n = 1|X_1, \ldots, X_{n-1}) = W_n = p_n^f(X_{(n-1)})$. It follows that for every $(x_1, x_2, \ldots) \in \{0,1\}^\infty$ and every $n \geq 1$, $P(X_{(n)} = x_{(n)}) = P(X_1 = x_1) \prod_{j=2}^n P(X_j = x_j|X_{(j-1)} = x_{(j-1)}) = k_0^{x_1}(1-k_0)^{1-x_1} \prod_{j=1}^n p_j^f(x_{(j-1)}) = P_n^f(\{x_{(n)}\})$ □

4. Two-dependent stationary determinantal processes

Let P^f be an SDP with

$$f(s) = \sum_{j=-2}^{2} k_j \exp(i2\pi js) \quad (s \in \mathbb{T}), \tag{8}$$

where $k_{-j} = \overline{k}_j$. By Lemma 1, we can assume $k_{-1} = k_1 \in \mathbb{R}$, without loss of generality.

Lemma 3. *Let P^f be a stationary, two-dependent determinantal probability measure with f as in (8), $k_0 \in (0,1), k_1 \in \mathbb{R}$, and $k_2 \in \mathbb{C}$. Then, for every $x \in \{0,1\}^{\mathbb{Z}}$ and every $n \geq 3$*

$$\begin{aligned} P_n^f(\{x_{(n)}\}) &= \sum_{j=2}^{6\wedge n} a_j (-1)^{\sum_{u=0}^{j-1} x_{n-u}} P_{n-j}^f(\{x_{(n-j)}\}) \\ &+ \sum_{j\in\{1,3\}} c_j (-1)^{\sum_{u=0}^{j-1} x_{n-u}} b(k_0; x_{n-j+1}) P_{n-j}^f(\{x_{(n-j)}\}) \\ &+ \sum_{j\in\{3,5\}, j\leq n} c_j (-1)^{\sum_{u=0}^{j-1} x_{n-u}} b(k_0; x_{n-j+2}) P_{n-j}^f(\{x_{(n-j)}\}) \end{aligned} \tag{9}$$

where $P_0^f(\{x_{(0)}\}) := 1$,

$$a_2 = |k_2|^2 - k_1^2 \quad a_3 = -2k_1^2|k_2|\cos\theta \quad a_4 = (|k_2|^2 - k_1^2)|k_2|^2 \quad a_5 = 0 \quad a_6 = -|k_2|^6 \tag{10}$$

$$b(p;x) = (-p)^x (1-p)^{1-x} \quad (p \in (0,1), x \in \{0,1\}) \tag{11}$$

$$c_j = (-|k_2|^2)^{(j-1)/2} \quad (j = 1,3,5). \tag{12}$$

Proof. Let $x_1, x_2, \ldots$ be a fixed $\{0,1\}$-valued sequence. Define the infinite dimensional matrix $[A_{jl}]_{j,l=1,2\ldots}$ by

$$A_{jl} = \begin{cases} k_0^{x_j}(1-k_0)^{1-x_j} & \text{if } l = j \\ (-1)^{x_j+1} k_1 & \text{if } l = j-1, j+1 \\ (-1)^{x_j+1}\overline{k_2} & \text{if } l = j-2 \\ (-1)^{x_j+1} k_2 & \text{if } l = j+2 \\ 0 \text{ otherwise,} \end{cases}$$

and set $\Delta_n = \det A|\{1,\ldots,n\}$ for $n \geq 1$, $\Delta_0 = 1$, and $\Delta_n = 0$ for $n < 0$. Then $P_n^f(\{x_{(n)}\}) = \Delta_n$ for $n \geq 0$. Denote by $[\Delta_n]_j^l$ the determinant of the

matrix that is obtained by deleting row j and column l from $A|\{1,\dots,n\}$ ($n \geq 1$, $j,l = 1,\dots,n$). Developing Δ_n along the last rows and columns, we obtain, for every $n \geq 3$,

