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# REVERSE CARLESON EMBEDDINGS FOR MODEL SPACES

ALAIN BLANDIGNÈRES, EMMANUEL FRICAIN, FRÉDÉRIC GAUNARD,  
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ABSTRACT. The classical embedding theorem of Carleson deals with finite positive Borel measures  $\mu$  on the closed unit disk for which there exists a positive constant  $c$  such that  $\|f\|_{L^2(\mu)} \leq c\|f\|_{H^2}$  for all  $f \in H^2$ , the Hardy space of the unit disk. Lefèvre et al. examined measures  $\mu$  for which there exists a positive constant  $c$  such that  $\|f\|_{L^2(\mu)} \geq c\|f\|_{H^2}$  for all  $f \in H^2$ . The first type of inequality above was explored with  $H^2$  replaced by one of the model spaces  $(\Theta H^2)^\perp$  by Aleksandrov, Baranov, Cohn, Treil, and Volberg. In this paper we continue this discussion.

## 1. INTRODUCTION

Let  $H^2$  be the classical Hardy space of the open unit disk  $\mathbb{D}$  [20, 24] with norm  $\|\cdot\|_2$ , and let  $H^\infty$  be the space of bounded analytic functions on  $\mathbb{D}$ . Let  $M_+(\mathbb{D})$  denote the finite positive Borel measures on  $\mathbb{D}$ , and, for  $\mu \in M_+(\mathbb{D})$ , let  $\|\cdot\|_\mu$  be the norm in  $L^2(\mu)$ . A beautiful theorem of L. Carleson [24] says that  $H^2$  can be continuously embedded into  $L^2(\mu)$ , i.e.,

$$(1.1) \quad \exists c > 0 \text{ such that } \|f\|_\mu \leq c\|f\|_2, \quad \forall f \in H^2$$

if and only if

$$(1.2) \quad \sup_I \frac{\mu(S(I))}{m(I)} < \infty,$$

where the above supremum is taken over all arcs  $I$  of the unit circle  $\mathbb{T} = \partial\mathbb{D}$ ,  $m := d\theta/2\pi$  is normalized Lebesgue measure on  $\mathbb{T}$ , and  $S(I)$  is the Carleson window

$$(1.3) \quad S(I) := \left\{ |z| \leq 1 : \frac{z}{|z|} \in I, 1 - |z| \leq \frac{m(I)}{2} \right\}.$$

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We will write  $H^2 \hookrightarrow L^2(\mu)$  to represent the fact that  $H^2$  can be continuously embedded into  $L^2(\mu)$  via the map  $f \mapsto f|E_\mu$ , where  $E_\mu$  is a carrier of  $\mu$ . Measures for which this is true are called *Carleson measures*. We will also write statements such as (1.1) in the more convenient form

$$\|f\|_\mu \lesssim \|f\|_2, \quad \forall f \in H^2.$$

Carleson's result can be extended to  $\mu \in M_+(\mathbb{D}^-)$  ( $\mathbb{D}^-$  is the closure of  $\mathbb{D}$ ) but, in the initial definition of embedding in (1.1), we need to change the phrase  $\forall f \in H^2$  into  $\forall f \in H^2 \cap C(\mathbb{D}^-)$ , owing to the fact that functions in  $H^2$  are defined  $m$ -almost everywhere on  $\mathbb{T}$  while functions in  $L^2(\mu)$  are defined  $\mu$ -almost everywhere on  $\mathbb{T}$ . However, condition (1.2) clearly implies that the restrictions of Carleson measures to the unit circle are absolutely continuous with respect to  $m$ , and so the initial concern in examining  $\|f\|_\mu$  for all  $f \in H^2$ , and not just the continuous ones, evaporates.

Lefèvre et al. [29] examined the *reverse embedding problem*, i.e., when is the above embedding injective with closed range, equivalently, when does (1.1) hold as well as the reverse inequality

$$(1.4) \quad \|f\|_2 \lesssim \|f\|_\mu, \quad \forall f \in H^2?$$

They proved that if  $\mu$  is a Carleson measure, then the reverse embedding happens if and only if

$$(1.5) \quad \inf_I \frac{\mu(S(I))}{m(I)} > 0.$$

In other words, the norms  $\|\cdot\|_\mu$  and  $\|\cdot\|_2$  are equivalent on  $H^2$  if and only if both (1.2) and (1.5) are satisfied.

We would like to point out that D. Luecking studied the question of reverse embeddings for Bergman spaces in [31, 32] and G. Chacón [11] has some related results for certain Dirichlet type spaces.

The purpose of this paper is to explore reverse embeddings for model spaces  $(\Theta H^2)^\perp = H^2 \ominus \Theta H^2$ , where  $\Theta$  is a non-constant inner function, that is  $\Theta \in H^\infty$  and  $\Theta$  admits radial limits of modulus one almost everywhere on  $\mathbb{T}$ . Aleksandrov [5] showed that if  $\mu \in M_+(\mathbb{D}^-)$  satisfies

$$(1.6) \quad \|f\|_\mu \lesssim \|f\|_2, \quad \forall f \in (\Theta H^2)^\perp \cap C(\mathbb{D}^-),$$

then each  $f \in (\Theta H^2)^\perp$  has a finite radial limit at  $\mu$ -almost every point of  $\mathbb{T}$  and the inequality in (1.6) holds for every  $f \in (\Theta H^2)^\perp$ . We point out that, amazingly,  $(\Theta H^2)^\perp \cap C(\mathbb{D}^-)$  is dense in  $(\Theta H^2)^\perp$  [3] (see also [12, p. 188]). Again we use the notation  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$  to denote the embedding  $f \mapsto f|E_\mu$ , where  $E_\mu \subset \mathbb{D}^-$  is a carrier of  $\mu$ .

Treil and Volberg [45], along with Cohn [14], examined when embedding of model spaces actually occurs. In particular, they showed that  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$  as soon as there is an  $\varepsilon \in (0, 1)$  such that  $\mu$  satisfies the condition (1.2) but the supremum is taken only over arcs  $I \subset \mathbb{T}$  satisfying

$$(1.7) \quad S(I) \cap L(\Theta, \varepsilon) \neq \emptyset,$$

where

$$(1.8) \quad L(\Theta, \varepsilon) := \{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\}$$

is a sub-level set for  $\Theta$ . In particular, we see that this condition is weaker than Carleson's one since  $(\Theta H^2)^\perp$  functions are much more regular than  $H^2$ -functions, particularly when we are far from  $L(\Theta, \varepsilon)$ . Using weighted Bernstein inequalities, Baranov [8] slightly improved the embedding result of Treil–Volberg.

