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A proof of Pillai's conjecture

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Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all ? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is entended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. Then, we generalize the proof to Pillai conjecture.

(Keywords : Diophantine equations ; Catalan equation ; Pillai conjecture ; Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers stricly greater than 1, $X > 1$ and $Y > 1$, for which with exponants strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3+1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai conjecture

$$Y^p = X^q + a$$

And prove that it has always a finite number of solutions for a fixed a.

The approach

Let $c=\frac{X^p-1}{Y^{\frac{p}{2}}},\quad c'=\frac{7-X^p}{Y^{\frac{p}{2}}}$ But $(c+c')Y^{\frac{p}{2}}=X^p-1+7-X^p=6\Rightarrow Y^{\frac{p}{2}}=\frac{6}{c+c'}$ And $X^p=cY^{\frac{p}{2}}+1=\frac{6c}{c+c'}+1=\frac{7c+c'}{c+c'}$ And $X^q=Y^p-1=\frac{36-(c+c')^2}{(c+c')^2}$ We have $(c+c')Y^{\frac{p}{2}}=6>0$ Thus $c+c'>0$. Also $cY^{\frac{p}{2}}=X^p-1>0$ Thus $c>0$. But $X^p\geq 4$, hence $cY^{\frac{p}{2}}=X^p-1\geq 7-X^p=c'Y^{\frac{p}{2}}$ hence $c\geq c'$ and $7-X^p=7-\frac{7c+c'}{c+c'}=\frac{6c'}{c+c'}$ If $c'>0$ Thus $X^p<7$ and $X^p=4$, it means that $c'<0$ and $(c+c')Y^{\frac{p}{2}}=6>0$ $(c+c')X^p=7c+c'>0$ thus $c+c'>0$ and $Y^{\frac{p}{2}}=\frac{6}{c+c'}\geq 3\Rightarrow c+c'\leq 2$

main results What we must retain is
$1)c'<0$
$2)0<c+c'<2$
$3)0<7c+c'$
And we will discuss two cases
$I)c^2<1$
$II)c^2\geq 1$

case $c^2<1$ We have

$$2X^q-X^{2p}=\frac{72-2(c+c')^2-(7c+c')^2}{(c+c')^2}$$

$$=\frac{72-51c^2-3c'^2-18cc'}{(c+c')^2}=\frac{51(1-c^2)-3c'(c+c')-15cc'+21}{(c+c')^2}>0$$

Thus $2X^q>X^{2p}\quad(1)$ So

$$X^{2p-q-1}<\frac{2}{X}\leq 1$$

Thus

$$2p\leq q+1$$

We will give two proofs : for $X>2$ $X^{4q}(36-(c+c')^2)^{2p-4}>32^{2p-4}X^{4q}>36^q$ and

$$X^{q(q-4)}=\frac{(36X^{-4}-(c+c')^2X^{-4})^q}{(c+c')^{2q}}<\frac{36^qX^{-4q}}{(c+c')^{2q}}$$

$$<X^{(2p-4)q}=\frac{(36-(c+c')^2)^{2p-4}}{(c+c')^{4p-8}}$$

Because $X^{2pq}=\frac{(7c+c')^{2q}}{(c+c')^{2q}}>\frac{36^q}{(c+c')^{2q}}$ Thus $2p+1>q\geq 2p-1$. We have then $q+1=2p$. Another proof : $c^2<1$, $\exists m>1$ for which $c^2=\frac{m-1}{m}$ thus $1-c^2=\frac{1}{m}<\frac{X-1}{X}$ thus $c^2X>1$ and $X^{2p+1}=\frac{(7c+c')^2X}{(c+c')^2}>\frac{36c^2X}{(c+c')^2}>\frac{36-(c+c')^2}{(c+c')^2}=X^q$ We deduce

$$X^{q-1}<X^{2p}$$

We have then

$$X^q\leq X^{2p}\leq X^{q+1}$$

As q is not even : $q+1=2p$ Thus

$$0>(c^2-1)Y^p=X^{2p}-2X^p-X^q=X^{q+1}-2X^p-X^q>0$$

we deduce $c^2=1$ and then $c^2-1=X^p(X^p-2)-X^q=0$ and $X^{p-1}-X^{q-p-1}=\frac{2}{X}$ This expression is an integer, hence $X=2$ and $2^{p-1}=1+2^{q-p-1}$ In one side an even number, in the other an odd one, it is possible if $p=2$, thus $q=p+1=3$ and $Y=\pm 3$

