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ABSTRACT. We prove that the sequence $(e_n^{\mathfrak{S}})_{n\geq 0}$ of excursions in the quarter plane corresponding to a nonsingular step set $\mathfrak{S} \subseteq \{0,\pm 1\}^2$ with infinite group does not satisfy any nontrivial linear recurrence with polynomial coefficients. Accordingly, in those cases, the trivariate generating function of the numbers of walks with given length and prescribed ending point is not D-finite. Moreover, we display the asymptotics of $e_n^{\mathfrak{S}}$.

1. INTRODUCTION

1.1. **General context.** Counting walks in a fixed region of the lattice $\mathbb{Z}^d$ is a classical problem in probability theory and in enumerative combinatorics [48, 52, 26, 47, 46, 19, 16, 27, 56, 57, 6, 38, 5]. In recent years, the case of walks restricted to the quarter plane $\mathbb{N}^2 = \{(i,j) \in \mathbb{Z}^2 \,|\, i \geq 0,\, j \geq 0\}$ has received special attention, and much progress has been done on this topic [35, 41, 32, 31, 24, 13, 12, 11, 43, 9, 34, 44, 45, 14, 22, 18, 23, 37, 36, 25, 49, 7]. Given a set $\mathfrak{S}$ of allowed steps, the general problem is to study $\mathfrak{S}$-walks in the quarter plane $\mathbb{N}^2$, that is walks confined to $\mathbb{N}^2$, starting at $(0,0)$ and using steps in $\mathfrak{S}$ only. Denoting by $f_{\mathfrak{S}}(i,j,n)$ the number of such walks that end at (i,j) and use exactly n steps, the main high-level objective is to *understand* the generating function

$$F_{\mathfrak{S}}(x,y,t) = \sum_{i,j,n\geq 0} f_{\mathfrak{S}}(i,j,n) x^i y^j t^n \ \in \mathbb{Q}[[x,y,t]],$$

since this *continuous* object captures a great amount of interesting combinatorial information about the *discrete* object $f_{\mathfrak{S}}(i,j,n)$. For instance, the specialization $F_{\mathfrak{S}}(1,1,t)$ counts $\mathfrak{S}$-walks with prescribed length, the specialization $F_{\mathfrak{S}}(1,0,t)$ counts $\mathfrak{S}$-walks ending on the horizontal axis, and the specialization $F_{\mathfrak{S}}(0,0,t)$ counts $\mathfrak{S}$-walks returning to the origin, called $\mathfrak{S}$-*excursions*.

1.2. **Questions.** From the combinatorial point of view, the ideal goal would be to find a closed form expression for $f_{\mathfrak{S}}(i,j,n)$, or at least for $F_{\mathfrak{S}}(x,y,t)$. This is not possible in general, even if one restricts to particular step sets $\mathfrak{S}$. Therefore, it is customary to address more modest, still challenging, questions such as: determine the asymptotic behavior of the sequence $f_{\mathfrak{S}}(i,j,n)$; determine the structural properties of $F_{\mathfrak{S}}(x,y,t)$: is it *algebraic* (that is, root of a polynomial in $\mathbb{Q}[x,y,t,T]$)? is it *D-finite*? (in one variable t this means solution of a linear differential equation with coefficients in $\mathbb{Q}[t]$; in several variables the appropriate generalization [40] is that the set of all partial derivatives spans a finite-dimensional vector space over $\mathbb{Q}(x,y,t)$). These two questions are related, since the asymptotic behavior of the coefficient sequence of a power series is well understood for algebraic and D-finite power series [29].

1.3. **Main result.** In this work, we prove that the generating function

$$F_{\mathfrak{S}}(0,0,t) = \sum_{n\geq 0} e_n^{\mathfrak{S}} t^n \ \in \mathbb{Q}[[t]]$$

of the sequence $(e_n^{\mathfrak{S}})_{n\geq 0}$ of $\mathfrak{S}$-excursions *is not D-finite* for a large class of walks in the quarter plane. Precisely, this large class corresponds to all *small step* sets $\mathfrak{S} \subseteq \{0,\pm 1\}^2$ for which a certain group $G(\mathfrak{S})$ of birational transformations *is infinite*, with the exception of a few cases for which

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$F_{\mathfrak{S}}(0,0,t)=1$ is trivially D-finite (these exceptional cases are called *singular*, see below.) If $\chi=\chi_{\mathfrak{S}}$ denotes the *characteristic polynomial* of the step set $\mathfrak{S}$ defined by

$$\chi(x,y)=\sum_{(i,j)\in\mathfrak{S}} x^i y^j \ \in \mathbb{Q}[x,x^{-1},y,y^{-1}],$$

then the group $G(\mathfrak{S})$ is defined [24, 14] as a group of rational automorphisms of $\mathbb{Q}(x,y)$ that leave invariant the (Laurent) polynomial $\chi(x,y)$. Up to some equivalence relations, there are 51 cases of nonsingular step sets in $\mathbb{N}^2$ with an infinite group. They are depicted in Table 1 in Appendix 3. With these definitions, our main result can be stated as follows.

Theorem 1. *Let $\mathfrak{S}\subseteq\{0,\pm1\}^2$ be any of the 51 nonsingular step sets in $\mathbb{N}^2$ whose group is infinite. Then the generating function $F_{\mathfrak{S}}(0,0,t)$ of $\mathfrak{S}$-excursions is not D-finite. Equivalently, the sequence of excursions $(e_n^{\mathfrak{S}})_{n\geq 0}$ does not satisfy any nontrivial linear recurrence with polynomial coefficients. In particular, the full generating function $F_{\mathfrak{S}}(x,y,t)$ is not D-finite.*

1.4. **Previous results.**

1.4.1. *Structural properties.* While it is known that unrestricted walks in $\mathbb{Z}^2$ have rational generating functions and walks restricted to a half-plane in $\mathbb{Z}^2$ have algebraic generating functions [3], a first intriguing result about walks in the quarter plane is that their generating functions need not be algebraic, and not even D-finite. For instance, Bousquet-Mélou and Petkovšek [12] proved that this is the case for the so-called *knight walk* with $\mathfrak{S}=\{(2,-1),(-1,2)\}$. (This actually constitutes one of the initial motivations to the study of walks evolving in the quarter plane.) It was later shown [45] that this remains true even if one restricts to *next nearest neighbor walks*: there still exist step sets $\mathfrak{S}\subseteq\{0,\pm1\}^2$, for instance $\mathfrak{S}=\{(-1,1),(1,1),(1,-1)\}$, such that the series $F_{\mathfrak{S}}(x,y,t)$ is not D-finite.

