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ON THE CAUCHY PROBLEM FOR HARTREE EQUATION IN THE WIENER ALGEBRA

RÉMI CARLES AND LOUNÈS MOUZAOUÏ

ABSTRACT. We consider the mass-subcritical Hartree equation with a homogeneous kernel, in the space of square integrable functions whose Fourier transform is integrable. We prove a global well-posedness result in this space. On the other hand, we show that the Cauchy problem is not even locally well-posed if we simply work in the space of functions whose Fourier transform is integrable. Similar results are proven when the kernel is not homogeneous, and is such that its Fourier transform belongs to some Lebesgue space.

1. INTRODUCTION

We consider the Cauchy problem for the following Hartree equation

$$(1.1) \quad i\partial_t u + \Delta u = (K * |u|^2) u, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^d; \quad u|_{t=0} = u_0,$$

where K denotes the Hartree kernel. We first deal with the case of a homogeneous kernel,

$$(1.2) \quad K(x) = \frac{\lambda}{|x|^\gamma}, \quad \lambda \in \mathbf{R}, \quad \gamma > 0.$$

In [12], it was proved that if $1 \leq d \leq 3$ and $\gamma < d$, the Cauchy problem (1.1) is locally well-posed in $L^2(\mathbf{R}^d) \cap W$, where W stands for the Wiener algebra (also called Fourier algebra, according to the context)

$$W = \left\{ f \in \mathcal{S}'(\mathbf{R}^d; \mathbf{C}), \quad \widehat{f} \in L^1(\mathbf{R}^d) \right\},$$

and the Fourier transform is defined, for $f \in L^1(\mathbf{R}^d)$, as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

In this paper, we investigate the global well-posedness for (1.1): we prove that if $d \geq 1$ and $\gamma < \min(2, d/2)$, then the solution to (1.1) is global in time in $L^2 \cap W$. In view of the classical result according to which (1.1) is globally well-posed in $L^2(\mathbf{R}^d)$, our result can be understood as a propagation of the Wiener regularity. On the other hand, the mere Wiener regularity does not suffice to ensure even local well-posedness for (1.1).

An advantage of working in W lies in the fact that $W \hookrightarrow L^\infty(\mathbf{R}^d)$, and, contrary to e.g. $H^s(\mathbf{R}^d)$, $s > d/2$, W scales like $L^\infty(\mathbf{R}^d)$ (note also that if $s > d/2$, $H^s(\mathbf{R}^d) \hookrightarrow W$). So in a way, W is the largest space included in $L^\infty(\mathbf{R}^d)$ on

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which the Schrödinger group $e^{it\Delta}$ acts continuously — see Remark 1.2 though. Recall that $e^{it\Delta}$ does not map $L^\infty(\mathbf{R}^d)$ to itself, as shown by the explicit formula $e^{i\Delta}(e^{-i|x|^2/4}) = \delta_{x=0}$, and the parabolic scale invariance.

Theorem 1.1. *Let $d \geq 1$, K given by (1.2) with $\lambda \in \mathbf{R}$ and $0 < \gamma < \min(2, d/2)$. If $u_0 \in L^2(\mathbf{R}^d) \cap W$, then (1.1) has a unique, global solution $u \in C(\mathbf{R}; L^2 \cap W)$. In addition, its L^2 -norm is conserved,*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbf{R}.$$

Remark 1.2. One might be tempted to consider a space included in L^∞ which also scales like L^∞ , and which is larger than $W = \mathcal{FL}^1$, namely the amalgam space $W(\mathcal{FL}^1; L^\infty)$. This space consists essentially of functions which are locally in W , and globally in L^∞ (see e.g. [6, 7] for a precise definition). Strichartz estimates in amalgam spaces have been established in [6] (even though the case $W(\mathcal{FL}^1; L^\infty)$ can never be considered). However, since the map $x \mapsto e^{i|x|^2}$ belongs to $W(\mathcal{FL}^1; L^\infty)$, we see that $e^{it\Delta}$ does not act continuously on $W(\mathcal{FL}^1; L^\infty)$.

We next show that (1.1) is not well-posed in the mere Wiener algebra. Precisely have the following theorem:

Theorem 1.3. *Let $d \geq 1$, K given by (1.2) with $0 < \gamma < d$. The Cauchy problem (1.1) is locally well-posed in $W \cap L^2$, but not in W : for all ball B of W , for all $T > 0$ the solution map $\varphi \in B \mapsto \psi \in C([0, T]; W)$ is not uniformly continuous.*

Remark 1.4. In the case of the nonlinear Schrödinger equation

$$(1.3) \quad i\partial_t u + \Delta u = \lambda|u|^{2\sigma}u,$$

where σ is an integer and $\lambda \in \mathbf{R}$, the Cauchy problem is locally well-posed in W (see [3]). From the above result, this is in sharp contrast with the case of the Hartree equation. On the other hand, it is not clear that the Cauchy problem for (1.3) is *globally* well-posed in $L^2(\mathbf{R}^d) \cap W$, even in the case $d = \sigma = 1$. We note that in [9], the authors study (1.3) in the one-dimensional case $d = 1$, with $\sigma < 2$ not necessarily an integer. They prove local well-posedness in

$$\hat{L}^p = \{f : \hat{f} \in L^{p'}\},$$

for p in some open neighborhood of 2. Global well-posedness results for initial data in \hat{L}^p are established, in spaces based on dispersive estimates.