$$\begin{aligned}\Delta_n =& (1-k_0)\left(\tfrac{k_0}{1-k_0}\right)^{x_n}\Delta_{n-1} - k_1^2(-1)^{x_n+x_{n-1}}\Delta_{n-2}\\ &- |k_2|^2(1-k_0)\left(\tfrac{k_0}{1-k_0}\right)^{x_{n-1}}(-1)^{x_n+x_{n-2}}\Delta_{n-3} + |k_2|^4(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-3}}\Delta_{n-4}\\ &+ k_1k_2(-1)^{x_n+x_{n-2}}[\Delta_{n-1}]_{n-2}^{n-1} + k_1\overline{k}_2(-1)^{x_{n-2}+x_{n-3}}[\Delta_{n-1}]_{n-1}^{n-2}\end{aligned} \tag{13}$$

Furthermore, developing $[\Delta_{n-1}]_{n-2}^{n-1}$ and $[\Delta_{n-1}]_{n-1}^{n-2}$ along the last row and column, we obtain

$$\begin{aligned}&k_1k_2(-1)^{x_n+x_{n-2}}[\Delta_{n-1}]_{n-2}^{n-1} + k_1\overline{k}_2(-1)^{x_{n-2}+x_{n-3}}[\Delta_{n-1}]_{n-1}^{n-2}\\ &= -k_1^2(k_2+\overline{k}_2)(-1)^{x_n+x_{n-1}+x_{n-2}}\Delta_{n-3} - 2k_1^2|k_2|^2(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-3}}\Delta_{n-4}\\ &+|k_2|^2(-1)^{x_n+x_{n-1}}\left(k_1k_2(-1)^{x_{n-2}+x_{n-4}}[\Delta_{n-3}]_{n-4}^{n-3} + k_1\overline{k}_2(-1)^{x_{n-2}+x_{n-3}}[\Delta_{n-3}]_{n-3}^{n-4}\right)\end{aligned}$$

Substituting, we get

$$\begin{aligned}\Delta_n =& (1-k_0)\left(\tfrac{k_0}{1-k_0}\right)^{x_n}\Delta_{n-1} - k_1^2(-1)^{x_n+x_{n-1}}\Delta_{n-2}\\ &- |k_2|^2(1-k_0)\left(\tfrac{k_0}{1-k_0}\right)^{x_{n-1}}(-1)^{x_n+x_{n-2}}\Delta_{n-3} + |k_2|^4(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-3}}\Delta_{n-4}\\ &- k_1^2(k_2+\overline{k}_2)(-1)^{x_n+x_{n-1}+x_{n-2}}\Delta_{n-3} - 2k_1^2|k_2|^2(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-3}}\Delta_{n-4}\\ &+ |k_2|^2(-1)^{x_n+x_{n-1}}\left(k_1k_2(-1)^{x_{n-2}+x_{n-4}}[\Delta_{n-3}]_{n-4}^{n-3} + k_1\overline{k}_2(-1)^{x_{n-2}+x_{n-3}}[\Delta_{n-3}]_{n-3}^{n-4}\right)\end{aligned} \tag{14}$$

Let us now compute $\Delta_n - |k_2|^2(-1)^{x_n+x_{n-1}}\Delta_{n-2}$, using equation (14) for Δ_n and equation (13) with $n-2$ in the place of n for Δ_{n-2}. Since the terms containing $[\Delta_{n-3}]_{n-4}^{n-3}$ and $[\Delta_{n-3}]_{n-3}^{n-4}$ cancel, we obtain