Case $c\geq 1$ We must retain here that
$1)c^2>1$
$2)c+c'<2$
$3)c'<0$
We have

$$2c^2X^q-X^{2p}=\frac{72c^2-2c^2(c+c')^2-(7c+c')^2}{(c+c')^2}$$

$$=\frac{72c^2-49c^2-c'^2-14cc'-2c^2(c^2+c'^2+2cc')}{(c+c')^2}$$

$$=\frac{c^2(23-2(c+c')^2)-c'(c+c')-13cc'}{(c+c')^2}>\frac{15c^2-c'(c+c')-13cc'}{(c+c')^2}>0$$

Or $2c^2X^q>X^{2p}>4X^p\Rightarrow c^2X^q>2X^p$ (5) And

$$c^2X^q-X^{2p}=\frac{36c^2-c^2(c+c')^2-(7c+c')^2}{(c+c')^2}$$

$$=\frac{(6c-7c-c')(6c+7c+c')-c^2(c+c')^2}{(c+c')^2}=\frac{-(c+c')(13c+c')-c^2(c+c')^2}{(c+c')^2}<0$$

We deduce $\frac{1}{2}X^{2p}<c^2X^q<X^{2p}<c^2X^{2p}$ (6) And $X^{2p-q}>c^2>1$ Or $2p\geq q+1$ We have seen that $2c^2X^{2p-1}\geq 2c^2X^q>X^{2p}$, we deduce $2c^2>X$, but $Y+(c+c')^2>Y>36$ or $Y^{2p}>(36-(c+c')^2)^{2p}$ And

$$\frac{32^qX^{2q}}{(36-(c+c')^2)^{2p}}>\frac{32^qX^{2q}}{Y^{2p}}=\frac{32^qX^{2q}}{(X^q+1)^2}>1$$

We deduce

$$X^{q(q+2)}=\frac{(36X^2-X^2(c+c')^2)^q}{(c+c')^{2q}}>\frac{32^qX^{2q}}{(c+c')^{4p}}$$

$$>\frac{(36-(c+c')^2)^{2p}}{(c+c')^{4p}}=X^{2pq}$$

Or $q+2>2p\geq q+1$ We can restrain the value of Y. for example : $Y\geq 5$ implies $Y^{\frac{5}{2}}>36$ and, by the same calculus $q+5>2p\geq q+1$. As q is odd, it leads to $q+1=2p$ or $q+3=2p$. And if $q+1=2p$ we have $c^2X^q=c^2X^{2p-1}<X^{2p}\Rightarrow X>c^2$ We pose $X^p>u_nc^{2p}$ With $u_0=1$ and $u_n<2^p$, we have

$$X^p-2^pc^{2p}=\frac{2^p}{u_n}X^p-2^pc^{2p}+(1-\frac{2^p}{u_n})X^p>\frac{u_n-2^p}{u_n}X^p$$

And $X^p-u_nc^{2p}=X^p-2^pc^{2p}+(2^p-u_n)c^{2p}>0$ Thus $X^p-2^pc^{2p}>(u_n-2^p)c^{2p}$ We add

$$2X^p-(u_n+2^p)c^{2p}>2X^p-2^{p+1}c^{2p}>(u_n-2^p)(\frac{X^p}{u_n}+c^{2p})$$

$$= (u_n - 2^p)(c^{2p} - \frac{X^p}{u_n}) + 2(u_n - 2^p)\frac{X^p}{u_n} > (2 - \frac{2^{p+1}}{u_n})X^p$$