The kernel K given by (1.2) is such that its Fourier transform belongs to no Lebesgue space, but to a weak Lebesgue space, from the following property (see e.g. [1, Proposition 1.29]):

Proposition 1.5. *Let $d \geq 1$ and $0 < \gamma < d$. There exists $C = C(\gamma, d)$ such that the Fourier transform of K defined by (1.2) is*

$$\widehat{K}(\xi) = \frac{\lambda C}{|\xi|^{d-\gamma}}.$$

The final result of this paper is concerned with the case where the kernel K is such that its Fourier transform belongs to some Lebesgue space.

Theorem 1.6. *Let $d \geq 1$.*

- Let $p \in [1, \infty]$, and suppose that K is such that $\widehat{K} \in L^p(\mathbf{R}^d)$. If $u_0 \in L^2(\mathbf{R}^d) \cap W$, then there exists $T > 0$ such that (1.1) has a unique solution $u \in C([-T, T]; L^2 \cap W)$. In addition, its L^2 -norm is conserved,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in [-T, T].$$

If $K \in W$ ($p = 1$), then the solution is global: $u \in C(\mathbf{R}; L^2 \cap W)$.

- Suppose that K is such that $\widehat{K} \in L^\infty(\mathbf{R}^d)$. If $u_0 \in W$, then there exists $T > 0$ such that (1.1) has a unique solution $u \in C([-T, T]; W)$.
- For any $p \in [1, \infty)$, one can find K with $\widehat{K} \in L^p(\mathbf{R}^d) \setminus L^\infty(\mathbf{R}^d)$, such that (1.1) is locally well-posed in $L^2(\mathbf{R}^d) \cap W$, but not in W : for all ball B of W , for all $T > 0$ the solution map $\varphi \in B \mapsto \psi \in C([0, T]; W)$ is not uniformly continuous.

2. STANDARD EXISTENCE RESULTS AND PROPERTIES

2.1. Main properties of the Wiener algebra. The space W enjoys the following elementary properties (see [5, 3]):

- (1) W is a Banach space, continuously embedded into $L^\infty(\mathbf{R}^d)$.
- (2) W is an algebra, and the mapping $(f, g) \mapsto fg$ is continuous from W^2 to W , with

$$\|fg\|_W \leq \|f\|_W \|g\|_W, \quad \forall f, g \in W.$$

- (3) For all $t \in \mathbf{R}$, the free Schrödinger group $e^{it\Delta}$ is unitary on W .

2.2. Existence results based on Strichartz estimates. For the sake of completeness, we recall standard definition and results.

Definition 2.1. A pair $(p, q) \neq (2, \infty)$ is admissible if $p \geq 2$, $q \geq 2$, and

$$\frac{2}{p} = d \left(\frac{1}{2} - \frac{1}{q} \right).$$

Proposition 2.2 (From [8, 11]). (1) For any admissible pair (p, q) , there exists C_q such that

$$\|e^{it\Delta}\varphi\|_{L^p(\mathbf{R}; L^q)} \leq C_q \|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\mathbf{R}^d).$$

(2) Denote

$$D(F)(t, x) = \int_0^t e^{i(t-\tau)\Delta} F(\tau, x) d\tau.$$

For all admissible pairs (p_1, q_1) and (p_2, q_2) , there exists $C = C_{q_1, q_2}$ such that for all interval $I \ni 0$,

$$\|D(F)\|_{L^{p_1}(I; L^{q_1})} \leq C \|F\|_{L^{p_2'}(I; L^{q_2'})}, \quad \forall F \in L^{p_2'}(I; L^{q_2'}).$$

Proposition 2.3. Let $d \geq 1$, K given by (1.2) with $\lambda \in \mathbf{R}$ and $0 < \gamma < \min(2, d)$. If $u_0 \in L^2(\mathbf{R}^d)$, then (1.1) has a unique, global solution

$$u \in C(\mathbf{R}; L^2) \cap L_{\text{loc}}^{8/\gamma}(\mathbf{R}; L^{4d/(2d-\gamma)}).$$

In addition, its L^2 -norm is conserved,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbf{R},$$

and for all admissible pair (p, q) , $u \in L_{\text{loc}}^p(\mathbf{R}; L^q(\mathbf{R}^d))$.

Proof. We give the main technical steps of the proof, and refer to [4] for details. By Duhamel's formula, we write (1.1) as

$$u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-\tau)\Delta}(K * |u|^2u)(\tau)d\tau =: \Phi(u)(t).$$

Introduce the space

$$Y(T) = \{\phi \in C([0, T]; L^2(\mathbf{R}^d)) : \|\phi\|_{L^\infty([0, T]; L^2(\mathbf{R}^d))} \leq 2\|u_0\|_{L^2(\mathbf{R}^d)}, \\ \|\phi\|_{L^{8/\gamma}([0, T]; L^{4d/(2d-\gamma)}(\mathbf{R}^d))} \leq 2C(8/\gamma)\|u_0\|_{L^2(\mathbf{R}^d)}\},$$

and the distance

$$d(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_{L^{8/\gamma}([0, T]; L^{4d/(2d-\gamma)}),}$$

where $C(8/\gamma)$ stems from Proposition 2.2. Then $(Y(T), d)$ is a complete metric space, as remarked in [10] (see also [4]). Hereafter, we denote by

$$q = \frac{8}{\gamma}, \quad r = \frac{4d}{2d-\gamma}, \quad \theta = \frac{8}{4-\gamma},$$

and $\|\cdot\|_{L^a([0, T]; L^b(\mathbf{R}^d))}$ by $\|\cdot\|_{L^a L^b}$ for simplicity. Notice that (q, r) is admissible and

$$\frac{1}{q'} = \frac{4-\gamma}{4} + \frac{1}{q} = \frac{1}{2} + \frac{1}{\theta} \quad ; \quad \frac{1}{r'} = \frac{\gamma}{2d} + \frac{1}{r} \quad ; \quad \frac{1}{2} = \frac{1}{\theta} + \frac{1}{q}.$$