Conversely, assuming that  $L(\Theta, \varepsilon)$  is connected for some  $\varepsilon > 0$ <sup>1</sup>, if  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ , then  $\mu$  satisfies (1.2) for arcs  $I$  satisfying (1.7).

Again, as was asked by Lefèvre et al. for Carleson measures in  $H^2$ , when is the embedding  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$  injective with closed range (equivalently, induces a reverse embedding)? For example, if  $\mu$  is any one of the *Clark measures* for  $\Theta$  (we will define these measures in a moment), then by results of Clark [13] and Poltoratskii [38] we have the isometric embedding  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ . The same is true when  $\mu$  is a Clark measure for an inner multiple of  $\Theta$ .

For another example, let  $(\lambda_n)_{n \geq 1} \subset \mathbb{D}$  be a *complete interpolating sequence* for  $(\Theta H^2)^\perp$ . This means that the sequence of reproducing kernels  $(k_{\lambda_n}^\Theta)_{n \geq 1}$  for  $(\Theta H^2)^\perp$  forms an unconditional basis in  $(\Theta H^2)^\perp$  [27, 34]. In this situation it turns out that for the measure

$$\mu = \sum_{n \geq 1} \frac{1}{\|k_{\lambda_n}\|_2^2} \delta_{\lambda_n},$$

the norms  $\|\cdot\|_\mu$  and  $\|\cdot\|_2$  are equivalent on  $(\Theta H^2)^\perp$  and, in particular, we have a reverse embedding.

For a third example, suppose  $M$  is a subspace of  $H^2$  satisfying  $f/z \in M$  whenever  $f \in M$  and  $f(0) = 0$ . Such subspaces  $M$  are called *nearly invariant* and were initially studied by Hitt and Sarason [26, 40]. If  $g \in M$  is the unique solution to the extremal problem,

$$\Re(g(0)) = \sup\{\Re(f(0)) : f \in M, \|f\|_2 \leq 1\},$$

---

<sup>1</sup>Such inner functions are said to satisfy the *connected level set condition* (CLS)

then there exists an inner function  $\Theta$  such that  $M = g(\Theta H^2)^\perp$  and  $g$  is an isometric multiplier of  $(\Theta H^2)^\perp$ . That is to say

$$\|gf\|_2 = \|f\|_2, \quad \forall f \in (\Theta H^2)^\perp.$$

Rephrasing this a bit we see that if  $d\mu = |g|^2 dm$  then the embedding  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$  is isometric. As a matter of fact, the measures  $\mu \in M_+(\mathbb{T})$  which ensure an isometric embedding of  $(\Theta H^2)^\perp$  have been characterized by Aleksandrov [4].

**Theorem 1.1** (Aleksandrov). *Let  $\mu \in M_+(\mathbb{T})$ . Then the following assertions are equivalent:*

- (i)  $(\Theta H^2)^\perp$  embeds isometrically into  $L^2(\mu)$ ;
- (ii) the function  $\Theta$  has non-tangential boundary values  $\mu$ -almost everywhere on  $\mathbb{T}$  and

$$\int_{\mathbb{T}} \left| \frac{1 - \overline{\Theta(z)}\Theta(\zeta)}{1 - \bar{z}\zeta} \right|^2 d\mu(\zeta) = \frac{1 - |\Theta(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D};$$

- (iii) there exists a  $\varphi \in H^\infty$  such that  $\|\varphi\|_\infty \leq 1$  and

$$(1.9) \quad \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta) = \Re \left( \frac{1 + \varphi(z)\Theta(z)}{1 - \varphi(z)\Theta(z)} \right), \quad z \in \mathbb{D}.$$

Note that in [18], de Branges has proven the result for meromorphic inner functions and in [25], Krein has obtained a characterization of isometric measures for  $(\Theta H^2)^\perp$  in a more operator-theoretic language.

In the particular case when  $\Theta(z) = \Theta_a(z) = \exp(-a\frac{1+z}{1-z})$ ,  $a > 0$ , the question of equivalence of norms in  $(\Theta_a H^2)^\perp$  was discussed in [28, 30, 36, 37]. In this case, the question is equivalent to the following one: for which (positive) measures  $\mu$  on the real line are the norms

$$\left( \int_{\mathbb{R}} |f(x)|^2 d\mu(x) \right)^{1/2} \quad \text{and} \quad \left( \int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2}$$

equivalent on the space of entire functions of exponential type not exceeding  $a/2$  and which belong to  $L^2(\mathbb{R})$ ?

In [44] Volberg generalized the previous results and gave a complete answer for general model spaces and absolutely continuous measures  $d\mu = w dm$ ,  $w \in L^\infty(\mathbb{T})$ ,

**Theorem 1.2** (Volberg). *Let  $d\mu = w dm$ , with  $w \in L^\infty(\mathbb{T})$ ,  $w \geq 0$ , and let  $\Theta$  be an inner function. Then the following assertions are equivalent:*

- (i) the norms  $\|\cdot\|_\mu$  and  $\|\cdot\|_2$  are equivalent on  $(\Theta H^2)^\perp$ ;

(ii) if  $(\lambda_n)_{n \geq 1} \subset \mathbb{D}$ , then

$$\lim_{n \rightarrow +\infty} \widehat{w}(\lambda_n) = 0 \implies \lim_{n \rightarrow +\infty} |\Theta(\lambda_n)| = 1;$$

(iii) we have

$$\inf_{\lambda \in \mathbb{D}} (\widehat{w}(\lambda) + |\Theta(\lambda)|) > 0.$$

In the above,  $\widehat{w}$  represents the harmonic continuation of the function  $w$  into the open unit disk, that is,

$$\widehat{w}(z) = \int_{\mathbb{T}} w(\zeta) \frac{1 - |z|^2}{|z - \zeta|^2} dm(\zeta), \quad z \in \mathbb{D}.$$

The aim of this paper is to prove a model space version of the Lefèvre et al. reverse Carleson embedding theorem for  $H^2$  where we weaken the condition (1.5) in the spirit of the result of Treil–Volberg for the direct embedding. Along the way, we will also develop a notion of dominating sets for model spaces and use this to state another reverse embedding theorem.

## 2. MAIN RESULTS

Our first result is a reverse embedding theorem along the lines of Treil–Volberg for which we need the following notation: given an arc  $I \subset \mathbb{T}$  and a number  $n > 0$  we define the amplified arc  $nI$  as the arc with same center as  $I$  and of length  $n \times m(I)$ .