We deduce $\frac{2^{p+1}}{u_n}X^p > (u_n + 2^p)c^{2p}$ Or

$$X^p > \frac{u_n(u_n + 2^p)}{2^{p+1}}c^{2p} = u_{n+1}c^{2p}$$

With

$$u_{n+1} = \frac{u_n(u_n + 2^p)}{2^{p+1}} < \frac{2^p(2^p + 2^p)}{2^{p+1}} = 2^p$$

For all n : in the infinity $\lim_{n\longrightarrow\infty}(u_n) = L$ And L verifies recurrence relation $L = \frac{L(L+2^p)}{2^{p+1}} \Rightarrow L = 2^p$ Hence

$$X^p > \lim_{n\longrightarrow\infty}(u_n)c^{2^p} = 2^p c^{2^p}$$

We have $2c^2 > X > 2c^2$ So $X = 2c^2$ and

$$X = \frac{X^{2p}}{X^q} = 2c^2 = \frac{2X^{2p} - 4X^p + 2}{X^q + 1}$$

$$\Rightarrow X - X^{q+1} + 4X^p = 2 \Rightarrow 1 - X^q + 4X^{p-1} = \frac{2}{X}$$

Thus $X = 2c^2 = 2$ and $c^2 = 1$ or $(2^p - 1)^2 = 2^q + 1$ and

$$2^p - 2 = 2^{q-p} = 2^{p-1} \Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm 3$$

Another proof : $Y^p = X^q + 1 = X^{2p-1} + 1 > X^{\frac{3p}{2}}$ Hence $Y^2 > X^3$ Let us pose $Y^2 = X^3 + u \geq X^3 + 1$ If we suppose p even

$$1 \leq u = Y^2 - X^3 = \frac{Y^p - X^{3p}}{Y^{p-2} + Y^{p-4}X^3 + ... + Y^2X^{3p-6} + X^{3p-3}}$$

$$= \frac{X^{2p-1} + 1 - X^{3p}}{Y^{p-2} + Y^{p-4}X^3 + ... + Y^2X^{3p-6} + X^{3p-3}} < 0$$

and it is impossible, it means that p is not greater than 2 thus $p = 2$ and $q = 2p - 1 = 3$ and $X = 2$ and $Y = \pm 3$. If p is odd and greater than 3, let $p = 2k + 1$

$$Y^{2k+1} - X^{3p} = (Y^2 - X^3)(Y^{2k-1} + Y^{2k-3}X^3 + ... + Y^3X^{3p-9} + YX^{3p-6}) + X^{3p-3}Y - X^{3p}$$

$$= (Y^2 - X^3)(X^{3p-3} + Y^2X^{3p-6} + ... + Y^{2k-2}X^3 + Y^{2k}) + Y^{2k+1} - Y^{2k+2}$$

$$= \frac{1}{2}((Y^2 - X^3)(Y^{2k-1} + Y^{2k-3}X^3 + ... + Y^3X^{3p-9} + YX^{3p-6}) +$$

$$+\frac{1}{2}((Y^2 - X^3)(X^{3p-3} + Y^2X^{3p-6} + ... + Y^{2k-2}X^3 + Y^{2k}) + Y^{2k+1} - X^{3p} - y^{2k+2} + X^{3p-3}Y)$$

Hence

$$0 > Y^{2k+1} - X^3 = (Y^2 - X^3)(Y^{2k-1} + ... + YX^{3p-6} + X^{3p-3} + ... + Y^{2k}) + YX^{3p-3} - Y^{2k+2}$$

We deduce $X^{3p-3)} < Y^{2k+1} = X^{2p-1} + 1$ It is impossible : it means $p = 3$ and $q = 2p - 1 = 5$, the equation becomes $Y^3 = X^5 + 1$ which does not have

solution! And if $q+3=2p$, we have $Y^p = X^{2p-3}+1 \geq X^{\frac{3p}{2}}+1$ it implies $p=2$ or $p=3$ and, respectively $q=1$ (impossible) or $q=3$ (also impossible!).