By using Strichartz estimates, Hölder inequality and Hardy–Littlewood–Sobolev inequality, we have, for $(\underline{q}, \underline{r}) \in \{(q, r), (\infty, 2)\}$:

$$\begin{aligned} \|\Phi(u)\|_{L^{\underline{q}}L^{\underline{r}}} &\leq C(\underline{q})\|u_0\|_{L^2} + C(\underline{q}, q) \|(K * |u|^2)u\|_{L^{q'}L^{r'}} \\ &\leq C(\underline{q})\|u_0\|_{L^2} + C(\underline{q}, q) \|K * |u|^2\|_{L^{4/(4-\gamma)}L^{2d/\gamma}} \|u\|_{L^q L^r} \\ &\leq C(\underline{q})\|u_0\|_{L^2} + C\|u\|_{L^\theta L^r}^2 \|u\|_{L^q L^r} \\ &\leq C(\underline{q})\|u_0\|_{L^2} + CT^{1-\gamma/2}\|u\|_{L^q L^r}^3, \end{aligned}$$

for any $u \in Y(T)$, with $C(\infty) = 1$ by the standard energy estimate. To show the contraction property of Φ , for any $v, w \in Y(T)$, we get

$$\begin{aligned} \|\Phi(v) - \Phi(w)\|_{L^q L^r} &\lesssim \|K * |v|^2\|_{L^{4/(4-\gamma)}L^{2d/\gamma}} \|v - w\|_{L^q L^r} \\ &\quad + \|K * |v|^2 - |w|^2\|_{L^2 L^{2d/\gamma}} \|w\|_{L^\theta L^r} \\ &\lesssim (\|v\|_{L^\theta L^r}^2 + \|w\|_{L^\theta L^r}^2) \|v - w\|_{L^q L^r} \\ &\leq CT^{1-\gamma/2}(\|v\|_{L^q L^r}^2 + \|w\|_{L^q L^r}^2) \|v - w\|_{L^q L^r}. \end{aligned}$$

Thus Φ is a contraction from $Y(T)$ to $Y(T)$ provided that T is sufficiently small. Then there exists a unique $u \in Y(T)$ solving (1.1). The global existence of the solution for (1.1) follows from the conservation of the L^2 -norm of u . The last property of the proposition then follows from Strichartz estimates applied with an arbitrary admissible pair on the left hand side, and the same pairs as above on the right hand side. \square

3. PROOF OF THEOREM 1.1

Throughout this section, we assume that the kernel K is given by (1.2).

3.1. Uniqueness. Uniqueness stems from the local well-posedness result established in [12], based on the following lemma, whose proof is recalled for the sake of completeness.

Lemma 3.1. *Let $0 < \gamma < d$. There exists C such that for all $f, g \in L^2(\mathbf{R}^d) \cap W$,*

$$\|(K * |f|^2) f - (K * |g|^2) g\|_{L^2 \cap W} \leq C (\|f\|_{L^2 \cap W}^2 + \|g\|_{L^2 \cap W}^2) \|f - g\|_{L^2 \cap W}.$$

Proof. Let $\kappa_1 = \mathbf{1}_{\{|\xi| \leq 1\}} \widehat{K}$ and $\kappa_2 = \mathbf{1}_{\{|\xi| > 1\}} \widehat{K}$. In view of Proposition 1.5, $\kappa_1 \in L^p(\mathbf{R}^d)$ for all $p \in [1, \frac{d}{d-\gamma})$ and $\kappa_2 \in L^q(\mathbf{R}^d)$ for all $q \in (\frac{d}{d-\gamma}, \infty]$. For $h \in L^1(\mathbf{R}^d) \cap W$, we have

$$\begin{aligned} \|K * h\|_W &\lesssim \|\kappa_1 \widehat{h}\|_{L^1} + \|\kappa_2 \widehat{h}\|_{L^1} \lesssim \|\kappa_1\|_{L^1} \|\widehat{h}\|_{L^\infty} + \|\kappa_2\|_{L^\infty} \|\widehat{h}\|_{L^1} \\ &\lesssim \|h\|_{L^1} + \|\widehat{h}\|_{L^1}, \end{aligned}$$

where we have used Hausdorff–Young inequality. Writing

$$(K * |f|^2) f - (K * |g|^2) g = (K * |f|^2) (f - g) + (K * (|f|^2 - |g|^2)) g,$$

the lemma follows, since W is a Banach algebra embedded into $L^\infty(\mathbf{R}^d)$. \square

We infer uniqueness in $L^2(\mathbf{R}^d) \cap W$ for (1.1) as soon as $0 < \gamma < d$:

Proposition 3.2. *Let $0 < \gamma < d$, $T > 0$, and $u, v \in C([0, T]; L^2 \cap W)$ solve (1.1), with the same initial datum $u_0 \in L^2(\mathbf{R}^d) \cap W$. Then $u \equiv v$.*