**Theorem 2.1.** *Let  $\Theta$  be a (CLS) inner function,  $L(\Theta, \varepsilon_1)$  its connected sublevel set, and  $\mu \in M_+(\mathbb{D}^-)$  such that  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ . There exists an  $N = N(\Theta, \varepsilon_1) > 1$  such that if*

$$(2.1) \quad \inf_I \frac{\mu(S(I))}{m(I)} > 0,$$

where the infimum is taken over all arcs  $I \subset \mathbb{T}$  with

$$S(NI) \cap L(\Theta, \varepsilon_1) \neq \emptyset,$$

then

$$(2.2) \quad \|f\|_2 \lesssim \|f\|_\mu, \quad \forall f \in (\Theta H^2)^\perp.$$

The proof of this theorem will show how  $N$  depends on  $\Theta$  and  $\varepsilon_1$ . There will also be a discussion in Section 6 of why the  $N$  is needed.

Our second reverse embedding result involves the notion of a dominating set for  $(\Theta H^2)^\perp$ . A (Lebesgue measurable) subset  $\Sigma \subset \mathbb{T}$ , with  $m(\Sigma) < 1$ , is called a *dominating set* for  $(\Theta H^2)^\perp$  if

$$\int_{\mathbb{T}} |f|^2 dm \lesssim \int_{\Sigma} |f|^2 dm, \quad \forall f \in (\Theta H^2)^\perp.$$

Notice how this is equivalent to saying that the measure  $d\mu = \chi_\Sigma dm$  satisfies the reverse embedding property for  $(\Theta H^2)^\perp$ . We will discuss some elementary properties of dominating sets in Section 5. In particular, we will show that dominating sets always exists for inner functions  $\Theta$  such that  $\sigma(\Theta) \neq \mathbb{T}$  – see (3.1) for the definition of  $\sigma(\Theta)$ . Moreover, if  $m(\sigma(\Theta)) = 0$ , then dominating sets can be of arbitrary small Lebesgue measure. This is the case for (CLS) inner functions. With regards to reverse embeddings, we will prove the following result.

**Theorem 2.2.** *Let  $\Theta$  be an inner function,  $\Sigma$  be a dominating set for  $(\Theta H^2)^\perp$ , and  $\mu \in M_+(\mathbb{D}^-)$  such that  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ . Suppose that*

$$\inf_I \frac{\mu(S(I))}{m(I)} > 0,$$

where the above infimum is taken over all arcs  $I \subset \mathbb{T}$  such that

$$I \cap \Sigma \neq \emptyset.$$

Then

$$(2.3) \quad \|f\|_2 \lesssim \|f\|_\mu, \quad \forall f \in (\Theta H^2)^\perp.$$

In both Theorems 2.1 and 2.2, the hypothesis  $(\Theta H^2)^\perp \subset L^2(\mu)$  ensures that the reverse inequality holds for every function in  $(\Theta H^2)^\perp$ . As already mentioned, when  $\mu$  is carried on  $\mathbb{T}$  we might not be able to define  $\|f\|_\mu$  properly for certain  $f \in (\Theta H^2)^\perp$ . Still, it will be clear from the proofs that the embeddings (2.2) and (2.3) still hold on the dense subspaces  $(\Theta H^2)^\perp \cap C(\mathbb{D}^-)$  under the reverse Carleson inequality even without the assumption on the embedding.

Finally we would like to discuss an alternate proof of Aleksandrov’s isometric embedding theorem. Our proof uses the theory of de Branges–Rovnyak spaces.

**Theorem 2.3** (Aleksandrov). *Suppose  $\Theta$  is an inner function and  $\mu \in M_+(\mathbb{T})$ . Then the embedding  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$  is isometric if and only if there is a function  $b$  in the closed unit ball of  $H^\infty$  such that*

$$\mu = \sigma_{\Theta b}^1,$$

where  $\sigma_{\Theta b}^1$  is a so-called Aleksandrov–Clark<sup>2</sup> measure associated with  $\Theta b$ .

In the following section, we recall some useful facts concerning model spaces  $(\Theta H^2)^\perp$  and de Branges–Rovnyak spaces  $\mathcal{H}(b)$ . In Section 4, we prove Theorem 2.3. Section 5 is devoted to dominating sets, and finally in Section 6 we prove Theorems 2.1 and 2.2.

<sup>2</sup>It satisfies (1.9) with  $\varphi$  replaced by  $b$ , see also Section 3

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### 3. MODEL SPACES AND DE BRANGES–ROVNYAK SPACES

Before getting underway, let us first gather up some well-known facts about the model spaces  $(\Theta H^2)^\perp$  that will be needed later. References for much of this can be found in [12]. Model spaces were initially studied as the typical invariant subspaces for the adjoint of the unilateral shift on  $H^2$  [19] but the subject has expanded in many ways since then.

By the Nevenlinna theory [24], an inner function  $\Theta$  can be factored as  $\Theta = e^{i\varphi} B_\Lambda S_\nu$ , where  $\varphi$  is a real constant,  $B_\Lambda$  is a Blaschke product with zeros (repeated according to multiplicity)  $\Lambda = (z_n)_{n \geq 1}$  and

$$S_\nu(z) = \exp \left( - \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\nu(\xi) \right),$$

where  $\nu \in M_+(\mathbb{T})$  with  $\nu \perp m$ , is a singular inner function. The *boundary spectrum* of  $\Theta$  is the set

$$(3.1) \quad \sigma(\Theta) := \left\{ \xi \in \mathbb{T} : \lim_{z \rightarrow \xi} |\Theta(z)| = 0 \right\}.^3$$

It turns out that  $\sigma(\Theta)$  is equal to  $(\Lambda^- \cap \mathbb{T}) \cup \text{supt}(\nu)$  – the union of cluster points of  $\Lambda$  on  $\mathbb{T}$  together with the support of the measure  $\nu$  [35, p. 63]. It is well known [33] that  $\Theta$ , along with every function in  $(\Theta H^2)^\perp$ , has an analytic continuation to an open neighborhood of  $\mathbb{T} \setminus \sigma(\Theta)$ .

Several authors [1, 5, 14] have examined the non-tangential limits of functions in  $(\Theta H^2)^\perp$  near  $\sigma(\Theta)$ , where analytic continuation fails. Indeed, let  $ADC_\Theta$  denote the set of points  $\xi \in \mathbb{T}$  where the angular derivative of  $\Theta$ , in the sense of Carathéodory, exists. More specifically,  $\xi \in ADC_\Theta$  if the radial limit of  $\Theta$  exists and is unimodular and the radial limit of  $\Theta'$  exists. We use the notation  $|\Theta'(\xi)|$  to denote the modulus of the angular derivative of  $\Theta$  at  $\xi$  (when it exists). Note that  $|\Theta'(\xi)| > 0$  whenever  $\xi \in ADC_\Theta$ . A result of Ahern-Clark [1] yields that  $\xi \in ADC_\Theta$  if and only if

$$(3.2) \quad |\Theta'(\xi)| = \sum_{n \geq 1} \frac{1 - |z_n|^2}{|\xi - z_n|^2} + 2 \int_{\mathbb{T}} \frac{d\nu(\zeta)}{|\xi - \zeta|^2} < \infty.$$

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<sup>3</sup>We should emphasize that this definition does not take into account the zeros of  $\Theta$  inside  $\mathbb{D}$ .



