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# $n$ -SUPERCYCLIC AND STRONGLY $n$ -SUPERCYCLIC OPERATORS IN FINITE DIMENSION

ERNST ROMUALD

ABSTRACT. We prove that on  $\mathbb{R}^n$ , there is no  $N$ -supercyclic operator with  $1 \leq N < \lfloor \frac{n+1}{2} \rfloor$  i.e. if  $\mathbb{R}^n$  has an  $N$  dimensional subspace whose orbit under  $T$  is dense in  $\mathbb{R}^n$ , then  $N$  is greater than  $\lfloor \frac{n+1}{2} \rfloor$ . Moreover, this value is optimal. We then consider the case of strongly  $N$ -supercyclic operators. An operator  $T$  is strongly  $N$ -supercyclic if  $\mathbb{R}^n$  has an  $N$ -dimensional subspace whose orbit under  $T$  is dense in  $\mathbb{P}_N(\mathbb{R}^n)$ , the  $N$ -th Grassmannian. We prove that strong  $N$ -supercyclicity does not occur non-trivially in finite dimension.

## 1. INTRODUCTION

Let  $T$  be a continuous linear operator on a completely separable Baire vector space  $X$ . The orbit of a set  $E \in X$  under  $T$  is defined by

$$\mathcal{O}(E, T) := \cup_{n \in \mathbb{N}} T^n(E).$$

Many authors have already studied some density properties of such orbits for different original sets  $E$ . If  $E$  is a singleton and  $\mathcal{O}(E, T)$  is dense in  $X$ , then  $T$  is said to be hypercyclic. Hypercyclicity has been first studied by Birkhoff in 1929 and has been a subject of great interest during the last twenty years, see [2] and [6] for a survey on hypercyclicity. Later, in 1974, Hilden and Wallen [8] worked on a different set  $E = \mathbb{K}x$  which is a one dimensional subspace of  $X$ , and if  $\mathcal{O}(E, T)$  is dense in  $X$ , then  $T$  is said to be supercyclic. Several generalisations of supercyclicity were proposed since. Not all of them were successful but the one introduced by Feldman [5] in 2002 is a natural generalisation of supercyclicity. Rather than considering orbits of lines, Feldman defines a  $n$ -supercyclic operator as being an operator for which there exists a  $n$ -dimensional subspace  $E$  such that  $\mathcal{O}(E, T)$  is dense in  $X$ . This notion has been mainly studied in [1], [3] and [5]. In 2004, Bourdon, Feldman and Shapiro proved in the complex case that non-trivial  $n$ -supercyclicity is purely infinite dimensional:

**Theorem Bourdon, Feldman, Shapiro.** Let  $n \geq 2$ . Then there is no  $(n - 1)$ -supercyclic operator on  $\mathbb{C}^n$ . In particular, there is no  $k$ -supercyclic operator on  $\mathbb{C}^n$  for  $1 \leq k \leq n - 1$ .

The last theorem extends a result proved by Hilden and Wallen for supercyclic operators in the complex setting. But, they also proved the non-existence of supercyclic operators on  $\mathbb{R}^n$  for  $n \geq 2$ . Therefore, it is natural to ask the question of the existence of  $n$ -supercyclic operators in the real setting. An open question regarding  $n$ -supercyclic operators is to know whether they satisfy the Ansari property: is it true that  $T^p$  is  $n$ -supercyclic for any  $p \geq 2$  provided  $T$  itself is  $n$ -supercyclic? This led Shkarin [12] in 2008 to introduce strongly  $n$ -supercyclic operators. This new property is stronger than Feldman's and moreover strongly  $n$ -supercyclic operators do satisfy the Ansari property. Actually, Shkarin asks if these two notions are equivalent. Indeed, this would solve the Ansari problem for  $n$ -supercyclic operators. Something interesting in Shkarin's definition is that he does not change the set  $E$  which is, as for Feldman's, a  $n$  dimensional subspace of  $X$ , he rather changes the space in which the orbit has to be dense for a more natural one (speaking of subspaces): the  $n$ -th Grassmannian. To be precise, we are going to remind the reader some basic facts before giving the formal definition of strong  $n$ -supercyclicity. The  $n$ -th Grassmannian is the set of all subspaces of dimension  $n$  in  $X$ . If  $\dim(X) \geq n$  then one may define a topology on the  $n$ -th Grassmannian. To do this, let us consider the open subset  $X_n$  of all linearly independent  $n$ -tuples with the topology induced from  $X^n$  and declare the map

$\pi_n : X_n \rightarrow \mathbb{P}_n(X)$ ,  $\pi_n(x) = \text{Span}\{x_1, \dots, x_n\}$ , continuous and open. The topology on  $\mathbb{P}_n(X)$  is the coarsest topology for which the map  $\pi_n$  is open and continuous.

Turn out to the definition of strong  $n$ -supercyclicity:  $L \in \mathbb{P}_n(X)$  is a strongly  $n$ -supercyclic subspace for  $T$  if every  $T^n(L)$  is  $n$ -dimensional and if  $\{T^n(L), n \in \mathbb{N}\}$  is dense in  $\mathbb{P}_n(X)$ . If such a subspace exists, then  $T$  is said to be strongly  $n$ -supercyclic. We denote by  $\mathcal{ES}_n(T)$  the set of strongly  $n$ -supercyclic subspaces for an operator  $T$ .

Unfortunately, Shkarin did not go further in the study of strongly  $n$ -supercyclic operators. A study of general properties of strongly  $n$ -supercyclic operators can be found in [4].

In Section 2, we are going to study the existence of  $n$ -supercyclic operators in the real setting. In fact we are going to prove the following theorem:

**Theorem 1.** Let  $n \geq 2$ . There is no  $(\lfloor \frac{n+1}{2} \rfloor - 1)$ -supercyclic operators on  $\mathbb{R}^n$ . Moreover there exist  $(\lfloor \frac{n+1}{2} \rfloor)$ -supercyclic operators on  $\mathbb{R}^n$ .

This theorem generalises Hilden and Wallen's result and is, in some sense, the best result one may wish. Actually, it is not difficult to prove that there exists an operator which is  $k$ -supercyclic but not  $(k-1)$ -supercyclic on  $\mathbb{R}^n$  for every  $\lfloor \frac{n+1}{2} \rfloor \leq k \leq n$ . The proof of Theorem 1 is not easy and one needs to get familiar with specific notations to fully understand it. The proof is progressing by steps from simplest matrices, which will be called primary, to general ones.

Then, in Section 3, we completely solve the question of the existence of non-trivial strongly  $n$ -supercyclic operators in finite dimensional vector spaces. In fact, we prove:

**Theorem 2.** For  $n \geq 3$ , there is no strongly  $k$ -supercyclic operator on  $\mathbb{R}^n$  for  $1 \leq k < n$ .

This result puts an end to the study of strong  $n$ -supercyclicity in finite dimension. This proves, in particular, that there exists  $n$ -supercyclic operators that are not strongly  $n$ -supercyclic and answers a question raised in [12]. The interested reader shall refer to [4] for other properties on strongly  $n$ -supercyclic operators in the infinite dimensional spaces setting.

## 2. PRELIMINARIES

It is known for years that in the real setting, supercyclic operators are completely characterised and they only appear on  $\mathbb{R}$  or  $\mathbb{R}^2$ . Moreover, on  $\mathbb{R}^2$ , if  $\pi$  and  $\theta$  are linearly independent over  $\mathbb{Q}$ , then  $R_\theta$ , the rotation with angle  $\theta$ , is supercyclic. Building on this, one may easily see that any rotation on  $\mathbb{R}^3$  around any one-dimensional subspace and with angle linearly independent with  $\pi$  over  $\mathbb{Q}$  is 2-supercyclic. This simple example proves that the real setting is completely different from the complex one and gives hope in finding similar examples in higher dimensions. It seems clear that rotations are making the difference between the real case and the complex case, because, in some sense rotations do not exist in the complex setting. The next part is devoted to the Jordan real decomposition and highlights the role played by rotations in the real setting.

### Jordan decomposition

In the complex setting, it is common to use the Jordan decomposition to obtain a matrix similar to  $T$  but with a better "shape". Bourdon, Feldman and Shapiro used this to prove that there is no  $(n-1)$ -supercyclic operator on  $\mathbb{C}^n$ . Recall that a Jordan block with eigenvalue  $\lambda$  and of size  $k$  is usually a  $k \times k$  matrix with  $\lambda$  along the main diagonal, ones on the first super-diagonal and zeros everywhere else. Note that a direct sum of Jordan blocks of size one forms a diagonal matrix. For convenience, all along this paper we follow another convention which improves slightly the notations but does not change the efficiency of this decomposition. Thus, in our convention, a classical Jordan block with eigenvalue  $\lambda$  and of size  $k$  will be a  $k \times k$  matrix with  $\lambda$  along the main diagonal and along the first super-diagonal and zeros elsewhere. This well-known decomposition for complex matrices cannot be applied without changes to the case of real matrices because of the existence of complex eigenvalues. However, there also exists a real version of the Jordan decomposition which is an improvement of the last one. In the real

case, every matrix is similar to a direct sum of diagonal blocks, classical Jordan blocks and real Jordan blocks, where a real Jordan block of modulus  $\lambda$  and of size  $k$  is usually a  $2k \times 2k$  matrix with  $\lambda R_\theta$  along the main diagonal, identity matrices along the first super-diagonal and zeros elsewhere. For the same reasons, our convention is different and for us the terms along the first super-diagonal are the same that those on the main diagonal i.e.  $\lambda R_\theta$ .

Let  $B$  be a classical (respectively real) Jordan block with eigenvalue (respectively modulus)  $\lambda$  and of size  $k$  and let  $A = \lambda$  (respectively  $A = \lambda R_\theta$ ). Then, powers of  $B$  are simple to compute. For all  $n \in \mathbb{N}^*$ ,

$$B^n = \begin{pmatrix} A^n & C_n^1 A^n & C_n^2 A^n & \dots & C_n^{k-1} A^n \\ 0 & A^n & C_n^1 A^n & C_n^2 A^n \dots & C_n^{k-2} A^n \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & C_n^1 A^n \\ 0 & \dots & 0 & 0 & A^n \end{pmatrix}.$$

Due to the fact that we are going to use repeatedly the Jordan decomposition, in both real and complex cases, we use the term modulus instead of eigenvalue. If the reader wishes more informations on the Jordan decomposition see [9] or [11] for a good review.

To summarise, every operator on  $\mathbb{R}^n$  is similar to one with the following shape:

$$R = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \lambda_p & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & \boxed{J_1} & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & \ddots & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & \boxed{J_q} & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & \boxed{\mathcal{A}_1} & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\mathcal{A}_r} \end{pmatrix}$$

where  $J_i$  are classical Jordan blocks:

$$J_i = \begin{pmatrix} \mu_i & \mu_i & 0 & 0 \\ 0 & \ddots & \ddots & \dots \\ \vdots & 0 & \ddots & \mu_i \\ 0 & \dots & 0 & \mu_i \end{pmatrix}$$

and  $\mathcal{A}_i$  are real Jordan blocks:

$$\mathcal{A}_i = \begin{pmatrix} \overline{A_i} & | & A_i & | & 0 & 0 \\ 0 & \ddots & \ddots & \dots & & \\ \vdots & 0 & \ddots & \overline{A_i} & & \\ 0 & \dots & 0 & A_i & & \end{pmatrix}.$$

### 3. $N$ -SUPERCYCLIC OPERATORS ON $\mathbb{R}^n$

**3.1. Introduction.** Bourdon, Feldman and Shapiro showed that there are  $k$ -supercyclic operators on  $\mathbb{C}^n$  if and only if  $k = n$ . This completely characterises  $n$ -supercyclic operators in the complex finite dimensional setting. In this section, we are going to apply the real Jordan decomposition to determine for which  $k \in \mathbb{N}$  there are  $k$ -supercyclic operators on  $\mathbb{R}^n$ .

Actually, the following examples show how to provide  $(\lfloor \frac{n+1}{2} \rfloor)$ -supercyclic operators on  $\mathbb{R}^n$ .

**Example 3.1.** For all  $n \geq 2$ :

- On  $\mathbb{R}^{2n}$ , endomorphisms represented by matrices of the form:

$$\begin{pmatrix} \overline{R_{\theta_1}} & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ \vdots & \cdots & 0 & \overline{R_{\theta_n}} \end{pmatrix}$$

are  $n$ -supercyclic if (and only if)  $\{\pi, \theta_1, \dots, \theta_n\}$  is a linearly independent family over  $\mathbb{Q}$ .

- On  $\mathbb{R}^{2n+1}$ , endomorphisms represented by matrices of the form:

$$\begin{pmatrix} \overline{R_{\theta_1}} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ \vdots & \cdots & 0 & \overline{R_{\theta_n}} & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

are  $(n+1)$ -supercyclic if (and only if)  $\{\pi, \theta_1, \dots, \theta_n\}$  is a linearly independent family over  $\mathbb{Q}$ .

The proof of this example relies on the fact that every rotation sub-matrix is supercyclic and the Kronecker Density Theorem [7] allows to consider each one separately.

These simple examples prove that our Theorem 1 is the best one may expect. In the following, we are going to study  $n$ -supercyclic operators on  $\mathbb{R}^n$  in order to prove Theorem 1. The first step is to study a real Jordan block with modulus one and of size two. We begin with this one mainly because the case of classical Jordan blocks have already been settled by Bourdon, Feldman and Shapiro in [3]. This one is the simplest example which has not already been solved. Moreover, it is the first step to deal with real Jordan blocks of arbitrary size.

