



**HAL**  
open science

# Entropy formulation of degenerate parabolic equation with zero-flux boundary condition

Boris Andreianov, Mohamed Karimou Gazibo

► **To cite this version:**

Boris Andreianov, Mohamed Karimou Gazibo. Entropy formulation of degenerate parabolic equation with zero-flux boundary condition. 2012. hal-00697593v1

**HAL Id: hal-00697593**

**<https://hal.science/hal-00697593v1>**

Preprint submitted on 15 May 2012 (v1), last revised 4 Oct 2012 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Entropy formulation of degenerate parabolic equation with zero-flux boundary condition

Boris Andreianov and Mohamed Karimou Gazibo

**Abstract.** We consider the general degenerate hyperbolic-parabolic equation:

$$u_t + \operatorname{div} f(u) - \Delta\phi(u) = 0 \text{ in } Q = (0, T) \times \Omega, \quad T > 0, \quad \Omega \subset \mathbb{R}^N; \quad (\text{E})$$

with initial condition and the zero flux boundary condition. Here  $\phi$  is a continuous non decreasing function. Following [11], we assume that  $f$  is compactly supported (this is the case in several applications) and we define an appropriate notion of entropy solution. Using vanishing viscosity approximation, we prove existence of entropy solution for any space dimension  $N \geq 1$  under a partial genuine nonlinearity assumption on  $f$ . Uniqueness is shown for the case  $N = 1$ , using the idea of [2], nonlinear semigroup theory and a specific regularity result for one dimension.

**Mathematics Subject Classification (2010).** Primary 35F31; Secondary 00A69.

**Keywords.** Degenerate hyperbolic-parabolic equation; Zero-flux boundary condition; Vanishing viscosity approximation; Nonlinear semigroup theory; Entropy solution; Integral solution.

## 1. Introduction

Let  $\Omega$  be a bounded open set of  $\mathbb{R}^N$  with a Lipschitz boundary  $\partial\Omega$  and  $\eta$  the unit normal to  $\partial\Omega$  outward to  $\Omega$ . We consider the zero-flux boundary problem:

$$(P) \begin{cases} u_t + \operatorname{div} f(u) - \Delta\phi(u) = 0 & \text{in } Q = (0, T) \times \Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \\ (f(u) - \nabla\phi(u)) \cdot \eta = 0 & \text{on } \Sigma = (0, T) \times \partial\Omega. \end{cases}$$

We assume that the convection flux  $f$  is a Lipschitz continuous function. Moreover, we require that

$$f(0) = 0, \quad f(u_{\max}) = 0 \text{ for some } u_{\max} > 0. \quad (1.1)$$

Accordingly, the initial datum is a measurable function taking values in the interval  $[0, u_{\max}]$ , which will be the invariant domain for the solutions of  $(P)$  under assumption (1.1). Further, the function  $\phi$  is continuous non decreasing on  $[0, u_{\max}]$ . This assumption means that the problem  $(P)$  is of degenerate parabolic-hyperbolic type. For the sake of simplicity, we will treat the case  $\phi(\cdot)$  is constant on  $[0, u_c]$  with  $0 \leq u_c \leq u_{\max}$  and  $\phi(\cdot)$  is strictly increasing on  $[u_c, u_{\max}]$ . The case of a general  $\phi$  can be treated without additional difficulty (see Carrillo [16]).

The framework (E) includes conservation law as a particular case and it is well known that in general, global classical solutions may not exist; and that weak solution in the sense of distributions may not be unique. The standard way to fix this problem is to work with the so-called entropy solutions (see Kruzhkov [17], for the case of conservation laws, and Carrillo [16], for the adaptation of this notion to the case of degenerate elliptic-parabolic-hyperbolic equation). There exist many papers in the literature dealing with Dirichlet boundary condition for (E). The main reference is the fundamental paper of Carrillo for homogenous Dirichlet boundary condition [16] which establishes

the uniqueness technique. In [23], Rouvre and Gagneux prove also existence and uniqueness for homogeneous Dirichlet condition under strong regularity requirement on the data. The general Dirichlet boundary condition received much attention, see Mascia and al [19], Michel and Vovelle [21], Vallet [24], see also the forthcoming work [3]. However, the Dirichlet boundary condition may not always provide the most natural setting for this kind of problem on bounded domains. Equation (E) occurs in several applications, for example it comes from the theory of porous media flow, phenomenological theory of sedimentation-consolidation processes, road traffic. In practice, it is often supplemented with the zero-flux (homogeneous Neumann boundary condition), at least on a part of the boundary (see [12]).

Let us describe more detail one application. Problem (P) is of interest in describing pressure filtration of flocculated suspensions. The domain  $\Omega$  is a filter medium, which lets only the liquid pass, by a piston which moves downwards due to an applied pressure. The material behavior of the suspension is described by two model functions, the flux density function or hindered settling factor  $f$  and the effective solid stress function  $\phi$ , both functions only of the local solids concentration  $u$ . Here  $f$  is a nonpositive Lipschitz continuous function with compact support in  $[0, u_{\max}]$ , where  $u_{\max} \leq 1$  is the maximum concentration and the function  $\phi$  satisfies  $\phi(u) = 0$  for  $u \leq u_c$ , with  $0 \leq u_c \leq u_{\max}$ , where  $u_c$  is a critical concentration value, and  $\phi'(u) > 0$  for  $u > u_c$ . Notice that these assumptions are exactly those that we have taken in this paper. According to the phenomenological sedimentation-consolidation theory [12], the evolution of the concentration distribution can be given with Neumann boundary condition, and this is our motivation.

In [11] Bürger and al. consider the problem (P) with  $\phi(u) \equiv 0$ . They introduce a notion of entropy solution based on the existence of strong trace  $u^\tau$  on  $\partial\Omega$  under some assumption on the boundary (see [25]) and the flux  $f$  which satisfied (1.1). They prove existence and uniqueness of entropy solution. The purpose of our paper is to extend the result of Bürger and al ([11]) to degenerate parabolic-hyperbolic equation. Since the total flux in (E) contains the diffusion flux term  $\nabla\phi(u)$  which is only  $L^2$ , we cannot ensure the existence of a strong trace for this term. Therefore, we suppose that this boundary condition is satisfied in the weak sense only. We propose a new entropy formulation that incorporates a boundary term which does not contain any trace of  $u$ . Its main advantage over the definition of [11] is that the stability under the  $L^1(Q)$  convergence of solutions is evident. Notice that we do not need existence of traces of entropy solutions  $u$  of (E), even it could be ensured using the method of [3].

To prove existence of entropy solution, we use a classical vanishing viscosity approximation and get the a priori estimates useful for passing to the limit in the approximate problem. The main point for passing to the limit is based on a rather involved local compactness argument of Panov [22]. We manage to apply this result in our case and prove that the limit of entropy solutions of approximate problem is an entropy solution of (P).

We focus on the question of uniqueness of entropy solution for (P). For this aims, we prove a version of an important proposition due to Carrillo [16]. This proposition identifies the entropy dissipation term which is a key ingredient of the uniqueness technique. Then, it is easy to prove uniqueness of solutions such that boundary condition is satisfied in the sense of strong boundary trace of the normal component of the flux ( $f(u) - \nabla\phi(u)$ ). Unfortunately, we are able to establish this additional solution regularity only for the stationary problem (S) associated to (P) (see section 4) and only in the case of one space dimension. Therefore, we adapt the hint from the paper [2] (see also [4]) and compare a general solution to (P) with a regular solution to (S). We conclude by a standard application of the notion of integral solution coming from the nonlinear semigroup theory [7]. Eventually, we prove the uniqueness result in one space dimension.

Let us stress that the problem of uniqueness is still open in multiple space dimensions. The definition of strong traces of the solution with respect to the lateral boundary of the domain  $\Omega$  is possible if for example the diffusion term  $\phi(u)$  is such that  $f(u) - \nabla\phi(u)$  is continuous up to the boundary  $\partial\Omega$ . If there existed “sufficiently many” solutions (in the sense of [2], [4]) having this regularity, uniqueness would follow. We leave the investigation of this regularity question to a future work. Another open question is how to define entropy solutions in the case where the assumption (1.1)

does not hold. Indeed, as in [11], the assumption (1.1) ensures that the zero-flux boundary condition is satisfied literally. When this assumption is dropped, we expect that the boundary condition should be relaxed, as in the case of Dirichlet boundary condition (see [6]). One example for the zero-flux hyperbolic problem is given in [5].

The rest of this paper is organized as follows. In section 2, we give some assumptions and preliminaries and state our definition of entropy solution. Section 3 is devoted to existence of approximate solutions and passage to the limit to prove existence of an entropy solution of  $(P)$ . Finally, in section 4 we study the abstract evolution equation associated with  $(P)$  and prove uniqueness of entropy solution in one space dimension.

## 2. Entropy Solutions

### 2.1. Assumptions and preliminaries

We introduce the sign function and its approximations:

$$\text{sign}(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0, \end{cases} \quad \text{sign}_\sigma(r) = \begin{cases} 1 & \text{if } r > \sigma, \\ \frac{r}{\sigma} & \text{if } |r| \leq \sigma, \\ -1 & \text{if } r < -\sigma, \end{cases}$$

$$\text{sign}^+(r) = \max(\text{sign}(r), 0), \quad \text{sign}^-(r) = \min(\text{sign}(r), 0),$$

$$\text{sign}_\sigma^+(r) = \frac{1}{\sigma} \min(r^+, \sigma) \quad \text{and} \quad \text{sign}_\sigma^-(r) = \frac{1}{\sigma} \max(-r^-, -\sigma).$$

We also introduce the cut-off function:

$$T_{a,b}(r) = \begin{cases} a & \text{if } r < a, \\ r & \text{if } a \leq r \leq b, \\ b & \text{if } r > b. \end{cases}$$

To apply a strong precompactness result needed for the proof of the existence of entropy solution, we assume that the couple  $(f(\cdot), \phi(\cdot))$  is non-degenerate in the sense of the following definition.

**Definition 2.1.** Panov [22]. Let  $\phi$  be zero on  $[0, u_c]$ , strictly increasing on  $[u_c, u_{\max}]$  and a vector  $f = (f_1, \dots, f_N)$ . A couple  $(f(\cdot), \phi(\cdot))$  is said to be non-degenerate if, for all  $\xi \in \mathbb{R}^N \setminus \{0\}$ , the functions  $\lambda \mapsto \sum_{i=1}^N \xi_i f_i(\lambda)$  are not “affine” on the non-degenerate sub intervals of  $[0, u_c]$ .

### 2.2. Definition of Entropy Solutions

In this section, we give our entropy formulation for the problem  $P$ .