Proof. Duhamel's formula yields

$$u(t) - v(t) = -i \int_0^t e^{i(t-\tau)\Delta} ((K * |u|^2) u - (K * |v|^2) v)(\tau) d\tau.$$

Since the Schrödinger group is unitary on L^2 and on W , Minkowski inequality and Lemma 3.1 yield, for $t \geq 0$,

$$\begin{aligned} \|u(t) - v(t)\|_{L^2 \cap W} &\lesssim \int_0^t (\|u(\tau)\|_{L^2 \cap W}^2 + \|v(\tau)\|_{L^2 \cap W}^2) \|u(\tau) - v(\tau)\|_{L^2 \cap W} d\tau \\ &\lesssim \left(\|u\|_{L^\infty([0, T]; L^2 \cap W)}^2 + \|v\|_{L^\infty([0, T]; L^2 \cap W)}^2 \right) \int_0^t \|u(\tau) - v(\tau)\|_{L^2 \cap W} d\tau. \end{aligned}$$

Gronwall lemma implies $u \equiv v$. \square

3.2. Existence. In view of Lemma 3.1, the standard fixed point argument yields:

Proposition 3.3. *Let $d \geq 1$, $\lambda \in \mathbf{R}$, $0 < \gamma < d$, and K given by (1.2). If $u_0 \in L^2(\mathbf{R}^d) \cap W$, then there exists $T > 0$ depending only on λ, γ, d and $\|u_0\|_{L^2 \cap W}$, and a unique $u \in C([0, T]; L^2 \cap W)$ to (1.1).*

Taking Proposition 2.3 into account, to establish Theorem 1.1, it suffices to prove that the Wiener norm of u cannot become unbounded in finite time.

Resuming the decomposition of \widehat{K} introduced in the proof of Lemma 3.1, we find

$$\begin{aligned} \|u(t)\|_W &\leq \|u_0\|_W + \int_0^t \|(K * |u(\tau)|^2) u(\tau)\|_W d\tau \\ &\leq \|u_0\|_W + \int_0^t \|K * |u(\tau)|^2\|_W \|u(\tau)\|_W d\tau \\ &\leq \|u_0\|_W + \int_0^t \left(\|\kappa_1\|_{L^1} \|u(\tau)\|_{L^2}^2 + \|\kappa_2\|_{L^p} \left\| \widehat{|u(\tau)|^2} \right\|_{L^{p'}} \right) \|u(\tau)\|_W d\tau, \end{aligned}$$

provided that $p > \frac{d}{d-\gamma}$. Using the conservation of the L^2 -norm of u and Hausdorff–Young inequality, we infer, if $p \leq 2$:

$$\|u(t)\|_W \lesssim \|u_0\|_W + \int_0^t (\|\kappa_1\|_{L^1} \|u_0\|_{L^2}^2 + \|\kappa_2\|_{L^p} \| |u(\tau)|^2 \|_{L^p}) \|u(\tau)\|_W d\tau.$$

To summarize, for all $1 < \frac{d}{d-\gamma} < p \leq 2$, there exists C such that

$$\|u(t)\|_W \leq \|u_0\|_W + C \int_0^t \|u(\tau)\|_W d\tau + C \int_0^t \|u(\tau)\|_{L^{2p}}^2 \|u(\tau)\|_W d\tau.$$

The above requirement on p can be fulfilled if and only if $0 < \gamma < d/2$. To take advantage of Proposition 2.3, introduce $\alpha > 1$ such that $(2\alpha, 2p)$ is admissible. This is possible provided that $2p < \frac{2d}{d-2}$ when $d \geq 3$: this condition is compatible with the requirement $p > \frac{d}{d-\gamma}$ if and only if $\gamma < 2$. Using Hölder inequality for the last integral, we have

$$\|u(t)\|_W \leq \|u_0\|_W + C \int_0^t \|u(\tau)\|_W d\tau + C \|u\|_{L^{2\alpha}([0,t];L^{2p})}^2 \|u\|_{L^{\alpha'}([0,t];W)}.$$

Set

$$\omega(t) = \sup_{0 \leq \tau \leq t} \|u(\tau)\|_W.$$

For a given $T > 0$, ω satisfies an estimate of the form

$$\omega(t) \leq \|u_0\|_W + C \int_0^t \omega(\tau) d\tau + C_0(T) \left(\int_0^t \omega(\tau)^{\alpha'} d\tau \right)^{1/\alpha'},$$

provided that $0 \leq t \leq T$, and where we have used the fact that α' is finite. Using Hölder inequality, we infer

$$\omega(t) \leq \|u_0\|_W + C_1(T) \left(\int_0^t \omega(\tau)^{\alpha'} d\tau \right)^{1/\alpha'}.$$

Raising the above estimate to the power α' , we find

$$\omega(t)^{\alpha'} \lesssim 1 + \int_0^t \omega(\tau)^{\alpha'} d\tau.$$

Gronwall lemma shows that $\omega \in L^\infty([0, T])$. Since $T > 0$ is arbitrary, $\omega \in L^\infty_{\text{loc}}(\mathbf{R})$, and the result follows.

4. ILL-POSEDNESS IN THE MERE WIENER ALGEBRA

In this section we still assume that K is given by (1.2). We show that the Cauchy problem (1.1) is ill-posed in the mere Wiener algebra, i.e without including L^2 . We recall the definition of well-posedness for the problem (1.1).