We restrict in the remaining of this text to *next nearest neighbor walks*, and reserve the wording *small step walk* to such particular walks in the quarter plane. Several sporadic cases of small step walks with D-finite generating functions have been known for a long time; among them, the most popular ones are Kreweras' walks [35, 11] for which $\mathfrak{S}=\{(-1,0),(0,-1),(1,1)\}$, and Gouyou-Beauchamps' walks [32] for which $\mathfrak{S}=\{(1,0),(-1,0),(-1,1),(1,-1)\}$. A whole class of small step walks with D-finite generating functions was first identified in [12, §2]: this class contains step sets that admit a vertical (or a horizontal) symmetry. Another class, including $\mathfrak{S}=\{(0,1),(-1,0),(1,-1)\}$ and $\mathfrak{S}=\{(0,1),(-1,0),(1,-1),(0,-1),(1,0),(-1,1)\}$, corresponds to step sets that are left invariant by a Weyl group and whose walks are confined to a corresponding Weyl chamber [31].

A first systematic classification of small step walks with respect to D-finiteness was then undertaken by Mishna [43, 44], but only for step sets of cardinality at most three. A complete, still conjectural, classification was obtained in [9] using computer algebra tools. Almost simultaneously, Bousquet-Mélou and Mishna [14] rigorously proved that among the 2^8 possible cases of small step walks there are exactly 79 inherently different cases of walks in the quarter plane, and they identified among them 22 cases of step sets $\mathfrak{S}$ having a D-finite generating function $F_{\mathfrak{S}}(x,y,t)$.

A 23rd case, namely $\mathfrak{G}=\{(1,0),(-1,0),(1,1),(-1,-1)\}$, known as *Gessel walks*, was discovered and proved to be D-finite, and even algebraic, in [10], using computer algebra techniques. It was proved afterwards in [22] by a different approach that for any fixed value $t_0\in(0,1/4)$, the bivariate generating function $F_{\mathfrak{G}}(x,y,t_0)$ for Gessel walks is algebraic over $\mathbb{R}(x,y)$. To our knowledge, there is currently no "purely mathematical" proof of the algebraicity (and even of the D-finiteness) of the full generating function $F_{\mathfrak{G}}(x,y,t)$.

Bousquet-Mélou and Mishna [14] showed that the 23 cases of step sets $\mathfrak{S}$ with D-finite generating function $F_{\mathfrak{S}}(x,y,t)$ correspond to walks possessing a finite group $G(\mathfrak{S})$. Informally speaking, the *group of a walk* is a notion that captures symmetries of the step set and that is used to generalize a

classical technique in lattice combinatorics called the "reflection principle" [26, Ch. III.1]. Moreover, Bousquet-Mélou and Mishna [14] conjectured that the 56 remaining models with infinite group have non-D-finite generating functions $F_{\mathfrak{S}}(x, y, t)$. This was recently proved in [37] for the 51 *nonsingular walks*, that is, for walks having at least one step from the set $\{(-1, 0), (-1, -1), (0, -1)\}$. It is worth mentioning that for these 51 models, the non-D-finiteness of $F_{\mathfrak{S}}(x, y, t)$ was obtained as a consequence of the non-D-finiteness of this series as a function of x, y. Further, before our work, nothing was known about the non-D-finiteness of $F_{\mathfrak{S}}(0, 0, t)$. Two out of the five singular walks were already shown to have non-D-finite generating series in [45]. Therefore, before this work, there still remained 3 cases of singular walks for which the generating function $F_{\mathfrak{S}}(x, y, t)$ was suspected, but not yet proved, to be non-D-finite [42].

1.4.2. *Closed form expressions.* Kreweras [35] and Gouyou-Beauchamps [32] found explicit counting formulas for the walks named after them. For Gessel walks, an explicit formula for excursions was conjectured around 2000 by Ira Gessel and proved by Kauers, Koutschan and Zeilberger in [34]. A closed form expression (in terms of nested radicals) for the full Gessel generating function $F_{\mathfrak{S}}(x, y, t)$ was proved in [10]. Some other explicit formulas for $f_{\mathfrak{S}}(i, j, n)$ and for $F_{\mathfrak{S}}(x, y, t)$, or some of their specializations, have been obtained in cases when $\mathfrak{S}$ admits a finite group [14]. A different type of explicit expressions (integral representations) for the generating function of Gessel walks was obtained in [36]. The approach of [36] was later generalized in [49] to all the 74 nonsingular walks. Finally, the article [7] provides unified formulas in terms of Gaussian hypergeometric functions for all D-finite transcendental series $F_{\mathfrak{S}}(x, y, t)$. Again, no "purely human" (computer-free) proof of this result is known yet.

1.4.3. *Asymptotics.* Concerning asymptotics, conjectural results were displayed in [9] for the coefficients of $F_{\mathfrak{S}}(1, 1, t)$ when this latter function is D-finite. Some of these conjectures have been proved in [14]. Explicit asymptotics for the coefficients of $F_{\mathfrak{S}}(0, 0, t)$ and $F_{\mathfrak{S}}(1, 1, t)$ were conjectured even in non-D-finite cases in some unpublished tables [8]. In a recent work, Denisov and Wachtel [18] obtained explicit expressions for the asymptotics of excursions $F_{\mathfrak{S}}(0, 0, t)$ in a much broader setting; in particular, their results provide (up to a constant) the dominating term in the asymptotics of the n-th coefficient of $F_{\mathfrak{S}}(0, 0, t)$ in terms of the step set. Even more recently, Fayolle and Raschel [25] showed that the dominant singularities of $F_{\mathfrak{S}}(0, 0, t)$, $F_{\mathfrak{S}}(1, 0, t)$ and $F_{\mathfrak{S}}(1, 1, t)$ are algebraic numbers, and announced more general and precise results about asymptotics of coefficients of $F_{\mathfrak{S}}(0, 0, t)$, $F_{\mathfrak{S}}(1, 0, t)$ and $F_{\mathfrak{S}}(1, 1, t)$ [21].