Generalization to Pillai equation

Pillai equation is $Y^p = X^q + a$. We will define Pillai numbers : we note them : b_a they depend of a and heve the same proprieties than b for $a=1$. Thus, we write $1_a + 7_a = 8_a$ and $7_a - 1_a = 6_a$: our goal is to follow the approach as we have done for Catalan equation. Of course, we have $1_a = a_1$ et $0_a = 0$. The equation becomes $Y^p = X^q + 1_a$. Let us pose for p, q, X, Y, a integers stricly greater to 1. Let $c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}}, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}$ And $(c+c')Y^{\frac{p}{2}} = X^p - 1_a + 7_a - X^p = 6_a \Rightarrow Y^{\frac{p}{2}} = \frac{6_a}{c+c'}$ So $X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c+c'} + 1_a = \frac{7_a c + 1_a c'}{c+c'}$ And $X^q = Y^p - 1_a = \frac{36_a 1_a - 1a(c+c')^2}{(c+c')^2}$ Also $(c+c')Y^{\frac{p}{2}} = 6_a > 0$ Thus $c+c' > 0$. And $cY^{\frac{p}{2}} = X^p - 1_a > 0$ So $c > 0$. But $X^p \geq 4_a$, then $cY^{\frac{p}{2}} = X^p - 1_a \geq 7_a - X^p = c'Y^{\frac{p}{2}}$ And $c \geq c'$ and $7_a - X^p = 7_a - \frac{7_a c + 1_a c'}{c+c'} = \frac{6_a c'}{c+c'}$ If $c' > 0$ Then $X^p < 7_a$ and we have a finite number of solutions, we have $c' < 0$, with $(c+c')Y^{\frac{p}{2}} = 6_a > 0$ and $(c+c')X^p = 7_a c + 1_a c' > 0$ with $c+c' > 0$. And $Y^{\frac{p}{2}} = \frac{6_a}{c+c'} \geq 3_a \Rightarrow c+c' \leq 2 < 2_a$

Main results We retain
$1) c' < 0$
$2) 0 < c + c' < 2_a$
$3) 0 < 7_a c + 1_a c'$
And we will discuss two cases
$I) c^2 < 1_a$
$II) c^2 \geq 1_a$
We have to not forget that $X^p - 1_a > 0$ and $7_a - X^p < 0$.

case $c^2 < 1_a$ Let

$$2_aX^q - X^{2p} = \frac{72_a1_a1_a - 2_a1_a(c+c')^2 - (7_ac + 1_ac')^2}{(c+c')^2}$$

$$= \frac{72_a1_a1_a - 1_a(49_a + 2_a)c^2 - 1_a(1_a + 2_a)c'^2 - 2_a(7_a + 2_a)cc'}{(c+c')^2}$$

$$= \frac{19_a1_a1_a + 51_a1_a(1_a - c^2) + 2_a1_a(1_a - c^2) - (2_a1_a + 1_a1_a)c'(c+c') - 16_a1_acc'}{(c+c')^2} > 0$$

Thus $2_aX^q - X^p > 0$ so $2_aX^q > X^{2p}$ (7) And also

$$c^2X^q - X^{2p} = \frac{36_a1_ac^2 - 1_ac^2(c+c')^2 - (7_ac + 1_ac')^2}{(c+c')^2}$$

$$= \frac{(6_ac - 7_ac - 1_ac')(6_a + 7_a + 1_ac') - 1_ac^2(c+c')^2}{(c+c')^2}$$

$$= \frac{-1_a(c+c')(13_ac + 1_ac') - 1_ac^2(c+c')^2}{(c+c')^2} < 0$$

So $c^2X^q < X^{2p}$ (8) But $X^{2p-q-1} < \frac{2}{X} \leq 1$ Thus $2p \leq q+1$ We will give two proofs : for $X > 2_a$ $X^{4q}(36_a1_a - (c+c')^2)^{2p-4} > (32_a1_a)^{2p-4}X^{4q} > (36_a1_a)^q$ and

$$X^{q(q-4)} = \frac{(36_a1_aX^{-4} - (c+c')^2X^{-4})^q}{(c+c')^{2q}} < \frac{(36_a1_a)^qX^{-4q}}{(c+c')^{4p}}$$