**Definition 2.2.** Let  $u_0$  be a measurable  $[0, u_{\max}]$ -valued function. A measurable function  $u$  taking values on  $[0, u_{\max}]$  is called weak solution of problem  $(P)$  if  $\phi(u) \in L^2(0, T; H^1(\Omega))$  and for all  $\xi \in L^2(0, T; H^1(\Omega))$  such that  $\xi_t \in L^1(Q)$  and  $\xi(T, \cdot) = 0$ , one has

$$\int_0^T \int_\Omega \left\{ u \xi_t + \left( f(u) - \nabla \phi(u) \right) \cdot \nabla \xi \right\} dx dt + \int_\Omega u_0 \xi(0, x) dx = 0. \quad (2.1)$$

**Definition 2.3.** A measurable function  $u$  taking values on  $[0, u_{\max}]$  is called an entropy solution of the initial-boundary value problem  $(P)$  if  $\phi(u) \in L^2(0, T; H^1(\Omega))$  and  $\forall k \in [0, u_{\max}]$ ,  $\forall \xi \in C^\infty([0, T] \times \mathbb{R}^N)$ , with  $\xi \geq 0$ , the following inequality hold

$$\int_0^T \int_\Omega \left\{ |u - k| \xi_t + \text{sign}(u - k) \left[ f(u) - f(k) - \nabla \phi(u) \right] \cdot \nabla \xi \right\} dx dt$$

$$+ \int_0^T \int_{\partial\Omega} |f(k) \cdot \eta(x)| \xi(t, x) d\mathcal{H}^{N-1} dt + \int_\Omega |u_0 - k| \xi(0, x) dx \geq 0. \quad (2.2)$$

If we remplace (2.2) by one of the following inequalities

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ (u-k)^+ \xi_t + \text{sign}^+(u-k) \left[ f(u) - f(k) - \nabla \phi(u) \right] \cdot \nabla \xi \right\} dx dt \\ & + \int_0^T \int_{\partial \Omega} (f(k) \cdot \eta(x))^+ \xi(t, x) d\mathcal{H}^{N-1} dt + \int_{\Omega} (u_0 - k)^+ \xi(0, x) dx \geq 0, \end{aligned} \quad (2.3)$$

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ (u-k)^- \xi_t + \text{sign}^-(u-k) \left[ f(u) - f(k) - \nabla \phi(u) \right] \cdot \nabla \xi \right\} dx dt \\ & + \int_0^T \int_{\partial \Omega} (f(k) \cdot \eta(x))^- \xi(t, x) d\mathcal{H}^{N-1} dt + \int_{\Omega} (u_0 - k)^- \xi(0, x) dx \geq 0, \end{aligned} \quad (2.4)$$

we obtain notions of entropy sub-solution and entropy super-solution respectively. Obviously, a function  $u$  is an entropy solution if and only if  $u$  is entropy sub-solution and entropy super-solution simultaneously.

*Remark 2.4.* 1. For the case  $\phi=0$ , solution of [11] is solution in our sense. The converse assertion is also true at least for  $N=1$ , this is the consequence of the uniqueness of a solution in the sense of Definition 2.3.

2. The entropy solution in the sense of Definition 2.3 is in particular a weak solution in the sense of Definition 2.2. Indeed, take in (2.2)  $k=0$  and  $k=u_{\max}$  and use (1.1); we find (2.1).
3. Let us stress that, in particular, the zero flux boundary condition  $(f(u) - \nabla \phi(u)) \cdot \eta = 0$  is verified literally in the weak sense. This contrasts with the properties of the Dirichlet problem (see [3]); we expect that the boundary condition should be relaxed if assumption (1.1) is dropped (see [5]).

### 3. Existence of Entropy Solutions

The main result of this section, is the following theorem:

**Theorem 3.1.** *Assume that (1.1) holds and that  $(f, \phi)$  is non-degenerate in the sense of Definition 2.3. Then, there exists an entropy solution  $u$  for the problem (P).*

#### 3.1. Viscosity Regularized Problem

To show the existence of entropy solutions, we approximate  $\phi(u)$  by  $\phi_{\epsilon}(u^{\epsilon}) = \phi(u^{\epsilon}) + \epsilon \text{Id}(u^{\epsilon})$  for each  $\epsilon > 0$ . We obtain the following regularized problem  $(P_{\epsilon})$ :

$$(P_{\epsilon}) \begin{cases} u^{\epsilon}_t + \text{div } f(u^{\epsilon}) - \Delta \phi_{\epsilon}(u^{\epsilon}) = 0 & \text{in } Q = (0, T) \times \Omega, \\ u^{\epsilon}(0, x) = u_0^{\epsilon}(x) & \text{in } \Omega, \\ (f(u^{\epsilon}) - \nabla \phi_{\epsilon}(u^{\epsilon})) \cdot \eta = 0 & \text{on } \Sigma = (0, T) \times \partial \Omega, \end{cases}$$

where  $(u_0^{\epsilon})_{\epsilon}$  is a sequence of smooth functions that converges to  $u_0$  a.e and respects the minimum/maximum values of  $u_0$ .

**Definition 3.2.** A function  $u^{\epsilon} \in L^2(0, T; H^1(\Omega))$  is called weak solution of the initial-boundary value problem  $(P_{\epsilon})$  if for all  $\xi \in L^2(0, T; H^1(\Omega))$  such that  $\xi_t \in L^1(Q)$  and  $\xi(T, \cdot) = 0$ , one has

$$\int_0^T \int_{\Omega} \left\{ u^{\epsilon} \xi_t + \left( f(u^{\epsilon}) - \nabla \phi_{\epsilon}(u^{\epsilon}) \right) \cdot \nabla \xi \right\} dx dt + \int_{\Omega} u_0^{\epsilon} \xi(0, x) dx = 0. \quad (3.1)$$

**Definition 3.3.** A measurable function  $u^{\epsilon} \in L^2(0, T; H^1(\Omega))$  taking values in  $[0, u_{\max}]$  is called an entropy solution of  $(P_{\epsilon})$  if  $\forall k \in [0, u_{\max}]$ ,  $\forall \xi \in C^{\infty}([0, T] \times \mathbb{R}^N)$ ,  $\xi \geq 0$ , the following inequality hold

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ |u^{\epsilon} - k| \xi_t + \text{sign}(u^{\epsilon} - k) \left[ f(u^{\epsilon}) - f(k) - \nabla \phi_{\epsilon}(u^{\epsilon}) \right] \cdot \nabla \xi \right\} dx dt \\ & + \int_0^T \int_{\partial \Omega} |f(k) \cdot \eta(x)| \xi(t, x) d\mathcal{H}^{N-1} dt + \int_{\Omega} |u_0^{\epsilon} - k| \xi(0, x) dx \geq 0. \end{aligned} \quad (3.2)$$

**Theorem 3.4.** For  $u_0 \in [0, u_{\max}]$  and (1.1) hold, the problem  $(P_\epsilon)$  admits a weak solution  $u^\epsilon$  which is also an entropy solution. In particular, we have  $0 \leq u^\epsilon \leq u_{\max}$ . In addition, there exists  $C$  independent on  $\epsilon$  such that

$$\|\sqrt{\epsilon} \nabla u^\epsilon\|_{L^2(Q)} \leq C; \quad (3.3)$$

$$\|\phi_\epsilon(u_\epsilon)\|_{L^2(0,T;H^1(\Omega))} \leq C. \quad (3.4)$$

### 3.2. Strong pre-compactness result and passage to the limit on $\epsilon$

**Theorem 3.5.** Panov [22]. Assume that  $(f, \phi)$  is non degenerate in the sense of Definition 2.1. Suppose  $u^\epsilon, \epsilon > 0$ , is a sequence such that

$$\exists d > 1, \forall a, b \in \mathbb{R} \text{ with } a < b$$

$$T_{a,b}(u^\epsilon)_t + \operatorname{div}\left(f(T_{a,b}(u^\epsilon)) - \nabla\phi(T_{a,b}(u^\epsilon))\right) \text{ is pre-compact in } W_{\text{Loc}}^{-1,d}(Q).$$

Moreover, suppose  $u^\epsilon, f(u^\epsilon), \phi_\epsilon(u^\epsilon)$  are equi-integrable locally on  $Q$ . Then, there exists subsequence  $(T_{a,b}(u^\epsilon))_\epsilon$  that converges in  $L^1_{\text{Loc}}(Q)$ .

To prove Theorem 3.5, we need the following result.

**Lemma 3.6.** Suppose  $(f, \phi)$  is non degenerate and let  $u^\epsilon = u^\epsilon(t, x)$  be an entropy solution of  $(P_\epsilon)$ . Then for all  $a, b \in \mathbb{R}$  such that  $0 \leq a < b \leq u_{\max}$ ,

$$T_{a,b}(u^\epsilon)_t + \operatorname{div}\left(f(T_{a,b}(u^\epsilon)) - \nabla\phi_\epsilon(T_{a,b}(u^\epsilon))\right) = \kappa_{a,b}^\epsilon \text{ in } \mathcal{D}'(Q);$$

with  $\kappa_{a,b}^\epsilon \in M_b(Q)$ . Here  $M_b(Q)$  represents the set of all Radon measures on  $Q$ . Moreover, for each compact set  $K \subset Q$ , we have  $\operatorname{Var}\kappa_{a,b}^\epsilon(K) \leq C(K, a, b)$ , uniformly in  $\epsilon \in (0, 1)$ .

*Proof.* By the well known representation property for non-negative distributions, we derive from (3.2) that for each  $k \in [0, u_{\max}]$

$$|u^\epsilon - k|_t + \operatorname{div}\left[\operatorname{sign}(u^\epsilon - k)(f(u^\epsilon) - f(k)) - \nabla|\phi_\epsilon(u^\epsilon) - \phi_\epsilon(k)|\right] = -\kappa_k^\epsilon \text{ in } \mathcal{D}'(Q)$$

where  $\kappa_k^\epsilon \in M_b(Q)$ ,  $\kappa_k^\epsilon \geq 0$ . Further, for a compact set  $K \subset Q$  we choose non-negative function  $\xi = \xi_K(t, x) \in \mathcal{C}_0^\infty(Q)$ , which equals 1 on  $K$ . Then, we have the estimate

$$\begin{aligned} \kappa_k^\epsilon(K) &\leq \int_0^T \int_\Omega \xi(t, x) d\kappa_k^\epsilon(t, x) \\ &= \int_Q \left\{ |u^\epsilon - k| \xi_t + \left[ \operatorname{sign}(u^\epsilon - k)(f(u^\epsilon) - f(k)) - \nabla|\phi_\epsilon(u^\epsilon) - \phi_\epsilon(k)| \right] \cdot \nabla \xi \right\} \\ &\leq \int_Q (J_1^\epsilon + J_2) \max(|\xi_t|, |\nabla \xi|, |\Delta \xi|) dxdt =: A(K, k, J_1^\epsilon); \end{aligned} \quad (3.5)$$

where  $J_1^\epsilon(t, x) = |u^\epsilon| + |f(u^\epsilon)| + |\phi_\epsilon(u^\epsilon)|$  and  $J_2 = |k| + |f(k)| + |\phi_\epsilon(k)|$  are bounded in  $L^1$  due to the fact that  $u^\epsilon, f(u^\epsilon), \phi_\epsilon(u^\epsilon)$  are bounded in  $L^\infty$ , uniformly on  $\epsilon$  (see Theorem 3.4). Therefore  $\kappa_k^\epsilon(K)$  can be upper bounded by some quantity  $A(K, k)$ . Further, notice that for each  $a, b \in \mathbb{R}$  and for any function  $g$

$$\begin{aligned} g(T_{a,b}(r)) &= \frac{1}{2} \left( \operatorname{sign}(r - a)(g(r) - g(a)) - \operatorname{sign}(r - b)(g(r) - g(b)) \right) \\ &\quad + \frac{1}{2} (g(a) + g(b)). \end{aligned}$$

From (3.5), we have with  $g = Id$ ,  $g = f$  and  $g = \phi_\epsilon$

$$T_{a,b}(u^\epsilon)_t + \operatorname{div}\left(f(T_{a,b}(u^\epsilon)) - \nabla\phi_\epsilon(T_{a,b}(u^\epsilon))\right) = \kappa_{a,b}^\epsilon \text{ in } \mathcal{D}'(Q), \quad (3.6)$$

with  $\kappa_{a,b}^\epsilon = \frac{1}{2}(\kappa_b - \kappa_a)$ . Moreover, we have

$$\text{Var}\kappa_{a,b}^\epsilon(K) \leq \frac{1}{2} \left( A(K, a) + A(K, b) \right) =: C(K, a, b).$$

This concludes the proof.  $\square$

Notice that for all  $a, b \in \mathbb{R}$ ,  $a < b$ , we have  $T_{a,b}(T_{0,u_{\max}}) = T_{\tilde{a},\tilde{b}}$  with  $\tilde{a} = \max(a, 0)$  and  $\tilde{b} = \min(b, u_{\max})$ . Now, we are able to prove Theorem 3.1.