The model space  $(\Theta H^2)^\perp$  is a reproducing kernel Hilbert space with kernel

$$(3.3) \quad k_\lambda^\Theta(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda}z}, \quad \lambda \in \mathbb{D}.$$

These kernels satisfy the reproducing property

$$(3.4) \quad \langle f, k_\lambda^\Theta \rangle_2 = f(\lambda), \quad \forall f \in (\Theta H^2)^\perp,$$

and have norm

$$(3.5) \quad \|k_\lambda^\Theta\|_2 = \sqrt{\frac{1 - |\Theta(\lambda)|^2}{1 - |\lambda|^2}}.$$

It is also possible to define  $k_\xi^\Theta$  for  $\xi \in ADC_\Theta$ . In this case  $k_\xi^\Theta \in (\Theta H^2)^\perp$ , every  $f \in (\Theta H^2)^\perp$  has a finite non-tangential limit  $f(\xi)$  at  $\xi$ ,  $\langle f, k_\xi^\Theta \rangle_2 = f(\xi)$ , and  $k_\xi^\Theta$  has norm

$$\|k_\xi^\Theta\|_2 = \sqrt{|\Theta'(\xi)|}.$$

For  $b$  in the closed unit ball of  $H^\infty$ , the associated *de Branges–Rovnyak space*  $\mathcal{H}(b)$  is defined by

$$\mathcal{H}(b) := (I - T_b T_b^*)^{1/2} H^2,$$

and is equipped with the following scalar product

$$\langle (I - T_b T_b^*)^{1/2} f, (I - T_b T_b^*)^{1/2} g \rangle_b = \langle f, g \rangle_2,$$

for any  $f, g \in (\ker(I - T_b T_b^*)^{1/2})^\perp$ , making  $\mathcal{H}(b)$  a Hilbert space contractively contained in  $H^2$ . Here  $T_b$  is the Toeplitz operator with symbol  $b$ ,  $T_b f = P_+(bf)$ ,  $f \in H^2$  and  $P_+ : L^2 \rightarrow H^2$  is the Riesz projection. When  $b = \Theta$  is an inner function,  $\mathcal{H}(\Theta) = (\Theta H^2)^\perp$  with equality of norms. The space  $\mathcal{H}(b)$  is also a reproducing kernel Hilbert space with kernel  $k_\lambda^b$  given by the same formula as (3.3) where  $\Theta$  is replaced by  $b$ . Formulas (3.4) and (3.5) are also valid in this  $\mathcal{H}(b)$  setting.

For each  $\alpha \in \mathbb{T}$ ,

$$\Re \left( \frac{\alpha + b(z)}{\alpha - b(z)} \right) > 0, \quad z \in \mathbb{D},$$

and hence, by Herglotz's theorem [20], there is a unique  $\sigma_b^\alpha \in M_+(\mathbb{T})$ , called the *Aleksandrov–Clark measure* [12, 39, 41] associated with  $b$  and  $\alpha$ , such that

$$(3.6) \quad \Re \left( \frac{\alpha + b(z)}{\alpha - b(z)} \right) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\sigma_b^\alpha(\zeta), \quad z \in \mathbb{D}.$$

The measure  $\sigma_b^\alpha$  is singular (with respect to the Lebesgue measure  $m$ ) if and only if  $b$  is an inner function. In this case,  $\sigma_b^\alpha$  is often called a *Clark measure* since they were initially studied by Clark [13].

Conversely, if we start with a positive Borel measure  $\mu$  on  $\mathbb{T}$ , then we can define the analytic function  $b$  by the formula

$$(3.7) \quad \frac{1+b(z)}{1-b(z)} = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta), \quad z \in \mathbb{D}.$$

Taking the real part of both sides, we see that the resulting function  $b$  is an element in the closed unit ball of  $H^\infty$ , with  $-1 < b(0) < 1$ , and (3.6) holds with  $\mu = \sigma_b^1$ . The special case  $b(0) = 0$  happens if and only if  $\mu$  is a probability measure.

Moreover, it follows easily from the uniqueness of the representation (3.6) that

$$(3.8) \quad \sigma_b^\alpha = \sigma_{\bar{a}b}^1.$$

Clark [13], for the inner case, and Ball [6, 7] for the general case (see also [41]), proved that for each fixed  $\alpha \in \mathbb{T}$  the operator

$$(3.9) \quad \omega_b^\alpha : L^2(\sigma_b^\alpha) \rightarrow \mathcal{H}(b), \quad \omega_b^\alpha h = (1 - \bar{\alpha}b) C_{\sigma_b^\alpha} h,$$

is an onto partial isometry whose kernel is  $(H^2(\sigma_b^\alpha))^\perp$ , where

$$(C_{\sigma_b^\alpha} h)(z) := \int_{\mathbb{T}} \frac{h(\xi)}{1 - \bar{\xi}z} d\sigma_b^\alpha(\xi), \quad z \in \mathbb{D},$$

denotes the Cauchy transform of the measure  $h d\sigma_b^\alpha$  (see [12] for more details) and  $H^2(\sigma_b^\alpha)$  is the closure of the polynomials in the  $L^2(\sigma_b^\alpha)$ -norm.