2. Probability, Number Theory and Algorithms

2.1. **Contributions.** In the present work, we prove the non-D-finiteness of the generating series of excursions $F_{\mathfrak{S}}(0, 0, t)$ for the 51 cases of nonsingular walks with infinite group. As a corollary, we deduce the non-D-finiteness of the full generating function $F_{\mathfrak{S}}(x, y, t)$ for those cases. Indeed, D-finite series are closed under Hadamard product [40] and $1/(1-t)$ is clearly D-finite with respect to x, y, t. This corollary has been already obtained in [37], but the approach here is at the same time simpler, and delivers a more accurate information. This new proof only uses asymptotic information about the coefficients of $F_{\mathfrak{S}}(0, 0, t)$, and arithmetic information about the constrained behavior of the asymptotics of these coefficients when their generating function is D-finite. More precisely, we first give consequences of the general results in [18] in the case of walks in the quarter plane. If $e_n = e_n^{\mathfrak{S}}$ denotes the number of excursions of length n using only steps in $\mathfrak{S}$, this analysis implies that, when n tends to infinity, e_n behaves like $K \cdot \rho^n \cdot n^\alpha$, where $K = K(\mathfrak{S}) > 0$ is a real number, $\rho = \rho(\mathfrak{S})$ is an algebraic number, and $\alpha = \alpha(\mathfrak{S})$ is a real number such that $c = \cos(\frac{\pi}{1+\alpha})$ is an algebraic number. Moreover, explicit real approximations for ρ, α and c can be determined to arbitrary precision, and exact minimal polynomials of ρ and c can be determined algorithmically starting from the step set $\mathfrak{S}$. For the 51 cases of nonsingular walks with infinite group, this enables

us to prove that the constant $\alpha = \alpha(\mathfrak{S})$ is not a rational number. The proof amounts to checking that some explicit polynomials in $\mathbb{Q}[t]$ are not cyclotomic. To conclude, we use a classical result in the arithmetic theory of linear differential equations [20, 2, 30] about the asymptotic behavior of an integer-valued, exponentially bounded D-finite sequence, stating that if such a sequence grows like $K \cdot \rho^n \cdot n^\alpha$, then α is necessarily a *rational number*.

> *In summary, our approach brings together (consequences of) a strong probabilistic result* [18] *and a strong arithmetic result* [20, 30]*, and demonstrates that this combination allows for the* algorithmic *certification of the non-D-finiteness of the generating series of excursions $F_{\mathfrak{S}}(0,0,t)$ in the 51 cases of nonsingular walks with infinite group.*

2.2. **Number theory.** It is classical that, in many cases, transcendence of a complex function can be recognized by simply looking at the local behavior around its singularities, or equivalently at the asymptotic behavior of its Taylor coefficients. This is a consequence of the Newton-Puiseux theorem and of transfer theorems based on Cauchy's integral formula, see, e.g., [28, §3] and [29, Ch. VII.7]. For instance, if $(a_n)_{n\geq 0}$ is a sequence whose asymptotic behaviour has the form $K\cdot\rho^n\cdot n^\alpha$ where either the *growth constant* ρ is transcendental, or the *singular exponent* α is irrational or negative integer, then the generating series $A(t) = \sum_{n\geq 0} a_n t^n$ is not algebraic.

A direct application of this criterion to our case allows to show at a glance that the generating series for excursions in the 51 cases of nonsingular walks with infinite group are *transcendental*.

Similar (stronger, though less known) results, originating from the arithmetic theory of linear differential equations, also allow to detect non-D-finiteness of power series by using asymptotics of their coefficients. This is a consequence of the theory of G-functions [1, 20], introduced by Siegel almost a century ago in his work on diophantine approximations [51].

We will only use a corollary of this theory, which is well-suited to applications in combinatorics.

Theorem 2. *Let $(a_n)_{n\geq 0}$ be an integer-valued sequence whose n-th term a_n behaves asymptotically like $K \cdot \rho^n \cdot n^\alpha$, for some real constant $K > 0$. If the* growth constant *ρ is transcendental, or if the* singular exponent *α is irrational, then the generating series $A(t) = \sum_{n\geq 0} a_n t^n$ is not D-finite.*

Proof. This result is more or less classical, but we could not find its exact statement in the literature.

Classical results by Birkhoff-Trjitzinsky [4] and Turrittin [53] imply that if the n-th coefficient of a D-finite power series is asymptotic to $K \cdot \rho^n \cdot n^\alpha$, then ρ and α are necessarily algebraic numbers.

The difficult part of Theorem 2 is that irrationality of the singular exponent implies non-D-finiteness, *under the integrality assumption* of coefficients. The only proof that we are aware of uses the fact that any D-finite power series with integer-valued and exponentially bounded coefficients is a *G-function*. It relies on the combination of several strong arithmetic results. First, the Chudnovsky-André theorem [15, 1] states that the minimal order linear differential operator satisfied by a G-function is *globally nilpotent*. Next, Katz's theorem [33] shows that the global nilpotence of a differential operator implies that all of its singular points are *regular singular* points with *rational exponents*.