**Proposition 3.2.**  $T = \begin{pmatrix} R_\theta & R_\theta \\ 0 & R_\theta \end{pmatrix}$  is not 2-supercyclic.

*Proof.* Suppose that  $T$  is 2-supercyclic to obtain a contradiction. Let  $L = \text{Span}(x, y)$  be a 2-supercyclic subspace for  $T$ . Then, one can suppose either  $x = (x_1, x_2, 0, 1)$  and  $y = (y_1, y_2, 1, 0)$  or  $x = (x_1, x_2, x_3, x_4)$  and  $y = (y_1, y_2, 0, 0)$  where  $(x_3, x_4) \neq (0, 0)$ .

- If  $x = (x_1, x_2, 0, 1)$  and  $y = (y_1, y_2, 1, 0)$ , then for any non-empty open sets  $U$  and  $V$  in  $\mathbb{R}^2$ , there exists a strictly increasing sequence  $(n_i)_{i \in \mathbb{N}}$  and two real sequences  $(\lambda_{n_i})_{i \in \mathbb{N}}, (\mu_{n_i})_{i \in \mathbb{N}}$  such that:

$$\begin{cases} R_\theta^{n_i} \left( \lambda_{n_i} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu_{n_i} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) + n_i R_\theta^{n_i} \begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in U \\ R_\theta^{n_i} \begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in V \end{cases}$$

This is equivalent to:

$$(1) \quad \begin{cases} \lambda_{n_i} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu_{n_i} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + n_i \begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in R_\theta^{-n_i}(U) \\ (2) \quad \begin{cases} \begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in R_\theta^{-n_i}(V) \end{cases} \end{cases}$$

Choose  $V = B\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varepsilon\right)$  with  $0 < \varepsilon < 1$  and  $U$  any non-empty bounded open set, then (2) implies that for all  $i \in \mathbb{N}$ ,  $0 \leq |\lambda_{n_i}|, |\mu_{n_i}| < 1 + \varepsilon$ . Dividing (1) by  $n_i$ , one gets:

$$\frac{\lambda_{n_i}}{n_i} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{\mu_{n_i}}{n_i} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in \frac{R_\theta^{-n_i}(U)}{n_i}$$

However, since  $\lambda_{n_i}$  and  $\mu_{n_i}$  are bounded,  $\frac{\lambda_{n_i}}{n_i} \xrightarrow{i \rightarrow +\infty} 0$  and  $\frac{\mu_{n_i}}{n_i} \xrightarrow{i \rightarrow +\infty} 0$ . Since  $U$  is bounded,  $\lambda_{n_i}$  and  $\mu_{n_i}$  have to go to zero. This contradicts  $\begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in R_\theta^{-n_i}(V)$ .

• If  $x = (x_1, x_2, x_3, x_4)$  and  $y = (y_1, y_2, 0, 0)$ , then one may suppose  $\|(x_3, x_4)\| = 1$ . For any non-empty open sets  $U, V$  in  $\mathbb{R}^2$ , there exists a strictly increasing sequence  $(n_i)_{i \in \mathbb{N}}$  and two real sequences  $(\lambda_{n_i})_{i \in \mathbb{N}}, (\mu_{n_i})_{i \in \mathbb{N}}$  such that:

$$\begin{cases} R_\theta^{n_i} \left( \lambda_{n_i} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu_{n_i} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) + n_i R_\theta^{n_i} \begin{pmatrix} \mu_{n_i} \\ \lambda_{n_i} \end{pmatrix} \in U \\ \lambda_{n_i} R_\theta^{n_i} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in V \end{cases}$$

One can rewrite:

$$(3) \quad \begin{cases} \lambda_{n_i} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu_{n_i} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + n_i \lambda_{n_i} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(U) \end{cases}$$

$$(4) \quad \begin{cases} \lambda_{n_i} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(V) \end{cases}$$

Let  $V = B\left(\begin{pmatrix} r \\ 0 \end{pmatrix}, \varepsilon\right)$  with  $0 < \varepsilon < 1, r > 1$ . According to (4),  $r - \varepsilon < \lambda_{n_i} < r + \varepsilon$  and dividing (3) by  $n_i \lambda_{n_i}$ , it comes:

$$\frac{\mu_{n_i}}{n_i \lambda_{n_i}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \xrightarrow{i \rightarrow +\infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From this we deduce that  $\frac{\mu_{n_i}}{n_i \lambda_{n_i}}$  is bounded and one may extract a subsequence converging to  $M \in \mathbb{R}$  satisfying:

$$M \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As  $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$  is non-zero, this last equation implies that  $\left\{ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\}$  is linearly dependent.

Thus choosing an appropriate linear combination of  $x$  and  $y$ , one may assume  $x = (x_1, x_2, x_3, x_4)$  and  $y = (x_3, x_4, 0, 0)$ , hence (3) and (4) give:

$$(5) \quad \begin{cases} \lambda_{n_i} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (\mu_{n_i} + n_i \lambda_{n_i}) \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(U) \end{cases}$$

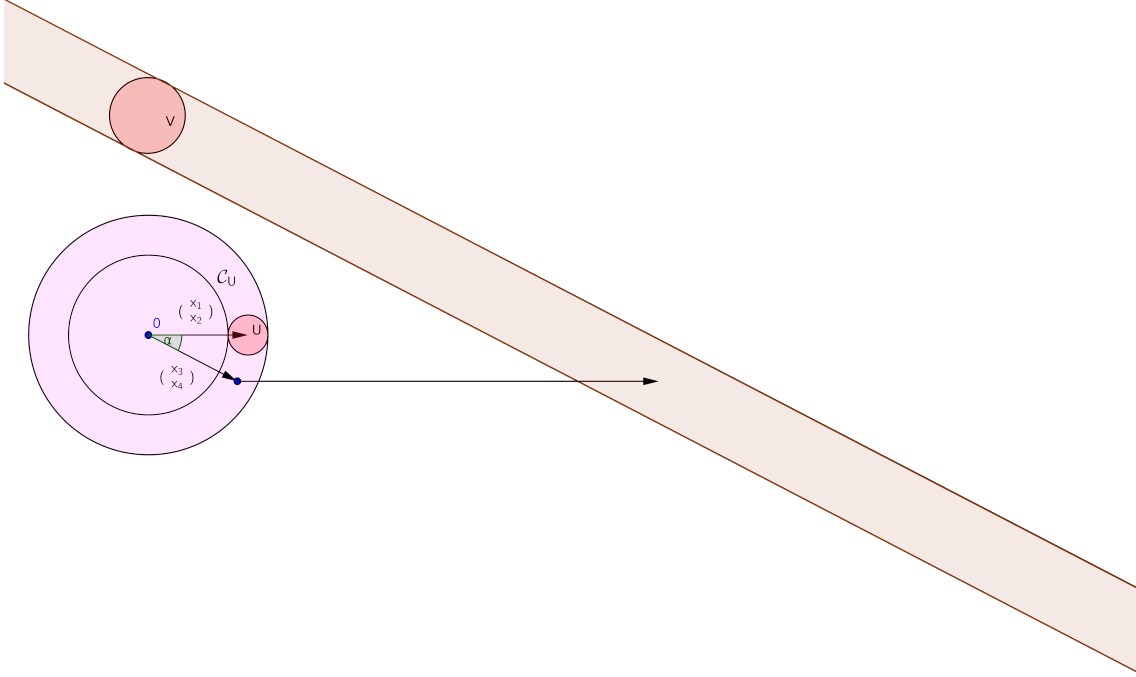
$$(6) \quad \begin{cases} \lambda_{n_i} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(V) \end{cases}$$

It is now clear that the vectors  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$  are linearly independent. Indeed, suppose, in order to obtain a contradiction, that they are linearly dependent. Then upon taking appropriate linear combinations and replacing  $x$ , we have  $x = (0, 0, x_3, x_4)$  and  $y = (x_3, x_4, 0, 0)$  and

$$\begin{cases} (\mu_{n_i} + n_i \lambda_{n_i}) \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(U) \\ \lambda_{n_i} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in R_\theta^{-n_i}(V) \end{cases}$$

Now if one chooses two non-empty open sets  $U$  and  $V$  such that there does not exist a line intersecting both  $U$  and  $V$ , then we have a contradiction!

So,  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$  being linearly independent, let  $\alpha$  denote the angle between these two vectors, then  $|\sin(\alpha)| > 0$ . Let  $0 < a < |\sin(\alpha)|(r - \varepsilon) \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|$ ,  $U = B\left(\begin{pmatrix} 0 \\ \frac{a}{2} \end{pmatrix}, \frac{a}{4}\right)$  and  $\mathcal{C}_U$  be the annulus obtained by rotations of  $U$ . One may easily notice that  $\left(\mathbb{R} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + [r - \varepsilon, r + \varepsilon] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \cap \mathcal{C}_U = \emptyset$  contradicting (5)!



So  $\begin{pmatrix} R_\theta & R_\theta \\ 0 & R_\theta \end{pmatrix}$  is not 2-supercyclic! □

**Remark 3.3.** One can easily notice that the previous matrix is 3-supercyclic if  $\pi$  and  $\theta$  are linearly independent over  $\mathbb{Q}$ .

In addition, the proof of this proposition is interesting in itself because it is divided into two parts depending on the "shape" of the basis. Actually, to be able to deal with such operators, we will constantly work on the basis' shape!

**3.2. A leading example.** We deal with an interesting example to show that we need some more tools if we want to go further in a clear and precise manner. The next result proves that  $n$  cannot be improved in the first part of Example 3.1. Moreover, in the following,  $T$  is a direct sum of rotations' multiples, every one of these acting on  $\mathbb{R}^2$ . Hence when one usually consider a vector component, we consider a vector bi-component instead, meaning that for the next result the natural way to define a vector is not as being in  $\mathbb{R}^{2n}$  but rather in  $(\mathbb{R}^2)^n$  because  $T$  is constructed with operators acting on  $\mathbb{R}^2$ .

**Proposition 3.4.**  $R_n := \begin{pmatrix} a_1 R_{\theta_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_n R_{\theta_n} \end{pmatrix}$  is not  $(n-1)$ -supercyclic on  $\mathbb{R}^{2n}$  for any choice of  $a_1, \dots, a_n \in \mathbb{R}$  and any choice of  $\theta_1, \dots, \theta_n \in \mathbb{R}$ .

*Proof.* First upon reordering blocks in  $R_n$  and taking a scalar multiple, one may suppose  $a_1 \leq \dots \leq a_{n-1} \leq a_n = 1$ . Indeed, the nullity of one of the  $a_i$  implies that  $R_n$  has not dense range and is not  $(n-1)$ -supercyclic.

We are going to prove that  $R_n$  is not  $(n-1)$ -supercyclic on  $\mathbb{R}^{2n}$  by induction.

- For  $n = 2$ , the result follows from Hilden and Wallen's result [8].
- Suppose that the induction hypothesis is proved for  $R_2, \dots, R_{n-1}$ . Let us prove it for  $R_n$ . Assume to the contrary that  $R_n$  is  $(n-1)$ -supercyclic and let  $L = \text{Span}(x^1, \dots, x^{n-1})$  be a  $(n-1)$ -supercyclic subspace for  $R_n$ . Once one have made a choice for the basis of  $L$ , then for any  $(w_1, \dots, w_n) \in \mathbb{R}^n$ , there exists a strictly increasing sequence  $(n_i)_{i \in \mathbb{N}}$  and  $((\lambda_1^{n_i})_{i \in \mathbb{N}}, \dots, (\lambda_{n-1}^{n_i})_{i \in \mathbb{N}}) \in (\mathbb{R}^{\mathbb{N}})^{n-1}$  such that:

$$\begin{pmatrix} a_1^{n_i} R_{\theta_1}^{n_i} \left( \sum_{j=1}^{n-1} \lambda_j^{n_i} \begin{pmatrix} x_1^j \\ x_2^j \end{pmatrix} \right) \\ a_2^{n_i} R_{\theta_2}^{n_i} \left( \sum_{j=1}^{n-1} \lambda_j^{n_i} \begin{pmatrix} x_3^j \\ x_4^j \end{pmatrix} \right) \\ \vdots \\ a_n^{n_i} R_{\theta_n}^{n_i} \left( \sum_{j=1}^{n-1} \lambda_j^{n_i} \begin{pmatrix} x_{2n-1}^j \\ x_{2n}^j \end{pmatrix} \right) \end{pmatrix} \xrightarrow{i \rightarrow \infty} \begin{pmatrix} w_1 \\ 0 \\ w_2 \\ 0 \\ \vdots \\ w_n \\ 0 \end{pmatrix}$$

Denote by  $(L_k)$  the  $k$ -th line of the preceding left member. Upon taking appropriate linear combinations of basis elements of  $L$ , reordering and relabelling, one may assume that the last bi-component is non-zero either for the first basis vector or for the first two basis vectors and is 0 for the other basis vectors. Moreover, in this last case, one may also require them to be  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . In the first case, upon taking appropriate linear combinations of the last  $n-2$  basis vectors, reordering and relabelling, one may get three different issues:

- ▷ the  $(n-1)$ -th bi-component is 0 for the last  $n-2$  basis vectors and this yields to a contradiction. Indeed, one may remark that the  $\begin{pmatrix} a_{n-1} R_{\theta_{n-1}} & 0 \\ 0 & a_n R_{\theta_n} \end{pmatrix}$  part of  $R_n$  is thus only acting on the last bicomponents of  $x^1$ . But this operator is not supercyclic by induction hypothesis. This contradicts the  $(n-1)$ -supercyclicity of  $R_n$ .
- ▷ the  $(n-1)$ -th bi-component is 0 for the last  $n-2$  basis vectors apart from the second one.
- ▷ the  $(n-1)$ -th bi-component is 0 for the last  $n-2$  basis vectors apart from the second and the third ones and these two components can be chosen to be  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

In the second case, upon taking appropriate linear combinations of the last  $n-3$  basis vectors, reordering and relabelling, one may get three different issues:

- ▷ the  $(n-1)$ -th bi-component is 0 for the last  $n-3$  basis vectors and this yields to a contradiction. Take  $w_n = 0$  and  $w_{n-1} = 1$ . Then,  $(L_n)$  implies  $\lambda_1^{n_i}, \lambda_2^{n_i} \xrightarrow{i \rightarrow +\infty} 0$ , but this leads to  $\|(L_{n-1})\| \xrightarrow{i \rightarrow +\infty} 0$  contradicting the choice of  $w_{n-1} = 1 \neq 0$ .
- ▷ the  $(n-1)$ -th bi-component is 0 for the last  $n-3$  basis vectors apart from the third one.
- ▷ the  $(n-1)$ -th bi-component is 0 for the last  $n-3$  basis vectors apart from the third and the fourth ones and these two components can be chosen to be  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .



Repeating this process allows us to assume that  $L$  can be expressed with a basis for which the  $k$ -th bi-component is 0 for at least the last  $\min(2k - n - 3, 0)$  vectors and at most the last  $\min(k - 2, 0)$  vectors. This new expression of the basis vectors now permits to conclude. Indeed, take  $w_1 = 1$  and  $w_2 = \dots = w_n = 0$ , then using  $(L_n), \dots, (L_2)$  in the same way we did it above, one deduces that for any  $k \in \{1, \dots, n-1\}$ , there exists  $j \in \{2, \dots, n\}$  such that  $|a_j^{n_i} \lambda_k^{n_i}| \xrightarrow{i \rightarrow +\infty} 0$ . So  $|a_1^{n_i} \lambda_k^{n_i}| \xrightarrow{i \rightarrow +\infty} 0$  for any  $k \in \{1, \dots, n-1\}$ , but this contradicts  $(L_1)$ .

Hence the result follows by induction.  $\square$

The key in the proof is the adaptation of the basis to the shape of  $R_n$  and we are going to make constant use of this method in what follows. This motivates us to detail this method in the next part.

**3.3. Basis reduction.** Let  $m, n \in \mathbb{N}^*$ ,  $T$  be a linear operator on  $\mathbb{R}^n$ ,  $\{x^1, \dots, x^m\}$  be a linearly independent family in  $\mathbb{R}^n$  and  $L$  be the subspace spanned by this family. Using the Jordan real decomposition one may suppose:

$$T = \begin{pmatrix} a_1 B_1 & 0 & \cdots & 0 \\ 0 & a_2 B_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_k B_k \end{pmatrix}$$

$$\text{where } B_i = \begin{pmatrix} A_i & A_i & 0 & \cdots & 0 \\ 0 & A_i & A_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & A_i \\ 0 & \cdots & & 0 & A_i \end{pmatrix} \text{ is a classical or real Jordan block for any } 1 \leq i \leq n$$

with  $A_i = 1$  or  $A_i = R_{\theta_i}$  respectively. Define  $d_i = 1$  when  $B_i$  is classical and  $d_i = 2$  when  $B_i$  is real and take also  $r_i$  such that  $d_i r_i$  is  $B_i$ 's size. Let also  $r = \sum_{i=1}^k r_i$ .

**Theorem 3.5.** *Let  $T$  be a linear operator on  $\mathbb{R}^n$  in the previous Jordan form. Let also  $\{x^1, \dots, x^m\}$  be a family of linearly independent vectors on  $\mathbb{R}^n$  and  $L = \text{Span}(x^1, \dots, x^m)$ . Then, there exist a basis  $\{y^1, \dots, y^m\}$  of  $L$  adapted to  $T$ , an increasing sequence of integers  $(K_i)_{i \in \mathbb{N}}$ , a set sequence  $(V_i)_{i \in \mathbb{N}}$  and for any  $i \in \{1, \dots, r\}$  a sequence  $(X_i^j)_{1 \leq j \leq m} \in (\mathbb{R})^m \cup (\mathbb{R}^2)^m$  satisfying:*

(a) For every  $i \in \{1, \dots, m\}$ ,  $y^i = (X_j^i)_{1 \leq j \leq r}$ .

(b)  $K_0 = 1$ ,  $V_0 = \{X_r^j, K_0 \leq j \leq m\}$ .

(c) For every  $i \in \mathbb{N}$ ,  $K_{i+1} = K_i + \dim(\text{Span}(V_i))$  and  $V_{i+1} = \{X_{r-(i+1)}^j, K_{i+1} \leq j \leq m\}$ .

(d) For every  $i \in \{0, \dots, r-1\}$ ,  $\{X_{r-i}^j, K_i \leq j < K_{i+1}\}$  is either empty or linearly independent.

(e)  $K_r = m + 1$ .

(f) For every  $p \in \{1, \dots, r\}$  and every  $i \in \{K_p, \dots, m\}$ ,  $X_{r-p+1}^i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

*Proof.* We want to construct a basis of  $L$  adapted to the decomposition of  $T$ .

For this purpose, denote  $x^1 = \begin{pmatrix} X_1^1 \\ \vdots \\ X_r^1 \end{pmatrix}, \dots, x^m = \begin{pmatrix} X_1^m \\ \vdots \\ X_r^m \end{pmatrix}$  with

$$X_j^i = \begin{cases} x^i_{\sum_{l=1}^{p-1} d_l r_l + j - \sum_{l=1}^{p-1} r_l} & \text{if } d_p = 1. \\ \begin{pmatrix} x^i_{\sum_{l=1}^{p-1} d_l r_l + 2(j - \sum_{l=1}^{p-1} r_l) - 1} \\ x^i_{\sum_{l=1}^{p-1} d_l r_l + 2(j - \sum_{l=1}^{p-1} r_l)} \end{pmatrix} & \text{if } d_p = 2. \end{cases}$$

and where  $p$  is the unique natural number satisfying:  $\sum_{l=1}^{p-1} r_l < j \leq \sum_{l=1}^p r_l$ . Roughly speaking  $p$  is the number of the block  $B_p$  of  $T$  which is acting on  $X_j^i$ . This probably seems a bit complicated at first sight but the underlying idea is natural: the operator  $T$  is trivially seen as almost a "sum" of operators  $A_i$  acting on either  $\mathbb{R}$  or  $\mathbb{R}^2$ . Then it is also natural to consider the vectors  $T$  is acting on, as a direct sum of vectors that the operators  $A_i$  are acting on. To summarise, on some parts (classical)  $T$  acts like if he was an operator on  $\mathbb{R}$  and on the others (real), it acts as on  $\mathbb{R}^2$ .

We now describe the adapted reduction we are going to make on the vectors  $x_1, \dots, x_m$ . Of course, this reduction heavily depends on  $T$ . We are going to create an increasing sequence of natural numbers  $(K_p)_{p \in \mathbb{N}}$  and a sequence of sets  $(V_p)_{p \in \mathbb{N}}$ . For every step of the reduction, the sequence of natural numbers marks the vector number up to which the reduction has been completed and the sequence of sets contains the part of the vectors we have to reduce to the next step.

First define  $K_0 = 1$  and  $V_0 = \{X_r^i, K_0 \leq i \leq m\}$ . By definition, for any  $1 \leq i \leq m$ ,  $X_r^i$  is either a subset of  $\mathbb{R}$  or of  $\mathbb{R}^2$ , then  $\dim(\text{Span}(V_0)) = 0, 1$  or  $2$ .

- If  $\dim(\text{Span}(V_0)) = 0$  : then  $\|X_r^i\| = 0$  for any  $1 \leq i \leq m$  and we set  $K_1 := K_0$ .
- If  $\dim(\text{Span}(V_0)) = 1$  : thus upon taking proper linear combinations of  $x^1, \dots, x^m$ , reordering and relabelling, one may suppose  $\|X_r^1\| = 1$  and  $\|X_r^i\| = 0$  for any  $K_0 + 1 \leq i \leq m$  and set  $K_1 := K_0 + 1$ .
- If  $\dim(\text{Span}(V_0)) = 2$  : thus upon taking proper linear combinations of  $x^1, \dots, x^m$ , reordering and relabelling, one may suppose  $X_r^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ,  $X_r^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\|X_r^i\| = 0$  for  $K_0 + 2 \leq i \leq m$

and set  $K_1 := K_0 + 2$ .

Then set also  $V_1 = \{X_{r-1}^i, K_1 \leq i \leq m\}$ . One may define in the same spirit by induction  $K_p$  and  $V_p$  until  $V_p = \emptyset$  ( $\Leftrightarrow K_p = m + 1$ ). Once the reduction is over, for technical conveniences for every  $q$  greater than  $p$ , we set  $K_q := K_p$ .

Then, (b), (c), (d), (e) and (f) are clearly satisfied by the proof. Moreover (a) is also satisfied if one notes that we have done operations on the columns only of the  $x^i$ , hence the vector space  $L$  they were spanning remains unchanged.  $\square$

**Remark 3.6.** The reduced basis we have described in the previous theorem has the following shape:

$$\begin{pmatrix} * & \dots & \dots & \dots \\ & \dots & \dots & \dots \\ & & \dots & \dots \\ \dots & & & \dots \end{pmatrix}$$

We keep these notations and vocabulary for the rest of this paper. The reader needs to have in mind these notations when we decompose an operator in his Jordan form or when we reduce a basis.

### 3.4. Primary matrices.

**Definition 3.7.** Let  $n, k \in \mathbb{N}^*$ . An operator  $T$  on  $\mathbb{R}^n$  is said to be primary of order  $k$  when  $T = \bigoplus_{i=1}^k A_i$  with  $A_i = 1$  or  $R_{\theta_i}$  with  $\theta_i \in \mathbb{R}$ .

**Remark 3.8.** One can see at first glance that if  $T$  is primary of order  $k$  on  $\mathbb{R}^n$ , then  $k \in \llbracket \lfloor \frac{n+1}{2} \rfloor, n \rrbracket$ .

We begin our study with primary matrices. However, even if the next result obviously generalises Proposition 3.4, the proof of the following proposition is independent from the other one. Moreover, this proof puts forward some useful ideas.

**Proposition 3.9.** *Let  $k \in \mathbb{N}^*$ . There is no  $(k-1)$ -supercyclic primary matrix of order  $k$ .*

*Proof.* Let  $T = \bigoplus_{i=1}^k A_i$  be a primary matrix of order  $k$ . Following the notations we introduced before, we denote  $r = k$  and  $r_i = 1$  for every  $1 \leq i \leq k$ . Now suppose, in order to obtain a contradiction, that  $T$  is  $(k-1)$ -supercyclic. Let  $L = \text{Span}(x^1, \dots, x^{k-1})$  be a  $(k-1)$ -supercyclic subspace for  $T$  and then reduce the basis of  $L$  with Theorem 3.5. First, it is worth noting that for any  $p < k$ ,  $K_p \neq K_{p+1}$ .

Assume to the contrary that there exists  $p < k$  such that  $K_p = K_{p+1}$  and let  $p$  be the smallest integer with this property, this implies  $\dim(\text{Span}(V_p)) = 0$  and thus for any  $K_p \leq j \leq k-1$ ,  $\|X_{k-p}^j\| = 0$ . But, for all  $i \in \mathbb{N}$  and all real sequence  $(\lambda_j)_{1 \leq j \leq k-1}$ ,

$$T^i \left( \sum_{j=1}^{k-1} \lambda_j x^j \right) = \begin{cases} A_1^i \sum_{j=1}^{k-1} \lambda_j X_1^j & (L_1) \\ \vdots & \vdots \\ A_{k-p}^i \sum_{j=1}^{K_{p-1}} \lambda_j X_{k-p}^j & (L_{k-p}) \\ A_{k-p+1}^i \sum_{j=1}^{K_{p-1}} \lambda_j X_{k-p+1}^j & (L_{k-p+1}) \\ \vdots & \vdots \\ A_k^i \sum_{j=1}^{K_1-1} \lambda_j X_k^j & (L_k) \end{cases}$$

Clearly, if  $p = 0$  then  $(L_k) = 0$  and  $L$  fails to be  $(k-1)$ -supercyclic for  $T$ . In the following, we may assume  $p > 0$  without loss of generality. Then, by  $(k-1)$ -supercyclicity of  $L$ , there exists  $(n_i)_{i \in \mathbb{N}}$  and real sequences  $(\lambda_j^{n_i})_{i \in \mathbb{N}, 1 \leq j \leq k-1}$  such that for any  $j \in \llbracket 1, k \rrbracket \setminus \{k-p\}$ ,

$$(L_j) \xrightarrow{i \rightarrow +\infty} 0 \text{ and } (L_{k-p}) \xrightarrow{i \rightarrow +\infty} Y \text{ with } \|Y\| = 1.$$

We shall prove that for any  $1 \leq j < K_p$ ,  $\lambda_j^{n_i} \xrightarrow{i \rightarrow +\infty} 0$ . Such an integer  $j$  belongs to a unique interval  $[K_q, K_{q+1}[$  and we shall prove this property by induction on  $q$ .