*Proof of Theorem 3.1.* Take some countable set of values  $\epsilon \rightarrow 0$ . We derive from Lemma 3.6 and the above remark that for all  $a, b \in \mathbb{R}$ ,  $a < b$ ,

$$T_{a,b}(u^\epsilon)_t + \text{div} \left( f(T_{a,b}(u^\epsilon)) - \nabla \phi(T_{a,b}(u^\epsilon)) \right) = \kappa_{a,b}^\epsilon + \epsilon \Delta T_{a,b}(u^\epsilon) \text{ in } \mathcal{D}'(Q);$$

here  $(\kappa_{a,b}^\epsilon)_\epsilon$  is a bounded sequence in  $M_b(K)$  for each compact set  $K \subset Q$ . Moreover, due to (3.3), we have  $\epsilon \Delta T_{a,b}(u^\epsilon)$  tends to zero in  $H^{-1}(Q)$ . By Sobolev embedding we have  $H^{-1}(Q) = W^{-1,2}(Q) \subset W^{-1,d}(Q)$ ,  $d \leq 2$ , therefore  $\epsilon \Delta T_{a,b}(u^\epsilon)$  tends to zero in  $W^{-1,d}(Q)$ . Since  $M(K)$  is compactly embedded in  $W^{-1,d}(K)$  for each  $d \in [1, \frac{N+1}{N})$ , we see that  $T_{a,b}(u^\epsilon)_t + \text{div} \left( f(T_{a,b}(u^\epsilon)) - \nabla \phi(T_{a,b}(u^\epsilon)) \right)$  is pre-compact in  $W^{-1,d}(K)$ . Since  $(f, \phi)$  is assumed to be non-degenerate, then applying Theorem 3.5, we find  $T_{0,u_{\max}}(u^\epsilon) \rightarrow T_{0,u_{\max}}(u)$  as  $\epsilon \rightarrow 0$  in  $L^1(K)$ , for a subsequence. Covering  $Q$  by a countable set of compacts  $K$  and using diagonal extraction argument, we get  $u^\epsilon \rightarrow u$  in  $L^1_{\text{Loc}}(Q)$  (and actually in  $L^1(Q)$ , because  $u^\epsilon$  take their values in  $[0, u_{\max}]$ ). Extracting a further subsequence if necessary, we can assume that  $u^\epsilon \rightarrow_{\epsilon \rightarrow 0} u$  a.e. in  $Q$ .

It remains to derive the entropy formulation (2.2) for  $u$ . Passing to the limit in relation (3.2), we claim that the limit function  $u = u(t, x)$  satisfies this relation for all  $k \in [0, u_{\max}]$  such that the level set  $u^{-1}(k)$  has zero measure. Indeed, by the continuity of  $f$ , we have  $\text{sign}(u^\epsilon - k)(f(u^\epsilon) - f(k)) \rightarrow \text{sign}(u - k)(f(u) - f(k))$  as  $\epsilon \rightarrow 0$  a.e. in  $Q$ , and since  $\phi_\epsilon(u_\epsilon)$  is bounded in  $L^2(0, T; H^1(\Omega))$  and converge a.e to the limit that is readily identified with  $\phi(u)$ , we deduce that  $\text{sign}(u^\epsilon - k)\nabla \phi_\epsilon(u^\epsilon) \rightarrow \text{sign}(u - k)\nabla \phi(u)$  a.e in  $Q$  for  $k$  such that  $u^{-1}(k)$  has zero measure. Notice that the set of such values of  $k$  is dense in  $[0, u_{\max}]$ . It is easy to see that the left-hand side of (2.2) is continuous with respect to  $k$ , because  $\nabla \phi(u) = 0$  a.e. on the set  $[u = k]$  (cf Lemma 4.4 below). Therefore, by density we inherit (2.2) for all  $k \in [0, u_{\max}]$ . We conclude that  $u(t, x)$  is an entropy solution of (P).  $\square$

#### 4. Uniqueness result of Entropy Solution in one space dimension

The main result of this section is the following theorem:

**Theorem 4.1.** *Suppose that  $\Omega = (a, b)$  is a bounded interval of  $\mathbb{R}$ , then (P) admits a unique entropy solution.*

Let us first recall an essential property of entropy solutions, based on the idea of J. Carrillo [16].

**Proposition 4.2.** *Let  $\xi \in C^\infty([0, T] \times \mathbb{R}^N)$ ,  $\xi \geq 0$ . Then for all  $k \in [u_c, u_{\max}]$ , for all  $D \in \mathbb{R}^N$  and for all entropy solution  $u$  of (P), we have*

$$\begin{aligned} & \int_0^T \int_\Omega \left\{ |u - k| \xi_t + \text{sign}(u - k) \left[ f(u) - f(k) - \nabla \phi(u) + D \right] \cdot \nabla \xi \right\} dx dt \\ & + \int_\Omega |u_0 - k| \xi(0, x) dx + \int_0^T \int_{\partial \Omega} |(f(k) - D) \cdot \eta(x)| \xi d\mathcal{H}^{N-1} dt \\ & \geq \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \iint_{Q \cap \{-\sigma < \phi(u) - \phi(k) < \sigma\}} \nabla \phi(u) \cdot (\nabla \phi(u) - D) \xi. \end{aligned} \quad (4.1)$$

*Remark 4.3.* Notice that this proposition makes explicit the information on the dissipation. Let us stress that in (2.2)  $D = 0$  and  $k \in [0, u_{\max}]$  while in (4.1)  $D \in \mathbb{R}^N$  but  $k \in [u_c, u_{\max}]$ .

Proposition 4.2 is a key ingredient of the uniqueness technique. To prove this proposition, we need the following remarks. Firstly, for all  $u \in [0, u_{\max}]$  and for all  $k \in [u_c, u_{\max}]$ , one has  $\text{sign}(u-k) = \text{sign}(\phi(u) - \phi(k))$ . Secondly, recall

**Lemma 4.4.** *For all entropy solution  $u$  of (P), one has:*

$\nabla\phi(u) = 0$   $\mathcal{L}^N$  a.e. on the set  $\{(t, x) \in Q \text{ such that } u(t, x) \in [0, u_c]\}$ .

*Proof.* This result comes from Marcus and Mizel lemma (cf. [20]) which states that for  $p \in (1, \infty)$ , and  $F$  in  $W^{1,p}$ ,  $\nabla F = 0$  a.e on  $F^{-1}(\mathcal{N})$ , where  $\mathcal{N}$  is a set of zero measure on  $\mathbb{R}$ . Apply this for a.e.  $t \in (0, T)$ , for  $u \in [u_c, u_{\max}]$ , we have  $\nabla\phi(u) = 0$  on  $[\phi(u)]^{-1}\{0\}$  with  $\phi(u) \in H^1(\Omega)$ . Let  $E_{N+1} = \{(t, x) \text{ such that } u(t, x) \in [0, u_c]\}$  and  $E_N(t) = \{x \text{ such that } u(t, x) \in [0, u_c]\}$ . Then, by Fubini theorem,  $|E_{N+1}| = \int_0^T |E_N(t)| dt = 0$ .  $\square$

*Proof of Theorem 4.2.* Since  $u$  is a weak solution of (P), then for all  $k \in [u_c, u_{\max}]$  and  $D \in \mathbb{R}^N$ ,  $u$  is also a weak solution of the following problem:

$$(P_k) \begin{cases} (u-k)_t + \text{div} \left[ (f(u) - \nabla\phi(u)) - (f(k) - D) \right] = 0 & \text{in } Q, \\ u(0, x) - k = u_0(x) - k & \text{on } \Omega, \\ \left( (f(u) - \nabla\phi(u)) - (f(k) - D) \right) \cdot \eta = -(f(k) - D) \cdot \eta & \text{on } \Sigma. \end{cases}$$

Take the test function  $\text{sign}_\sigma(\phi(u) - \phi(k))\xi$  in the weak formulation of this problem with  $\xi \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^N)$ . We have, using the formalism of [1],

$$\begin{aligned} & \int_0^T \langle (u-k)_t, \text{sign}_\sigma(\phi(u) - \phi(k))\xi \rangle_{H^1(\Omega)^*, H^1(\Omega)} dt \\ & - \int_0^T \int_\Omega \text{sign}_\sigma(\phi(u) - \phi(k)) \left[ (f(u) - \nabla\phi(u)) - (f(k) - D) \right] \cdot \nabla \xi \\ & - \int_0^T \int_\Omega \xi \left[ (f(u) - \nabla\phi(u)) - (f(k) - D) \right] \cdot \nabla \text{sign}_\sigma(\phi(u) - \phi(k)) \\ & - \int_0^T \int_{\partial\Omega} \text{sign}_\sigma(\phi(u) - \phi(k)) (f(k) - D) \cdot \eta \xi = 0. \end{aligned} \quad (4.2)$$

By the chain rule (see [1], [15]) the first integral of (4.2) give:

$$- \int \int_Q \int_k^u \text{sign}_\sigma(\phi(r) - \phi(k)) dr \xi_t - \int_\Omega \int_k^{u_0} \text{sign}_\sigma(\phi(r) - \phi(k)) dr \xi(0, x).$$

Using the fact that  $k \in [u_c, u_{\max}]$  and passing to the limit as  $\sigma$  goes to 0, we obtain:

$$\begin{aligned} & \int_0^T \int_\Omega \left\{ \int_k^u \text{sign}_\sigma(\phi(r) - \phi(k)) dr \right\} \xi_t dx dt \longrightarrow \int_0^T \int_\Omega |u-k| \xi_t dx dt, \\ & \int_\Omega \left\{ \int_k^{u_0} \text{sign}_\sigma(\phi(r) - \phi(k)) dr \right\} \xi(0, x) dx \longrightarrow \int_\Omega |u_0 - k| \xi(0, x) dx. \end{aligned}$$

After passing to the limit as  $\sigma$  goes to 0 in the second integral of (4.2), and using the fact that  $k \in [u_c, u_{\max}]$ , we obtain the expression

$$\int_0^T \int_\Omega \text{sign}(u-k) \left[ (f(u) - \nabla\phi(u)) - (f(k) - D) \right] \cdot \nabla \xi.$$

The third integral of (4.2) writes

$$\iint_Q \xi (f(u) - f(k)) \cdot \nabla \text{sign}_\sigma(u-k) - \iint_Q \xi (\nabla\phi(u) - D) \cdot \nabla \text{sign}_\sigma(\phi(u) - \phi(k)).$$



By passing to the limit, the integral in the first term goes to 0, and the second term becomes

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_0^T \int_{\Omega} \xi (\nabla \phi(u) - D) \cdot \nabla \text{sign}_{\sigma}(\phi(u) - \phi(k)) \\ &= \lim_{\sigma \rightarrow 0} \int_0^T \int_{\Omega} \xi (\nabla \phi(u) - D) \text{sign}'_{\sigma}(\phi(u) - \phi(k)) \cdot \nabla \phi(u) \\ &= \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \int_{Q \cap \{-\sigma < \phi(u) - \phi(k) < \sigma\}} \nabla \phi(u) \cdot (\nabla \phi(u) - D) \xi. \end{aligned}$$

The limit of last integral of (4.2) can be upper bounded as follows

$$\lim_{\sigma \rightarrow 0} \int_0^T \int_{\partial \Omega} \text{sign}_{\sigma}(\phi(u) - \phi(k)) (f(k) - D) \cdot \eta \xi \leq \int_0^T \int_{\partial \Omega} |(f(k) - D) \cdot \eta(x)| \xi.$$

Then, we obtain the required inequality (4.1).  $\square$

Now, we consider the stationary problem associated to problem (P):

$$(S) \begin{cases} u + \text{div}(f(u) - \nabla \phi(u)) = g & \text{in } \Omega, \\ (f(u) - \nabla \phi(u)) \cdot \eta = 0 & \text{on } \partial \Omega. \end{cases}$$

*Remark 4.5.* If  $u(x)$  independent of  $t$  is solution of (S) then  $u(t, x) = u(x)$  is solution of (P) with the source term  $g - u$ . Then, we can deduce from Definition 2.3 and Proposition 4.2 their equivalent form for the stationary problem.