Definition 4.1. *Let $(S, \|\cdot\|_S)$ be a Banach space of initial data, and $(D, \|\cdot\|_D)$ be a Banach space of space-time functions. The Cauchy problem (1.1) is well posed from D to S if, for all bounded subset $B \subset D$, there exist $T > 0$ and a Banach space $X_T \hookrightarrow C([0, T]; S)$ such that:*

- (1) *For all $\varphi \in B$, (1.1) has a unique solution $\psi \in X_T$ with $\psi|_{t=0} = \varphi$.*
- (2) *The mapping $B \ni \varphi \mapsto \psi \in C([0, T]; S)$ is uniformly continuous.*

Proof of Theorem 1.3. In view of [2], it suffices to prove that one term in the Picard iterations of Φ defined in Section 2 does not verify the Definition 4.1. We argue by contradiction and assume that (1.1) is well-posed from W to W . Then, let $T > 0$ be the local time existence of the solution. We recall that for $0 < \gamma < d$, (1.1) is well-posed from $W \cap L^2$ to $W \cap L^2$ (see [12, Theorem 2.1], recalled in Proposition 3.3). We define the following operator associated to the second Picard iterate:

$$D(f)(t, x) = -i \int_0^t e^{i(t-\tau)\Delta} (K * |\psi|^2 \psi)(\tau, x) dx,$$

where ψ is the solution of the free equation

$$i\partial_t \psi + \Delta \psi = 0; \quad \psi|_{t=0} = f,$$

that is $\psi(t) = e^{it\Delta} f$. We denote $e^{it\Delta} f = U(t)f$. By [2, Proposition 1] the operator D is continuous from W to W , that is,

$$(4.1) \quad \|D(f)(t)\|_W \leq C \|f\|_W^3, \quad \forall t \in [0, T],$$

for some positive constant C . Let $f \in \mathcal{S}(\mathbf{R}^d)$ be an element of the Schwartz space. We define a family of functions indexed by $h > 0$ by

$$f^h(x) = f(hx).$$

For all $h > 0$, $\|f^h\|_{L^2} = \frac{1}{h^{d/2}} \|f\|_{L^2}$ so for $h > 0$ close to 0 the family $(f^h)_{h>0}$ leaves any compact of L^2 . Remark that $\widehat{f^h}(\xi) = h^{-d} \widehat{f}(\frac{\xi}{h})$ and $\|f^h\|_W = \|f\|_W < \infty$, so (4.1) yields

$$(4.2) \quad \|D(f^h)(t)\|_W \leq C \|f\|_W^3.$$

We develop the expression of $\|D(f^h)(t)\|_W$. We have

$$\begin{aligned} & \mathcal{F}(K * |U(\tau)f^h|^2 U(\tau)f^h)(\xi) = \\ &= (2\pi)^{d/2} \mathcal{F}(K * |U(\tau)f^h|^2) * \mathcal{F}(U(\tau)f^h)(\xi) \\ &= (2\pi)^{d/2} \int_{\mathbf{R}^d} \mathcal{F}(K * |U(\tau)f^h|^2)(\xi - y) \mathcal{F}(U(\tau)f^h)(y) dy \\ &= \int_{\mathbf{R}^d} e^{i\tau|y|^2} \widehat{K}(\xi - y) \mathcal{F}(|U(\tau)f^h|^2)(\xi - y) \widehat{f^h}(y) dy \\ &= (2\pi)^{d/2} \iint_{\mathbf{R}^{2d}} e^{i\tau|y|^2} e^{i\tau|\xi - y - z|^2} e^{-i\tau|z|^2} \widehat{K}(\xi - y) \widehat{f^h}(\xi - y - z) \widehat{f^h}(z) \widehat{f^h}(y) dy dz \\ &= \frac{(2\pi)^{d/2}}{h^{3d}} \iint_{\mathbf{R}^{2d}} e^{i\tau|y|^2} e^{i\tau|\xi - y - z|^2} e^{-i\tau|z|^2} \widehat{K}(\xi - y) \widehat{f}\left(\frac{\xi - y - z}{h}\right) \widehat{f}\left(\frac{z}{h}\right) \widehat{f}\left(\frac{y}{h}\right) dy dz. \end{aligned}$$

Taking the W -norm gives

$$\begin{aligned} \|D(f^h)(t)\|_W &= \|\widehat{D(f^h)}(t)\|_{L^1} \\ &= \int_{\mathbf{R}^d} \left| \int_0^t \mathcal{F}(U(t-\tau)(K * |U(\tau)f^h|^2 U(\tau)f^h)(\xi) d\tau \right| d\xi \\ &= \int_{\mathbf{R}^d} \left| \int_0^t e^{i(t-\tau)|\xi|^2} \mathcal{F}(K * |U(\tau)f^h|^2 U(\tau)f^h)(\xi) d\tau \right| d\xi. \end{aligned}$$

We replace $\mathcal{F}(K * |U(\tau)f^h|^2 U(\tau)f^h)(\xi)$ by its integral formula above and apply the following changes of variable: $\xi' = \xi/h, y' = y/h, z' = z/h, \tau' = \tau h^2$. We obtain

$$(4.3) \quad \|D(f^h)(t)\|_W = \frac{1}{h^{d-\gamma+2}} \|D(f)(th^2)\|_W.$$

Let $s \in (0, T)$. We examine more closely the term $F(s) := \int_0^s U(s-\tau)g(\tau)d\tau$ where $g(s) := (K * |U(s)f|^2)U(s)f$. Taylor formula yields

$$F(s) = F(0) + F'(0)s + \frac{s^2}{2} \int_0^1 (1-\theta)F''(s\theta) d\theta.$$