If  $1 \leq j < K_1$ , then  $\{X_k^j\}_{1 \leq j < K_1} \neq \emptyset$  is a linearly independent family and  $A_k$  being an isometry,  $(L_k)$  gives :  $\sum_{j=1}^{K_1-1} \lambda_j^{n_i} X_k^j \xrightarrow{i \rightarrow +\infty} 0$ , hence  $\lambda_j^{n_i} \xrightarrow{i \rightarrow +\infty} 0$  for every  $1 \leq j < K_1$ .

We assume that the induction hypothesis is true for  $1 \leq q < p$ . We have to prove it for  $q+1$  too. Since  $(L_{k-q})$  converges to 0 and  $A_{k-q}$  being an isometry, we have:  $\sum_{j=1}^{K_{q+1}-1} \lambda_j^{n_i} X_{k-q}^j \xrightarrow{i \rightarrow +\infty} 0$ .

The recurrence hypothesis implies  $\lambda_j^{n_i} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j < K_q$ . Hence  $\sum_{j=K_q}^{K_{q+1}-1} \lambda_j^{n_i} X_{k-q}^j \xrightarrow{i \rightarrow +\infty} 0$ . However,  $\{X_{k-q}^j\}_{K_q \leq j < K_{q+1}} \neq \emptyset$  is a linearly independent family, so  $\lambda_j^{n_i} \xrightarrow{i \rightarrow +\infty} 0$  for every  $1 \leq j < K_{q+1}$ . This ends the induction step.

Thus, for any  $1 \leq j < K_p$ ,  $\lambda_j^{n_i} \xrightarrow{i \rightarrow +\infty} 0$ . Considering these limits in  $(L_{k-p})$  and the fact that  $A_{k-p}$  is an isometry yields :

$$A_{k-p}^i \sum_{j=1}^{K_p-1} \lambda_j^{n_i} X_{k-p}^j \xrightarrow{i \rightarrow +\infty} 0.$$

But this contradicts the convergence of  $(L_k)$  to some unit vector concluding that  $K_p \neq K_{p+1}$  for some  $p < k$ .

Considering that  $K_0 = 1$  and that the sequence  $(K_p)_{0 \leq p \leq k}$  is strictly increasing, one obtains  $K_{k-1} \geq k$ , hence by Theorem 3.5  $K_k = K_{k-1}$ . This contradiction proves that  $T$  is not  $(k-1)$ -supercyclic.  $\square$

**3.5. For a single real Jordan block.** The aim of this section is to generalise Proposition 3.2 to the case of a real Jordan block of arbitrary dimension. The two following lemmas are useful to express in another way the iterates of a subspace by a real Jordan block.

**Lemma 3.10.** *Define  $\Delta_n(i) := \binom{i}{n} - \sum_{k=1}^{n-1} \Delta_k(i) \binom{i}{n-k}$  for any  $n \geq 0$  and  $i \geq 0$ . Then,  $\Delta_n$  is a polynomial in  $i$  of degree  $n$  and his leading coefficient is  $\delta_n := \frac{(-1)^{n+1}}{n!}$ .*

*Proof.* We prove it by induction on  $n \geq 0$ . The lemma is obviously true for  $n = 1$  because  $\Delta_1(i) = i$ .

Assume that we have verified the induction hypothesis for  $1 \leq k < n$ . Let us prove it for  $k = n$ . The leading coefficient of  $\binom{i}{k}$  is  $\frac{1}{k!}$  for all  $k \in \mathbb{N}$ . Combining with the induction hypothesis, one gets:

$$\delta_n = \frac{1}{n!} - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{(n-k)!k!} = \frac{1}{n!} \left( 1 - \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n}{k} \right).$$

Now, it is easy to check that:

$$\sum_{k=1}^{n-1} \binom{n}{k} (-1)^{k+1} = \begin{cases} 0 & \text{if } n \text{ is odd.} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

This yields:

$$\delta_n = \begin{cases} \frac{1}{n!} & \text{if } n \text{ is odd.} \\ -\frac{1}{n!} & \text{if } n \text{ is even.} \end{cases}.$$

This ends the induction and the proof of the lemma.  $\square$

In order to fully understand the interest of introducing the sequence  $\Delta_n$ , we also need the following lemma:

**Lemma 3.11.** *Let  $i, n \in \mathbb{N}$  with  $i \geq n$  and let  $(l_k)_{1 \leq k \leq n}$  be a sequence of real numbers. For every  $1 \leq k \leq n$  define  $L_k := \sum_{j=0}^{n-k} \binom{i}{j} l_{k+j}$ . Then,  $L_k = l_k + \sum_{j=1}^{n-k} \Delta_j(i) L_{k+j}$ .*

*Proof.* Once again, we prove this result by induction on  $n$ . For  $n = 1$ , the result is straightforward since  $L_1 = \sum_{j=0}^{1-1} \binom{i}{j} l_{1+j} = l_1$ .

Now, assume that the induction hypothesis is true for any natural number strictly smaller than  $n$  and let us prove it for  $n$ . For  $1 \leq k < n$ , set  $\mathcal{L}_k = \sum_{j=0}^{n-k-1} \binom{i}{j} l_{k+j} = L_k - \binom{i}{n-k} l_n$ . Then the induction hypothesis gives:

$$\mathcal{L}_k = l_k + \sum_{j=1}^{n-k-1} \Delta_j(i) \mathcal{L}_{k+j}$$

and so for  $1 \leq k \leq n-1$ :

$$L_k = \mathcal{L}_k + \binom{i}{n-k} l_n = l_k + \sum_{j=1}^{n-k-1} \Delta_j(i) \mathcal{L}_{k+j} + \binom{i}{n-k} l_n.$$

Finally, use the definition of  $\mathcal{L}_k$  and  $\Delta_{n-k}(i)$  and note that  $L_n = l_n$ , we obtain:

$$\begin{aligned}
L_k &= l_k + \sum_{j=1}^{n-k-1} \Delta_j(i) \left( L_{k+j} - \binom{i}{n-k-j} l_n \right) + \binom{i}{n-k} l_n \\
&= l_k + \sum_{j=1}^{n-k-1} \Delta_j(i) L_{k+j} + \left( \binom{i}{n-k} - \sum_{j=1}^{n-k-1} \Delta_j(i) \binom{i}{n-k-j} \right) l_n \\
&= l_k + \sum_{j=1}^{n-k-1} \Delta_j(i) L_{k+j} + \Delta_{n-k}(i) l_n \\
&= l_k + \sum_{j=1}^{n-k} \Delta_j(i) L_{k+j}
\end{aligned}$$

This completes the proof of this lemma. □

Here comes now the generalisation of Proposition 3.2, its proof is of significant importance to an understanding of the mechanisms involved in later proofs.

**Proposition 3.12.** *Let  $n > 1$  and  $\theta \in \mathbb{R}$ .  $J_n := \begin{pmatrix} R_\theta & R_\theta & 0 & \cdots & 0 \\ 0 & R_\theta & R_\theta & 0 \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & R_\theta \\ 0 & \cdots & 0 & 0 & R_\theta \end{pmatrix}$  is not  $n$ -*

*supercyclic on  $\mathbb{R}^{2n}$ .*

*Proof.* As we already noticed, Proposition 3.2 proves the case  $n = 2$ . Let then  $n \geq 3$  and assume to the contrary that  $J_n$  is  $n$ -supercyclic. Let also  $L = \text{Span}(x^1, \dots, x^n)$  be a  $n$ -supercyclic subspace which basis  $x^1, \dots, x^n$  is reduced. Then, as we already pointed out in Theorem 3.5,  $K_n = n + 1$ . Moreover, Proposition 3.2 claims that  $J_2$  is not 2-supercyclic, thus  $\text{Span} \left( \begin{pmatrix} X_{n-1}^1 \\ X_n^1 \end{pmatrix}, \begin{pmatrix} X_{n-1}^2 \\ X_n^2 \end{pmatrix}, \begin{pmatrix} X_{n-1}^3 \\ X_n^3 \end{pmatrix} \right)$  has to span a subspace of dimension 3, yielding  $K_2 \geq 4$ . In addition,  $n$ -supercyclicity of  $L$  implies the existence of a sequence of natural numbers  $(n_i)_{i \in \mathbb{N}}$  and  $((\lambda_1^{n_i})_{i \in \mathbb{N}}, \dots, (\lambda_n^{n_i})_{i \in \mathbb{N}}) \in (\mathbb{R}^{\mathbb{N}})^n$  such that:

$$T^{n_i} \left( \sum_{j=1}^n \lambda_j^{n_i} x^j \right) = \begin{pmatrix} \sum_{k=0}^{n-1} \binom{n_i}{k} \left( \sum_{j=1}^n \lambda_j^{n_i} \begin{pmatrix} x_{2k+1}^j \\ x_{2(k+1)}^j \end{pmatrix} \right) \\ \sum_{k=0}^{n-2} \binom{n_i}{k} \left( \sum_{j=1}^n \lambda_j^{n_i} \begin{pmatrix} x_{2(k+1)+1}^j \\ x_{2(k+2)}^j \end{pmatrix} \right) \\ \vdots \\ \sum_{j=1}^n \lambda_j^{n_i} \begin{pmatrix} x_{2n-1}^j \\ x_{2n}^j \end{pmatrix} \end{pmatrix} \xrightarrow{i \rightarrow +\infty} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}.$$

Denote by  $(L_1), \dots, (L_n)$  the lines appearing in  $T^{n_i} \left( \sum_{j=1}^n \lambda_j^{n_i} x^j \right)$  above, and define  $(l_k) = \sum_{j=1}^n \lambda_j^{n_i} \begin{pmatrix} x_{2k-1}^j \\ x_{2k}^j \end{pmatrix}$ . Then, using Lemma 3.11 one may rewrite the preceding identity:

$$(7) \quad \begin{cases} \|(l_1) + \sum_{j=1}^{n-1} \Delta_j(n_i)(L_{j+1})\| & \xrightarrow{i \rightarrow +\infty} 0 \\ \vdots & \\ \|(l_k) + \sum_{j=1}^{n-k} \Delta_j(n_i)(L_{j+k})\| & \xrightarrow{i \rightarrow +\infty} 0 \\ \vdots & \\ \|(l_n)\| & \xrightarrow{i \rightarrow +\infty} 1 \end{cases}$$

where  $\Delta_n$  is defined in Lemma 3.10.

We come now to the key point of the proof: we prove by induction on  $k$  that  $\frac{\lambda_j^{n_i}}{\Delta_k(n_i)} \xrightarrow{i \rightarrow +\infty} 0$ , for every  $1 \leq k \leq n-1$  and every  $1 \leq j \leq K_k-1$ .  
 If  $k=1$ : Divide  $(L_n)$  by  $\Delta_1(n_i)$  and take the limit:

$$\left\| \sum_{j=K_0}^{K_1-1} \frac{\lambda_j^{n_i}}{\Delta_1(n_i)} X_n^j \right\| \xrightarrow{i \rightarrow +\infty} 0.$$

In addition the fact that  $\{X_n^j\}_{K_0 \leq j \leq K_1-1}$  is either linearly independent or empty leads to:

$$\frac{\lambda_j^{n_i}}{\Delta_1(n_i)} \xrightarrow{i \rightarrow +\infty} 0 \text{ for any } K_0 \leq j \leq K_1-1.$$

We now assume that the induction hypothesis is true for any natural number smaller than  $k$  and we prove it for  $k+1$ . First, divide  $(L_{n-k})$  by  $\Delta_{k+1}(n_i)$  and take the limit:

$$\left\| \sum_{j=K_0}^{K_k-1} \frac{\lambda_j^{n_i}}{\Delta_{k+1}(n_i)} X_{n-k}^j + \sum_{j=K_k}^{K_{k+1}-1} \frac{\lambda_j^{n_i}}{\Delta_{k+1}(n_i)} X_{n-k}^j + \sum_{j=1}^k \frac{\Delta_j(n_i)}{\Delta_{k+1}(n_i)} (L_{n-k+j}) \right\| \xrightarrow{i \rightarrow +\infty} 0.$$

The induction hypothesis provides that the sum on the left of the preceding line converges to 0. Moreover, as  $(L_j)$  is bounded for all  $1 \leq j \leq n$ , then the sum on the right of the previous line converges also to 0 providing:

$$\left\| \sum_{j=K_k}^{K_{k+1}-1} \frac{\lambda_j^{n_i}}{\Delta_k(n_i)} X_{n-k}^j \right\| \xrightarrow{i \rightarrow +\infty} 0.$$

Moreover,  $\{X_{n-k}^j\}_{K_k \leq j \leq K_{k+1}-1}$  is either linearly independent or empty, and taking this into account in the line above we conclude:

$$\frac{\lambda_j^{n_i}}{\Delta_1(n_i)} \xrightarrow{i \rightarrow +\infty} 0 \text{ for any } K_0 \leq j \leq K_{k+1}-1.$$

So, for every  $1 \leq k \leq n-1$  and  $1 \leq j \leq K_k-1$ ,  $\frac{\lambda_j^{n_i}}{\Delta_k(n_i)} \xrightarrow{i \rightarrow +\infty} 0$ .