We refer to [20] for more details on this topic, and to [30] for a brief and elementary account. □

2.3. **Probability.** Random processes in cones have arisen a great interest in the mathematical community, as they appear in several distinct domains: quantum random walks, random matrices, noncolliding random walks, etc. Accordingly, many results are available concerning the exit times from cones of the Brownian motion, and more recently for random walks in discrete time. For our purpose, the setting is the following: given a sequence $(X_1(k), X_2(k))_{k\geq 1}$ of independent and identically distributed (i.i.d.) random variables such that for all step $s \in \mathfrak{S}$ and all $k \geq 1$, $\mathbb{P}[(X_1(k), X_2(k)) = s] = 1/|\mathfrak{S}|$, the exit time τ is the hitting time of the boundary of the (translated

positive) quarter plane $(\{-1\}\cup\mathbb{N})^2$, i.e., $\tau=\inf\{n\geq 1:\sum_{k=1}^n X_1(k)=-1 \text{ or } \sum_{k=1}^n X_2(k)=-1\}$. Now, the standard relation between probability and counting reads

$$\mathbb{P}\left[\sum_{k=1}^{n}(X_1(k),X_2(k))=(i,j),\ \tau>n\right]=\frac{f_{\mathfrak{S}}(i,j,n)}{|\mathfrak{S}|^n}. \tag{1}$$

The asymptotic behavior of this probability, as the time n goes to infinity, is called a *local limit theorem*. This is now well understood for a very large class of random walks and cones [18, 55]. To the best of our knowledge, even in the special case of the small step nonsingular random walks in the quarter plane, the precise asymptotic behavior of (1) was not known before [18]. The corresponding result is the following.

Theorem 3 ([18]). *Let $\mathfrak{S}\subseteq\{0,\pm1\}^2$ be the step set of a nonsingular walk in the quarter plane $\mathbb{N}^2$. Let $e_n=e_n^{\mathfrak{S}}$ denote the number of excursions of length n using only steps in $\mathfrak{S}$, and let $\chi=\chi_{\mathfrak{S}}$ denote the characteristic polynomial $\sum_{(i,j)\in\mathfrak{S}}x^iy^j\in\mathbb{Q}[x,x^{-1},y,y^{-1}]$ of the step set $\mathfrak{S}$. Then*

- *if the walk is aperiodic,*
$$e_n\sim K\cdot\rho^n\cdot n^{\alpha},$$
- *if the walk is periodic (then of period 2),*
$$e_{2n}\sim K\cdot\rho^{2n}\cdot(2n)^{\alpha},\quad e_{2n+1}=0,$$

where in both cases, $K=K_{\mathfrak{S}}$ is a real constant, $\rho=\rho_{\mathfrak{S}}$ is an algebraic number, and $\alpha=\alpha_{\mathfrak{S}}$ is a real number such that $\cos(\frac{\pi}{1+\alpha})$ is an algebraic number. Moreover, the system

$$\frac{\partial\chi}{\partial x}=\frac{\partial\chi}{\partial y}=0 \tag{2}$$

has a unique solution $(x_0,y_0)\in\mathbb{R}^2_{>0}$. It is such that $\rho=\chi(x_0,y_0)$ and $\alpha=-1-\pi/\arccos(-c)$, where c is the algebraic number

$$c=\frac{\frac{\partial^2\chi}{\partial x\partial y}}{\sqrt{\frac{\partial^2\chi}{\partial x^2}\cdot\frac{\partial^2\chi}{\partial y^2}}}(x_0,y_0). \tag{3}$$

Note that actually, the precise lower and upper bounds for the probability in (1) obtained previously by Varopoulos [55] (see in particular [55, Theorem 4]) would be sufficient for our needs, as they already provide the values of ρ and α.

Proof of Theorem 3. This is a rewriting of results in [18]. Consider first an aperiodic step set $\mathfrak{S}$ and a random walk with increments $(Y_1(k),Y_2(k))_{k\geq1}$ such that for all $s\in\mathfrak{S}$ and all $k\geq1$,

$$\mathbb{P}[(Y_1(k),Y_2(k))=s]=\frac{\exp(\langle h,s\rangle)}{R(h)},\qquad R(h)=\sum_{s\in\mathfrak{S}}\exp(\langle h,s\rangle), \tag{4}$$

where $h=(h_1,h_2)$ is fixed so that $\mathbb{E}[(Y_1(k),Y_2(k))]=0$. It turns out that for all nonsingular cases, this uniquely determines h (the uniqueness follows from the strict convexity of the Laplace transform $R(h)$, see [57, Ch. 8]). With this notation, [18, Formula (9)] reads

$$f_{\mathfrak{S}}(i,j,n)=C((0,0),(i,j))R(h)^n n^{-p-d/2}(1+o(1)),\qquad n\to\infty, \tag{5}$$

where

- $C((0,0),(i,j))$ is some constant (in fact, the product of certain harmonic functions evaluated at the starting and ending points $(0,0)$ and (i,j));
- d is the dimension of the lattice where the random walk evolves, here equal to 2;
- $p=\pi/\arccos(-r)$, with $r=\mathbb{E}[Y_1(k)Y_2(k)]/\sqrt{\mathbb{E}[Y_1(k)^2]\cdot\mathbb{E}[Y_2(k)^2]}$, see [18, Example 2].

To prove Theorem 3, we thus have to show that $R(h) = \chi(x_0, y_0)$, and that $r = c$, with c as in (3).

We start with proving that $R(h) = \chi(x_0, y_0)$. The law of the increments $(Y_1(k), Y_2(k))$ of the random walk has the expectation

$$\mathbb{E}[(Y_1(k), Y_2(k))] = \frac{1}{R(h)} \sum_{(i,j)\in\mathfrak{S}} (i,j) \exp(\langle (h_1,h_2),(i,j)\rangle).$$

Accordingly, $\mathbb{E}[(Y_1(k), Y_2(k))] = 0$ if and only if

$$\sum_{(i,j)\in\mathfrak{S}} i(\exp(h_1))^i(\exp(h_2))^j = \sum_{(i,j)\in\mathfrak{S}} j(\exp(h_1))^i(\exp(h_2))^j = 0.$$

This exactly means that $(x_0, y_0) = (\exp(h_1), \exp(h_2))$ satisfies the system (2), and then that $R(h) = \chi(x_0, y_0)$.