Poltoratskii [38] (see also [12]) went on further to show that for each  $h \in L^2(\sigma_b^\alpha)$ ,

$$\lim_{r \rightarrow 1^-} (\omega_b^\alpha h)(r\xi) = h(\xi), \quad (\sigma_b^\alpha)_s\text{-a.e.},$$

where  $(\sigma_b^\alpha)_s$  is the singular part of  $\sigma_b^\alpha$ . In particular, if  $b = \Theta$  is an inner function, then  $\sigma_\Theta^\alpha$  is singular,  $H^2(\sigma_\Theta^\alpha) = L^2(\sigma_\Theta^\alpha)$  and putting this all together, we have the isometric, in fact unitary, embedding

$$(\Theta H^2)^\perp \hookrightarrow L^2(\sigma_\Theta^\alpha),$$

i.e.,

$$(3.10) \quad \int_{\mathbb{T}} |f|^2 d\sigma_\Theta^\alpha = \|f\|_2^2, \quad \forall f \in (\Theta H^2)^\perp.$$

For  $a \in \mathbb{D}$ , let

$$\phi_a(z) = \frac{a-z}{1-\bar{a}z}$$

be the (involutive) conformal automorphism of the disk which sends  $a$  to 0. Straightforward computations [2] show that

$$(3.11) \quad \sigma_{\phi_a \circ b}^{\phi_a(\alpha)} = \frac{1}{|\phi_a'(\alpha)|} \sigma_b^\alpha.$$

If  $b = \Theta$  is an inner function, the *Crofoot transform*

$$(3.12) \quad (U_\Theta^a f)(z) := \frac{\sqrt{1-|a|^2}}{1-\bar{a}\Theta(z)} f(z)$$

is a unitary operator from  $(\Theta H^2)^\perp$  onto  $((\phi_a \circ \Theta) H^2)^\perp$  [17].

The Clark measure  $\sigma_\Theta^\alpha$  is carried by the set

$$(3.13) \quad \left\{ \xi \in \mathbb{T} : \lim_{r \rightarrow 1^-} \Theta(r\xi) = \alpha \right\}.$$

By this last statement we mean that the  $\sigma_\Theta^\alpha$ -measure of the complement of this set is zero. The carrier is not to be confused with the support of  $\sigma_\Theta^\alpha$  – which is different. A Clark measure  $\sigma_\Theta^\alpha$  has a point mass at  $\xi \in \mathbb{T}$  if and only if  $|\Theta'(\xi)| < \infty$  and  $\Theta(\xi) = \alpha$ . Moreover, in this case  $\sigma_\Theta^\alpha(\{\xi\}) = |\Theta'(\xi)|^{-1}$ . Thus if  $\sigma_\Theta^\alpha$  is discrete then

$$\sigma_\Theta^\alpha = \sum_{\{\Theta(\xi)=\alpha, |\Theta'(\xi)| < \infty\}} \frac{1}{|\Theta'(\xi)|} \delta_\xi.$$

When the carrier of  $\sigma_\Theta^\alpha$  in (3.13) is discrete, say  $(\xi_n)_{n \geq 1} \subset \mathbb{T}$ , Clark [13] showed that the system

$$(3.14) \quad \left\{ \frac{k_{\xi_n}^\Theta}{\sqrt{|\Theta'(\xi_n)|}} : n \geq 1 \right\}$$

forms an orthonormal basis for  $(\Theta H^2)^\perp$  – called the *Clark basis*. In [22] it is shown that this situation cannot occur in the general setting of de Branges–Rovnyak spaces.

An inner function  $\Theta$  is said to have the *connected level set* property (written  $\Theta \in (CLS)$ ), if there exists  $\eta \in (0, 1)$  such that the sub-level set

$$L(\Theta, \eta) := \{z \in \mathbb{D} : |\Theta(z)| < \eta\}$$

is connected. For example, the atomic inner function

$$\Theta(z) = \exp\left(\frac{z+1}{z-1}\right)$$

satisfies this condition since the sub-level set is a disk tangent to  $\mathbb{T}$  at  $\zeta = 1$ . Though more difficult to see, one can show that the Blaschke

products with zeros

$$\left\{1 - \frac{1}{n^2} : n \geq 1\right\} \quad \text{or} \quad \left\{1 - \frac{1}{2^n} : n \geq 1\right\}$$

are CLS inner functions. The Blaschke product whose zeros are

$$\left\{1 - \frac{1}{2^{n^2}} : n \geq 1\right\}$$

is not a CLS inner function. If  $\Theta \in (CLS)$ , Aleksandrov [5] showed that  $m(\sigma(\Theta)) = 0$  and moreover, for every  $\alpha \in \mathbb{T}$ ,  $\sigma_{\Theta}^{\alpha}(\sigma(\Theta)) = 0$ . Since  $\mathbb{T} \setminus \sigma(\Theta)$  is a countable union of arcs where  $\Theta$  continues analytically, the carrier of  $\sigma_{\Theta}^{\alpha}$ ,

$$\{\xi \in \mathbb{T} \setminus \sigma(\Theta) : \Theta(\xi) = \alpha\}$$

is a discrete set  $(\xi_n)_{n \geq 1}$ . Hence, as discussed above, every  $\Theta \in (CLS)$  has a Clark basis (3.14).

#### 4. ISOMETRIC EMBEDDINGS

In this section, we propose an alternate proof of Aleksandrov's isometric embedding theorem. The main idea is to first prove the result when  $\Theta$  is a Blaschke product with simple zeros and then use an approximation argument based on Frostman shifts.

*Proof of Theorem 2.3.* The first implication is Aleksandrov's. We flesh out more of the details. First assume that  $\mu = \sigma_{\Theta b}^1$  for some function  $b$  in the closed unit ball of  $H^{\infty}$ . Then, by Carathéodory's theorem (see [24, p.6]), there exists a sequence of (finite) Blaschke products  $(B_n)_{n \geq 1}$  which converges pointwise to  $b$ . In particular, if for  $z \in \mathbb{D}$ , we denote by

$$P_z(\zeta) = \frac{1 - |z|^2}{|\zeta - z|^2}, \quad \zeta \in \mathbb{T},$$

the associated Poisson kernel, then according to (3.6), we get that for every  $z \in \mathbb{D}$ ,

$$\int_{\mathbb{T}} P_z(\zeta) d\sigma_{\Theta B_n}^1(\zeta) \rightarrow \int_{\mathbb{T}} P_z(\zeta) d\mu(\zeta), \quad n \rightarrow +\infty.$$

Since  $\|\sigma_{\Theta B_n}^1\|$  is bounded (apply (3.6) to  $b = \Theta B_n$  and  $z = 0$  and use the fact that  $(\Theta B_n)(0) \rightarrow \Theta(0)b(0)$ ) and since the closed linear span of  $\{P_z : z \in \mathbb{D}\}$  is dense in  $C(\mathbb{T})$ , we get that  $\sigma_{\Theta B_n}^1 \rightarrow \mu$  in the weak- $*$  topology. In others words, for any  $f \in C(\mathbb{T})$ , we have

$$(4.1) \quad \int_{\mathbb{T}} f(\zeta) d\sigma_{\Theta B_n}^1(\zeta) \rightarrow \int_{\mathbb{T}} f(\zeta) d\mu(\zeta), \quad n \rightarrow +\infty.$$

Moreover, Clark's theorem says that the embedding  $(B_n \Theta H^2)^\perp \hookrightarrow L^2(\sigma_{\Theta B_n}^1)$  is isometric. Since  $(\Theta H^2)^\perp \subset (\Theta B_n H^2)^\perp$ , we also have the isometric embedding  $(\Theta H^2)^\perp \hookrightarrow L^2(\sigma_{\Theta B_n}^1)$ . Thus, according to (4.1), for any  $f \in (\Theta H^2)^\perp \cap C(\mathbb{T})$ , we have

$$\|f\|_2^2 = \int_{\mathbb{T}} |f(\zeta)|^2 d\sigma_{\Theta B_n}^1(\zeta) \rightarrow \int_{\mathbb{T}} |f(\zeta)|^2 d\mu(\zeta).$$

Then, each function  $f \in (\Theta H^2)^\perp$  has a finite radial limit at  $\mu$ -almost every point of  $\mathbb{T}$  and we have  $\|f\|_2 = \|f\|_\mu$ .