**Definition 4.6.** Let  $g$  a measurable function taking values in  $[0, u_{\max}]$ . A measurable function  $u$  taking values in  $[0, u_{\max}]$  is an entropy solution of (S), if  $\phi(u) \in H^1(\Omega)$  and for all  $\xi \in C^{\infty}(\mathbb{R}^N)^+$ ,  $\forall k \in [0, u_{\max}]$ ,

$$\begin{aligned} & - \int_{\Omega} \text{sign}(u - k) u \xi dy + \int_{\Omega} \text{sign}(u - k) [f(u) - f(k) - \nabla \phi(u)] \cdot \nabla \xi dy \\ & + \int_{\partial \Omega} |f(k) \cdot \eta(y)| \xi d\mathcal{H}^{N-1} + \int_{\Omega} \text{sign}(u - k) g \xi dy \geq 0. \end{aligned} \quad (4.3)$$

**Proposition 4.7.** Let  $\xi \in C^{\infty}(\mathbb{R}^N)$ ; then for all  $k \in [u_c, u_{\max}]$ , for all  $D \in \mathbb{R}^N$ , for all entropy solution  $u$  of (S), we have:

$$\begin{aligned} & - \int_{\Omega} \text{sign}(u - k) u \xi dy + \int_{\Omega} \text{sign}(u - k) [f(u) - f(k) - \nabla \phi(u) + D] \cdot \nabla \xi dy \\ & + \int_{\partial \Omega} |(f(k) - D) \cdot \eta(y)| \xi d\mathcal{H}^{N-1} + \int_{\Omega} \text{sign}(u - k) g \xi dy \\ & \geq \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \int_{\Omega \cap \{-\sigma < \phi(u) - \phi(k) < \sigma\}} \nabla \phi(u) \cdot (\nabla \phi(u) - D) \xi. \end{aligned} \quad (4.4)$$

From now on, we will suppose that  $\Omega = (a, b)$  is a bounded interval of  $\mathbb{R}$ .

**Proposition 4.8.** For all measurable function  $g$  taking values in  $[0, u_{\max}]$  the problem (S) admits a solution  $u$  such that  $(f(u) - \phi(u)_y)$  is continuous up the boundary, i.e.,  $(f(u) - \phi(u)_y) \in C([a, b])$ . Moreover,  $f(u) - \phi(u)_y$  is zero at  $y = a$  and  $y = b$ .

*Proof.* For existence of entropy solution, we can refer to [18, Chap 2], using Galerkin approximations, in a way similar to and Theorem 3.4 and 3.1.

Now, if  $u$  is a weak solution of (S), this means that  $(f(u) - \phi(u)_y)_y = g - u$  in  $\mathcal{D}'$ . Then  $(f(u) - \phi(u)_y)_y \in L^{\infty}([a, b])$ , which implies that  $(f(u) - \phi(u)_y) \in W^{1, \infty}([a, b]) \subset C([a, b])$ .

Now, as in Remarks 2.4, item 3, from (4.3) we deduce that  $(f(u) - \phi(u)_y)|_{\partial \Omega} = 0$  in the weak sense. Therefore  $f(u) - \phi(u)_y \in \mathcal{C}_0([a, b])$ .  $\square$

Let us define the (possibly multivalued) operator  $A_{f,\phi}$  by its resolvent

$$(u, z) \in A_{f,\phi} = \{ u \text{ such that } u \text{ is an entropy solution of } (S), \text{ with } g = u + z \}.$$

For an operator  $A : L^1(\Omega) \rightarrow L^1(\Omega)$ , denote by  $R(A)$  its range, by  $D(A)$  its domain and by  $\overline{R(A)}$ ,  $\overline{D(A)}$  their closures in  $L^1(\Omega)$  respectively.

Let us stress that for  $u \in D(A)$ ,  $f(u) - \phi(u)_y \in \mathcal{C}([a, b])$  due to Proposition 4.8.

Recall (cf. [7]) that an operator  $A$  is accretive if  $[\beta - \hat{\beta}, \alpha - \hat{\alpha}]_{L^1(\Omega)} \geq 0$  for all  $(\beta, \alpha), (\hat{\beta}, \hat{\alpha}) \in A$ , where for  $\beta, \alpha \in L^1(\Omega)$  the bracket  $[\cdot, \cdot]_{L^1(\Omega)}$  is defined by  $[\beta, \alpha]_{L^1(\Omega)} = \int_{\Omega} \text{sign}(\beta)\alpha + \int_{[\beta=0]} |\alpha|$ . If  $A$  is accretive and  $R(I + \lambda A) = L^1(\Omega)$  for some  $\lambda > 0$ , then  $A$  is m-accretive.

**Proposition 4.9.** *Let  $(u, z) \in A_{f,\phi}$ ,  $(\hat{u}, \hat{z}) \in A_{f,\phi}$ . Then for  $\xi \in \mathcal{C}^\infty(\overline{\Omega})^+$*

$$\begin{aligned} & \int_{\Omega} |u - \hat{u}| \xi dy + \int_{\Omega} \text{sign}(u - \hat{u}) \left[ f(u) - f(\hat{u}) - \phi(u)_y + \phi(\hat{u})_y \right] \cdot \xi_y dy \\ & \leq \int_{\Omega} \text{sign}(u - \hat{u})(g - \hat{g}) \xi dy + \int_{[u=\hat{u}]} |g - \hat{g}| \xi dy = [u - \hat{u}, g - \hat{g}]_{L^1(\Omega)}. \end{aligned} \quad (4.5)$$

*Proof.* (Sketched) The proof of Proposition 4.5 is actually contained in the proof of Theorem 4.13 below, due to Remark 2.4. Actually a simpler argument applies, because both  $f(\hat{u}) - \phi(\hat{u})_y$  and  $f(u) - \phi(u)_y$  have strong trace in the context of the stationary problem (S).  $\square$

Somewhat abusively, we will write  $L^1(\Omega; [0, u_{\max}])$  for the set of all measurable functions from  $[a, b]$  to  $[0, u_{\max}]$ .

**Proposition 4.10.** 1.  $A_{f,\phi}$  is accretive in  $L^1(\Omega)$ .

2. For all  $\lambda$  sufficiently small,  $R(I + \lambda A_{f,\phi})$  contains  $L^1(\Omega; [0, u_{\max}])$ .

3.  $\overline{D(A_{f,\phi})} = L^1(\Omega; [0, u_{\max}])$ .

*Proof.* 1. Let  $(u, z) \in A_{f,\phi}$ ,  $(\hat{u}, \hat{z}) \in A_{f,\phi}$ . Applying Proposition 4.9 with  $\xi = 1$  in (4.5) and the standard properties of the bracket, we get

$$\begin{aligned} \|u - \hat{u}\|_{L^1(\Omega)} & \leq [u - \hat{u}, g - \hat{g}]_{L^1(\Omega)} \\ & \leq [u - \hat{u}, u - \hat{u} + z - \hat{z}]_{L^1(\Omega)} \\ & \leq \|u - \hat{u}\|_{L^1(\Omega)} + [u - \hat{u}, z - \hat{z}]_{L^1(\Omega)}. \end{aligned}$$

We deduce that  $[u - \hat{u}, z - \hat{z}]_{L^1(\Omega)} \geq 0$ , do that  $A_{f,\phi}$  is accretive.

2. For  $\lambda > 0$ , consider the problem

$$(S_\lambda) \begin{cases} u_\lambda + \lambda(f(u_\lambda) - (\phi(u_\lambda))_y)_y = g & \text{in } \Omega, \\ \lambda(f(u_\lambda) - \phi(u_\lambda)_y) \cdot \eta(y) = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that the notion of solution for  $(S_\lambda)$  is like in Definition 4.6. Let  $g \in L^1(\Omega; [0, u_{\max}])$ , and  $\lambda > 0$  then, there exists  $u_\lambda$  entropy solution of  $(S_\lambda)$  such that  $(u_\lambda, \frac{g - u_\lambda}{\lambda}) \in A_{f,\phi}$ . Hence  $g \in R(I + \lambda A_{f,\phi})$  and therefore  $R(I + \lambda A_{f,\phi}) \supset L^1(\Omega; [0, u_{\max}])$ , which was to be shown.

3. Let  $PC([a, b]; [0, u_{\max}])$  be the set of piecewise constant functions from  $[a, b]$  to  $[0, u_{\max}]$ . Then  $PC([a, b]; [0, u_{\max}])$  is dense in  $L^1([a, b]; [0, u_{\max}])$ . Take  $g \in PC([a, b]; [0, u_{\max}])$ ,  $g = \sum_i c_i \mathbf{1}_{(a_i, b_i)}$

where the  $(a_i, b_i)$  are disjoint intervals. There exists  $u_n \in L^\infty(a, b)$  entropy solution of  $(S_{\frac{1}{n}})$ , i.e, we have  $(u_n, n(g - u_n)) \in A_{f,\phi}$ . For  $k \in [0, u_{\max}]$ , for all  $\xi \in \mathcal{C}_0^\infty(\mathbb{R})$  we get

$$\begin{aligned} & \frac{1}{n} \int_a^b \text{sign}(u_n - k) \left( f(u_n) - f(k) - \partial_y \phi(u_n) \right) \cdot \partial_y \xi dy \\ & + \int_a^b \text{sign}(u_n - k)(g - u_n) \xi dy + \frac{1}{n} \int_{a,b} |f(k) \cdot \eta(y)| \xi d\sigma \geq 0. \end{aligned} \quad (4.6)$$

For every  $i$ , one can construct  $\xi_i^n$  such that  $\xi_i^n \rightarrow \mathbf{1}_{(a_i, b_i)}$ , as  $n \rightarrow \infty$ ,  $\text{supp}\xi_i^n \subset (a_i, b_i)$ ,  $\|\partial_y \xi\|_{L^\infty} \leq 2\sqrt[3]{n}$  and,  $\xi_i^n \equiv 1$  in  $(a_i + \delta_n^i, b_i - \delta_n^i)$  with  $\delta_n^i = \frac{b_i - a_i}{2\sqrt[3]{n}}$ .