As we claimed before, notice that  $K_2 \geq 4$  and  $K_n = n+1$ . It follows that there exists  $2 \leq p \leq n-1$  such that  $K_p = K_{p+1}$ . Divide then  $(L_{n-p})$  by  $\Delta_p(n_i)$  to get:

$$\left\| \sum_{j=K_0}^{K_{p+1}-1} \frac{\lambda_j^{n_i}}{\Delta_p(n_i)} X_{n-p}^j + \sum_{j=1}^p \frac{\Delta_j(n_i)}{\Delta_p(n_i)} (L_{n-p+j}) \right\| \xrightarrow{i \rightarrow +\infty} 0.$$

The sum on the left above tends to 0 because  $\frac{\lambda_j^{n_i}}{\Delta_p(n_i)} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq K_p - 1$ . Furthermore, combine Lemma 3.10 and the boundedness of  $(L_k)$  to deal with the second sum:

$$\left\| \frac{(-1)^{p+1}}{p!} (L_n) \right\| \xrightarrow{i \rightarrow +\infty} 0.$$

This contradicts  $\|(L_n)\| \xrightarrow{i \rightarrow +\infty} 1$ . So,  $J_n$  is not  $n$ -supercyclic.  $\square$

**3.6. Sum of Jordan blocks with different moduli.** The key is to separate the behaviour of one block from the others. The main idea is to prove that the coefficients already used for previous blocks are approaching 0 in this new one and that the only coefficients affecting this block are all new.

**Lemma 3.13.** *Let  $h \in \mathbb{N}$  and  $k, n, N \in \mathbb{N}^*$  with  $h < n$ . Let also  $T = aC$  be an operator on  $\mathbb{R}^N$  with  $0 < |a| < 1$  and where  $C = \bigoplus_{i=1}^k B_i$ ,  $B_i$  being a Jordan block of modulus 1 and with  $A_i = 1$  or  $R_{\theta_i}$ . Let  $x^1, \dots, x^n \in \mathbb{R}^N$  where  $x^{h+1}, \dots, x^n$  denotes a reduced basis (adapted to  $T$  with Theorem 3.5), and let also  $0 < |a| < |b| \leq 1$ . Assume that there exists  $(n_i)_{i \in \mathbb{N}}$  strictly increasing, real sequences  $(\lambda_j^{n_i})_{i \in \mathbb{N}, 1 \leq j \leq n}$  and  $q \in \mathbb{N}$  such that:  $\frac{b^{n_i} \lambda_j^{n_i}}{n_i^q} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq h$  and  $T^{n_i}(\sum_{j=1}^n \lambda_j^{n_i} x^j) \xrightarrow{i \rightarrow +\infty} 0$ . Then, there is  $q' \in \mathbb{N}$  satisfying  $\frac{a^{n_i} \lambda_j^{n_i}}{n_i^{q'}} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq n$ .*

*Proof.* Denote as usual  $d_i r_i$  the size of the block  $B_i$  with  $d_i = 1, 2$  and set  $r = \sum_{i=1}^k r_i$ . Then,

$$T^{n_i} \left( \sum_{j=1}^n \lambda_j^{n_i} x^j \right) = \begin{cases} a^{n_i} A_1^{n_i} \left( \sum_{j=1}^n \lambda_j^{n_i} X_1^j + \sum_{j=1}^{r_1-1} \binom{n_i}{j} \sum_{g=1}^{h+K_r-j-1} \lambda_g^{n_i} X_{1+j}^g \right) & (L_1) \\ \vdots & \vdots \\ a^{n_i} A_k^{n_i} \sum_{j=1}^{h+K_1-1} \lambda_j^{n_i} X_r^j & (L_r) \end{cases}$$

It is worth noting that only the  $(n - h)$  last vectors are reduced thus we are going to make a constant difference between the  $h$  first vectors and the  $(n - h)$  last vectors and one has to keep in mind that the common notations for a reduced basis only refers to a reduction on the last vectors.

Let us prove the lemma by decreasing induction on  $l \in \{1, \dots, r\}$ . Our induction hypothesis is: there exists  $q' \in \mathbb{N}$  such that for any  $1 \leq j \leq h + K_{r+1-l} - 1$ ,  $\frac{a^{n_i} \lambda_j^{n_i}}{n_i^{q'}} \xrightarrow{i \rightarrow +\infty} 0$ .

We begin by proving the induction hypothesis with  $l = r$ . Observing that  $A_k$  is an isometry, then  $(L_r)$  gives:

$$(8) \quad a^{n_i} \left( \sum_{j=1}^h \lambda_j^{n_i} X_r^j + \sum_{j=h+K_0}^{h+K_1-1} \lambda_j^{n_i} X_r^j \right) \xrightarrow{i \rightarrow +\infty} 0.$$

According to the assumptions, for any  $1 \leq j \leq h$ ,  $\frac{b^{n_i} \lambda_j^{n_i}}{n_i^q} \xrightarrow{i \rightarrow +\infty} 0$ . Recall that  $0 < |a| < |b| \leq 1$ , we deduce  $a^{n_i} \lambda_j^{n_i} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq h$ . Substitute this result into (8):

$$a^{n_i} \sum_{j=h+K_0}^{h+K_1-1} \lambda_j^{n_i} X_r^j \xrightarrow{i \rightarrow +\infty} 0.$$

Moreover,  $\{X_r^j\}_{j=h+K_0}^{h+K_1-1}$  is linearly independent or empty so for every  $1 \leq j \leq h + K_1 - 1$ ,  $a^{n_i} \lambda_j^{n_i} \xrightarrow{i \rightarrow +\infty} 0$ .

Assume that  $l \in \{1, \dots, r\}$  and that the induction hypothesis is true for natural numbers greater than  $l$  and smaller than  $r$ . By induction hypothesis, there exists  $q' \in \mathbb{N}$  so that:

$$\frac{a^{n_i} \lambda_j^{n_i}}{n_i^{q'}} \xrightarrow{i \rightarrow +\infty} 0 \text{ for any } 1 \leq j \leq h + K_{r-l} - 1.$$

If  $K_{r-l} = K_{r+1-l}$ , then the same  $q'$  answers.

If  $K_{r-l} < K_{r+1-l}$ , then  $(L_l)$  gives:

$$a^{n_i} A_f^{n_i} \left( \sum_{j=1}^{h+K_{r-l}-1} \lambda_j^{n_i} X_l^j + \sum_{j=h+K_{r-l}}^{h+K_{r+1-l}-1} \lambda_j^{n_i} X_l^j + \sum_{j=1}^e \binom{n_i}{j} \sum_{g=1}^{h+K_{r+1-l}-j-1} \lambda_g^{n_i} X_{l+j}^g \right) \xrightarrow{i \rightarrow +\infty} 0.$$

with  $f \in \llbracket 1, k \rrbracket$  and  $e \in \llbracket 0, r_f - 1 \rrbracket$ . Note that  $A_f$  is an isometry and the second sum from the preceding equation tends to 0 by induction hypothesis:

$$\frac{1}{n_i^{q'}} a^{n_i} \left( \sum_{j=h+K_{r-l}}^{h+K_{r+1-l}-1} \lambda_j^{n_i} X_l^j + \sum_{j=1}^e \binom{n_i}{j} \sum_{g=1}^{h+K_{r+1-l}-j-1} \lambda_g^{n_i} X_{l+j}^g \right) \xrightarrow{i \rightarrow +\infty} 0.$$

Now, combining the boundedness of  $\frac{\binom{n_i}{j}}{n_i^e}$  for any  $1 \leq j \leq e$  and the induction hypothesis for the last sum in the equation above, we have:

$$\frac{1}{n_i^{q'+e}} a^{n_i} \left( \sum_{j=h+K_{r-l}}^{h+K_{r+1-l}-1} \lambda_j^{n_i} X_l^j \right) \xrightarrow{i \rightarrow +\infty} 0$$

Hence, as  $K_{r-l} < K_{r+1-l}$ , then  $\{X_l^j\}_{j=h+K_{r-l}}^{j=h+K_{r+1-l}-1}$  is linearly independent, and therefore:

$$\frac{a^{n_i} \lambda_j^{n_i}}{n_i^{q'+e}} \xrightarrow{i \rightarrow +\infty} 0 \text{ for any } 1 \leq j \leq h + K_{r+1-l} - 1.$$

This ends the induction and also the proof of the lemma because Theorem 3.5 implies  $K_r = n - h + 1$ , thus  $h + K_r - 1 = n$ .  $\square$

**3.7. Sum of Jordan blocks with the same modulus.** This lemma deals with the growth of coefficients as we did before but in the case of Jordan blocks with the same modulus. It actually depends on the size of the two bigger blocks. The proof is close to the one of Lemma 3.13.

**Lemma 3.14.** *Let  $k, n, N \in \mathbb{N}^*$ ,  $k \geq 2$ . Let also  $T = \bigoplus_{i=1}^k B_i$  be an operator on  $\mathbb{R}^N$ , where  $B_i$  is a Jordan block of modulus one and  $A_i = 1$  or  $R_{\theta_i}$ . Moreover, assume that one of the following conditions holds:*

$$(9) \quad r_1 \geq \dots \geq r_k$$

$$(10) \quad r_2 \geq \dots \geq r_k \text{ and } r_1 = r_2 - 1$$

Let  $x^1, \dots, x^n \in \mathbb{R}^N$  denote a reduced basis with Theorem 3.5 for which there exists a strictly increasing sequence of natural numbers  $(n_i)_{i \in \mathbb{N}}$  and real sequences  $(\lambda_j^{n_i})_{i \in \mathbb{N}, 1 \leq j \leq n}$  such that  $T^{n_i}(\sum_{j=1}^n \lambda_j^{n_i} x^j) \xrightarrow{i \rightarrow +\infty} \underbrace{(1, \dots, 1, 0, \dots, 0)}_{r_1 \text{ times}}$ .

Then:

- If (9) holds, then:  $\frac{\lambda_j^{n_i}}{n_i^{r_1}} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq n$ .

- If (10) holds, then:  $\frac{\lambda_j^{n_i}}{n_i^{r_2-1}} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq n$ .



*Proof.* Denote by  $d_i r_i$  the size of the block  $B_i$  with  $d_i = 1$  or  $2$  and  $r = \sum_{i=1}^k r_i$ . Then,

$$T^{n_i} \left( \sum_{j=1}^n \lambda_j^{n_i} x^j \right) = \begin{cases} A_1^{n_i} \left( \sum_{j=1}^n \lambda_j^{n_i} X_1^j + \sum_{j=1}^{r_1-1} \binom{n_i}{j} \sum_{g=1}^{K_{r-j}-1} \lambda_g^{n_i} X_{1+j}^g \right) & (L_1) \\ \vdots & \vdots \\ A_1^{n_i} \left( \sum_{j=1}^{K_{r+1-r_1}-1} \lambda_j^{n_i} X_{r_1}^j \right) & (L_{r_1}) \\ \vdots & \vdots \\ A_k^{n_i} \left( \sum_{j=1}^{K_{r_k}-1} \lambda_j^{n_i} X_{r+1-r_k}^j + \sum_{j=1}^{r_k-1} \binom{n_i}{j} \sum_{g=1}^{K_{r_k-j}-1} \lambda_g^{n_i} X_{r-r_k+1+j}^g \right) & (L_{r-r_k+1}) \\ \vdots & \vdots \\ A_k^{n_i} \sum_{j=1}^{K_1-1} \lambda_j^{n_i} X_r^j & (L_r) \end{cases}$$

becomes

$$T^{n_i} \left( \sum_{j=1}^n \lambda_j^{n_i} x^j \right) = \begin{cases} A_1^{n_i} \left( \sum_{j=1}^n \lambda_j^{n_i} X_1^j + \sum_{j=1}^{r_1-1} \Delta_j(n_i)(L_{1+j}) \right) & (L_1) \\ \vdots & \vdots \\ A_1^{n_i} \left( \sum_{j=1}^{K_{r+1-r_1}-1} \lambda_j^{n_i} X_{r_1}^j \right) & (L_{r_1}) \\ \vdots & \vdots \\ A_k^{n_i} \left( \sum_{j=1}^{K_{r_k}-1} \lambda_j^{n_i} X_{r+1-r_k}^j + \sum_{j=1}^{r_k-1} \Delta_j(n_i) \right) (L_{r-r_k+1+j}) & (L_{r-r_k+1}) \\ \vdots & \vdots \\ A_k^{n_i} \sum_{j=1}^{K_1-1} \lambda_j^{n_i} X_r^j & (L_r) \end{cases}$$

with the help of Lemma 3.10 and Lemma 3.11. Moreover, remark that every  $l \in \{1, \dots, r\}$  can be written  $l = r - \sum_{j=k-j+1}^k r_i - e$  with  $j \in \llbracket 0, k-1 \rrbracket$  and  $e \in \llbracket 0, r_{k-j}-1 \rrbracket$  in a unique way.

Thus for every  $l \in \{1, \dots, r\}$ ,  $D_l = \begin{cases} e+1 & \text{if } j=0 \\ \max(e+1, r_{k-j+1}) & \text{if } j \neq 0 \end{cases}$  is well-defined. Roughly speaking  $D_l$  is the size of the bigger Jordan block of  $T$  under the  $l$ -th line ( we need to precise that a line has to be understood as a normal line in a classical Jordan block but as two lines for a real Jordan block) of  $T$ . In addition, denote  $E_l = \begin{cases} D_l & \text{if (9) holds} \\ D_l - 1 & \text{if (10) holds} \end{cases}$ .

We are going to prove that for any  $1 \leq j \leq K_{r+1-l} - 1$ ,  $\frac{\lambda_j^{n_i}}{n_i^{E_l}} \xrightarrow{i \rightarrow +\infty} 0$  by decreasing induction on  $l \in \{1, \dots, r\}$ .