Conversely, suppose that  $\mu \in M_+(\mathbb{T})$  and suppose that the embedding  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$  is isometric. In view of (3.7), we know that there exists a function  $b_0$  in the closed unit ball of  $H^\infty$  such that

$$(4.2) \quad \mu = \sigma_{b_0}^1.$$

We will now show that  $\Theta$  divides  $b_0$ . We first do this when  $\Theta = B$  is a Blaschke product with simple zeros  $\Lambda = (\lambda_n)_{n \geq 1}$ . By our discussion of the Clark theory (generalized by Ball) we have the onto partial isometry

$$\omega_{b_0} : L^2(\mu) \rightarrow \mathcal{H}(b_0), \quad \omega_{b_0} h = (1 - b_0) C_\mu h.$$

Note that for any  $n \geq 1$ ,

$$k_{\lambda_n}^B(z) = k_{\lambda_n}(z) := \frac{1}{1 - \bar{\lambda}_n z} \in (BH^2)^\perp \cap C(\mathbb{D}^-),$$

which yields, since  $k_{\lambda_n} \in H^2(\mu) = (\ker \omega_{b_0})^\perp$ ,

$$(4.3) \quad \langle \omega_{b_0} k_{\lambda_n}, \omega_{b_0} k_{\lambda_\ell} \rangle_{b_0} = \langle k_{\lambda_n}, k_{\lambda_\ell} \rangle_\mu = \langle k_{\lambda_n}, k_{\lambda_\ell} \rangle_2, \quad n, \ell \geq 1.$$

A standard computation (see for instance [42, III.6]) shows that

$$\omega_{b_0} k_{\lambda_n} = (1 - \overline{b_0(\lambda_n)})^{-1} k_{\lambda_n}^{b_0}, \quad n \geq 1.$$

Apply formula (4.3) to the above identity and use the reproducing property to get

$$\begin{aligned} \langle k_{\lambda_n}, k_{\lambda_\ell} \rangle_2 &= (1 - \overline{b_0(\lambda_n)})^{-1} (1 - b_0(\lambda_\ell))^{-1} \langle k_{\lambda_n}^{b_0}, k_{\lambda_\ell}^{b_0} \rangle_{b_0} \\ &= (1 - \overline{b_0(\lambda_n)})^{-1} (1 - b_0(\lambda_\ell))^{-1} k_{\lambda_n}^{b_0}(\lambda_\ell) \\ &= (1 - \overline{b_0(\lambda_n)})^{-1} (1 - b_0(\lambda_\ell))^{-1} (1 - \overline{b_0(\lambda_n)} b_0(\lambda_\ell)) k_{\lambda_n}(\lambda_\ell). \end{aligned}$$

Since

$$\langle k_{\lambda_n}, k_{\lambda_\ell} \rangle_2 = k_{\lambda_n}(\lambda_\ell) = (1 - \overline{\lambda_n} \lambda_\ell)^{-1} \neq 0,$$

we obtain

$$(1 - \overline{b_0(\lambda_n)})(1 - b_0(\lambda_\ell)) = 1 - \overline{b_0(\lambda_n)} b_0(\lambda_\ell),$$

which, after a little algebra, gives us

$$\overline{b_0(\lambda_n)} - 2\overline{b_0(\lambda_n)}b_0(\lambda_\ell) + b_0(\lambda_\ell) = 0.$$

The above can be re-arranged as

$$\overline{b_0(\lambda_n)}(1 - b_0(\lambda_\ell)) = -b_0(\lambda_\ell)(1 - \overline{b_0(\lambda_n)}), \quad (n, \ell \geq 1).$$

Setting  $f := b_0 / (1 - b_0)$ , the last equality implies that, for every  $n \geq 1$ ,

$$(4.4) \quad f(\lambda_n) = -\overline{f(\lambda_1)} = c,$$

and so

$$(4.5) \quad b_0(\lambda_n) = \frac{c}{1+c} =: \delta, \quad n \geq 1.$$

Setting  $\phi_\delta(z) := (\delta - z) / (1 - \overline{\delta}z)$ , we see that  $\phi_\delta \circ b_0$  vanishes on  $\Lambda$  and so  $B$ , since it has *simple* zeros, divides  $\phi_\delta \circ b_0$ . This implies the existence of a function  $\vartheta$  in the closed unit ball of  $H^\infty$  such that  $B\vartheta = \phi_\delta \circ b_0$ . Since  $\phi_\delta \circ \phi_\delta$  is the identity we get  $b_0 = \phi_\delta \circ (B\vartheta)$ , which shows that

$$\mu = \sigma_{\phi_\delta \circ (B\vartheta)}^1.$$

To finish this off, we use (3.11) and (3.8) to get

$$\mu = \frac{1}{|\phi'_\delta(\phi_\delta(1))|} \sigma_{B\vartheta}^{\phi_\delta(1)} = \frac{1 - |\delta|^2}{|1 - \delta|^2} \sigma_{B\vartheta}^{\phi_\delta(1)} = \frac{1 - |\delta|^2}{|1 - \delta|^2} \sigma_{Bb}^1,$$

where  $b$  is the function in the closed unit ball of  $H^\infty$  defined by  $b = \overline{\phi_\delta(1)}\vartheta$  (note that  $|\phi_\delta(1)| = 1$ ). Finally, using (4.4), we easily see that  $c$  is imaginary. Thus from the definition of  $\delta$  in (4.5) we have

$$\frac{1 - |\delta|^2}{|1 - \delta|^2} = 1,$$

which concludes the proof in the case when  $\Theta = B$  is a Blaschke product with simple zeros.

