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A note on Variance Bounding for continuous time Markov chains

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Abstract

In this paper the notion of Variance Bounding introduced by Roberts and Rosenthal (2008) is extended to continuous time Markov Chains. Moreover, it is proven that, as in the discrete time case, the notion of Variance Bounding for reversible Markov Chains is equivalent to the existence of a central limit theorem. A connection with the continuous time Peskun ordering, introduced by Leisen and Mira (2008), concludes the paper.

Keywords: Variance Bounding, Central Limit Theorems, Peskun Ordering.

1. Introduction

Roberts and Rosenthal (2008) introduced the notion of Variance Bounding for discrete time Markov chains (DTMC). Let $v(h, P)$ be the asymptotic variance of a DTMC with transition kernel P , stationary w.r.t. a distribution π . Let $Var_{\pi}(h)$ be the stationary variance. The transition kernel P is Variance Bounding if there exists $K < \infty$ such that for all real valued functions h we have $v(h, P) \leq K Var_{\pi}(h)$. For a reversible P , Roberts et al. (2008) proved that the variance bounding of the DTMC is a necessary and sufficient condition to get a central limit theorem (CLT). In this paper, the notion of variance bounding is extended to continuous time Markov Chains and it is shown that a reversible continuous time Markov Chain (CTMC) satisfies a usual CLT if and only if it is variance bounding. It is well known that in the Markov Chain Monte Carlo (MCMC) setting we have a degree of freedom in the choice of the Markov Chain that we run in the simulation. Many criterias have been thus developed to maximize the efficiency of the MCMC simulation. One of the most notable is the Peskun ordering (see Peskun, 1973; Tierney, 1998 for discrete time), which allows to compare the asymptotic

variances of different Markov Chains. In Roberts et al. (2008) it is shown that if a Markov Chain dominates [another](#) one in the Peskun [ordering](#), and the latter is variance bounding then the former is variance bounding as well. [Peskun ordering was extended to CTMCs by Leisen and Mira \(2008\). The last Section of the paper gives a connection between variance bounding in continuous time and the continuous time Peskun ordering.](#)

2. Preliminaries

Let $(X_t)_{t \in \mathbb{R}^+}$ be a continuous time Markov Chain (CTMC) with values in a measurable space $(\mathcal{X}, \mathcal{F})$ and with generator G . Throughout the paper we assume that the CTMC is *reversible* with respect to π , i.e.

$$\int_{x \in A} \pi(dx) P^t(x, B) = \int_{x \in B} \pi(dx) P^t(x, A) \quad \text{for any } A, B \in \mathcal{F} \quad \text{and } t \in \mathbb{R}^+,$$

where $P^t(x, dy)$ is the t -step transition kernel.

Definition 1. (see [Bhattacharya \(1982\)](#)) For every $h : \mathcal{X} \rightarrow \mathbb{R}$, the asymptotic variance of the CTMC $(X_t)_{t \in \mathbb{R}^+}$ is defined as

$$v(h, G) := \lim_{n \rightarrow +\infty} \frac{1}{n} \text{Var} \left(\int_0^{nt} h(X_s) ds \right).$$

[Bhattacharya \(1982\)](#) shows that a CLT holds true for functions $h : \mathcal{X} \rightarrow \mathbb{R}$ in the range of the generator G . In the paper, we thus will refer to functions h in this class. In particular for functions h in the range of the generator, we have

$$v(h, G) = -2 \langle h, G^{-1} h \rangle,$$

where $\langle f, g \rangle = \int f(x)g(x)\pi(x)$ denotes the scalar product in $L^2(\pi)$. Suppose that the chain starts in stationarity and that $\text{Var}_\pi(h) = E[(h(X_0) - E[h(X_0)])^2]$ is the stationary variance. Then, the natural extension of the definition of variance bounding to CTMC is the following

Definition 2. The generator G is Variance Bounding if there exists $K < \infty$ such that for all h [in the range of the generator](#) we have

$$v(h, G) \leq K \text{Var}_\pi(h).$$

Let $\sigma(P)$ be the spectra of an operator P . In particular, the spectra of a generator G is a subset of \mathbb{R}^- . We denote [by](#) E_P the resolution of the

identity of an operator P given in the spectral theorem (see Conway, 1985; Rosenthal 2003; Mira and Leisen, 2009). We thus have

$$g(P) = \int_{\sigma(P)} g(\lambda) E_P(d\lambda),$$

for every bounded and measurable function $g : \sigma(P) \rightarrow \mathbb{R}$. Moreover, given a bounded and measurable function g , let $E_{g,P}$ be the spectral measure associated with g and P , so that $E_{g,P}(A) = \langle g, E_P(A)g \rangle$ and

$$\langle g, h(P)g \rangle = \int_{\sigma(P)} h(\lambda) E_{g,P}(d\lambda)$$

for every bounded and measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$. Let G be the generator of a CTMC. Let

$$\Lambda = \Lambda(G) := \sup_{\lambda \in \sigma(G)} \lambda.$$

Theorem 1. *Let G be the generator of a CTMC reversible w.r.t. π . G is variance bounding if and only if $\Lambda < 0$.*