Consider first that  $l = r$ ,  $A_k$  being an isometry,  $(L_r)$  gives:

$$(11) \quad \left( \sum_{j=1}^{K_1-1} \lambda_j^{n_i} X_r^j \right) \xrightarrow{i \rightarrow +\infty} 0.$$

And by independence of the family  $\{X_r^j\}_{K_0}^{K_1-1}$ , we conclude that for any  $1 \leq j \leq K_1 - 1$ ,  $\lambda_j^{n_i} \xrightarrow{i \rightarrow +\infty} 0$ .

Now we assume that the induction hypothesis is satisfied for  $l+1, \dots, r$  and we prove it for  $l$ . The induction hypothesis yields:

$$\frac{\lambda_j^{n_i}}{n_i^{E_{l+1}}} \xrightarrow{i \rightarrow +\infty} 0, \text{ for any } 1 \leq j \leq K_{r-l} - 1.$$

If  $K_{r-l} = K_{r+1-l}$ , then the same  $E_{l+1}$  gives the result and therefore  $E_l$  gives it too because  $E_{l+1} \leq E_l$ .

If  $K_{r-l} < K_{r+1-l}$ , then remember  $(L_l)$ :

$$\left\| A_f^{n_i} \left( \sum_{j=1}^{K_{r-l}-1} \lambda_j^{n_i} X_l^j + \sum_{j=K_{r-l}}^{K_{r+1-l}-1} \lambda_j^{n_i} X_l^j + \sum_{j=1}^e \Delta_j(n_i) L_{l+j} \right) \right\| \xrightarrow{i \rightarrow +\infty} 0 \text{ or } 1$$

where  $f \in \llbracket 1, k \rrbracket$  and  $e \in \llbracket 0, r_f - 1 \rrbracket$ . Keep in mind that  $A_f$  is an isometry and divide the preceding line by  $n_i^{E_{l+1}}$ , then by induction the first sum tends to zero and we get:

$$\frac{1}{n_i^{E_{l+1}}} \left( \sum_{j=K_{r-l}}^{K_{r+1-l}-1} \lambda_j^{n_i} X_l^j + \sum_{j=1}^e \Delta_j(n_i) (L_{l+j}) \right) \xrightarrow{i \rightarrow +\infty} 0.$$

It suffices then to use Lemma 3.10 and to compare  $E_{l+1}$  and  $e$  to obtain:

- if (9) holds:

$$\begin{cases} \frac{1}{n_i^{E_{l+1}}} \left( \sum_{j=K_{r-l}}^{K_{r+1-l}-1} \lambda_j^{n_i} X_l^j \right) \xrightarrow{i \rightarrow +\infty} 0 & \text{if } e < E_{l+1} \text{ i.e. } D_l = D_{l+1}. \\ \frac{1}{n_i^{E_{l+1}}} \left( \sum_{j=K_{r-l}}^{K_{r+1-l}-1} \lambda_j^{n_i} X_l^j \right) + \frac{(-1)^{e+1}}{e!} (L_{l+e}) \xrightarrow{i \rightarrow +\infty} 0 & \text{if } e = E_{l+1} \text{ i.e. } D_l = D_{l+1} + 1. \end{cases}$$

Because of  $K_{r-l} < K_{r+1-l}$ , the family  $\{X_l^j\}_{K_{r-l}}^{K_{r+1-l}-1}$  is linearly independent and we deduce that for any  $1 \leq j \leq K_{r+1-l} - 1$ ,

$$\frac{\lambda_j^{n_i}}{n_i^{D_l}} \xrightarrow{i \rightarrow +\infty} 0.$$

- if (10) holds:

$$\begin{cases} \frac{1}{n_i^{E_{l+1}}} \left( \sum_{j=K_{r-l}}^{K_{r+1-l}-1} \lambda_j^{n_i} X_l^j \right) \xrightarrow{i \rightarrow +\infty} 0 & \text{if } e < E_{l+1} \text{ hence } D_l = D_{l+1}. \\ \frac{1}{n_i^{E_{l+1}}} \left( \sum_{j=K_{r-l}}^{K_{r+1-l}-1} \lambda_j^{n_i} X_l^j \right) + \frac{(-1)^{e+1}}{e!} (L_{l+e}) \xrightarrow{i \rightarrow +\infty} 0 & \text{if } e = E_{l+1} \text{ hence } D_l = D_{l+1}. \\ \frac{1}{n_i^{E_{l+1}+1}} \left( \sum_{j=K_{r-l}}^{K_{r+1-l}-1} \lambda_j^{n_i} X_l^j \right) + \frac{(-1)^{e+1}}{e!} (L_{l+e}) \xrightarrow{i \rightarrow +\infty} 0 & \text{if } e = E_{l+1} + 1, \text{ i.e. } D_l = D_{l+1} + 1. \end{cases}$$

However, in the second and third cases just above, we are not working on the block  $B_1$  yet because the condition  $r_1 = r_2 - 1$  is not compatible with  $e = D_{l+1} - 1$  or  $e = D_{l+1}$  given by the second and third cases. Thus,  $(L_{l+e}) \xrightarrow{i \rightarrow +\infty} 0$ . Moreover,  $K_{r-l} < K_{r+1-l}$  yields to the independence of the family  $\{X_l^j\}_{K_{r-l}}^{K_{r+1-l}-1}$ . Hence, we deduce that for any  $1 \leq j \leq K_{r+1-l} - 1$ ,

$$\frac{\lambda_j^{n_i}}{n_i^{D_l-1}} \xrightarrow{i \rightarrow +\infty} 0.$$

This ends the induction process.

Then, by Theorem 3.5  $K_r - 1 = n$  and this proves the lemma.  $\square$

From this lemma, we deduce a general result for operators which are given as a direct sum of Jordan blocks with the same moduli.

**Theorem 3.15.** *Let  $T$  be an operator on  $\mathbb{R}^N$  which is a direct sum of Jordan blocs of modulus 1. Then  $T$  is not  $(r - 1)$ -supercyclic.*

*Proof.* Let us define the natural number  $D := \sum_{i=1}^k (r_i - 1)$  as the degree of  $T$ .

The proof is done by induction on the degree  $D$  of  $T$ . If  $D = 0$ , then  $T$  is a primary matrix of order  $k = r$  and Proposition 3.9 claims that  $T$  is not  $(k - 1)$ -supercyclic.

Assume that the induction hypothesis is true from 0 to  $D - 1$ , let us prove it for  $D$ .

Suppose in order to obtain a contradiction that  $T$  is  $(r - 1)$ -supercyclic. Without loss of generality, we can assume that  $T = \bigoplus_{i=1}^k B_i$ , where  $B_i$  is a Jordan block of modulus one and  $r_1 \geq \dots \geq r_k$ .  $T$  is clearly not  $(r - 1)$ -supercyclic when  $T$  contains only one block due to either

Proposition 3.12 if the block is real or [3] if it is a classical Jordan block. Otherwise, one can write:

$$T = \begin{pmatrix} A_1 & A_1 & \cdots & 0 \\ 0 & C_1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & B_k \end{pmatrix}$$

with  $B_1 = \begin{pmatrix} A_1 & A_1 \\ 0 & C_1 \end{pmatrix}$ . We may warn the reader that the preceding matrix notation is not proper in the sense that  $A_1$  and  $C_1$  may not be of same size. Denote also by  $S$  the diagonal block matrix being the direct sum of  $C_1, B_2, \dots, B_k$ . Denote  $L = \text{Span}(x^1, \dots, x^{r-1})$  a  $(r-1)$ -supercyclic subspace for  $T$  where the basis is reduced with Theorem 3.5.

According to the induction hypothesis,  $S$  is not  $(r-2)$ -supercyclic, and thus there exists  $p < r$  such that  $K_p = r$ .

Indeed, suppose by way of contradiction that  $K_{r-1} < r$ , this means that  $X_j^{r-1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  for every  $2 \leq j \leq r$ . Thus,  $d : \dim(\text{Span}((X_j^1)_{2 \leq j \leq r}, \dots, (X_j^{r-1})_{2 \leq j \leq r})) \leq r-2$ . Hence, as  $L$  is  $(r-1)$ -supercyclic for  $T$ , then  $S$  is  $d$ -supercyclic. This contradicts the fact that  $S$  is not  $(r-2)$ -supercyclic.

Therefore, there exists a strictly increasing sequence of natural numbers  $(n_i)_{i \in \mathbb{N}}$  and real sequences  $(\lambda_j^{n_i})_{i \in \mathbb{N}, 1 \leq j \leq r-1}$  such that:  $T^{n_i} \left( \sum_{j=1}^{r-1} \lambda_j^{n_i} x^j \right) \xrightarrow{i \rightarrow +\infty} \underbrace{(1, \dots, 1)}_{r_1 \text{ times}}, 0, \dots, 0$ . Then, there are

two options: either  $r_1 - 1 \geq r_2$  or  $r_1 = r_2$ . Moreover, in both cases, Lemma 3.14 applied to  $T = S$  leads to  $\frac{\lambda_j^{n_i}}{n_i^{r_1-1}} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq n$  because there exists  $p < r$  such that  $K_p = r$ .

Applying Lemma 3.11 to the first line of  $T^{n_i} \left( \sum_{j=1}^{r-1} \lambda_j^{n_i} x^j \right)$  and taking the limit provides:

$$\left\| A_1^{n_i} \sum_{j=1}^{r-1} \lambda_j^{n_i} X_1^j + \sum_{j=1}^{r_1-1} \Delta_j(n_i)(L_{1+j}) \right\| \xrightarrow{i \rightarrow +\infty} 1.$$

Now, divide this by  $n_i^{r_1-1}$  and recall that  $A_1$  is an isometry:

$$\sum_{j=1}^{r-1} \frac{\lambda_j^{n_i}}{n_i^{r_1-1}} X_1^j + \sum_{j=1}^{r_1-1} \frac{\Delta_j(n_i)}{n_i^{r_1-1}} (L_{1+j}) \xrightarrow{i \rightarrow +\infty} 0.$$

Using both  $\frac{\lambda_j^{n_i}}{n_i^{r_1-1}} \xrightarrow{i \rightarrow +\infty} 0$  and Lemma 3.10, one may observe that all terms above are tending to 0 apart from the term in  $j = r_1 - 1$  in the last sum:

$$\frac{(-1)^{r_1}}{(r_1 - 1)!} \xrightarrow{i \rightarrow +\infty} 0.$$

This last result is obviously contradictory, thus  $T$  is not  $(r-1)$ -supercyclic and this achieves the induction process.  $\square$

**3.8. General matrix.** The next theorem reduces the study of  $k$ -supercyclic operators on  $\mathbb{R}^n$  to that of operators which are direct sums of Jordan blocks of modulus one.

**Theorem 3.16.** *Let  $T$  be such that  $T = \oplus_{i=1}^k a_i C_i$  where  $|a_1| < \dots < |a_k| \leq 1$ , and  $C_i$  is a direct sum of Jordan blocks of modulus one for any  $1 \leq i \leq k$ . Assume that for any  $1 \leq i \leq k$ ,  $C_i$  is  $s_i$ -supercyclic and that  $s_i$  is optimal. Then,  $T$  is not  $\left( \left( \sum_{i=1}^k s_i \right) - 1 \right)$ -supercyclic.*

*Proof.* Let  $T_p := T = \oplus_{i=k+1-p}^k a_i C_i$  and  $t(p)$  denotes this matrix's size. We may prove by induction that for any  $1 \leq p \leq k$ ,  $T_k$  is not  $\left( \left( \sum_{i=k+1-p}^k s_i \right) - 1 \right)$ -supercyclic. Actually, we prove

a little bit more:

(12)

For any  $1 \leq p \leq k$ ,  $T_p$  is not  $((\sum_{i=k+1-p}^k s_i) - 1)$ -supercyclic.

Moreover, for any  $b$ -supercyclic subspace with reduced basis  $L = \text{Span}(x^1, \dots, x^b)$

and with  $\sum_{i=k+1-p}^k s_i \leq b \leq t(p)$ , if  $T_p^{n_i} \left( \sum_{j=1}^b \lambda_j^{n_i} x^j \right) \xrightarrow{i \rightarrow +\infty} 0$ , then there exists  $q \in \mathbb{N}$  so that:

$$\frac{a_{k+1-p}^{n_i} \lambda_j^{n_i}}{n_i^q} \xrightarrow{i \rightarrow +\infty} 0 \text{ for any } 1 \leq j \leq b.$$

Assume that  $p = 1$ , then  $T_p = a_k C_k$  and by definition,  $T_p$  is  $s_k$ -supercyclic and  $s_k$  is the minimum supercyclic constant. Let  $L = \text{Span}(x^1, \dots, x^b)$  be a  $b$ -supercyclic subspace with reduced basis and  $s_k \leq b \leq t(1)$ . Let also,  $T_p^{n_i} \left( \sum_{j=1}^b \lambda_j^{n_i} x^j \right) \xrightarrow{i \rightarrow +\infty} 0$ . Applying Lemma 3.13 with  $h = 0$ ,

$C = C_k$ ,  $a = a_k$ ,  $n = b$ ,  $N = t(p)$ , we see that there exists  $q \in \mathbb{N}$  such that  $\frac{a_k^{n_i} \lambda_j^{n_i}}{n_i^q} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq b$ .

Assume the induction hypothesis true for integers lower than  $p$ , and let us prove it for  $p$ . Write  $T_p = a_{k+1-p} C_{k+1-p} \oplus T_{p-1}$  and  $C_{k+1-p} = \bigoplus_{i=1}^t B_i$  where  $B_i$  is a Jordan block of modulus one and of size  $d_i r_i$  with  $d_i = 1$  or  $2$  and define  $r := \sum_{i=1}^t r_i$ .