We have $F(0) = 0$, so for $s \in [0, 1]$,

$$\begin{aligned} \|F(s) - F'(0)s\|_W &\leq s^2 \left\| \int_0^1 (1-\theta)F''(s\theta) d\theta \right\|_W \\ &\leq s^2 \int_0^1 \|F''(s\theta)\|_W d\theta \leq s^2 \|F''\|_{L^\infty([0,1];W)}. \end{aligned}$$

The first and second derivatives of F are given by

$$\begin{aligned} F'(s) &= g(s) + \int_0^s U(s-\tau)i\Delta g(\tau) d\tau, \\ F''(s) &= g'(s) + i\Delta g(s) - \int_0^s U(s-\tau)\Delta^2 g(\tau) d\tau, \end{aligned}$$

so $F'(0) = g(0)$ and

$$\|F''\|_{L^\infty([0,1];W)} \leq \|g'\|_{L^\infty([0,1];W)} + \|\Delta g\|_{L^\infty([0,1];W)} + \|\Delta^2 g\|_{L^\infty([0,1];W)}.$$

From the formula of g , since $f \in \mathcal{S}$ we can see easily that $F'' \in L^\infty([0,1];W)$ (uniformly in $h \in (0, 1]$). We obtain

$$\|F(s) - sg(0)\|_W \leq Cs^2,$$

where C is a positive constant independent of s (it depends on f and γ). Thus,

$$\|F(s)\|_W = s\|g(0)\|_W + \mathcal{O}(s^2).$$

In particular, for all $t, h > 0$

$$\|D(f)(th^2)\|_W = \|F(th^2)\|_W = th^2\|g(0)\|_W + \mathcal{O}(t^2h^4).$$

This implies that

$$\|D(f^h)(t)\|_W = \frac{1}{h^{d-\gamma+2}} \|D(f)(th^2)\|_W = \frac{t}{h^{d-\gamma}} \|g(0)\|_W + \frac{1}{h^{d-\gamma}} \mathcal{O}(t^2h^2).$$

Fix $t > 0$:

$$\lim_{h \rightarrow 0^+} \frac{t}{h^{d-\gamma}} \|g(0)\|_W + \frac{1}{h^{d-\gamma}} \mathcal{O}(t^2h^2) = +\infty.$$

We deduce that for $h > 0$ sufficiently close to 0,

$$\|D(f^h)(t)\|_W > C\|f\|_W^3.$$

This contradicts (4.1), and Theorem 1.3 follows. \square

5. PROOF OF THEOREM 1.6

We decompose the proof of Theorem 1.6 according to the three cases considered.

5.1. **Well-posedness in $L^2 \cap W$.** We assume that K is such that $\widehat{K} \in L^p$ for some $p \in [1, \infty]$, and we consider an initial data $u_0 \in L^2 \cap W$. For $T > 0$ we define the following space

$$E_T = \{u \in C([0, T]; L^2 \cap W), \quad \|u\|_{L^\infty([0, T]; L^2 \cap W)} \leq 2\|u_0\|_{L^2 \cap W}\}.$$

It is a complete space metric when equipped with the metric

$$d(u, v) = \|u - v\|_{L^\infty([0, T]; L^2 \cap W)}.$$

Lemma 5.1. *Let $p \in [1, \infty]$, and K such that $\widehat{K} \in L^p$. There exists C such that for all $f, g \in L^2 \cap W$,*

$$\|K * (fg)\|_W \leq C\|\widehat{K}\|_{L^p}\|f\|_{L^2 \cap W}\|g\|_{L^2 \cap W}.$$

Proof. Let $f, g \in L^2 \cap W$. We denote by p' the Hölder conjugate exponent of p . We have

$$\begin{aligned} \|K * (fg)\|_W &= (2\pi)^{d/2}\|\widehat{K} \widehat{fg}\|_{L^1} \leq C\|\widehat{K}\|_{L^p}\|\widehat{fg}\|_{L^{p'}} \leq C\|\widehat{K}\|_{L^p}\|\widehat{f} * \widehat{g}\|_{L^{p'}} \\ &\leq C\|\widehat{K}\|_{L^p}\|\widehat{f}\|_{L^q}\|\widehat{g}\|_{L^q}, \end{aligned}$$

where $1 + \frac{1}{p'} = \frac{2}{q}$. Since $q \in [1, 2]$ there exists $\theta \in [0, 1]$ such that $\frac{1}{2} + \frac{\theta}{2} = \frac{1}{q}$ and

$$\|\widehat{f}\|_{L^q} \leq \|\widehat{f}\|_{L^1}^\theta \|\widehat{f}\|_{L^2}^{1-\theta} = \|f\|_W^\theta \|f\|_{L^2}^{1-\theta} \leq \|f\|_{L^2 \cap W}.$$

The lemma follows. \square

Let Φ as defined in Section 2. In view of the previous lemma we have, for $u \in E_T$ and $t \in [0, T]$,

$$\begin{aligned} \|\Phi(u)(t)\|_{L^2 \cap W} &\leq \|u_0\|_{L^2 \cap W} + \int_0^t \|K * |u(\tau)|^2 u(\tau)\|_{L^2 \cap W} \, d\tau \\ &\leq \|u_0\|_{L^2 \cap W} + \int_0^t \|K * |u(\tau)|^2\|_W \|u(\tau)\|_{L^2 \cap W} \, d\tau \\ &\leq \|u_0\|_{L^2 \cap W} + C\|\widehat{K}\|_{L^p} \int_0^t \|u(\tau)\|_{L^2 \cap W}^3 \, d\tau. \end{aligned}$$