$$\begin{aligned}&\Delta_n - |k_2|^2(-1)^{x_n+x_{n-1}}\Delta_{n-2}\\ &= (1-k_0)\left(\tfrac{k_0}{1-k_0}\right)^{x_n}\Delta_{n-1} - k_1^2(-1)^{x_n+x_{n-1}}\Delta_{n-2}\\ &-|k_2|^2(1-k_0)\left(\tfrac{k_0}{1-k_0}\right)^{x_{n-1}}(-1)^{x_n+x_{n-2}}\Delta_{n-3} + |k_2|^4(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-3}}\Delta_{n-4}\\ &-k_1^2(k_2+\overline{k}_2)(-1)^{x_n+x_{n-1}+x_{n-2}}\Delta_{n-3} - 2k_1^2|k_2|^2(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-3}}\Delta_{n-4}\\ &-|k_2|^2(-1)^{x_n+x_{n-1}}(1-k_0)\left(\tfrac{k_0}{1-k_0}\right)^{x_{n-2}}\Delta_{n-3} + k_1^2|k_2|^2(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-3}}\Delta_{n-4}\\ &+|k_2|^4(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-4}}(1-k_0)\left(\tfrac{k_0}{1-k_0}\right)^{x_{n-3}}\Delta_{n-5}\\ &-|k_2|^6(-1)^{x_n+x_{n-1}+x_{n-2}+x_{n-3}+x_{n-4}+x_{n-5}}\Delta_{n-6}\end{aligned}$$

Equation (9) is obtained by rearranging the terms in the above equation. □

The following theorem gives recursive equations for the predictive probabilities of P^f.

Theorem 2. *Let P^f be a stationary, two-dependent determinantal probability measure on the cylinder sigma-algebra of $\{0,1\}^Z$ with f as in (8), $k_0 \in (0,1), k_1 \in \mathbb{R}$, and $k_2 \in \mathbb{C}$. Then, for every $x \in \{0,1\}^Z$ and every $n \geq 3$*

$$\begin{aligned} p_n^f(x_{(n-1)}) &= k_0 - \sum_{j=2}^{6\wedge n} a_j \prod_{u=1}^{j-1} \Big(b(p_{n-u}^f(x_{(n-u-1)}); x_{n-u})\Big)^{-1} \\ &+ |k|_2^2 b(k_0; x_{n-2}) \prod_{u=1}^{2} \Big(b(p_{n-u}^f(x_{(n-u-1)}); x_{n-u})\Big)^{-1} \\ &- \sum_{j\in\{3,5\}, j\leq n} c_j b(k_0; x_{n-j+2}) \prod_{u=1}^{j-1} \Big(b(p_{n-u}^f(x_{(n-u-1)}); x_{n-u})\Big)^{-1} \end{aligned} \tag{15}$$

where $p_1^f(x_{(0)}) := P_1^f(\{1\})$, a_j $(2 \leq j \leq 6)$, b and c_j $(j = 3, 5)$ are defined as in (10), (11), and (12), respectively.

Proof. To prove (15), it is sufficient to compute

$$p_n^f(x_{(n-1)}) = P_n^f(\{(x_1, \ldots, x_{n-1}, 1)\})/P_{n-1}^f(\{(x_1, \ldots, x_{n-1})\}),$$

using (9), and notice that

$$\frac{P_n^f(\{(x_1, \ldots, x_n)\})}{P_{n-j}^f(\{(x_1, \ldots, x_{n-j})\})} = \prod_{u=0}^{j-1} p_{n-u-1}^f(x_{(n-u-2)})^{x_{n-u-1}} (1 - p_{n-u-1}^f(x_{(n-u-2)}))^{1-x_{n-u-1}}$$

for every $n = 1, 2, \ldots$ and $j < n$. □

The next corollary allows us to construct samples from P_n^f starting from independent random variables. The proof is similar to that of Corollary 2.