We now show that $r = c$, with c as in (3). Using (4) yields

$$\frac{\mathbb{E}[Y_1(k)Y_2(k)]}{\sqrt{\mathbb{E}[Y_1(k)^2]\cdot\mathbb{E}[Y_2(k)^2]}} = \frac{\sum_{(i,j)\in\mathfrak{S}} ij(\exp(h_1))^i(\exp(h_2))^j}{\sqrt{(\sum_{(i,j)\in\mathfrak{S}} i^2(\exp(h_1))^i(\exp(h_2))^j)\cdot(\sum_{(i,j)\in\mathfrak{S}} j^2(\exp(h_1))^i(\exp(h_2))^j)}}.$$

The numerator above is precisely $x_0y_0\frac{\partial^2\chi}{\partial x\partial y}(x_0, y_0)$. As for the denominator, we notice that

$$\frac{\partial^2\chi}{\partial x^2}(x_0, y_0) = \sum_{(i,j)\in\mathfrak{S}} i(i-1)x_0^{i-2}y_0^j = \frac{1}{x_0^2}\sum_{(i,j)\in\mathfrak{S}} i(i-1)x_0^iy_0^j = \frac{1}{x_0^2}\sum_{(i,j)\in\mathfrak{S}} i^2x_0^iy_0^j,$$

where the last equality comes from $\frac{\partial\chi}{\partial x}(x_0, y_0) = 0$. A similar computation for $\frac{\partial^2\chi}{\partial y^2}(x_0, y_0)$ shows that $r = c$.

We now consider the periodic step sets (they are marked with a star in Tables 1 and 2). By examining the different cases, it is clear that the period is necessarily 2. In particular, we have $e_{2n+1} = 0$ for all $n \geq 0$, and (5) does not hold anymore. However, a slight adaptation of the local limit theorem [18, Theorem 6] (from which the asymptotic formula (5) is a simple consequence) gives

$$f_{\mathfrak{S}}(i,j,2n) = \widetilde{C}((0,0),(i,j))R(h)^{2n}(2n)^{-p-d/2}(1+o(1)), \qquad n\to\infty,$$

where $\widetilde{C}((0,0),(i,j))$ is some constant (different from that in (5)). The proof of Theorem 3 is complete. □

2.4. **Algorithmic irrationality proof.** Let $\mathfrak{S} \subseteq \{0,\pm1\}^2$ be one of the 51 nonsingular step sets with infinite group (see Table 1 in Appendix 3). By Theorem 3, the singular exponent α in the asymptotic expansion of the excursion sequence $(e_n^{\mathfrak{S}})_{n\geq0}$ is equal to $-1-\pi/\arccos(-c)$, where c is an algebraic number. Therefore, if $\arccos(c)/\pi$ is an irrational number, then by Theorem 2, the generating series $F_{\mathfrak{S}}(0,0,t)$ is not D-finite.

We now explain how, starting from the step set $\mathfrak{S}$ one can *algorithmically* prove that $\arccos(c)/\pi$ is irrational. This effective proof decomposes into two main steps, solved by two different algorithms. The first algorithm computes the minimal polynomial $\mu_c(t) \in \mathbb{Q}[t]$ of c starting from $\mathfrak{S}$. The second one performs computations on $\mu_c(t)$ showing that $\arccos(c)/\pi$ is irrational.

2.4.1. *Computing the minimal polynomial of the correlation coefficient.* Given $\chi = \chi_{\mathfrak{S}}$ the characteristic polynomial of the step set $\mathfrak{S}$, Theorem 3 shows that the exponential growth ρ and the correlation coefficient c are algebraic numbers, for which algebraic equations can be obtained by eliminating x and y from the equations

$$\frac{\partial\chi}{\partial x} = 0, \qquad \frac{\partial\chi}{\partial y} = 0, \qquad \rho - \chi = 0, \qquad c^2 - \frac{\left(\frac{\partial^2\chi}{\partial x\partial y}\right)^2}{\frac{\partial^2\chi}{\partial x^2}\cdot\frac{\partial^2\chi}{\partial y^2}} = 0.$$

This elimination is a routine task in effective algebraic geometry, usually performed with Gröbner bases for lexicographic or elimination orders [17]. For any zero (x_0, y_0) of the system $\frac{\partial \chi}{\partial x}(x_0, y_0) = \frac{\partial \chi}{\partial y}(x_0, y_0) = 0$ and any polynomials $P(x, y)$ and $Q(x, y)$ such that $Q \notin I$, the algebraic number $P(x_0, y_0)/Q(x_0, y_0)$ is a root of a generator of the ideal $I + \langle P(x, y) - tQ(x, y)\rangle \cap \mathbb{Q}[t]$.

This is summarized in the following algorithm.

Input: A step set $\mathfrak{S}$ satisfying the assumptions of Theorem 3
Output: The minimal polynomials of ρ and c defined in Theorem 3

(1) Set $\chi(x, y) := \sum_{(i,j)\in\mathfrak{S}} x^i y^j$, and compute $\chi_x := \mathsf{numer}(\frac{\partial \chi}{\partial x})$, $\chi_y := \mathsf{numer}(\frac{\partial \chi}{\partial y})$.
(2) Compute the Gröbner basis of the ideal generated by $(\chi_x, \chi_y, \mathsf{numer}(t - \chi))$ in $\mathbb{Q}[x, y, t]$ for a term order that eliminates x and y. Isolate the unique polynomial in this basis that is free of x and y, factorize it, and identify its factor μ_ρ that annihilates ρ.
(3) Compute the polynomial

$$P(x, y, t) := \mathsf{numer}\left(t^2 - \frac{\left(\frac{\partial^2 \chi}{\partial x \partial y}\right)^2}{\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}}\right)$$

and eliminate x and y by computing a Gröbner basis of (χ_x, χ_y, P) for a term order that eliminates x and y. Isolate the unique polynomial in this basis that is free of x and y, factorize it, and identify its factor μ_c that annihilates c.