Proof. We prove the sufficiency part of the theorem by contradiction. Thus suppose that $\sup_{\lambda \in \sigma(G)} \lambda = 0$ and let $r < 0$. Then $E_G((r, 0])$ is non zero, so

there exists $h \in L_0^2(\pi)$ in the range of $E_G((r, 0])$ such that

$$\begin{aligned} v(h, G) &= -2\langle h, G^{-1}h \rangle = -\int_{\sigma(G)} \frac{2}{\lambda} E_{h,G}(d\lambda) \geq -\int_r^0 \frac{2}{\lambda} E_{h,G}(d\lambda) \\ &\geq \frac{-2}{r} E_{h,G}((r, 0]) = \frac{-2}{r} \langle h, E_G((r, 0])h \rangle = \frac{-2}{r} \langle h, h \rangle = -\frac{2}{r} \text{Var}_\pi(h). \end{aligned}$$

Hence

$$\frac{v(h, G)}{\text{Var}_\pi(h)} \geq -\frac{2}{r}$$

and

$$\sup_{h \in L_0^2(\pi)} \frac{v(h, G)}{\text{Var}_\pi(h)} \geq \sup_{r < 0} -\frac{2}{r} = +\infty.$$

Thus G is not variance bounding, and we get a contradiction. Hence $\Lambda = \Lambda(G) = \sup_{\lambda \in \sigma(G)} \lambda < 0$.

Conversely, suppose that $\Lambda = \Lambda(G) = \sup_{\lambda \in \sigma(G)} \lambda < 0$. Hence for $h \in L_0^2(\pi)$ in the range of the generator

$$\begin{aligned} v(h, G) &= -2\langle h, G^{-1}h \rangle = - \int_{\sigma(G)} \frac{2}{\lambda} E_{h,G}(d\lambda) \leq -\frac{2}{\Lambda} \int_{\sigma(G)} \langle h, E_G(d\lambda)h \rangle \\ &= -\frac{2}{\Lambda} \langle h, h \rangle = -\frac{2}{\Lambda} \text{Var}_\pi(h). \end{aligned}$$

Thus G is variance bounding with $K = -\frac{2}{\Lambda}$. \square

3. Relation with central limit theorems

The following theorem shows that for reversible CTMC, the hypothesis of variance bounding is a [necessary](#) and sufficient condition to get a usual CLT. A similar result holds for DTMC (see Roberts et al. (2008)).

Theorem 2. *Let G be reversible w.r.t. π . G is variance bounding if and only if every $h \in L^2(\pi)$ in the range of the generator G satisfies a usual CLT for G .*

Proof. If G is variance bounding, then by Theorem 1 we have $\Lambda = \Lambda(G) = \sup_{\lambda \in \sigma(G)} \lambda < 0$. Hence,

$$v(h, G) \leq -\frac{2}{\Lambda} \langle h, h \rangle < +\infty,$$

and a [CLT](#) holds (see Bhattacharya, 1982; Kipnis and Varadhan, 1986).

Conversely, suppose that G satisfies a CLT for every $h \in L^2(\pi)$ in the range of G , but G is not variance bounding. Then by Theorem 1 we have $\Lambda = \Lambda(G) = \sup_{\lambda \in \sigma(G)} \lambda = 0$. Thus $E_G((r, 0]) \neq 0$ for any $r < 0$, and there must

exist infinitely many $m \in \mathbb{N}$ such that $E_G((-2^{-m}, -2^{-m-1}])$ is non zero. Let $m_1 < m_2 < \dots$ (and thus $m_i \geq i$) such that $E_G((-2^{-m_i}, -2^{-m_i-1}])$ is non zero. Let g_i in the range of $E_G((-2^{-m_i}, -2^{-m_i-1}])$ with $\|g_i\| = \langle g_i, g_i \rangle = 1$. The g_i are thus orthonormal and furthermore

$$\begin{aligned} \langle g_i, G^{-1}g_i \rangle &= \int_{\sigma(G)} \lambda^{-1} E_{g_i,G}(d\lambda) = \int_{-2^{-m_i}}^{-2^{-m_i-1}} \lambda^{-1} E_{g_i,G}(d\lambda) \\ &\leq -2^{m_i} \int_{-2^{-m_i}}^{-2^{-m_i-1}} E_{g_i,G}(d\lambda) = -2^{m_i} \langle g_i, g_i \rangle = -2^{m_i}. \end{aligned}$$

Define $h = \sum_{i=1}^{+\infty} 2^{-\frac{i}{2}} g_i$. From the orthonormality

$$Var_{\pi}(h) = \langle h, h \rangle = \left\langle \sum_{i=1}^{+\infty} 2^{-\frac{i}{2}} g_i, \sum_{i=1}^{+\infty} 2^{-\frac{i}{2}} g_i \right\rangle = \sum_{i=1}^{+\infty} (2^{-\frac{i}{2}})^2 \langle g_i, g_i \rangle = \sum_{i=1}^{+\infty} 2^{-i} = 1.$$