Corollary 3. *Let $k_0 \in (0,1), k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{C}$ be such that, for every $s \in \mathbb{T}$, $f(s) := \sum_{j=-2}^{2} k_j \exp(i2\pi js) \in [0,1]$. Let a_j $(2 \leq j \leq 6)$, b and c_j $(j = 3, 5)$ be defined as in (10), (11) and (12), respectively. Furthermore, let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and uniformly distributed on $[0,1]$.*

If $X_n = 1_{[0,W_n]}(Z_n)$ $(n \geq 1)$, $W_1 = k_0$, $W_2 = k_0 + k_1^2(1-k_0)^{-1}(1-1/k_0)^{X_1}$ *and, for* $n \geq 3$,

$$\begin{aligned} W_n &= k_0 - \sum_{j=2}^{6\wedge n} a_j \prod_{u=1}^{j-1} \frac{(1-1/W_{n-u})^{X_{n-u}}}{(1-W_{n-u})} \\ &+ |k_2|^2 \frac{(1-1/k_0)^{X_{n-2}}}{(1-k_0)} \prod_{u=1}^{2} \frac{(1-1/W_{n-u})^{X_{n-u}}}{(1-W_{n-u})} \\ &- \sum_{j\in\{3,5\} j\leq n} c_j \frac{(1-1/k_0)^{X_{n-j+2}}}{(1-k_0)} \prod_{u=1}^{j-1} \frac{(1-1/W_{n-u})^{X_{n-u}}}{(1-W_{n-u})}, \end{aligned}$$

then, for every $n \geq 1$, *the probability distribution of* $X_{(n)}$ *coincides with* P_n^f.

Equations (9) and (15) simplify when k_2 is real. Actually, recursive equations of lower degree hold in this case. It can be proved with techniques similar to the ones used in the proof of Theorem 2 that

$$\begin{aligned} p_n^f(\{x_{(n-1)}\}) &= k_0 - k_2 \left[1 - b(k_0; x_{n-1}) \left(b(p_{n-1}^f(x_{(n-2)}); x_{n-1})\right)^{-1}\right] \\ &- \sum_{j=2}^{5\wedge n} d_j \prod_{u=1}^{j-1} \left(b(p_{n-u}^f(x_{(n-u-1)}); x_{n-u})\right)^{-1} \\ &+ \sum_{j\in\{3,4\}, j\leq n} e_j b(k_0; x_{n-j+2}) \prod_{u=1}^{j-1} \left(b(p_{n-u}^f(x_{(n-u-1)}); x_{n-u})\right)^{-1} \end{aligned} \tag{16}$$

where $p_1^f(x_{(0)}) := P_1^f(\{1\})$, b is defined as in (11),

$$d_1 = k_2 \quad d_2 = -k_1^2 \quad d_3 = -k_1^2 k_2 \quad d_4 = k_2^4 \quad d_5 = -k_2^5,$$

$$e_j = (-k_2)^{j-1} \quad (1 \leq j \leq 4).$$

Furthermore, one can construct a two-dependent SDP $X_{(n)}$ with $k_2 \in \mathbb{R}$

through $X_n = 1_{[0,W_n]}(Z_n)$ $(n \geq 1)$, $W_1 = k_0$,

$$\begin{aligned} W_n &= k_0 - k_2\left[1 - \frac{1-k_0}{1-W_{n-1}}\left(\frac{1-1/W_{n-1}}{1-1/k_0}\right)^{X_{n-1}}\right] \\ &- \sum_{j=2}^{5\wedge n} d_j \prod_{u=1}^{j-1} \frac{(1-1/W_{n-u})^{X_{n-u}}}{1-W_{n-u}} \\ &+ \sum_{j\in\{3,4\}, j\leq n} e_j \frac{1-k_0}{(1-1/k_0)^{X_{n-j+2}}} \prod_{u=1}^{j-1} \frac{(1-1/W_{n-u})^{X_{n-u}}}{1-W_{n-u}}, \end{aligned} \tag{17}$$

$(n \geq 2)$, $(Z_n)_{n\in\mathbb{N}}$ independent and uniformly distributed on $[0,1]$.

The above results can be used either to simulate two-dependent SDP's sequentially with efficient algorithms or to explore the properties of the predictive distributions of P^f. The next example shows that the predictive probabilities of a non-degenerate two-dependent SDP converge nowhere a.s.