Table 2 in Appendix 3 displays the minimal polynomials of ρ and of c obtained using this algorithm.

2.4.2. *Proving that the arccosine of the correlation coefficient is not commensurable with π.* Given the minimal polynomial μ_c of the correlation coefficient c, we now want to check that $\arccos(c)/\pi$ is irrational. General classification results exist, e.g., [54], but they are not sufficient for our purpose. Instead, we rather prove that $\arccos(c)/\pi$ is irrational in an algorithmic way. This is based on the observation that if $\arccos(c)/\pi$ were rational, then c would be of the form $(x + 1/x)/2$ with x a root of unity. This implies that the numerator of the rational function $\mu_c(\frac{x^2+1}{2x})$ would possess a root which is a root of unity. In other words, the polynomial $R(x) = x^{\deg \mu_c}\mu_c(\frac{x^2+1}{2x})$ would be divisible by a cyclotomic polynomial. This possibility can be discarded by analyzing the minimal polynomials μ_c displayed in Table 2 in Appendix 3.

Indeed, in all the 51 cases, the polynomial $R(x)$ is irreducible and has degree $2\deg(\mu_c)$, thus at most 28. Now, it is known that if the cyclotomic polynomial Φ_N has degree at most 30, then N is at most 150 [50, Theorem 15], and the coefficients of Φ_N belong to the set $\{-2, -1, 0, 1, 2\}$ [39]. Computing R in the 51 cases shows that it has at least one coefficient of absolute value greater than 3. This allows to conclude that R is not a cyclotomic polynomial, and therefore that $\arccos(c)/\pi$ is irrational, and finishes the proof of Theorem 1.

2.5. **Example.** We now illustrate the systematic nature of our algorithms on Example 23 of Table 1, i.e., walks with step set $\mathfrak{S} = \{(-1, 0), (0, 1), (1, 0), (1, -1), (0, -1)\}$. For ease of use, we give explicit Maple instructions.

Step 1. The characteristic polynomial of the step set is

```
S:=[[-1,0],[0,1],[1,0],[1,-1],[0,-1]]:
chi:=add(x^s[1]*y^s[2],s=S);
```

$$\chi := \frac{1}{x} + \frac{1}{y} + x + y + \frac{x}{y},$$

whose derivatives have numerators

```
chi_x:=numer(diff(chi,x));chi_y:=numer(diff(chi,y));
```

$$\chi_x := x^2 + x^2y - y, \qquad \chi_y := y^2 - x - 1. \tag{6}$$

These define the system (2).

Step 2. We now compute a polynomial that vanishes at $\rho = \chi(x_0, y_0)$ when (x_0, y_0) is a solution of (6). To this aim, we eliminate x and y in $\{\chi_x, \chi_y, \mathsf{numer}(\chi) - t\mathsf{denom}(\chi)\}$ by a Gröbner basis computation using an elimination order with $(x, y) > t$. In Maple, this is provided by the command

```
G:=Groebner[Basis]([chi_x,chi_y,numer(t-chi)],lexdeg([x,y],[t])):
```

which returns four polynomials, only one of which is free of x and y, namely

```
p:=factor(op(remove(has,G,{x,y})));
```

$$p := (t+1)(t^3 + t^2 - 18t - 43).$$

Since we know that $\rho > 0$, we identify its minimal polynomial as $\mu_\rho = t^3 + t^2 - 18t - 43$, which gives the entry in Column 3 of Table 2. The numerical value for ρ in Table 1 is given by

```
fsolve(p,t,0..infinity);
```

$$4.729031538.$$

Step 3. Next, we obtain a polynomial which vanishes at c by a very similar computation:

```
G:=Groebner[Basis]([chi_x,chi_y,
   numer(t^2-diff(chi,x,y)^2/diff(chi,x,x)/diff(chi,y,y))],lexdeg([x,y],[t]));
```

Again, this command returns four polynomials, with one of them free of x and y, namely

```
p:=factor(op(remove(has,G,{x,y})));
```

$$p := (4t^2+1)(8t^3 + 8t^2 + 6t + 1)(8t^3 - 8t^2 + 6t - 1).$$

This polynomial has only two real roots, $\pm c$. Since $c < 0$, we identify its minimal polynomial as $\mu_c = 8t^3 + 8t^2 + 6t + 1$, which gives the entry in Column 4 of Table 2. Again, the numerical value for α in Table 1 is given by

```
evalf(-1-Pi/arccos(-fsolve(p,t,-infinity..0)));
```

$$-3.320191962.$$

Step 4. To conclude, we compute the polynomial

$$R(x) = x^3 \mu_c\left(\frac{x^2+1}{2x}\right) = x^6 + 2x^5 + 6x^4 + 5x^3 + 6x^2 + 2x + 1.$$

This polynomial does not have any root that is a root of unity, since it is irreducible and not cyclotomic:

```
irreduc(R),numtheory[iscyclotomic](R,x);
```

$$\mathit{true}, \mathit{false}$$

This completes the proof that the generating function for this walk is not D-finite.

2.6. **Open problems.** As we mentioned in Section 2.1, our approach brings together a strong arithmetic result (Theorem 2) and a strong probabilistic result (Theorem 3). It therefore appears natural to search for alternative simpler proofs of these results.

Proving that α is transcendental. In Section 2.4, we are able to prove that for the 51 nonsingular models, the exponent α in the asymptotic expansion of the excursion sequence is irrational. It is worth mentioning that if it was possible to prove that α is not only irrational, but also transcendental, then Theorem 2 would not be needed.

Simpler proof of Theorem 2. In Section 2.2, we proposed a proof of Theorem 2 based on several strong results from arithmetic theory [15, 1, 33, 20]. It would be interesting to know whether Theorem 2 admits a simpler, direct proof.

Combinatorial proof of Theorem 3. Last but not least, finding a combinatorial proof of Theorem 3 appears as an interesting challenge.

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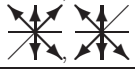
3. Appendix

Table 1 Non-singular walks in the quarter plane with an infinite group, and the asymptotics of their excursions. The labels used in the columns "Tag" correspond to those from Table 4 in [14]. Entries marked with a star correspond to *periodic* walks.