In order to obtain a contradiction, assume that  $N = (\sum_{i=k+1-p}^k s_i) - 1$  and let  $L = \text{Span}(x^1, \dots, x^N)$  be a  $N$ -supercyclic subspace with reduced basis by Theorem 3.5. Then for any  $1 \leq i \leq N$ , decompose  $x_i = y_i \oplus z_i$  relatively to the direct sum decomposition of  $T_p$  stated above. A straightforward use of the induction hypothesis provides  $h := \dim(\text{Span}(z^1, \dots, z^N)) \geq \sum_{i=k+2-p}^k s_i$ . Furthermore, we can show that  $\dim(\text{Span}(y^{h+1}, \dots, y^N)) \geq s_{k+1-p}$ , indeed it suffices to prove that  $\text{Span}(y^{h+1}, \dots, y^N)$  is supercyclic for  $a_{k+1-p} C_{k+1-p}$ . Thus take any  $u \in \mathbb{R}^r$ , then there exists  $(n_i)_{i \in \mathbb{N}}$  and  $(\lambda_j^{n_i})_{i \in \mathbb{N}, 1 \leq j \leq N}$  so that  $T_p^{n_i} \left( \sum_{j=1}^N \lambda_j^{n_i} x^j \right) \xrightarrow{i \rightarrow +\infty} u \oplus 0$  by  $N$ -supercyclicity of  $T_p$ . Moreover the induction hypothesis implies that for any  $1 \leq j \leq h$ , there exists  $q \in \mathbb{N}$  such that  $\frac{a_{k+1-p}^{n_i} \lambda_j^{n_i}}{n_i^q} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq h$  and since  $a_{k+1-p} < a_{k+2-p}$ , we obtain:

$$(13) \quad a_{k+1-p}^{n_i} \lambda_j^{n_i} P(n_i) \xrightarrow{i \rightarrow +\infty} 0 \text{ for any polynomial } P.$$

Now we come back to  $T_p^{n_i} \left( \sum_{j=1}^N \lambda_j^{n_i} x^j \right) \xrightarrow{i \rightarrow +\infty} u \oplus 0$ , projecting onto the firsts components, and separating the sum, we get:

$$(14) \quad (a_{k+1-p} C_{k+1-p})^{n_i} \left( \sum_{j=1}^h \lambda_j^{n_i} y_j \right) + (a_{k+1-p} C_{k+1-p})^{n_i} \left( \sum_{j=h+1}^N \lambda_j^{n_i} y_j \right) \xrightarrow{i \rightarrow +\infty} u.$$

Then, focus on the first sum of the preceding line:

$$(a_{k+1-p}C_{k+1-p})^{n_i}(\sum_{j=1}^h \lambda_j^{n_i} y_j) = \begin{cases} a_{k+1-p}A_1^{n_i}(\sum_{j=1}^h \lambda_j^{n_i} X_1^j + \sum_{j=1}^{r_1-1} \binom{n_i}{j} \sum_{g=1}^h \lambda_g^{n_i} X_{1+j}^g) & (L_1) \\ \vdots & \vdots \\ a_{k+1-p}A_1^{n_i} \sum_{j=1}^h \lambda_j^{n_i} X_{r_1}^j & (L_{r_1}) \\ \vdots & \vdots \\ a_{k+1-p}A_t^{n_i}(\sum_{j=1}^h \lambda_j^{n_i} X_{m+1-r_t}^j + \sum_{j=1}^{r_t-1} \binom{n_i}{j} \sum_{g=1}^h \lambda_g^{n_i} X_{m+1-r_t+j}^g) & (L_{m+1-r_t}) \\ \vdots & \vdots \\ a_{k+1-p}A_t^{n_i} \sum_{j=1}^h \lambda_j^{n_i} X_m^j & (L_m) \end{cases}$$

It is an easy observation to notice that  $(L_1), \dots, (L_m)$  are linear combinations of polynomials in  $n_i$  multiplied by some  $\lambda_j^{n_i}$ ,  $1 \leq j \leq h$ . Since  $A_j$  is an isometry, (13) shows that for any  $1 \leq j \leq h$ ,  $(L_j) \xrightarrow{i \rightarrow +\infty} 0$ . Hence, the first sum into (14) converges to 0 leading to:

$$(a_{k+1-p}C_{k+1-p})^{n_i}(\sum_{j=h+1}^N \lambda_j^{n_i} y_j) \xrightarrow{i \rightarrow +\infty} u.$$

Thus,  $\text{Span}(y^{h+1}, \dots, y^N)$  is supercyclic for  $a_{k+1-p}C_{k+1-p}$  and  $\dim(\text{Span}(y^{h+1}, \dots, y^N)) \geq s_{k+1-p}$ .

Hence  $h := \dim(\text{Span}(z^1, \dots, z^N)) \geq \sum_{i=k+2-p}^k s_i$  and  $\dim(\text{Span}(y^{h+1}, \dots, y^N)) \geq s_{k+1-p}$ .

Then, as the basis is reduced by Theorem 3.5, we have  $x^j = y^j \oplus 0$  for any  $h+1 \leq j \leq N$ , and so  $N = \dim(\text{Span}(x^1, \dots, x^N)) \geq \sum_{i=k+2-p}^k s_i + s_{k+1-p} = N+1$ . This contradiction proves that  $T_p$  is not  $((\sum_{i=k+1-p}^k s_i) - 1)$ -supercyclic.

Let us focus now on the second part of the induction hypothesis. For this purpose, let  $L = \text{Span}(x^1, \dots, x^b)$  be a  $b$ -supercyclic subspace whose basis is reduced and  $\sum_{i=k+1-p}^k s_i \leq b \leq t(p)$ . Let also  $T_p^{n_i}(\sum_{j=1}^b \lambda_j^{n_i} x^j) \xrightarrow{i \rightarrow +\infty} 0$ . We just have to invoke Lemma 3.13 and the result comes:

there exists  $q \in \mathbb{N}$  such that:  $\frac{a_{k+1-p} \lambda_j^{n_i}}{n_i^q} \xrightarrow{i \rightarrow +\infty} 0$  for any  $1 \leq j \leq b$ .

This achieves the proof of the induction and of the theorem.  $\square$

We are now ready to state global results of supercyclicity for operators on  $\mathbb{R}^n$ . These results follow straightforwardly from Theorem 3.16 above and Theorem 3.15 and generalise Hilden and Wallen's result for supercyclic operators.

**Corollary 3.17.** *Let  $n \geq 2$  and  $T$  be an operator on  $\mathbb{R}^n$ , then  $T$  is not  $(r-1)$  supercyclic.*

*Proof.* Without loss of generality, one may assume that  $T$  is in Jordan form and also upon reordering blocks and considering a multiple of  $T$  instead of  $T$  itself, one may assume that the sequence formed with each Jordan block's modulus satisfies:  $a_1 \leq \dots \leq a_k \leq 1$ . As a consequence, one may realise  $T$  as a direct sum of matrices  $M_1, \dots, M_t$ , where  $M_1$  contains all Jordan blocks with the smallest modulus and so on. Let  $r_{M_j}$  denote the  $r$  from the Jordan decomposition of the matrix  $M_j$ ,  $j = 1, \dots, t$ . First, Theorem 3.15 implies that every matrix  $M_j$  is no less than  $r_{M_j}$ -supercyclic. Then, one shall use Theorem 3.16 to come back to  $T$ , hence  $T$  is no less than  $(\sum_{j=1}^t r_{M_j})$ -supercyclic. Then, one just have to recall the definition of  $r$  and of  $r_{M_j}$  providing  $(\sum_{j=1}^t r_{M_j}) = r$ . This proves the corollary.  $\square$

A direct application of the preceding corollary yields to a more concrete result:

**Corollary 3.18.** *Let  $n \geq 2$ . There is no  $(\lfloor \frac{n+1}{2} \rfloor - 1)$ -supercyclic operator on  $\mathbb{R}^n$ . Moreover, there always exists a  $(\lfloor \frac{n+1}{2} \rfloor)$ -supercyclic operator on  $\mathbb{R}^n$ .*

*Proof.* The first part follows from Corollary 3.17 and the second part is due to Example 3.1.  $\square$

**Question.** Does there exist a theorem similar to Theorem 3.16 in the case of a direct sum of Jordan blocks of modulus one?

**Question.** Does there exist a  $(2n - 2)$ -supercyclic real Jordan block on  $\mathbb{R}^{2n}$ ?  
If so, what is the best supercyclic constant for a real Jordan block on  $\mathbb{R}^{2n}$ ?

#### 4. STRONG $n$ -SUPERCYCLICITY

The aim of this section is to study the existence of strong  $n$ -supercyclicity in finite dimension. Of course, this is interesting only if  $n < \dim(E)$ . Bourdon, Feldman and Shapiro [3] answer this question for the complex case. Indeed, they prove that  $n$ -supercyclicity cannot occur non-trivially in finite complex dimension, and thus strong  $n$ -supercyclicity cannot either. However, in the real setting, we noticed in the previous section that  $n$ -supercyclicity can occur and thus the question for strong  $n$ -supercyclicity is still open. For this purpose we need the following Proposition from [4]. It provides a more concrete definition of strongly  $n$ -supercyclic operators:

**Proposition 4.1.** (Proposition 1.13 [4]) *Let  $X$  be a completely separable Baire vector space. The following are equivalent:*

- (i)  $T$  is strongly  $n$ -supercyclic.
- (ii) There exists a  $n$ -dimensional subspace  $L$  such that for every  $i \in \mathbb{N}$ ,  $T^i(L)$  is  $n$ -dimensional and:  $\mathcal{B} := \cup_{i=1}^{\infty} \pi_n^{-1}(\tilde{T}^i(L))$  is dense in  $X^n$ .
- (iii) There exists a  $n$ -dimensional subspace  $L$  such that for every  $i \in \mathbb{N}$ ,  $T^i(L)$  is  $n$ -dimensional and:  $\mathcal{E} := \cup_{i=1}^{\infty} (T^i(L) \times \cdots \times T^i(L))$  is dense in  $X^n$ .

**Remark 4.2.** Moreover, from the definition of strong  $n$ -supercyclicity or with the previous Proposition, one may observe that if  $T$  is a strongly  $k$ -supercyclic operator on  $\mathbb{R}^n$  with  $k \leq n$ , then  $T$  is bijective,

We turn out to the case of strongly  $n$ -supercyclic operators on a real finite dimensional vector space. Our first result is interesting and provides a partial answer to the question:

**Proposition 4.3.** *Let  $k < n$ . An operator  $T$  on  $\mathbb{R}^n$  is strongly  $k$ -supercyclic if and only if  $(T^{-1})^*$  is also strongly  $(n - k)$ -supercyclic and the strong  $k$ -supercyclic subspaces for  $T$  are orthogonal to  $(T^{-1})^*$ 's strongly  $(n - k)$ -supercyclic subspaces.*

This duality property is very useful if one combines it with Corollary 3.18, by the way we get:

**Corollary 4.4.** *There is no strongly  $k$ -supercyclic operators on  $\mathbb{R}^{2n+1}$  for any  $n \geq 1$  and any  $1 \leq k < n$ .*

*There is no strongly  $k$ -supercyclic operators on  $\mathbb{R}^{2n}$  for any  $n \geq 2$  and any  $1 \leq k < n$ ,  $k \neq \frac{n}{2}$ .*

This corollary provides many examples of  $k$ -supercyclic operators that are not strongly  $k$ -supercyclic and thus answers the question of Shkarin [12] that  $n$ -supercyclicity and strong  $n$ -supercyclicity are not equivalent.

**Counterexample 4.5.** A 2-supercyclic operator that is not strongly 2-supercyclic  
Any rotation on  $\mathbb{R}^3$  around any one-dimensional subspace and with angle linearly independent with  $\pi$  on  $\mathbb{Q}$  is 2-supercyclic but not strongly 2-supercyclic.

We are now turning to the proof of Proposition 4.3. We need the two following well-known lemmas:

**Lemma 4.6.** *Let  $L$  be a subspace of  $\mathbb{R}^n$  and  $T$  be an automorphism on  $\mathbb{R}^n$ . Then, for any  $i \in \mathbb{N}$ ,  $(T^i(L))^\perp = (T^{-i})^*(L^\perp)$ .*

**Lemma 4.7.** *Let  $\Phi : \mathbb{P}_k(\mathbb{R}^n) \rightarrow \mathbb{P}_{n-k}(\mathbb{R}^n)$  be defined such that  $\Phi(F) = F^\perp$ . Then  $\Phi$  is a homeomorphism.*

The first lemma is a classical one and the second may be found and proven in [10]. Then, the proof of the proposition is straightforward:

**Proof of proposition 4.3.** The combination of Lemma 4.6 and Lemma 4.7, when  $L$  is a strongly  $k$ -supercyclic subspace for  $T$ , implies that  $\Phi(\{T^i(L)\}_{i \in \mathbb{N}}) = \{(T^{-i})^*(L^\perp)\}_{i \in \mathbb{N}}$  is dense in  $\mathbb{P}_{n-k}(\mathbb{R}^n)$ .

**4.1. Strongly 2-supercyclic operators on  $\mathbb{R}^4$ .** The idea is to prove by induction that there is no strongly  $k$ -supercyclic operator on  $\mathbb{R}^n$  for  $n \geq 3$  and  $1 \leq k < n$ . The first step is to prove this for  $n$  small. It is already done for  $n = 3$ , we now focus on the case  $n = 4$ . To this purpose, we begin with characterizing 2-supercyclic subspaces for a direct sum of rotations and we prove then that none are strongly 2-supercyclic.