Example 3. Let $(X_n)_{n\in Z}$ be an SDP with f as in (8), $k_0 \in (0,1), k_1, k_2 \in \mathbb{R}$. We will prove that $0 < \liminf_{n\to\infty} p_n^f(X_{(n-1)}) < \limsup_{n\to\infty} p_n^f(X_{(n-1)}) < 1$ *a.s.*

Let us first show that $\liminf_{n\to\infty} p_n^f(X_{(n-1)}) > 0$ a.s. Since for every $x_{(n-1)} \in \{0,1\}^{n-1}$, $p_n^f(x_{(n-1)}) \leq p_n^f(1_{(n-1)})$ (see Lyons, 2003), it holds $\liminf_{n\to\infty} p_n^f(X_{(n-1)}) \geq \liminf_{n\to\infty} p_n^f(1_{(n-1)})$.

On the other hand, $(p_n^f(1_{(n-1)}))_{n\geq 2}$ is a decreasing sequence converging to $p := \exp\int_{\mathbb{T}} \log f d\lambda$ (see Lyons, 2003). To show that $p > 0$ without evaluating the integral, we can use (16). In fact p must satisfy $p^5 = (k_0-k_1)p^4 + (k_0k_2 - k_1^2)p^3 - k_2(k_0k_2 - k_1^2)p^2 - k_2^3(k_0-k_2)p + k_2^5$ and $p = 0$ is not a solution of this equation. By duality, $q := \lim_{n\to\infty} p_n^f(0_{(n-1)}) < 1$.

Let us now prove that $\liminf_{n\to\infty} p_n^f(X_{(n-1)}) < \limsup_{n\to\infty} p_n^f(X_{(n-1)})$ *a.s.* Let $M = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} \{x \in \{0,1\}^Z : x_{k-1} = 1, \ldots, x_{k-5} = 1\}$ and let $N = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} \{x \in \{0,1\}^Z : x_{k-1} = 1, \ldots, x_{k-4} = 1, x_{k-5} = 0)$. Since P^f is a Bernoulli shift (see Theorem 3.1 in Lyons and Steif, 2003), it is ergodic. Hence $P^f(M)$ and $P^f(N)$ are either equal to zero or equal to one. On the other hand, P^f has full support (see Theorem 4.2 in Lyons and Steif, 2003). Hence $P^f(M) = P^f(N) = 1$. Let now $Q = \{x \in \{0,1\}^{\mathbb{Z}} : \liminf_{n\to\infty} p_n^f(x_{(n-1)}) = \limsup_{n\to\infty} p_n^f(x_{(n-1)})\}$. Then $P^f(Q \cap M \cap N) = P^f(Q)$. Suppose that there exists $x \in M \cap N \cap Q$ and let

$p(x) = \lim_{n\to\infty} p_n^f(x_{(n-1)})$. Then, by (16),

$$\begin{aligned} p(x) &= k_0 - k_2 + \tfrac{k_0k_2-k_1^2}{p(x)} - k_2\tfrac{k_0k_2-k_1^2}{p(x)^2} - k_2^3\tfrac{k_0-k_2}{p(x)^3} + \tfrac{k_2^5}{p(x)^4} \\ &= k_0 - k_2 + \tfrac{k_0k_2-k_1^2}{p(x)} - k_2\tfrac{k_0k_2-k_1^2}{p(x)^2} - k_2^3\tfrac{k_0-k_2}{p(x)^3} - \tfrac{k_2^5}{p(x)^3(1-p(x))}. \end{aligned}$$

Simplifying, we find $\frac{k_2^5}{p(x)^4} = \frac{k_2^5}{p(x)^3(p(x)-1)}$ which leads to a contradiction since $k_2 \neq 0$. We deduce that $M \cap N \cap Q = \emptyset$ and, therefore $P^f(Q) = 0$.

We close by noting that the problem is still open whether fixed order recursive equations hold for m-dependent SDP's when $m > 2$.

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