In the general case, the nice little fact that will contribute here is Frostman's result [23] (see also [16]) which says that for an arbitrary inner function  $\Theta$ , the *Frostman shift*  $\phi_\lambda \circ \Theta$  is a Blaschke product for every  $\lambda \in \mathbb{D}$  with the possible exception of a set of logarithmic capacity zero. This result can be refined to fit to our situation:  $\phi_\lambda \circ \Theta$  is a Blaschke product with *simple* zeros for all  $\lambda \in \mathbb{D}$  with the possible exception of a set of Lebesgue area measure zero [21, p. 677]. So let  $(\lambda_n)_{n \geq 1}$  be a sequence in  $\mathbb{D}$ ,  $\lambda_n \rightarrow 0$ , such that  $B_n := \phi_{\lambda_n} \circ (-\Theta)$  is a Blaschke product with simple zeros. A trivial estimate shows that

$$(4.6) \quad \|B_n - \Theta\|_\infty \leq \frac{2|\lambda_n|}{1 - |\lambda_n|} \rightarrow 0, \quad n \rightarrow +\infty.$$

Fix  $n \geq 1$ . Since  $\phi_{\lambda_n} \circ \phi_{\lambda_n}$  is the identity, remember from (3.12) the Crofoot transform  $U_{B_n}^{\lambda_n}$  which is a unitary operator from  $(B_n H^2)^\perp$  onto  $(\Theta H^2)^\perp$ . If we define  $\mu'_n \in M_+(\mathbb{T})$  by

$$(4.7) \quad d\mu'_n := \frac{1 - |\lambda_n|^2}{|1 - \overline{\lambda_n} B_n|^2} d\mu,$$

then  $(B_n H^2)^\perp$  embeds isometrically into  $L^2(\mu'_n)$ . Indeed, for  $g \in (B_n H^2)^\perp$ , note that  $U_{B_n}^{\lambda_n} g \in (\Theta H^2)^\perp$  and  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$  isometrically. Thus,

$$\int_{\mathbb{T}} |g|^2 dm = \int_{\mathbb{T}} |U_{B_n}^{\lambda_n} g|^2 dm = \int_{\mathbb{T}} |U_{B_n}^{\lambda_n} g|^2 d\mu = \int_{\mathbb{T}} |g|^2 d\mu'_n.$$

By the first case of the proof, there is a function  $b_n$  in the closed unit ball of  $H^\infty$  such that  $\mu'_n = \sigma_{B_n b_n}^1$ , that is

$$(4.8) \quad \mu = \frac{|1 - \overline{\lambda_n} B_n|^2}{1 - |\lambda_n|^2} \sigma_{B_n b_n}^1, \quad n \geq 1.$$

By the Banach-Alaoglu theorem, there is a subsequence  $(b_{n_\ell})_{\ell \geq 1}$  and a function  $b$  in the closed unit ball of  $H^\infty$  such that  $b_{n_\ell}$  converges to  $b$  in the weak-\* topology, which means that

$$\int_{\mathbb{T}} f b_{n_\ell} dm \rightarrow \int_{\mathbb{T}} f b dm, \quad \ell \rightarrow +\infty,$$

for any  $f \in L^1(\mathbb{T})$ . In particular, for any  $z \in \mathbb{D}$ ,  $b_{n_\ell}(z) \rightarrow b(z)$  as  $\ell \rightarrow +\infty$ . Now we argue that  $\mu = \sigma_{\Theta b}^1$ . Indeed according to (4.8), we have

$$\begin{aligned} \int_{\mathbb{T}} P_z(\zeta) d\mu(\zeta) &= \int_{\mathbb{T}} P_z(\zeta) \left( \frac{|1 - \overline{\lambda_{n_\ell}} B_{n_\ell}(\zeta)|^2}{1 - |\lambda_{n_\ell}|^2} - 1 \right) d\sigma_{B_{n_\ell} b_{n_\ell}}^1(\zeta) + \\ &+ \int_{\mathbb{T}} P_z(\zeta) d\sigma_{B_{n_\ell} b_{n_\ell}}^1(\zeta). \end{aligned}$$

Observe that

$$\frac{|1 - \overline{\lambda_n} B_n(\zeta)|^2}{1 - |\lambda_n|^2} - 1 \rightarrow 0, \quad n \rightarrow +\infty,$$

uniformly on  $\mathbb{T}$  and, according to (4.6),  $B_{n_\ell}$  tends to  $\Theta$  uniformly on  $\mathbb{D}^-$  and  $b_{n_\ell}(z) \rightarrow b(z)$ ,  $z \in \mathbb{D}$ , as  $\ell \rightarrow +\infty$ . Hence,

$$\int_{\mathbb{T}} P_z(\zeta) d\sigma_{B_{n_\ell} b_{n_\ell}}^1(\zeta) = \frac{1 - |B_{n_\ell}(z) b_{n_\ell}(z)|^2}{|1 - B_{n_\ell}(z) b_{n_\ell}(z)|^2}$$

tends to

$$\frac{1 - |\Theta(z) b(z)|^2}{|1 - \Theta(z) b(z)|^2} = \int_{\mathbb{T}} P_z(\zeta) d\sigma_{\Theta b}^1(\zeta).$$

Thus, we obtain that for any  $z \in \mathbb{D}$ ,

$$\int_{\mathbb{T}} P_z(\zeta) d\mu(\zeta) = \int_{\mathbb{T}} P_z(\zeta) d\sigma_{\Theta b}^1(\zeta).$$

But the closed linear span of  $\{P_z : z \in \mathbb{D}\}$  is dense in  $C(\mathbb{T})$ , which yields that

$$\mu = \sigma_{\Theta b}^1,$$

and thus concludes the proof.  $\square$

## 5. DOMINATING SETS

In this section, we introduce and discuss the notion of dominating sets for  $(\Theta H^2)^\perp$  where  $\Theta$  is a non-constant inner function. This terminology is perhaps reminiscent of the concept of a dominating sequence for  $H^\infty$  [10].

**Definition 5.1.** A (Lebesgue) measurable subset  $\Sigma \subset \mathbb{T}$ , with  $m(\Sigma) < 1$ , is called a *dominating set* for  $(\Theta H^2)^\perp$  if

$$\|f\|_2^2 \lesssim \int_{\Sigma} |f|^2 dm, \quad \forall f \in (\Theta H^2)^\perp.$$

Necessarily, in the above definition,  $m(\Sigma) > 0$ . Also notice that the hypothesis  $m(\Sigma) < 1$  is crucial since otherwise the whole matter becomes trivial. Another observation is that if  $\Sigma$  is a measurable subset of  $\mathbb{T}$  such that  $m(\Sigma) < 1$ , then  $\Sigma$  is a dominating set if and only if the measure  $d\mu = \chi_{\Sigma} dm$  yields a reverse embedding for  $(\Theta H^2)^\perp$  (and hence equivalent norms). Therefore, by Volberg's theorem (Theorem 1.2), we obtain a criterion for dominating sets  $\Sigma$  in terms of the harmonic continuation of  $\chi_{\Sigma}$  into the open unit disc. However, this criterion is not so easy to deal with. In particular, it is not clear that dominating sets even exist – for *any* model space.