On the other hand, since $\langle g_i, G^{-1} g_i \rangle \leq -2^{m_i}$ and $\langle g_i, G^{-1} g_j \rangle = 0$ for all $i \neq j$

$$\begin{aligned} v(h, G) &= -2 \langle h, G^{-1} h \rangle = -2 \left\langle \sum_{i=1}^{+\infty} 2^{-\frac{i}{2}} g_i, G^{-1} \sum_{i=1}^{+\infty} 2^{-\frac{i}{2}} g_i \right\rangle = -2 \sum_{i=1}^{+\infty} (2^{-\frac{i}{2}})^2 \langle g_i, G^{-1} g_i \rangle \\ &\geq -2 \sum_{i=1}^{+\infty} 2^{-i} (-2^{m_i}) = 2 \sum_{i=1}^{+\infty} 2^{-i} (2^{m_i}) \geq 2 \sum_{i=1}^{+\infty} 2^{-i} 2^i = +\infty. \end{aligned}$$

Thus G does not satisfy a CLT, but this is in contrast with the hypothesis. Hence G is variance bounding. \square

4. Relation with Peskun ordering

In Leisen and Mira (2008), the Peskun ordering has been extended to CTMC. For MC with finite or countable state space, a natural definition for continuous time Peskun ordering is the following

Definition 3. Suppose that $G_1 = (g(1))_{ij}$ and $G_2 = (g(2))_{ij}$ are the generators of two CTMCs, both stationary with respect to a distribution π (i.e., $\pi G_1 = 0$, $\pi G_2 = 0$), and suppose that the two chains are uniformizables. Then we say that G_1 dominates G_2 in the continuous time Peskun ordering and we write $G_1 \succeq_{EP} G_2$ if

$$g(1)_{ij} \geq g(2)_{ij}, \quad \forall i \neq j.$$

For general state space, we have to consider generators G which can be written as

$$Gf(x) = \int f(y) Q(x, dy), \quad (1)$$

where the kernel Q is defined in terms of the transition kernel P as

$$Q(x, dy) = \frac{\partial}{\partial t} P(t, x, dy)|_{t=0}$$

In this case the Peskun ordering in a general state space $(\mathcal{X}, \mathcal{F})$ is defined in the following way

Definition 4. Let G_1 and G_2 be the generators of two CTMCs admitting the representation (1) with kernels Q_1 and Q_2 respectively, both stationary with respect to a common distribution π on \mathcal{X} . Assume $\sup_x Q_i(x, \mathcal{X} \setminus \{x\}) < \infty$, for $i = 1, 2$. Then G_1 dominates G_2 in the continuous time Peskun ordering, $G_1 \succeq_{EP} G_2$, if

$$Q_1(x, A \setminus \{x\}) \geq Q_2(x, A \setminus \{x\}), \quad \forall A \in \mathcal{F}.$$

In Leisen et al. (2008) the following theorem for continuous time Peskun ordering is proven

Theorem 3. If $G_1 \succeq_{EP} G_2$ and if the corresponding CTMCs are reversible, then $G_2 - G_1$ is a positive operator.

As a consequence of Theorem 3, the following theorem holds.

Theorem 4. Let G_1, G_2 be the generators of two CTMCs reversible w.r.t. π , which admit a representation as in equation (1). Let $G_1 \succeq_{EP} G_2$, and let G_2 be variance bounding. Then G_1 is variance bounding.

Proof. By Theorem 3, the operator $G_2 - G_1$ is positive, thus $\sup \sigma(G_2) \geq \sup \sigma(G_1)$. Applying twice Theorem 1 the result easily follows. \square

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References

- Bhattacharya R., *On the functional central limit theorem and the law of the iterated logarithm for Markov processes*, Z. Wahrsch. Verw. Gebiete, 60(2): 185–201, 1982.
- Conway J. B., *A course in Functionals Analysis*, Springer Verlag, 1985.
- Kipnis C, and Varadhan R.S. *Central Limit Theorem for Additive Functionals of Reversible Markov Process and Applications to Simple Exclusions*, Commun. Math. Phys., 104: 1–19, 1986.
- Leisen F. and Mira A., *An extension of Peskun and Tierney orderings to continuous time Markov chains*, Statistica Sinica, 18: 1641–1651, 2008.

- Mira A. and Leisen F., *Covariance Ordering for discrete and continuous time Markov Chains*, Statistica Sinica, 19: 651–666, 2009.
- Peskun P.H., *Optimum Monte Carlo sampling using Markov Chains*, Biometrika, 60: 607–612, 1973.
- Rosenthal J. S., *Asymptotic Variance and convergence rates of nearly periodic MCMC algorithm*, JASA, 98: 169–177, 2003.
- Roberts G. O. and Rosenthal J. S., *Geometric ergodicity and hybrid Markov chains*, Electronic Comm. Prob., 2: 13–25, 1997.
- Roberts G. O. and Rosenthal J. S., *Variance Bounding Markov Chains*, Ann. of App. Prob., 18: 1201–1214, 2008.
- Tierney L., *A Note on Metropolis Hastings Kernels for General state spaces*, Ann. of App. Prob., 8: 1–9, 1998.