Tag	Steps	First terms	Asymptotics	Tag	Steps	First terms	Asymptotics
3		1, 0, 1, 2, 2, 13, 21, 67, 231	$\frac{3.799605^n}{n^{2.610604}}$	4		1, 0, 0, 2, 2, 0, 16, 44, 28	$\frac{3.608079^n}{n^{2.720448}}$
5		1, 0, 1, 2, 2, 14, 21, 76, 252	$\frac{3.799605^n}{n^{2.318862}}$	6		1, 0, 1, 2, 2, 13, 21, 67, 231	$\frac{3.799605^n}{n^{2.610604}}$
7⋆		1, 0, 1, 0, 4, 0, 29, 0, 230	$\frac{3.800378^n}{n^{2.521116}}$	8		1, 0, 1, 1, 2, 7, 10, 38, 89	$\frac{3.799605^n}{n^{3.637724}}$
9		1, 0, 1, 1, 2, 7, 10, 38, 89	$\frac{3.799605^n}{n^{3.637724}}$	10		1, 0, 0, 1, 2, 0, 5, 26, 28	$\frac{3.608079^n}{n^{3.388025}}$
11⋆		1, 0, 0, 0, 2, 0, 6, 0, 42	$\frac{3.800378^n}{n^{3.918957}}$	12		1, 0, 0, 1, 0, 1, 5, 1, 18	$\frac{3.799605^n}{n^{5.136154}}$
14		1, 0, 0, 1, 2, 0, 5, 26, 28	$\frac{3.608079^n}{n^{3.388025}}$	16		1, 0, 1, 2, 2, 14, 21, 76, 252	$\frac{3.799605^n}{n^{2.318862}}$
17⋆		1, 0, 1, 0, 4, 0, 29, 0, 230	$\frac{3.800378^n}{n^{2.521116}}$	18		1, 0, 0, 2, 2, 0, 16, 44, 28	$\frac{3.608079^n}{n^{2.720448}}$
19⋆		1, 0, 0, 0, 2, 0, 6, 0, 42	$\frac{3.800378^n}{n^{3.918957}}$	20		1, 0, 1, 2, 4, 14, 45, 120, 468	$\frac{4.372923^n}{n^{2.482876}}$
21		1, 0, 1, 1, 4, 7, 25, 64, 201	$\frac{4.214757^n}{n^{3.347502}}$	23		1, 0, 2, 1, 10, 14, 75, 178, 738	$\frac{4.729032^n}{n^{3.320192}}$
24		1, 0, 2, 2, 10, 26, 86, 312, 1022	$\frac{4.729032^n}{n^{2.757466}}$	25		1, 0, 2, 2, 11, 27, 101, 348, 1237	$\frac{4.729032^n}{n^{2.397625}}$
26		1, 0, 2, 2, 11, 27, 101, 348, 1237	$\frac{4.729032^n}{n^{2.397625}}$	27⋆		1, 0, 2, 0, 13, 0, 124, 0, 1427	$\frac{4.569086^n}{n^{2.503534}}$
28		1, 0, 1, 2, 4, 13, 36, 111, 343	$\frac{4.214757^n}{n^{2.742114}}$	29⋆		1, 0, 1, 0, 5, 0, 35, 0, 313	$\frac{4.569086^n}{n^{3.985964}}$
30		1, 0, 1, 1, 6, 17, 58, 202, 749	$\frac{5^n}{n^{2.722859}}$	31		1, 0, 0, 1, 2, 1, 11, 27, 60	$\frac{4.372923^n}{n^{4.070925}}$
32⋆		1, 0, 2, 0, 13, 0, 124, 0, 1427	$\frac{4.569086^n}{n^{2.503534}}$	33		1, 0, 1, 1, 4, 7, 25, 64, 201	$\frac{4.214757^n}{n^{3.347502}}$
34⋆		1, 0, 1, 0, 5, 0, 35, 0, 313	$\frac{4.569086^n}{n^{3.985964}}$	35		1, 0, 1, 1, 3, 8, 19, 65, 177	$\frac{4.729032^n}{n^{4.514931}}$
36		1, 0, 0, 1, 2, 1, 11, 27, 60	$\frac{4.372923^n}{n^{4.070925}}$	37		1, 0, 1, 2, 4, 13, 36, 111, 343	$\frac{4.214757^n}{n^{2.742114}}$
38		1, 0, 2, 2, 10, 26, 86, 312, 1022	$\frac{4.729032^n}{n^{2.757466}}$	39		1, 0, 1, 1, 3, 8, 19, 65, 177	$\frac{4.729032^n}{n^{4.514931}}$
40		1, 0, 0, 2, 4, 8, 28, 108, 372	$\frac{5^n}{n^{3.383396}}$	41		1, 0, 1, 2, 4, 14, 45, 120, 468	$\frac{4.372923^n}{n^{2.482876}}$
42		1, 0, 0, 2, 4, 8, 28, 108, 372	$\frac{5^n}{n^{3.383396}}$	43		1, 0, 2, 2, 13, 27, 140, 392, 1882	$\frac{5.064419^n}{n^{2.491053}}$
44		1, 0, 2, 3, 15, 51, 208, 893, 3841	$\frac{5.891838^n}{n^{2.679783}}$	45		1, 0, 1, 1, 5, 8, 40, 91, 406	$\frac{5.064419^n}{n^{4.036441}}$
46		1, 0, 1, 2, 8, 22, 101, 364, 1618	$\frac{5.799605^n}{n^{2.959600}}$	47		1, 0, 1, 3, 7, 29, 101, 404, 1657	$\frac{5.891838^n}{n^{3.471058}}$