Take  $k = c_i$  and  $\xi = \xi_i^n$  in (4.6).

$$\begin{aligned} \int_{a_i + \delta_n^i}^{b_i - \delta_n^i} |u_n - c_i| &\leq \frac{1}{n} \int_{a_i + \delta_n^i}^{b_i - \delta_n^i} \text{sign}(u_n - c_i) (f(u_n) - f(c_i) - \partial_y \phi(u_n)) \partial_y \xi_i^n \\ &\leq \frac{2}{n} |b - a| \|f\|_{L^\infty} \cdot \|\partial_y \xi_i^n\|_{L^1} + \left\| \frac{1}{\sqrt{n}} \partial_y \phi(u_n) \right\|_{L^2} \cdot \|\partial_y \xi_i^n\|_{L^2}. \end{aligned}$$

Then, for all  $\delta > \delta_n^i$ ,  $u_n \rightarrow g$  a.e on  $\bigcup_i (a_i + \delta, b_i - \delta)$ . We conclude by the Lebesgue theorem that

$u_n \rightarrow g$  in  $L^1([a, b])$ .

In conclusion,  $D(A_{f,\phi})$  is dense in  $PC([a, b]; [0, u_{\max}])$  and therefore, also on  $L^1(\Omega; [0, u_{\max}])$ .  $\square$

Now, we can exploit the notion of integral solution (see, e.g., [7], [8]).

**Definition 4.11.** Suppose that  $h \in L^1(Q)$ ,  $u_0 \in L^1(\Omega)$ . A function  $v \in \mathcal{C}([0, T]; L^1([a, b]; [0, u_{\max}]))$  is an integral solution of the problem

$$v_t + A_{f,\phi}(v) \ni h, \quad v(t=0) = u_0, \quad (4.7)$$

if  $v(0, \cdot) = u_0(\cdot)$  and for all  $(u, z) \in A_{f,\phi}$

$$\frac{d}{dt} \|v(t) - u\|_{L^1(\Omega)} \leq [v(t) - u, h(t) - z]_{L^1(\Omega)} \text{ in } \mathcal{D}'(0, T)$$

By Proposition 4.10, the operator  $A_{f,\phi}$  is m-accretive<sup>1</sup> densely defined in  $L^1(\Omega; [0, u_{\max}])$ , by the general theory of non-linear semigroups (cf. [7], [8], [9]), we have the following result.

**Corollary 4.12.** Let  $\Omega = [a, b]$ ,  $u_0, \hat{u}_0 \in L^1(\Omega)$  and  $h, \hat{h} \in L^1(Q)$ . Let  $v, \hat{v}$  be integral solutions of (4.7) associated with the data  $(u_0, h)$  and  $(\hat{u}_0, \hat{h})$ , respectively. Then for a.e.  $t \in [0, T]$ .

$$\|v(t) - \hat{v}(t)\|_{L^1} \leq \|u_0 - \hat{u}_0\|_{L^1} + \int_0^t \|h(\tau) - \hat{h}(\tau)\|_{L^1} dt.$$

In particular, the integral solution is unique.

**Theorem 4.13.** Let  $\Omega = [a, b]$ . Let  $v$  be an entropy solution of (P) and  $u$  be an entropy solution of (S). Then

$$\frac{d}{dt} \|v(t) - u\|_{L^1(\Omega)} \leq \int_{\Omega} \text{sign}(v - u)(u - g) dx \text{ in } \mathcal{D}'(0, T). \quad (4.8)$$

In particular,  $v$  is an integral solution of (4.7) with  $h = 0$ .

First, note the following auxiliary result.

**Lemma 4.14.** Let  $\delta$  be a positive function with support in  $[-1, 1]$  and  $\|\delta\|_{L^1} = 1$ . Let for all  $z \in [-1, 1]$ ,  $w_n(\cdot, z) \rightarrow w(\cdot)$  and  $h_n(\cdot, z) \rightarrow h(\cdot)$  in  $L^1(\mathbb{R})$  as  $n \rightarrow \infty$ . If in addition  $\|h_n(\cdot, z)\|_{L^1(\mathbb{R})}$  is bounded uniformly in  $n$  and  $z$ , then

$$\limsup_{n \rightarrow \infty} \iint \text{sign} w_n(x, z) h_n(x, z) \delta(z) \leq [w, h]. \quad (4.9)$$

Moreover, if for all  $n \in \mathbb{N}$  and a.e.  $z \in [-1, 1]$ ,  $h_n(\cdot, z) = 0$  a.e. on  $\{w_n(\cdot, z) = 0\}$  and if  $h = 0$  a.e. on  $\{w = 0\}$ , then there exists

$$\lim_{n \rightarrow \infty} \iint \text{sign} w_n(x, z) h_n(x, z) \delta(z) = \int \text{sign}(w) h. \quad (4.10)$$

<sup>1</sup>Rigourously speaking, this statement is false because  $L^1(\Omega; [0, u_{\max}])$  is not a Banach space, but it's convex subset. Nonetheless, this subset is invariant for the stationary problem (S), therefore the nonlinear semi-groupe theory applies without change in our case.

*Proof.* Inequality (4.9) claim follows from the definition and the upper semicontinuity of the bracket, the definition of  $\delta$  and the Fatou lemma. Inequality (4.10) follows by applying the first one to  $w_n, h_n$  and to  $-w_n, h_n$ .  $\square$

*Proof of Theorem 4.13.* To start with, note that by the result of [14] an entropy solution  $v$  of (P) is automatically times-continuous with values in  $L^1(\Omega; [0, u_{\max}])$ .

Now, we apply the doubling of variables [17] in the way of [2]. We consider  $v = v(t, x)$  an entropy solution of (P) and  $u = u(y)$  an entropy solution of (S). Consider nonnegative functions  $\xi = \xi(t, x, y)$  having the property that  $\xi(\cdot, \cdot, y) \in C^\infty([0, T] \times \bar{\Omega})$  for each  $y \in \bar{\Omega}$ ,  $\xi(t, x, \cdot) \in C_0^\infty(\bar{\Omega})$  for each  $(t, x) \in [0, T] \times \bar{\Omega}$ .

We denote  $\Omega_x = \{x \in \Omega; v(t, x) \in [0, u_c]\}$ ;  $\Omega_y = \{y \in \Omega; u(y) \in [0, u_c]\}$  and  $\Omega_x^c, \Omega_y^c$  their complementary in  $\Omega$ . In (4.1), take  $\xi = \xi(t, x, y)$ ,  $k = u(y)$ ,  $D = \phi(u)_y$  and integrate over  $\Omega_y^c$ . We get

$$\begin{aligned} & \int_{\Omega_y^c} \int_0^T \int_{x \in \Omega} \left\{ |v - u| \xi_t + \text{sign}(v - u) \left[ f(v) - \phi(v)_x - f(u) + \phi(u)_y \right] \cdot \xi_y \right\} \\ & + \int_{\Omega_y^c} \int_0^T \int_{x \in \partial\Omega} |(f(u) - \phi(u)_y) \cdot \eta(x)| \xi d\sigma dt dy + \int_{\Omega_y^c} \int_{x \in \Omega} |v_0 - u| \xi(0, x, y) \\ & \geq \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \int_{\Omega_y^c} \int_0^T \int_{x \in \Omega \cap \{-\sigma < \phi(v) - \phi(u) < \sigma\}} \phi(v)_x (\phi(v)_x - \phi(u)_y) \xi. \end{aligned} \quad (4.11)$$

In the same way, in (2.2) take  $\xi = \xi(t, x, y)$ ,  $k = u(y)$ , integrate over  $\Omega_y$ , and use the fact that  $\phi(u)_y = 0$  in  $\Omega_y$ . We get

$$\begin{aligned} & \int_{\Omega_y} \int_0^T \int_{x \in \Omega} \left\{ |v - u| \xi_t + \text{sign}(v - u) \left[ f(v) - \phi(v)_x - f(u) + \phi(u)_y \right] \cdot \xi_y \right\} \\ & + \int_{\Omega_y} \int_0^T \int_{x \in \partial\Omega} |(f(u) - \phi(u)_y) \cdot \eta(x)| \xi + \int_{\Omega_y} \int_{x \in \Omega} |v_0 - u| \xi(0, x, y) \geq 0. \end{aligned} \quad (4.12)$$

Since  $\Omega = \Omega_x \cup \Omega_x^c$ , by adding (4.11) to (4.12) we obtain:

$$\begin{aligned} & \int_{\Omega} \int_0^T \int_{\Omega} \left\{ |v - u| \xi_t + \text{sign}(v - u) \left[ f(v) - \phi(v)_x - f(u) + \phi(u)_y \right] \cdot \xi_x \right\} \\ & + \int_{\Omega} \int_0^T \int_{x \in \partial\Omega} |(f(u) - \phi(u)_y) \cdot \eta(x)| \xi + \int_{\Omega} \int_{\Omega} |v_0 - u| \xi(0, x, y) \\ & \geq \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \int_{\Omega_y^c} \int_0^T \int_{x \in \Omega \cap \{-\sigma < \phi(v) - \phi(u) < \sigma\}} \phi(v)_x (\phi(v)_x - \phi(u)_y) \xi. \end{aligned} \quad (4.13)$$

We proceed in the same way, exchanging the roles of  $v$  and  $u$ . Starting from (4.4) and (4.3), we deduce

$$\begin{aligned} & \int_0^T \int_{\Omega} \int_{\Omega} \text{sign}(v - u) \left[ f(v) - \phi(v)_x - f(u) + \phi(u)_y \right] \cdot \xi_y dy dx dt \\ & + \int_0^T \int_{\Omega} \int_{y \in \partial\Omega} |(f(v) - \phi(v)_x) \cdot \eta(y)| \xi d\sigma dx dt \\ & + \int_0^T \int_{\Omega} \int_{\Omega} \text{sign}(v - u) (u - g(y)) \xi dx dt dy \\ & \geq \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \int_{\Omega_x^c} \int_0^T \int_{y \in \Omega \cap \{-\sigma < \phi(u) - \phi(v) < \sigma\}} \phi(u)_y (\phi(u)_y - \phi(v)_x) \xi. \end{aligned} \quad (4.14)$$

Now, sum (4.13) and (4.14) to obtain

$$\begin{aligned}
& \int_0^T \int_{\Omega} \int_{\Omega} |v - u| \xi_t dy dx dt + \int_{\Omega} \int_{\Omega} |v_0 - u| \xi(0, x, y) dx dy \\
& + \int_0^T \int_{\Omega} \int_{\Omega} \text{sign}(v - u) \left[ (f(v) - \phi(v)_x) - (f(u) + \phi(u)_y) \right] \cdot (\xi_x + \xi_y) \\
& + \int_0^T \int_{x \in \partial\Omega} \int_{\Omega} |(f(u) - \phi(u)_y) \cdot \eta(x)| \xi d\sigma dt dy \\
& + \int_0^T \int_{\Omega} \int_{y \in \partial\Omega} |(f(v) - \phi(v)_x) \cdot \eta(y)| \xi dy d\sigma dt \\
& + \int_0^T \int_{\Omega} \int_{\Omega} \text{sign}(v - u) (u - g(y)) \xi \\
& \geq \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \int_0^T \iint_{\Omega_x^c \times \Omega_y^c \cap \{-\sigma < \phi(v) - \phi(u) < \sigma\}} |\phi(v)_x - \phi(u)_y|^2 \xi dy dx dt \geq 0. \tag{4.15}
\end{aligned}$$