We obtain

$$\|\Phi(u)(t)\|_{L^\infty([0, T]; L^2 \cap W)} \leq \|u_0\|_{L^2 \cap W} + C\|\widehat{K}\|_{L^p}\|u_0\|_{L^\infty([0, T]; L^2 \cap W)}^3 T.$$

For T sufficiently small (depending on $\|u_0\|_{L^2 \cap W}$) $\|\Phi(u)\|_{L_T^\infty L^2} \leq 2\|u_0\|_{L^2 \cap W}$. Let $u, v \in E_T$. From Lemma 5.1 we have

$$\begin{aligned} &\|\Phi(u)(t) - \Phi(v)(t)\|_{L^2 \cap W} \\ &\leq \int_0^t \|K * (|u(\tau)|^2 - |v(\tau)|^2)u(\tau)\|_{L^2 \cap W} \, d\tau \\ &\quad + \int_0^t \|K * |v(\tau)|^2(u(\tau) - v(\tau))\|_{L^2 \cap W} \, d\tau \\ &\leq \int_0^t \|K * (|u(\tau)|^2 - |v(\tau)|^2)\|_W \|u(\tau)\|_{L^2 \cap W} \, d\tau \\ &\quad + \int_0^t \|K * |v(\tau)|^2\|_W \|u(\tau) - v(\tau)\|_{L^2 \cap W} \, d\tau \\ &\leq \int_0^t C\|\widehat{K}\|_{L^p} (\|u(\tau)\|_{L^2 \cap W}^2 + \|v(\tau)\|_{L^2 \cap W}^2) \|u(\tau) - v(\tau)\|_{L^2 \cap W} \, d\tau. \end{aligned}$$

We deduce that

$$\|\Phi(u) - \Phi(v)\|_{L^\infty([0,T];L^2 \cap W)} \leq C \|\widehat{K}\|_{L^p} \|u_0\|_{L^2 \cap W}^2 T \|u - v\|_{L^\infty([0,T];L^2 \cap W)}.$$

For T possibly smaller (still depending on $\|u_0\|_{L^2 \cap W}$) Φ is a contraction from E_T to E_T , so admits a unique fixed point in E_T , which is a solution for the Cauchy problem. By resuming the same arguments as in Proposition 3.2 we deduce that the fixed point obtained before is the unique solution for the Cauchy problem.

For $p = 1$, the solution constructed before is global in time. In view of the conservation of the L^2 -norm, we have

$$\begin{aligned} \|u(t)\|_W &\leq \|u_0\|_W + \int_0^t \|K * |u(\tau)|^2\|_W \|u(\tau)\|_{L^2 \cap W} \, d\tau \\ &\leq \|u_0\|_W + \int_0^t C \|\widehat{K}\|_{L^1} \|u(\tau)\|_{L^2}^2 \|u(\tau)\|_W \, d\tau \\ &\leq \|u_0\|_W + C \|\widehat{K}\|_{L^1} \|u_0\|_{L^2}^2 \int_0^t \|u(\tau)\|_W \, d\tau, \end{aligned}$$

and by the Gronwall lemma, we conclude that $\|u(t)\|_W$ remains bounded on finite time intervals. This completes the proof of the first point in Theorem 1.6.

5.2. Well-posedness in W . Let $\widehat{K} \in L^\infty$ and consider the Cauchy problem (1.1) with an initial data in W . For $T > 0$ we define the following space

$$Y_T = \{u \in C([0, T]; W), \|u\|_{L^\infty([0, T]; W)} \leq 2\|u_0\|_W\}.$$

This later is a complete metric space when equipped with the metric

$$d(u, v) = \|u - v\|_{L^\infty([0, T]; W)}.$$

As previously, the local existence of a solution is easily shown by a fixed point argument, since Φ is a contraction from Y_T to Y_T , and we show that it is unique. The proof relies on the following lemma:

Lemma 5.2. *There exists C such that for all $f, g \in W$,*

$$\|K * (fg)\|_W \leq C \|\widehat{K}\|_{L^\infty} \|f\|_W \|g\|_W.$$

Proof. Let $f, g \in W$. We have

$$\begin{aligned} \|K * (fg)\|_W &= (2\pi)^{d/2} \|\widehat{K} \widehat{fg}\|_{L^1} \leq (2\pi)^{d/2} \|\widehat{K}\|_{L^\infty} \|\widehat{f} * \widehat{g}\|_{L^1} \\ &\leq (2\pi)^{d/2} \|\widehat{K}\|_{L^\infty} \|\widehat{f}\|_{L^1} \|\widehat{g}\|_{L^1} \\ &\leq (2\pi)^{d/2} \|\widehat{K}\|_{L^\infty} \|f\|_W \|g\|_W. \end{aligned}$$

□

It then suffices to reproduce the proof given in the previous subsection in order to prove the second point of Theorem 1.6, by replacing Lemma 5.1 with Lemma 5.2.

5.3. Ill-posedness in W . For $\gamma \in (0, d)$ we consider the Cauchy problem (1.1) with the kernel K given by its Fourier transform

$$(5.1) \quad \widehat{K}(\xi) = \frac{1}{|\xi|^{d-\gamma}} \mathbf{1}_{\{|\xi| \leq 1\}}.$$

Then, $\widehat{K} \in L^p$ for all $p \in [1, \frac{d}{d-\gamma})$. Conversely, for $p \in [1, \infty)$, we can always find $\gamma \in (0, d)$ such that K , defined by (5.1), satisfies $\widehat{K} \in L^p$.