$$< X^{(2p-4)q} = \frac{(36_a1_a - (c+c')^2)^{2p-4}}{(c+c')^{4p-8}}$$

Because $X^{2pq} = \frac{(7_ac+1_ac')^{2q}}{(c+c')^{2q}} > \frac{(36_a1_a)^q}{(c+c')^{2q}}$ Thus $2p > q \geq 2p-1$. We have then $q+1 = 2p$. Another proof : But $c^2 < 1_a$, $\exists m > 1_a$ for which $c^2 = \frac{1_am - 1_a}{m}$ thus $1_a - c^2 = \frac{1_a}{m} < \frac{1_aX - 1}{X}$ thus $c^2X > 1$ and $X^{2p+1} = \frac{(7_ac+1_ac')^2X}{(c+c')^2} > \frac{36_a1_ac^2X}{(c+c')^2} > \frac{36_a1_a-(c+c')^2}{(c+c')^2} = X^q$ We deduce $X^{q-1} < X^{2p}$ We have then $X^q \leq X^{2p} \leq X^{q+1}$ As q is not even : $q+1 = 2p$ Thus

$$0 > (c^2 - 1_a)Y^p = X^{2p} - 2_aX^p - X^q = X^{q+1} - 2_aX^p - X^q > 0$$

we deduce $c^2 = 1_a$ and then $X^p - 2_a = X^{q-p} \geq 2$ and $X^{p-1} - X^{q-p-1} = \frac{2_a}{X}$ is an integer and there is a finite number of solutions.

Case $c^2 \geq 1_a$ We will retain
1)$c^2 > 1_a$
2)$c + c' < 2_a$
3)$c' < 0$
As precendently :

$$2c^2X^q - X^{2p} = \frac{72_a1_ac^2 - 2_ac^2(c+c')^2 - (7_ac + 1_ac')^2}{(c+c')^2}$$

$$= \frac{-1_a 1_a c'(14_a + 1_a c') + 1_a c^2(23_a - 2(c+c')^2)}{(c+c')^2} > 0$$

And $2c^2X^q > X^{2p}$ (11) And $2c^2X^q > X^{2p} > 4X^p \Rightarrow c^2X^q > 2X^p$ (11′) Also

$$c^2X^q - X^{2p} = \frac{36_a 1_a c^2 - 1_a c^2(c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2}$$

$$< \frac{36_a 1_a c^2 - 50_a 1_a c^2 - 2_a 1_a c'^2 - 16_a 1_a cc' + (1_a 1_a - 1_a c^2)(c+c')^2}{(c+c')^2}$$

$$= \frac{-2_a(7_a c + 1_a c')(c+c') + (c+c')^2(1_a 1_a - 1_a c^2)}{(c+c')^2} < 0$$

Thus $\frac{1}{2}X^{2p} < c^2X^q < X^{2p} < c^2X^{2p}$ (12) Or $X^{2p-q} > c^2 > 1_a > 1$ We have here $2p \geq q+1$ We have seen that $2c^2X^{2p-1} \geq 2c^2X^q > X^{2p}$, we deduce $2c^2 > X$, but $Y + (c+c')^2 > Y > 36_a 1_a$ Hence $Y^{2p} > (36_a 1_a - (c+c')^2)^{2p}$ Or

$$\frac{32^q X^{2q}}{(36_a 1_a - (c+c')^2)^{2p}} > \frac{32_a^q 1_a^q X^{2q}}{Y^{2p}} = \frac{32_a^q 1_a^q X^{2q}}{(X^q + 1_a)^2} > 1$$