Next, following the idea of [2] we take the test function  $\xi(t, x, y) = \theta(t) \rho_n(x, y)$ , where  $\theta \in \mathcal{C}_0^\infty(0, T)$ ,  $\theta \geq 0$ ,  $\rho_n(x, y) = \delta_n(\Delta)$  and  $\Delta = (1 - \frac{1}{n(b-a)})x - y + \frac{a+b}{2n(b-a)}$ . Then,  $\rho_n \in \mathcal{D}(\bar{\Omega} \times \bar{\Omega})$  and  $\rho_n|_{\Omega \times \partial\Omega}(x, y) = 0$ . Due to this choice

$$\int_0^T \int_{x \in \Omega} \int_{y \in \partial\Omega} |(f(v) - \phi(v)_x) \cdot \eta(y)| \rho_n \theta dy d\sigma dt = 0.$$

By Proposition 4.8,  $(f(u) - \phi(u)_y) \in \mathcal{C}_0([a, b])$ . Therefore we have  $|(f(u) - \phi(u)_y) \cdot \eta(x)| \rightarrow 0$  when  $x \rightarrow y$ , i.e, as  $n \rightarrow \infty$ . We conclude that

$$\lim_{n \rightarrow \infty} \int_0^T \int_{x \in \partial\Omega} \int_{y \in \Omega} |(f(u) - \phi(u)_y) \cdot \eta(x)| \rho_n \theta dy d\sigma dt = 0.$$

It remains to study the limit, as  $n \rightarrow \infty$

$$I = \int_0^T \int_{\Omega} \int_{\Omega} \theta \text{sign}(v - u) \left[ (f(v) - \phi(v)_x) - (f(u) - \phi(u)_y) \right] \cdot ((\rho_n)_x + (\rho_n)_y) dy dx dt.$$

By the change of variable  $(x, y) \mapsto (x, z)$  with  $z = n(x - y) - \frac{1}{b-a}x + \frac{a+b}{b-a}$ ,

$$\begin{aligned}
I &= \frac{2}{b-a} \int_{-1}^1 \int_0^T \int_{\Omega} \text{sign}(v - u) \left[ (f(v) - \phi(v)_x) - (f(u) - \phi(u)_y) \right] \cdot \delta'_n(z) \theta \\
&= \frac{2}{b-a} \int_{-1}^1 \int_0^T \int_a^b \text{sign}(v(t, x) - u_n(x, z)) \left[ p(t, x) - q_n(x, z) \right] \delta'_n(z) \theta(t),
\end{aligned}$$

where  $u_n(x, z) := u(y)$ ,  $p(t, x) := f(v) - \phi(v)_x$  and  $q_n := f(u) - \phi(u)_y$ . For  $z$  given,  $u_n(\cdot, z)$  converge to  $u(\cdot)$  in  $L^1$  and  $q_n(\cdot, z)$  converges to  $q(\cdot) := f(u) - \phi(u)_x$  in  $L^1$ . From Lemma 4.14, we deduce that for all  $z \in [-1, 1]$

$$K_n(z) := \int_Q \text{sign}(v_n(t, x, z)) h_n(t, x, z) dx dt \xrightarrow{n \rightarrow \infty} \int_Q \text{sign}(v) h dx dt =: K = \text{const},$$

where  $v_n := v - u_n$ ,  $h_n := p - q_n$  and  $h := p - q$ . Then  $K_n(\cdot)$  converge to  $K$  independently of  $z$ . Moreover, from the definition of  $K_n$  ones finds easily the uniform  $L^\infty$  bound  $|K_n| \leq 2(\|p\|_{L^1(Q)} + T\|q\|_{L^1(\Omega)})$ , for  $n$  large enough. Hence by the Lebesgue theorem,

$$\lim_{n \rightarrow \infty} \int_{-1}^1 K_n(z) \delta'(z) = K \int_{-1}^1 \delta'(z) = 0.$$

We have shown that the limit  $I \rightarrow 0$ . The passage to the limit in other terms in (4.15) is straightforward. Finally (4.15) gives for  $n \rightarrow \infty$

$$\int_0^T \int_{\Omega} |v(t, x) - u(y)| \theta'(t) dx dt + \int_0^T \int_{\Omega} \text{sign}(v - u)(u - g) \theta \geq 0.$$

Hence

$$\frac{d}{dt} \|v(t) - u\|_{L^1(\Omega)} \leq \int_{\Omega} \text{sign}(v - u)(u - g) dx \text{ in } \mathcal{D}'(0, T).$$

Thus,  $v$  is an integral solution of (4.7).  $\square$

Now, the claim of Theorem 4.1 is a direct consequence of the fact that the entropy solution is also an integral solution, and of Corollary 4.12.

## 5. Appendix: Existence of entropy solutions for the viscosity regularized problem

For the sake of completeness, we give a full proof of Theorem 3.4. By  $C$ , we denote a generic constant independent of the approximation parameters  $\epsilon$  and  $m$ , otherwise, the dependence of  $C$  is made explicit in the notation.

*Proof of Theorem 3.4.* We need four steps for this proof.

**First step:** By Faedo-Galerkin method (see e.g., [18]), we construct a sequence of approximate solutions. We choose  $V_m = \langle e_1(x), \dots, e_m(x) \rangle$  with  $(e_i)_{i=1}^{\infty}$  a regular Hilbert basis of  $H^1(\Omega)$  and formulate the problem in terms of the new unknown  $w^\epsilon = \phi_\epsilon(u^\epsilon)$ . We seek  $w_m^\epsilon(t) = \sum_{i=1}^m c_{im}(t) e_i(x)$ , then we require

$$\text{Proj}_{V_m} \left( \phi_\epsilon^{-1}(w_m^\epsilon)_t + \text{div } f(\phi_\epsilon^{-1}(w_m^\epsilon)) - \Delta w_m^\epsilon \right) = 0.$$

Here  $\text{Proj}_{V_m}$  is the orthogonal projection, in  $L^2(\Omega)$ , on the subspace  $V_m$ . The function  $\phi_\epsilon^{-1}$  is Lipschitz continuous, and  $(\phi_\epsilon^{-1})' \leq \frac{1}{\epsilon}$ . To start with, we assume that  $\phi$  is Lipschitz continuous; then  $(\phi_\epsilon^{-1})' \geq \alpha > 0$ . Then the equation rewrites as

$$\text{Proj}_{V_m} \left( (\phi_\epsilon^{-1})'(w_m^\epsilon) \sum_{i=1}^m c'_{im} e_i(x) + \text{div } \tilde{f}(w_m^\epsilon) - \sum_{i=1}^m c_{im}(t) \Delta e_i(x) \right) = 0;$$

where  $\tilde{f} = f \circ \phi_\epsilon^{-1}$ . To determine the family  $\{c_{im}\}_i \subset \mathcal{C}^1([0, T])$ , we write the weak formulation of the above equation in  $\Omega$  with  $e_j$  as test function: we get

$$\sum_{i=1}^m c'_{im}(t) \int_{\Omega} (\phi_\epsilon^{-1})'(w_m^\epsilon) e_i \cdot e_j dx - \int_{\Omega} \left( \tilde{f}(w_m^\epsilon) - \sum_{i=1}^m c_{im} \nabla e_i(x) \right) \cdot \nabla e_j dx = 0;$$

$1 \leq j \leq m$ . Recall that  $w_m^\epsilon$  depends on  $x$  and on  $(c_{im})_i$ . Notice that the matrix  $M(c_{i1}, \dots, c_{im}) = \left( \int_{\Omega} (\phi_\epsilon^{-1})'(w_m^\epsilon) e_i \cdot e_j dx \right)_{i,j}$  is invertible due to the fact that for all  $b = (b_1, \dots, b_m) \in \mathbb{R}^m$

$$(Mb, b) = \int_{\Omega} (\phi_\epsilon^{-1})'(w_m^\epsilon) \left| \sum_{i=1}^m b_i e_i \right|^2 \geq \text{const}(m) \alpha \|b\|^2.$$

We obtain a system of non-linear differential equations, which is completed with initial condition  $w_m^\epsilon(0) = w_{0m}^\epsilon$ ;  $w_{0m}^\epsilon = \sum_{i=1}^m \beta_{im} e_i$  where  $\beta_{im}$  are chosen to ensure that  $w_{0m}^\epsilon \rightarrow \phi_\epsilon(u_0^\epsilon)$  in  $L^2(\Omega)$ .

By the Cauchy-Peano theorem of the classical ODE theory, we have existence of solution  $w_m^\epsilon(t)$  in some interval  $[0, t_m]$ ,  $t_m > 0$ . Note that existence of  $u_m^\epsilon$  is ensured by the fact that  $\phi_\epsilon$  is bijective, moreover,  $w_m^\epsilon = \phi_\epsilon(u_m^\epsilon)$  and  $u_m^\epsilon$  is in  $\mathcal{C}^1(0, t_m; V_m)$ .

Now, we have to prove that  $t_m = T$ .

**Second step:** a priori estimates.

We can take  $w_m^\epsilon$  as a test function and we integrate over  $[0, t]$ ; we get

$$\int_{\Omega} \int_0^t \theta_\epsilon(w_m^\epsilon)_s ds dx - \int_0^t \int_{\Omega} f(u_m^\epsilon) \cdot \nabla w_m^\epsilon dx ds + \|\nabla w_m^\epsilon\|_{L^2(0,t;\Omega)}^2 = 0. \quad (5.1)$$

Here  $\theta_\epsilon(w_m^\epsilon) = \int_0^{w_m^\epsilon} r d\phi_\epsilon^{-1}(r)$  and we have used the chain rule for  $C^1$  functions of variable  $t$ . It is also possible to rewrite the function  $\theta_\epsilon(w_m^\epsilon)$  as  $\psi_\epsilon(u_m^\epsilon) = \int_0^{u_m^\epsilon} \phi_\epsilon(r) dr$ . Since  $f$  has its support in  $[0, u_{\max}]$ , the second integral of (5.1) can be upper bounded as follows

$$\begin{aligned} \left| \int_0^t \int_{\Omega} \operatorname{div} \left( \int_0^{u_m^\epsilon} f(r) d\phi_\epsilon(r) \right) dx ds \right| &= \left| \int_0^t \int_{\partial\Omega} \left( \int_0^{u_m^\epsilon} f(r) d\phi_\epsilon(r) \right) \cdot \eta d\mathcal{H}^{N-1} ds \right| \\ &\leq d\phi_\epsilon([0, u_{\max}]) \|f\|_{L^\infty} |\partial\Omega| T \\ &\leq (\phi(u_{\max}) + 1) \|f\|_{L^\infty} |\partial\Omega| T. \\ &\leq C = C(T, \partial\Omega, \|f\|_{L^\infty}) \end{aligned}$$

with  $d\phi_\epsilon([0, u_{\max}])$  the measure of  $[0, u_{\max}]$  with respect to the Stieltjes measure  $d\phi_\epsilon$ . Hence,

$$\int_{\Omega} \theta_\epsilon(w_m^\epsilon)(t) dx + \|\nabla w_m^\epsilon\|_{L^2(0,t;\Omega)}^2 \leq C + \int_{\Omega} \theta_\epsilon(w_m^\epsilon)(0) dx. \quad (5.2)$$

The last term in the right-hand side of inequality (5.2) is bounded uniformly in  $m$  by  $\frac{1}{\epsilon} \sup_m \|w_{0m}^\epsilon\|_{L^2}^2$ .

In fact,

$$\int_{\Omega} \theta_\epsilon(w_m^\epsilon)(0) dx = \int_{\Omega} \int_0^{w_m^\epsilon(0)} r (\phi_\epsilon^{-1})'(\sigma) dr dx \leq \frac{1}{2} \|(\phi_\epsilon^{-1})'\|_{\infty} \|w_m^\epsilon\|_{L^2(\Omega)}^2 \leq \frac{1}{\epsilon} \sup_m \|w_{0m}^\epsilon\|_{L^2(\Omega)}^2.$$

Then  $\nabla w_m^\epsilon$  is bounded in  $L^2(\Omega)$  uniformly in  $m$ .