From the first point of Theorem 1.6, the Cauchy problem (1.1) is locally well-posed in $L^2 \cap W$. We suppose that (1.1) is well posed in W : let $T > 0$ be the local time existence of the solution. Arguing as in Section 4, this implies that there exists $C > 0$ such that for all $f \in W$ and for all $t \in [0, T]$,

$$(5.2) \quad \|D(f)(t)\|_W \leq C \|f\|_W^3,$$

where D is the operator defined in Section 4. Let $f \in \mathcal{S}(\mathbf{R}^d)$. As in Section 4 we define the family of functions $(f^h)_{0 < h \leq 1}$ by

$$f^h(x) = f(hx).$$

From (5.2) we also have for all $t \in [0, T]$

$$(5.3) \quad \|D(f^h)(t)\|_W \leq C \|f^h\|_W^3 = C \|f\|_W^3.$$

We define K_h by setting $\widehat{K}_h(\xi) = \frac{1}{|\xi|^{d-\gamma}} \mathbf{1}_{\{|\xi| > \frac{1}{h}\}}$. We use the following identity:

$$\begin{aligned} \mathcal{F}(K * |U(\tau)f^h|^2 U(\tau)f^h)(\xi) &= \\ &= \frac{(2\pi)^{d/2}}{h^{3d}} \iint_{\mathbf{R}^{2d}} e^{i\tau|\xi-y|^2} e^{i\tau|y-z|^2} e^{-i\tau|z|^2} \widehat{K}(y) \widehat{f}\left(\frac{\xi-y}{h}\right) \widehat{f}\left(\frac{z}{h}\right) \widehat{f}\left(\frac{y-z}{h}\right) dy dz \\ &= \frac{(2\pi)^{d/2}}{h^{3d}} \iint_{\mathbf{R}^{2d}} e^{i\tau|\xi-y|^2} e^{i\tau|y-z|^2} e^{-i\tau|z|^2} \frac{1}{|y|^{d-\gamma}} \widehat{f}\left(\frac{\xi-y}{h}\right) \widehat{f}\left(\frac{z}{h}\right) \widehat{f}\left(\frac{y-z}{h}\right) dy dz \\ &\quad - \frac{(2\pi)^{d/2}}{h^{3d}} \iint_{\mathbf{R}^{2d}} e^{i\tau|\xi-y|^2} e^{i\tau|y-z|^2} e^{-i\tau|z|^2} \widehat{K}_h(y) \widehat{f}\left(\frac{\xi-y}{h}\right) \widehat{f}\left(\frac{z}{h}\right) \widehat{f}\left(\frac{y-z}{h}\right) dy dz. \end{aligned}$$

We inject the above formula into the expression of $\|D(f^h)(t)\|_W$ and perform the same changes of variables as in Section 4. We obtain

$$\|D(f^h)(t)\|_W \geq \frac{t}{h^{d-\gamma}} (\|(K * |f|^2)f\|_W + \mathcal{O}(th^2)) - \frac{1}{h^{d-\gamma+2}} X^h,$$

where

$$X^h = \left\| \int_0^{th^2} U(t-\tau) (K_h * |U(\tau)f|^2 U(\tau)f) d\tau \right\|_W.$$

For $q \in (\frac{d}{d-\gamma}, \infty)$,

$$\begin{aligned} X^h &\leq \int_0^{th^2} \|(K_h * |U(\tau)f|^2) U(\tau)f\|_W d\tau \leq \int_0^{th^2} \|K_h * |U(\tau)f|^2\|_W \|U(\tau)f\|_W d\tau \\ &\leq \|f\|_W \int_0^{th^2} \|\widehat{K}_h\|_{L^q} \|\mathcal{F}(|U(\tau)f|^2)\|_{L^{q'}} d\tau \leq C th^2 \|\widehat{K}_h\|_{L^q}, \end{aligned}$$

for some constant C independent of $h \in (0, 1]$ and $t \in [0, T]$ (recall that $f \in \mathcal{S}(\mathbf{R}^d)$). Moreover,

$$\|\widehat{K}_h\|_{L^q} = \left(\int_{\mathbf{R}^d} \frac{1}{|y|^{(d-\gamma)q}} \mathbf{1}_{\{|y| > \frac{1}{h}\}} dy \right)^{1/q} = C \left(\int_{1/h}^{+\infty} \frac{r^{d-1}}{r^{(d-\gamma)q}} dr \right)^{1/q},$$

so, $\|\widehat{K}_h\|_{L^p} = Ch^{d-\gamma-\frac{d}{q}}$ and

$$\|D(f^h)(t)\|_W \geq \frac{t}{h^{d-\gamma}} \left(\|(K * |f|^2)f\|_W + \mathcal{O}(th^2) - Ch^{d-\gamma-\frac{d}{q}} \right).$$

Fix $t > 0$:

$$\lim_{h \rightarrow +\infty} \frac{t}{h^{d-\gamma}} \left(\|(K * |f|^2)f\|_W + \mathcal{O}(th^2) - Ch^{d-\gamma-\frac{d}{q}} \right) = +\infty.$$

So, for $h > 0$ small enough we have

$$\|D(f^h)(t)\|_W > C\|f\|_W^3,$$

hence a contradiction.

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