We deduce

$$X^{q(q+2)} = \frac{(36_a 1_a X^2 - X^2(c+c')^2)^q}{(c+c')^{2q}} > \frac{32_a^q 1_a^q X^{2q}}{(c+c')^{4p}}$$

$$> \frac{(36_a 1_a - (c+c')^2)^{2p}}{(c+c')^{4p}} = X^{2pq}$$

And $q+2 > 2p \geq q+1$ And $q+1 = 2p$ but $c^2X^q = c^2X^{2p-1} < X^{2p} \Rightarrow X > c^2$ We pose

$$X^p > u_n c^{2p}$$

With $u_0 = 1$ and $u_n < 2^p$, we have

$$X^p - 2^p c^{2p} = \frac{2^p}{u_n}X^p - 2^p c^{2p} + (1 - \frac{2^p}{u_n})X^p > \frac{u_n - 2^p}{u_n}X^p$$

And $X^p - u_n c^{2p} = X^p - 2^p c^{2p} + (2^p - u_n)c^{2p} > 0$ Thus $X^p - 2^p c^{2p} > (u_n - 2^p)c^{2p}$ We add

$$2X^p - (u_n + 2^p)c^{2p} > 2X^p - 2^{p+1}c^{2p} > (u_n - 2^p)(\frac{X^p}{u_n} + c^{2p})$$

$$= (u_n - 2^p)(c^{2p} - \frac{X^p}{u_n}) + 2(u_n - 2^p)\frac{X^p}{u_n} > (2 - \frac{2^{p+1}}{u_n})X^p$$

We deduce $\frac{2^{p+1}}{u_n}X^p > (u_n + 2^p)c^{2p}$ Or $X^p > \frac{u_n(u_n+2^p)}{2^{p+1}}c^{2p} = u_{n+1}c^{2p}$ With $u_{n+1} = \frac{u_n(u_n+2^p)}{2^{p+1}} < \frac{2^p(2^p+2^p)}{2^{p+1}} = 2^p$ For all n : in the infinity $\lim_{n\longrightarrow\infty}(u_n) = L$ And L verifies recurrence relation $L = \frac{L(L+2^p)}{2^{p+1}} \Rightarrow L = 2^p$ Hence $X^p > \lim_{n\longrightarrow\infty}(u_n)c^{2^p} = 2^p c^{2^p}$ We have $2c^2 > X > 2c^2$ So $X = 2c^2$ and

$$X = 2c^2 = \frac{2X^{2p} - 4_a X^p + 2_a 1_a}{X^q + 1_a} \Rightarrow 1_a X - X^{q+1} + 4_a X^p = 2_a 1_a \Rightarrow 1_a - X^q + 4_a X^{p-1} = \frac{2_a 1_a}{X}$$

We deduce a finite number of solutions.

$$X = \frac{X^{2p}}{X^q} = 2c^2 = \frac{2X^{2p} - 4X^p + 2}{X^q + 1}$$

$$\Rightarrow X - X^{q+1} + 4X^p = 2 \Rightarrow 1 - X^q + 4X^{p-1} = \frac{2}{X}$$

Thus $X = 2c^2 = 2$ and $c^2 = 1$ or $(2^p - 1)^2 = 2^q + 1$ and

$$2^p - 2 = 2^{q-p} = 2^{p-1} \Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm 3$$

Another proof : $Y^p = X^q + 1_a = X^{2p-1} + 1_a > X^{\frac{3p}{2}}$ Hence $Y^2 > X^3$ We pose $Y^2 = X^3 + u \geq X^3 + 1_a$ If we suppose p even

$$1_a \leq u = Y^2 - X^3 = \frac{Y^p - X^{3p}}{Y^{p-2} + Y^{p-4}X^3 + ... + Y^2X^{3p-6} + X^{3p-3}}$$

$$= \frac{X^{2p-1} + 1_a - X^{3p}}{Y^{p-2} + Y^{p-4}X^3 + ... + Y^2X^{3p-6} + X^{3p-3}} < 0$$

and it is impossible, it means that p is not greater than 2 thus $p = 2$ and $q = 2p - 1 = 3$ and $X = 2$ and $Y = \pm 3$. In all cases, Pillai equation has a finite number of solutions, it is the formulation of the conjecture. Thus, we have proved it.

Conclusion

Catalan equation is solved, an original solution exists ! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this.

Références

[1] P. MIHAILESCU A class number free criterion for catalan's conjecture, *Journal of Number theory* , **99** (2003).