Without loss of restriction, we can assume  $\phi \equiv 0$  on  $(-\infty, 0]$  and  $\phi \equiv \phi(u_{\max})$  on  $[u_{\max}, +\infty)$ . (Indeed, we show in the last step that  $u$  takes values in  $[0, u_{\max}]$ , therefore the values of  $\phi$  outside  $[0, u_{\max}]$  do not matter.) Then  $\phi'_\epsilon = \epsilon$  outside  $[0, u_{\max}]$ . Hence, for  $w \notin [0, \phi(u_{\max}) + 1]$ , we have  $(\phi_\epsilon^{-1})'(w) = \frac{1}{\epsilon}$ . Therefore

$$|w_m^\epsilon|^2 \leq C(\epsilon)(1 + \theta_\epsilon(w_m^\epsilon)). \quad (5.3)$$

This means that  $t_m = T$  and  $w_m^\epsilon$  is bounded in  $L^2(0, T, H^1(\Omega))$  uniformly in  $m$ .

Now, fix  $\delta$  and consider  $t$  such that  $[t, t + \delta] \subset [0, T]$ . We integrate over  $s \in [t, t + \delta]$ . Then, we take (by approximation)  $(w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot)) \mathbf{1}_{[0, T - \delta t]}$  as test function

$$\begin{aligned} &\int_0^{T - \delta t} \int_t^{t + \delta t} \int_{\Omega} (u_m^\epsilon(s, \cdot))' \left( w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot) \right) dx dt ds \\ &- \int_0^{T - \delta t} \int_t^{t + \delta t} \int_{\Omega} f(u_m^\epsilon) \cdot \nabla \left( w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot) \right) dx ds dt \\ &+ \int_0^{T - \delta t} \int_t^{t + \delta t} \int_{\Omega} \nabla w_m^\epsilon(s, x) \cdot \nabla \left( w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot) \right) dx dt ds = 0. \end{aligned} \quad (5.4)$$

Denote the three terms in the left-hand side of (5.4) by  $A$ ,  $B$  and  $D$  respectively. We calculate

$$\begin{aligned}
A &= \int_0^{T-\delta t} \int_t^{t+\delta t} \int_{\Omega} (u_m^\epsilon(s, \cdot))' (w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot)) dx dt ds \\
&= \int_0^{T-\delta t} \int_{\Omega} (w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot)) \int_t^{t+\delta t} (u_m^\epsilon(s, \cdot))' ds dx dt \\
&= \int_0^{T-\delta t} \int_{\Omega} (w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot)) (u_m^\epsilon(t + \delta t, \cdot) - u_m^\epsilon(t, \cdot)) dx dt; \tag{5.5}
\end{aligned}$$

$$\begin{aligned}
|B| &= \left| - \int_0^{T-\delta t} \int_t^{t+\delta t} \int_{\Omega} f(u_m^\epsilon(s, \cdot)) \cdot \nabla (w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot)) dx dt ds \right| \\
&\leq \left| \int_0^{T-\delta t} \int_t^{t+\delta t} \int_{\Omega} f(u_m^\epsilon(s, \cdot)) \cdot \nabla (w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot)) dx dt ds \right| \\
&\leq C(\epsilon) \delta t \left[ \int_0^{T-\delta t} \int_{\Omega} |\nabla w_m^\epsilon(t + \delta t, \cdot)| + \int_0^{T-\delta t} \int_{\Omega} |\nabla w_m^\epsilon(t, \cdot)| \right].
\end{aligned}$$

By a change of variable in the first integral ( $\tau = t + \delta t$ ), using Cauchy-Schwarz inequality we obtain

$$|B| \leq 2C(\epsilon) \delta t \|\nabla w_m^\epsilon\|_{L^2(Q)} \|1_Q\|_{L^2(Q)} \leq C(\epsilon) \delta t. \tag{5.6}$$

The last integral of (5.4) is treated similarly

$$\begin{aligned}
|D| &\leq \left| \int_0^{T-\delta t} \int_t^{t+\delta t} \int_{\Omega} \nabla w_m^\epsilon(s, \cdot) \cdot \nabla (w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot)) \right| \\
&\leq 2 \int_{\Omega} \int_0^{T-\delta t} |\nabla w_m^\epsilon(\tau, \cdot)| \left( \int_t^{t+\delta t} |\nabla w_m^\epsilon(s, \cdot)| ds \right) d\tau dx \\
&\leq 2 \|\nabla w_m^\epsilon\|_{L^2(Q)} \left( \int_{\Omega} \int_0^{T-\delta t} \left( \int_t^{t+\delta t} |\nabla w_m^\epsilon(s, \cdot)| ds \right)^2 d\tau dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Using Jensen's Inequality,

$$\left( \int_t^{t+\delta t} |\nabla(w_m^\epsilon(s, \cdot))| ds \right)^2 \leq \int_t^{t+\delta t} |\nabla(w_m^\epsilon(s, \cdot))|^2 ds.$$

We find

$$|D| \leq 2 \|\nabla w_m^\epsilon\|_{L^2(Q)} \sqrt{\int_{\Omega} \int_0^T \int_t^{t+\delta t} |\nabla w_m^\epsilon(s, \cdot)|^2} \leq C(\epsilon) \sqrt{\delta t}. \tag{5.7}$$

The sum of (5.5), (5.6) and (5.7) gives

$$\int_0^{T-\delta t} \int_{\Omega} \left| (w_m^\epsilon(t + \delta t, \cdot) - w_m^\epsilon(t, \cdot)) (u_m^\epsilon(t + \delta t, \cdot) - u_m^\epsilon(t, \cdot)) \right| \leq C(\epsilon) (\delta t + \sqrt{\delta t}).$$

Now, using the fact  $\phi_\epsilon^{-1}$  is Lipschitz, there exist another constant  $C(\epsilon)$  such that

$$\|u_m^\epsilon(t + \delta t, \cdot) - u_m^\epsilon(t, \cdot)\|_{L^2(0, T-\delta t; \Omega)} \leq C(\epsilon) (\sqrt{\delta t} + \delta t). \tag{5.8}$$

By the characterization theorem for  $H^1(\Omega)$  (see [10]), since  $V_m \subset H^1(\Omega)$ , and  $\|\nabla w_m^\epsilon\|_{L^2(Q)} \leq C$ , we have for all open subset  $\omega \subset \subset \Omega$

$$\|u_m^\epsilon(t, x + \delta x) - u_m^\epsilon(t, x)\|_{L^2(0, T; \omega)} \leq C(\epsilon) \delta x. \tag{5.9}$$



Finally, we obtain

$$\|u_m^\epsilon(t + \delta t, x + \delta x) - u_m^\epsilon(t, x)\|_{L^2(0, T - \delta t; \omega)} \leq C(\epsilon)(\sqrt{\delta t} + \delta t + \delta x). \quad (5.10)$$

**Third step:** Passage to the limit ( $m \rightarrow +\infty$ ).

By the estimate  $\|w_m^\epsilon\|_{L^2(0, T; H^1(\Omega))} \leq C(\epsilon)$ , by (5.10) and Fréchet-Kolmogorov compactness criterion,  $(u_m^\epsilon)_m$  is relatively compact in the space  $L^2(0, T; H^1(\Omega))$  weakly and in  $L^1_{\text{Loc}}((0, T) \times \Omega)$  strongly. Moreover,  $\|u_m^\epsilon\|_{L^2(Q)} \leq C(\epsilon)$  then,  $u_m^\epsilon$  is relatively compact in  $L^1(Q)$  strongly.

We can take  $\xi(t)e_i(x)$  as test function in the weak formulation where  $\xi \in \mathcal{D}[0, T]$ , for  $m \geq 0$ , we have

$$\int_0^T \int_\Omega u_m^\epsilon \xi' e_i dx dt + \int_0^T \int_\Omega \left( f(u_m^\epsilon) - \nabla w_m^\epsilon \right) \cdot \nabla e_i \xi dx dt + \int_\Omega u_{0m}^\epsilon \xi(0) e_i dx = 0.$$

We can extract a subsequence  $w_m^\epsilon$  such that  $\nabla w_m^\epsilon \rightharpoonup \nabla w^\epsilon$  in  $L^2(Q)$  and  $u_m^\epsilon \rightarrow u^\epsilon$  in  $L^1(Q)$  and a.e.. The Lebesgue theorem, continuity and boundedness of  $f$  permit at last to pass to the limit. Finally we conclude that (2.2) holds, this means that  $u^\epsilon$  is a weak solution of  $(P_\epsilon)$  by the density of the linear span of  $\mathcal{D}(0, T) \times \mathcal{D}(\Omega)$  in  $\mathcal{D}([0, T] \times \Omega)$ .

At this point, we can also drop the Lipschitz continuity assumption on  $\phi$ . Indeed, approximating  $\phi$  with a sequence of Lipschitz continuous functions  $\phi_\alpha$ , on as uniform in  $\alpha$  estimates (5.10). Then one can pass to the limit as  $\alpha$  goes to zero in the equation corresponding to  $\phi = \phi_\alpha$ , with the same argument as above.

It remains to prove that  $u^\epsilon$  is an entropy solution.

**Fourth step:** Now, we prove that weak solution of  $(P_\epsilon)$  is also an entropy sub-solution and entropy super-solution.

Since  $u^\epsilon$  is a weak solution of  $(P_\epsilon)$ , then  $u^\epsilon$  is also a weak solution of the following problem

$$(P_{k, \epsilon}) \begin{cases} (u^\epsilon - k)_t + \operatorname{div} \left[ f(u^\epsilon) - \nabla \phi_\epsilon(u^\epsilon) - f(k) \right] = 0 & \text{in } Q, \\ u^\epsilon(0, x) - k = u_0^\epsilon(x) - k & \text{in } \Omega, \\ \left( f(u^\epsilon) - \nabla \phi_\epsilon(u^\epsilon) - f(k) \right) \cdot \eta = -f(k) \cdot \eta & \text{on } \Sigma. \end{cases}$$

Take  $\operatorname{sign}_\sigma^+(u^\epsilon - k)\xi$  in the weak formulation of this problem with  $\xi \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^N)$ ,  $\xi \geq 0$ .