Tag	Steps	First terms	Asymptotics	Tag	Steps	First terms	Asymptotics
48		1, 0, 1, 1, 5, 8, 40, 91, 406	$\frac{5.064419^n}{n^{4.036441}}$	49		1, 0, 2, 2, 13, 27, 140, 392, 1882	$\frac{5.064419^n}{n^{2.491053}}$
50		1, 0, 2, 3, 15, 51, 208, 893, 3841	$\frac{5.891838^n}{n^{2.679783}}$	51		1, 0, 1, 3, 7, 29, 101, 404, 1657	$\frac{5.891838^n}{n^{3.471058}}$
52		1, 0, 1, 1, 8, 18, 90, 301, 1413	$\frac{5.799605^n}{n^{3.042101}}$	53		1, 0, 1, 2, 8, 22, 101, 364, 1618	$\frac{5.799605^n}{n^{2.959600}}$
54		1, 0, 3, 5, 30, 111, 548, 2586, 13087	$\frac{6.729032^n}{n^{2.667986}}$	55		1, 0, 2, 4, 16, 64, 266, 1210, 5630	$\frac{6.729032^n}{n^{3.497037}}$
56		1, 0, 2, 4, 16, 64, 266, 1210, 5630	$\frac{6.729032^n}{n^{3.497037}}$				

Table 2 Non-singular walks in the quarter plane with an infinite group, and the minimal polynomials of the growth constants ρ and of the correlation coefficients c.

Tag	Steps	Minimal polynomial μ_ρ of ρ	Minimal polynomial μ_c of $c = -\cos(\frac{\pi}{1+\alpha})$
3, 6		$t^4+t^3-8t^2-36t-11$	$t^8+\frac{1}{4}t^6-\frac{3}{16}t^4+\frac{3}{64}t^2-\frac{1}{256}$
8, 9			$t^8+\frac{1}{4}t^6-\frac{3}{16}t^4+\frac{3}{64}t^2-\frac{1}{256}$
5, 16			$t^4-\frac{9}{2}t^3+\frac{27}{4}t^2-\frac{35}{8}t+\frac{17}{16}$
12			$t^4+\frac{9}{2}t^3+\frac{27}{4}t^2+\frac{35}{8}t+\frac{17}{16}$
7*, 17*		$t^6-11t^4-32t^2-256$	$t^6+\frac{3}{4}t^4+2t^2-\frac{1}{2}$
11*, 19*			$t^6+\frac{3}{4}t^4+2t^2-\frac{1}{2}$
4, 18		$t^5+t^4+t^3-30t^2-96t-91$	$t^{10}+2t^8+t^6-\frac{1}{64}t^4+\frac{3}{256}t^2-\frac{1}{1024}$
10, 14			$t^{10}+2t^8+t^6-\frac{1}{64}t^4+\frac{3}{256}t^2-\frac{1}{1024}$
20, 41		$t^5-2t^4-4t^3-31t^2+23t-41$	$t^{10}+t^8+\frac{157}{32}t^6+\frac{145}{128}t^4+\frac{1681}{512}t^2-\frac{2209}{2048}$
31, 36			$t^{10}+t^8+\frac{157}{32}t^6+\frac{145}{128}t^4+\frac{1681}{512}t^2-\frac{2209}{2048}$
21, 33		$t^5+2t^4-7t^3-46t^2-116t-131$	$t^{10}+\frac{3}{2}t^8+\frac{13}{16}t^6+\frac{5}{64}t^4+\frac{3}{256}t^2-\frac{1}{1024}$
28, 37			$t^{10}+\frac{3}{2}t^8+\frac{13}{16}t^6+\frac{5}{64}t^4+\frac{3}{256}t^2-\frac{1}{1024}$
23		$t^3+t^2-18t-43$	$t^3+t^2+\frac{3}{4}t+\frac{1}{8}$
24, 38			$t^3-t^2+\frac{3}{4}t-\frac{1}{8}$
25, 26			$t^6-t^4+\frac{7}{16}t^2-\frac{5}{64}$
35, 39			$t^6-t^4+\frac{7}{16}t^2-\frac{5}{64}$
27*, 32*		$t^6-20t^4-16t^2-48$	$t^6+2t^4+\frac{5}{2}t^2-\frac{3}{4}$
29*, 34*			$t^6+2t^4+\frac{5}{2}t^2-\frac{3}{4}$

Tag	Steps	Minimal polynomial μ_ρ of ρ	Minimal polynomial μ_c of $c = -\cos(\frac{\pi}{1+\alpha})$
30		$t-5$	$t-\frac{1}{4}$
40, 42			$t+\frac{1}{4}$
43, 49		$t^6+2t^5-18t^4-67t^3-108t^2-40t-19$	$t^{12}+\frac{11}{4}t^{10}+\frac{107}{16}t^8+\frac{145}{32}t^6+\frac{455}{128}t^4-\frac{2859}{1024}t^2+\frac{1521}{4096}$
45, 48			$t^{12}+\frac{11}{4}t^{10}+\frac{107}{16}t^8+\frac{145}{32}t^6+\frac{455}{128}t^4-\frac{2859}{1024}t^2+\frac{1521}{4096}$
44, 50		$t^7+3t^6-18t^5-127t^4-328t^3-560t^2-704t-448$	$t^{14}+\frac{23}{4}t^{12}+\frac{25}{2}t^{10}+\frac{971}{64}t^8+\frac{421}{32}t^6+\frac{307}{64}t^4+\frac{107}{64}t^2-\frac{49}{256}$
47, 51			$t^{14}+\frac{23}{4}t^{12}+\frac{25}{2}t^{10}+\frac{971}{64}t^8+\frac{421}{32}t^6+\frac{307}{64}t^4+\frac{107}{64}t^2-\frac{49}{256}$
46, 53		$t^4-7t^3+10t^2-24t+37$	$t^4-\frac{1}{2}t^3+\frac{55}{4}t^2-\frac{19}{8}t+\frac{1}{16}$
52			$t^4+\frac{1}{2}t^3+\frac{55}{4}t^2+\frac{19}{8}t+\frac{1}{16}$
54		$t^3-5t^2-10t-11$	$t^3+\frac{11}{4}t-\frac{7}{8}$
55, 56			$t^3+\frac{11}{4}t+\frac{7}{8}$

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