**Proposition 4.8.** *Characterisation of 2-supercyclic subspaces for a direct sum of two rotations*  
Let  $R$  be a direct sum of two rotations:

$$R = \left( \begin{array}{cc|cc} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ \hline 0 & 0 & \cos(\theta_2) & -\sin(\theta_2) \\ 0 & 0 & \sin(\theta_2) & \cos(\theta_2) \end{array} \right) \text{ with } (\theta_1, \theta_2, \pi) \text{ linearly independent over } \mathbb{Q}.$$

Then:

$$\mathcal{ES}_2(R) = \left\{ \text{Span} \left( \left( \begin{array}{c} x_1 \\ x_2 \\ ay_1 \\ ay_2 \end{array} \right), \left( \begin{array}{c} bx_1 \\ bx_2 \\ y_1 \\ y_2 \end{array} \right) \right) \right\},$$

$$\text{where } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\} \text{ and } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \text{ with } ab \neq -1.$$

*Proof.* We proceed by double inclusion.

$$\bullet \supseteq \text{“: Let } x = \begin{pmatrix} x_1 \\ x_2 \\ ay_1 \\ ay_2 \end{pmatrix} \text{ and } y = \begin{pmatrix} bx_1 \\ bx_2 \\ y_1 \\ y_2 \end{pmatrix} \text{ satisfying the above hypothesis.}$$

Then,  $x - ay$  is non-zero but his third and fourth components are zeros and  $y - bx$  is also non-zero but his two first components are null.

Let  $U \times V$  be an open set in  $\mathbb{R}^4$ , with  $U, V$  non-empty open sets in  $\mathbb{R}^2$ . As  $\theta_1, \theta_2, \pi$  are linearly independent over  $\mathbb{Q}$ , there exists  $n \in \mathbb{N}$  and  $c_1, c_2 \in \mathbb{R}$  such that

$$c_1 R_{\theta_1}^n(x - ay) \in U \text{ and } c_2 R_{\theta_2}^n(-bx + y) \in V.$$

Thus  $R^n(c_1(x - ay) + c_2(-bx + y)) \in U \times V$ , hence  $\text{Span}(x, y)$  is a 2-supercyclic subspace of  $R$ .

$\bullet \subseteq \text{“: Now, let us suppose that there exists a two dimensional subspace } L = \text{Span}(x, y)$  which is 2-supercyclic for  $R$  and that is not satisfying the conditions stated in the proposition.

Hence, one may suppose that the family  $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\}$  is linearly independent. Set  $\mathcal{C}_1 := \{z \in \mathbb{R}^2 : \|z\| < 1\}$  and define also  $\mathcal{C}_2 := \{z \in \mathbb{R}^2 : \|z\| > M\}$  where  $M$  is a positive real number we may choose later. Then, as  $L$  is a 2-supercyclic subspace for  $R$ ,  $\{R^i(\lambda x + \mu y)\}_{i \in \mathbb{N}, (\lambda, \mu) \in \mathbb{R}^2}$  is dense in  $\mathbb{R}^4$ . As a result, there exists  $i \in \mathbb{N}$  and  $(\lambda, \mu) \in \mathbb{R}^2$  such that

$$R^i(\lambda x + \mu y) \in \mathcal{C}_1 \times \mathcal{C}_2 \Leftrightarrow \begin{pmatrix} \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ \lambda \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \mu \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \end{pmatrix} \in R_{\theta_1}^{-i}(\mathcal{C}_1) \times R_{\theta_2}^{-i}(\mathcal{C}_2) = \mathcal{C}_1 \times \mathcal{C}_2$$

Then, define

$$\Gamma_1 := \left\{ (\lambda, \mu) \in \mathbb{R}^2 : \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{C}_1 \right\}$$

Since  $\mathcal{C}_1$  is bounded and the family  $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\}$  is linearly independent, one may deduce that  $\Gamma_1$  is bounded too. Note that  $\Omega := \left\{ \lambda \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \mu \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}, (\lambda, \mu) \in \Gamma_1 \right\}$  is also bounded because of the boundedness of  $\Gamma_1$ . We now define  $M$  being an upper bound for  $\Omega$ , then one deduces that  $\mathcal{C}_2 \cap \Omega = \emptyset$ . This contradicts the fact that  $L$  is a 2-supercyclic subspace for  $R$ . So  $L$  satisfies the proposition's conditions. □

**Corollary 4.9.**  *$R$  is not strongly 2-supercyclic on  $\mathbb{R}^4$ .*

*Proof.* Assume that  $R$  is strongly 2-supercyclic on  $\mathbb{R}^4$ , then any strongly 2-supercyclic subspace is given by Proposition 4.8. Thus if  $x, y \in \mathbb{R}^4$  span a strongly 2-supercyclic subspace for  $R$ , then

$$R^i(\lambda x + \mu y) = \begin{pmatrix} (\lambda + b\mu)R_{\theta_1}^i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ (\lambda a + \mu)R_{\theta_2}^i \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{pmatrix}.$$

Moreover, according to Proposition 4.1 this means that for any two non-empty open sets  $U_1, U_2 \subset \mathbb{R}^2$ , there exists  $i \in \mathbb{N}$  and  $\lambda, \mu, \alpha, \beta \in \mathbb{R}$  such that:

$$(\lambda + b\mu)R_{\theta_1}^i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in U_1 \qquad (\alpha + b\beta)R_{\theta_1}^i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in U_2$$

But this cannot happen if we choose  $U_1$  and  $U_2$  such that there does not exist a line intersecting both  $U_1$  and  $U_2$  and 0. Therefore,  $R$  is not strongly 2-supercyclic. □

We are now going to deal with two other different-shaped operators on  $\mathbb{R}^4$ .

**Proposition 4.10.**  *$R = \left( \begin{array}{c|c} A & A \\ \hline 0 & A \end{array} \right)$  and  $R = \left( \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right)$  with  $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$  and  $B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ ,  $(a, b, c, d) \in \mathbb{R}^4$  are not strongly 2-supercyclic.*

*Proof.* Without loss of generality, one may assume  $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} c \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  because strongly 2-supercyclic operators have dense range.

▷ If  $R = \left( \begin{array}{c|c} A & A \\ \hline 0 & A \end{array} \right)$ , then  $R$  is not strongly 2-supercyclic according to Proposition 3.2 .

▷ If  $R = \left( \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right)$ , we assume, in order to obtain a contradiction, that  $R$  is strongly 2-supercyclic. Upon considering a scalar multiple, relabelling and rearranging blocks, one can suppose:  $R = \left( \begin{array}{c|c} R_\theta & 0 \\ \hline 0 & C \end{array} \right)$  with  $C = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$  and  $c^2 + d^2 \leq 1$ . Let  $L = \text{Span}\{x, y\}$  be a strongly 2-supercyclic subspace for  $R$ .

- If  $c^2 + d^2 = 1$ , then Corollary 4.9 implies that  $R$  is not strongly 2-supercyclic.



• If  $c^2 + d^2 < 1$ ,  $L$  being strongly 2-supercyclic for  $R$  and using Proposition 4.1, for any non-empty open sets  $U_1, U_2, V_1, V_2$  in  $\mathbb{R}^2$ , there exists  $i \in \mathbb{N}$  and  $(\lambda, \mu, \alpha, \beta) \in \mathbb{R}^4$  such that:

$$\begin{cases} R^i(\lambda x + \mu y) \in U_1 \times U_2 \\ R^i(\alpha x + \beta y) \in V_1 \times V_2 \end{cases} \Leftrightarrow \begin{cases} \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in R_\theta^{-i}(U_1) \\ \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in R_\theta^{-i}(V_1) \\ \lambda \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \mu \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \in C^{-i}(U_2) \\ \alpha \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \beta \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \in C^{-i}(V_2) \end{cases}$$

From this last identity, we deduce that the family  $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\}$  is linearly independent.

Indeed, if not, then we choose  $U_1$  and  $V_1$  such that there does not exist a line intersecting both  $U_1, V_1$  and  $0$  to obtain a contradiction with the previous identity.

The family being linearly independent, let  $U_1 = \{z \in \mathbb{R}^2 : \|z\| < 1\}$  and  $U_2 = \{z \in \mathbb{R}^2 : \|z\| > M\}$ . Set also

$$\Gamma = \left\{ (\lambda, \mu) \in \mathbb{R}^2 : \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in U_1 \right\}.$$

Since the family  $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\}$  is linearly independent and  $U_1$  is bounded, one may deduce

that  $\Gamma$  is bounded in  $\mathbb{R}^2$  and then  $\Omega = \left\{ \lambda \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \mu \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}, (\lambda, \mu) \in \Gamma \right\}$  is obviously bounded too and we define  $M$  being an upper bound for  $\Omega$ .

On the other hand, we have that  $C^{-1} = \frac{1}{c^2+d^2} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \frac{1}{\sqrt{c^2+d^2}} R_\varphi$ , thus  $C^{-i}(U_2) \subseteq U_2$  for any  $i \in \mathbb{N}$ . Hence,  $\Omega \cap \bigcup_{i \in \mathbb{N}} C^{-i}(U_2) = \emptyset$ , contradicting the strong 2-supercyclicity of  $R$ .  $\square$

**4.2. General result.** We are going to prove that there is no non-trivial strongly  $k$ -supercyclic operator by induction on the space dimension. The following proposition inspired from Bourdon, Feldman, Shapiro [3] is useful for the induction step:

**Proposition 4.11.** *Let  $X$  be a Hausdorff topological vector space. Let also  $T : X \rightarrow X$  be a bounded operator and  $K$  a closed invariant subspace for  $T$ . If  $T$  is strongly  $k$ -supercyclic with supercyclic subspace  $L$ , then the quotient map  $T_K : \frac{X}{K} \rightarrow \frac{X}{K}$  has supercyclic subspace  $\frac{L}{K}$  and thus is strongly  $m$ -supercyclic where  $m = \dim(\frac{L}{K})$ .*

The proof is just a verification using the characterisation of strongly  $n$ -supercyclic given in Proposition 4.1.

**Theorem 4.12.** *For any  $n \geq 3$ , there is no strongly  $k$ -supercyclic operator on  $\mathbb{R}^n$  for  $1 \leq k < n$ .*

*Proof.* We give a proof by induction on the space dimension. First, according to Corollary 4.4 and Hilden and Wallen's result [8], there is no strongly  $k$ -supercyclic operator on  $\mathbb{R}^3$  with  $k = 1, 2$ .

So, let  $n \geq 4$ , and assume that for all  $m < n$ , there is no strongly  $k$ -supercyclic operator on  $\mathbb{R}^m$  with  $k < m$ . Let  $R$  be an operator on  $\mathbb{R}^n$ , using the Jordan real decomposition, one may

suppose:

$$R = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \lambda_p & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & \boxed{J_1} & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & \ddots & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & \boxed{J_q} & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & \boxed{\mathcal{A}_1} & 0 & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\mathcal{A}_r} \end{pmatrix}$$

where  $J_i$  is a classical Jordan block:

$$J_i = \begin{pmatrix} \mu_i & 1 & 0 & 0 \\ 0 & \ddots & \ddots & \dots \\ \vdots & 0 & \ddots & 1 \\ 0 & \dots & 0 & \mu_i \end{pmatrix}$$

and  $\mathcal{A}_i$  is a real Jordan block:

$$\mathcal{A}_i = \begin{pmatrix} \overline{A_i} & | & A_i & | & 0 & 0 \\ 0 & \ddots & \ddots & \dots & & \\ \vdots & 0 & \ddots & \overline{A_i} & & \\ 0 & \dots & 0 & A_i & & \end{pmatrix}.$$

If  $n = 4$ , then according to Proposition 4.3, it suffices to prove that  $R$  is not strongly  $k$ -supercyclic with  $k = 2$ .

And if  $n \geq 5$ , it suffices to prove that  $R$  is not strongly  $k$ -supercyclic for all  $\lfloor \frac{n+1}{2} \rfloor \leq k < n - 2$ . Assume to the contrary that there exists such a  $k$  and let  $L$  be a strongly  $k$ -supercyclic subspace for  $R$ .

▷ If either  $p \neq 0$  or  $q \neq 0$ , then  $K = \text{Span}\{(1, 0, \dots, 0)\}$  is  $R$ -invariant. Consider then the quotient  $\frac{\mathbb{R}^n}{K}$ , applying Proposition 4.11, one deduces that  $R_K$  is strongly  $\dim(\frac{L}{K})$ -supercyclic on  $\mathbb{R}^{n-1}$ . In addition,  $1 \leq \dim(\frac{L}{K}) \leq \dim(L) < n - 1$  contradicting the induction hypothesis.

▷ If  $p = q = 0$ , then  $K = \text{Span}\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0)\}$  is  $R$ -invariant.

• If  $n = 4$ , then Corollary 4.9 and Proposition 4.10 implies that  $R$  is not strongly 2-supercyclic.

• If  $n \geq 5$ , one shall consider the quotient by  $K$ , and apply Proposition 4.11 providing that  $R_K$  is strongly  $\dim(\frac{L}{K})$ -supercyclic on  $\mathbb{R}^{n-2}$ . Moreover,  $1 \leq \dim(\frac{L}{K}) \leq \dim(L) < n - 2$  contradicting the induction hypothesis. This proves the induction hypothesis and the theorem.  $\square$

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