We get (see [1] and [15] for the use of  $H^1(\Omega)^* - H^1(\Omega)$  duality)

$$\begin{aligned} & \int_0^T \langle (u^\epsilon - k)_t, \operatorname{sign}_\sigma^+(u^\epsilon - k)\xi \rangle_{H^1(\Omega)^*, H^1(\Omega)} - \int_0^T \int_{\partial\Omega} \operatorname{sign}_\sigma^+(u^\epsilon - k) f(k) \cdot \eta \xi \\ & - \int_0^T \int_\Omega \left[ f(u^\epsilon) - \nabla \phi_\epsilon(u^\epsilon) - f(k) \right] \xi \nabla \operatorname{sign}_\sigma^+(u^\epsilon - k) \\ & - \int_0^T \int_\Omega \operatorname{sign}_\sigma^+(u^\epsilon - k) \left[ f(u^\epsilon) - \nabla \phi_\epsilon(u^\epsilon) - f(k) \right] \cdot \nabla \xi = 0. \end{aligned} \quad (5.11)$$

The first integral of 5.11 gives (see [1] and [15] for the use of Chain rule)

$$\begin{aligned} \int_0^T \langle (u^\epsilon - k)_t, \operatorname{sign}_\sigma^+(u^\epsilon - k)\xi \rangle &= - \int_0^T \int_\Omega \left\{ \int_k^{u^\epsilon} \operatorname{sign}_\sigma^+(r - k) dr \right\} \xi_t dx dt \\ & - \int_\Omega \left\{ \int_k^{u_0^\epsilon} \operatorname{sign}_\sigma^+(r - k) dr \right\} \xi(0, x) dx. \end{aligned}$$

Passing to the limit as  $\sigma$  goes to 0, we obtain:

$$\begin{aligned} & \int_0^T \int_\Omega \left\{ \int_k^{u^\epsilon} \operatorname{sign}_\sigma^+(r - k) dr \right\} \xi_t dx dt \longrightarrow \int_0^T \int_\Omega (u^\epsilon - k)^+ \xi_t dx dt, \\ & \int_\Omega \left\{ \int_k^{u_0^\epsilon} \operatorname{sign}_\sigma^+(r - k) dr \right\} \xi(0, x) dx \longrightarrow \int_\Omega (u_0^\epsilon - k)^+ \xi(0, x) dx. \end{aligned}$$

The limit of the second integral of (5.11) can be upper bounded as follows

$$\lim_{\sigma \rightarrow 0} \int_0^T \int_{\partial\Omega} \text{sign}_\sigma^+(u^\epsilon - k) f(k) \cdot \eta \xi \leq \int_0^T \int_{\partial\Omega} (f(k) \cdot \eta(x))^+ \xi.$$

The third integral of (5.11) writes

$$- \iint_{\Omega} \xi \nabla \phi_\epsilon(u^\epsilon) \cdot \nabla \text{sign}_\sigma^+(u^\epsilon - k) + \iint_{\Omega} \xi (f(u^\epsilon) - f(k)) \cdot \nabla \text{sign}_\sigma^+(u^\epsilon - k).$$

Here, the first integral is non-positive. Moreover, the second one tends to zero as  $\sigma \rightarrow 0$ . In fact, to see this, we set  $F_\sigma(r) = \int_{k-\sigma}^r (f(s) - f(k)) \text{sign}_\sigma^+(s - k) ds$ . We have  $|F_\sigma(r)| \leq 2 \sup_{|k-s| < \sigma} |f(s) - f(k)|$ . Using now the Green-Gauss formula, we find

$$\begin{aligned} & \left| \int_{\Omega} \text{sign}_\sigma^+(u^\epsilon - k) \nabla u^\epsilon \cdot (f(u^\epsilon) - f(k)) \xi \right| = \left| \int_{\Omega} \text{div}(F_\sigma(u^\epsilon)) \xi \right| \\ & \leq 2 \sup_{|k-s| < \sigma} |f(s) - f(k)| \left( \int_{\Omega} |\nabla \xi| + \int_{\partial\Omega} |\xi| \right) \rightarrow 0 \text{ as } \sigma \rightarrow 0. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ (u^\epsilon - k)^+ \xi_t + \text{sign}^+(u^\epsilon - k) [f(u^\epsilon) - \nabla \phi_\epsilon(u^\epsilon) - f(k)] \cdot \nabla \xi \right\} dx dt \\ & + \int_{\Omega} (u_0 - k)^+ \xi(0, x) dx + \int_0^T \int_{\partial\Omega} (f(k) \cdot \eta(x))^+ \xi d\mathcal{H}^{n-1} dt \geq 0. \end{aligned}$$

Therefore  $u^\epsilon$  is entropy sub-solution of  $(P_\epsilon)$ . In the same way, we prove that  $u^\epsilon$  is entropy super-solution of  $(P_\epsilon)$ .

Now we prove that  $u^\epsilon$  is bounded. To this aim take  $\xi = \xi(t)$ , (i.e.,  $\nabla \xi = 0$ ), take  $k = 0$  in (2.3), and use (1.1) and the fact that  $u_0^\epsilon \in [0, u_{\max}]$ . We get

$$\int_{\Omega} (u^\epsilon)^-(0, \cdot) \xi(0, \cdot) + \int_0^T \int_{\Omega} (u^\epsilon)^-(t, x) \xi_t = \int_0^T \left( \int_{\Omega} \left( (u^\epsilon)^-(t, x) - (u_0^\epsilon)^-(x) \right) \right) \xi_t \geq 0.$$

Let us introduce the function

$$G(t) \leq \begin{cases} \int_{\Omega} \left( (u^\epsilon)^-(t, x) - (u_0^\epsilon)^-(x) \right) dx & \text{for } t \in (0, T), \\ 0 & \text{for } t \in (-T, 0). \end{cases}$$

We have  $\frac{dG}{dt} \leq 0$  in  $\mathcal{D}'(-T, T)$ , and therefore since  $G(t)$  vanishes for  $t < 0$ , we deduce

$$\int_{\Omega} (u^\epsilon)^-(t, x) dx \leq \int_{\Omega} (u_0^\epsilon)^-(x) dx = 0, \text{ i.e., } u^\epsilon(t, x) \geq 0.$$

In the same way, we get  $u^\epsilon(t, x) \leq u_{\max}$ .

Now, we go back to the technique used to get (5.2), recall that we can rewrite  $\theta_\epsilon(w^\epsilon)$  as  $\psi(u^\epsilon)$ . We find

$$\int_{\Omega} \psi_\epsilon(u^\epsilon)(t) + \|\nabla w^\epsilon\|_{L^2(0,t;\Omega)}^2 \leq C + \int_{\Omega} \psi_\epsilon(u^\epsilon)(0). \quad (5.12)$$

The last term is now bounded uniformly in  $\epsilon$ , due to the  $L^\infty$  bound on  $u^\epsilon$ . Therefore,

$$\|w^\epsilon\|_{L^2(0,T;H^1(\Omega))} \leq C, \quad (5.13)$$

with  $C$  that is now  $\epsilon$ -independent.

Finally, if we take  $u^\epsilon$  as test function in (3.1), we find

$$\frac{1}{2} \|u^\epsilon(t)\|_{L^2(\Omega)}^2 + \|\sqrt{\epsilon} \nabla u^\epsilon\|_{L^2(Q)}^2 \leq C + \frac{1}{2} \|u_0^\epsilon\|_{L^2(\Omega)}^2.$$

Then

$$\|\sqrt{\epsilon}\nabla u^\epsilon\|_{L^2(Q)}^2 \leq C. \quad (5.14)$$

This concludes the proof of the Theorem  $\square$

## References

- [1] H.W. Alt, S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*. Math. Z. **183** (3) (1983) 311-341.
- [2] B. Andreianov, F. Bouhsiss, *Uniqueness for an elliptic-parabolic problem with Neumann boundary condition*. J. Evol. Equ. **4** (2004) 273-295.
- [3] B. Andreianov, M. Karimou Gazibo, G. Vallet *Strong boundary conditions for entropy solutions of the Dirichlet problem for parabolic-hyperbolic balance law in preparation*.
- [4] B. Andreianov, N. Igbida, *On uniqueness techniques for degenerate convection-diffusion problems*. Int. J. Dyn. Syst. Diff. Equ. , accepted ; available at <http://hal.archives-ouvertes.fr/hal-00553819>, version 1.
- [5] B. Andreianov, K. Shibi, *Scalar conservation laws with nonlinear boundary conditions* C. R. Acad. Paris. **345** (8) (2007) 431-434.
- [6] C. Bardos, A.Y. Le Roux, J.C. Nedelec, *First order quasilinear equations with boundary conditions*. Comm. PDE. **4** (1979) 1017-1034.
- [7] Ph. Bénilan, Crandall, M. G. and Pazy, A., *Nonlinear evolution equations in Banach spaces*. Preprint book.
- [8] Ph., Bénilan, H., Toure, *Sur l'équation générale  $u_t = a(\cdot, u, \phi(\cdot, u)_x)_x + v$  dans  $L^1$* . I. Etude du probleme stationnaire, In Evolution equations (Baton Rouge, LA, 1992). pp. 35.62. Dekker, New York, 1995.
- [9] Ph. Bénilan, P. Wittbold, *On mild and weak solutions of elliptic-parabolic problems*. Adv. Differential Equations. **1** (6) (1996) 1053-1073.
- [10] H. Brezis, *Functional Analysis, Sobolev Space and Partial Differential Equation*. Springer-Verlag New York Inc, (2010).
- [11] R. Bürger, H. Frid, K. H. Karlsen, *On the well-posedness of entropy solution to conservation laws with a zero-flux boundary condition*. J. Math. Anal. Appl. **326** (2007), 108-120.
- [12] R. Bürger, H. Frid, K. H. Karlsen, *On a free boundary problem for a strongly degenerate quasilinear parabolic equation with an application to a model of presssure filtration*. SIAM J. Math. Anal. **34** (2003) 611-635.
- [13] R. Bürger, S. Evje and K.H. Karlsen, *On strongly degenerate convection diffusion problems modeling sedimentation-consolidation process*. J.Math. Anal. Appli. **247** (2000), 517-556.
- [14] C. Cancés, T. Gallouët, *On the time continuity of entropy solutions*. J. Evol Equ **11** (2011) 43-45.
- [15] G. Gagneux , M. Madaune-Tort, *Analyse mathématique de modeles nonlinéaires de l'ingénerie pétroliere; Mathématique et Applications, 22*. Springer-Verlag, Berlin, (1996).
- [16] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*. Arch. Ration. Mech. Anal. **147** (4) (1999) 269-361.
- [17] S.N. Kruzhkov, *First order quasi-linear equations in several independent variables*. Math. USSR Sb. **10** (2) (1970) 217-243.
- [18] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, (1969).
- [19] C. Mascia, A. Porretta, A. Terracina, *Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equation*. Arch. Ration. Mech. Anal. **163** (2) (2002) 87-124.
- [20] M. Marcus, V.J. Mizel, *Absolute continuity on Tracks and Mappings of Sobolev Spaces*. Arch. Ration Mech. Anal. **45**, (4) (1972), 294-320.
- [21] A. Michel, J. Vovelle, *Entropy formulation for parabolic degenerate equations with general Dirichlet boundary conditions and application to the convergence of FV methods*. SIAM J. Numer. Anal. **41** (2003) 2262-2293.
- [22] E.Yu. Panov, *On the strong pre-compactness property for entropy solutions of a degenerate elliptic equation with discontinuous flux*. J. Differential Equations. **247** (2009) 2821-2870.
- [23] E. Rouvre and G. Gagneux, *Formulation forte entropique de lois scalaires hyperboliques-paraboliques dgn-res*. An. Fac. Sci. Toulouse. **X** (1) (2001) 163-183.

- [24] G. Vallet, *Dirichlet problem for a degenerated hyperbolic-parabolic equation*. Advance in Math. Sci. Appl. **15** (2) (2005)
- [25] A. Vasseur, *Strong traces for solutions of multidimensional scalar conservation laws*. Arch. Ration. Mech. Anal. **160** (2001) 181-193.

Boris Andreianov  
Laboratoire de Mathématiques  
CNRS : UMR 6623 - Université de Franche-Comte  
16, route de Gray  
25030 Besancon France  
e-mail: [boris.andreianov@univ-fcomte.fr](mailto:boris.andreianov@univ-fcomte.fr)

Mohamed Karimou Gazibo  
Laboratoire de Mathématiques  
CNRS : UMR 6623 - Université de Franche-Comte  
16, route de Gray  
25030 Besancon France  
e-mail: [mgazibok@univ-fcomte.fr](mailto:mgazibok@univ-fcomte.fr)