The aim of this section is to give simpler necessary or sufficient conditions for dominating sets. More precisely, we would like to highlight some interesting relationship between dominating sets for  $(\Theta H^2)^\perp$  and the spectrum of the inner function  $\Theta$ . Indeed, a simple look at the definition of dominating set tells us that such a set cannot be too ‘far’ from the spectrum or from the points where the modulus of the angular derivative is infinite. We make this more precise with the next two results. For sets  $A$  and  $B$  and a point  $x$ , let

$$\begin{aligned} d(A, B) &:= \inf\{|a - b| : a \in A, b \in B\}, \\ d(x, A) &:= d(\{x\}, A). \end{aligned}$$



**Proposition 5.2.** *If  $\Sigma$  is a dominating set for  $(\Theta H^2)^\perp$ , and  $\zeta \in \mathbb{T} \setminus ADC_\Theta$  then  $d(\zeta, \Sigma) = 0$ .*

*Proof.* Let  $\zeta \in \mathbb{T}$  such that  $|\Theta'(\zeta)| = \infty$  and  $(\lambda_n)_{n \geq 1} \subset \mathbb{D}$  with radial limit  $\zeta$ . Let us suppose, towards a contradiction, that  $d(\zeta, \Sigma) > 0$ . Then, for  $n$  large enough, we see that  $d(\lambda_n, \Sigma) \gtrsim 1$ . If we consider the reproducing kernels  $k_{\lambda_n}^\Theta \in (\Theta H^2)^\perp$ , the defining property of  $\Sigma$  along with (3.5) gives us

$$\begin{aligned} \frac{1 - |\Theta(\lambda_n)|^2}{1 - |\lambda_n|^2} &= \int_{\mathbb{T}} |k_{\lambda_n}^\Theta|^2 dm \lesssim \int_{\Sigma} |k_{\lambda_n}^\Theta|^2 dm \\ &\lesssim \int_{\Sigma} \left| \frac{\Theta(\xi) - \Theta(\lambda_n)}{\lambda_n - \xi} \right|^2 dm(\xi) \\ &\lesssim \frac{1}{d(\lambda_n, \Sigma)} \\ &\lesssim 1, \end{aligned}$$

whereas, by Ahern-Clark [1],

$$\frac{1 - |\Theta(\lambda_n)|^2}{1 - |\lambda_n|^2} \longrightarrow \infty, \quad n \rightarrow \infty,$$

This yields the required contradiction.  $\square$

The following consequence is obvious.

**Corollary 5.3.** *If  $\Sigma$  is a dominating set for  $(\Theta H^2)^\perp$ , then*

$$d(\Sigma, \mathbb{T} \setminus ADC_\Theta) = 0.$$

Now, since  $\mathbb{T} \setminus ADC_\Theta$  is a subset of the spectrum  $\sigma(\Theta)$ , the previous corollary directly implies the following one.

**Proposition 5.4.** *If  $\Sigma$  is a dominating set for  $(\Theta H^2)^\perp$ , then*

$$d(\Sigma, \sigma(\Theta)) = 0.$$

We will see in Corollary 5.10 that if the spectrum of the inner function is the whole circle, then a dominating set — if it exists — must be dense. However, dense sets can be of measure zero. Here is a little fact that states that any neighborhood of a point in the spectrum has to contain portions of the dominating set  $\Sigma$  with non negligible Lebesgue measure. First let us introduce the *Privalov shadow*: for  $\lambda \in \mathbb{D}$  and  $\alpha > 0$ , let  $I_\lambda^\alpha$  be the arc in  $\mathbb{T}$  centered at  $\lambda/|\lambda|$  with length  $\alpha(1 - |\lambda|)$ .

For  $\alpha = 1$  we simply write  $I_\lambda := I_\lambda^1$ , (see Figure 1). Observe that  $I_\lambda^\alpha$  is the  $\alpha$ -amplification of  $I_\lambda$ .

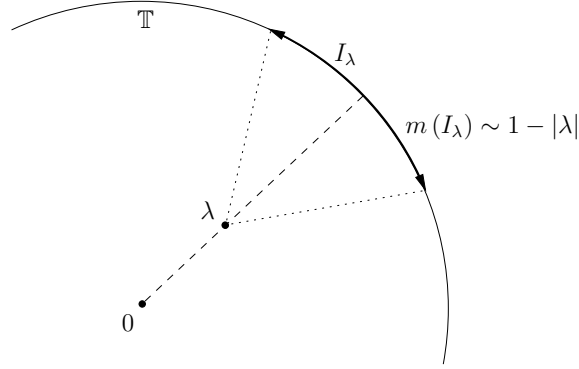


FIGURE 1. Privalov shadow

**Proposition 5.5.** *Let  $\zeta \in \sigma(\Theta)$  and  $\Sigma$  dominating. Then there exists an  $\alpha > 0$  such that for every sequence  $\lambda_n \rightarrow \zeta$  with  $\Theta(\lambda_n) \rightarrow 0$ , there is an integer  $N$  with*

$$m(\Sigma \cap I_{\lambda_n}^\alpha) \gtrsim m(I_{\lambda_n}^\alpha), \quad n \geq N.$$

*Proof.* Since  $\Sigma$  is dominating we have

$$c \leq \frac{\int_\Sigma |k_{\lambda_n}^\Theta|^2 dm}{\|k_{\lambda_n}^\Theta\|_2^2}.$$

Now, since  $\Theta(\lambda_n) \rightarrow 0$ , there exists an  $N$  such that  $|\Theta(\lambda_n)| \leq 1/2$  for  $n \geq N$ , so that in the above inequality we can replace  $k_{\lambda_n}^\Theta$  by  $k_{\lambda_n}$ ,  $n \geq N$ . Hence (with a change of constant from  $c$  to  $c_1$ ) for  $n \geq N$ ,

$$c_1 \leq \int_\Sigma \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|^2} dm(\xi).$$

Now with an appropriate choice of  $\alpha$  we have

$$\int_{\mathbb{T} \setminus I_{\lambda_n}^\alpha} \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|^2} dm(\xi) < \frac{c_1}{2}.$$

Hence

$$\begin{aligned}
c_1 &\leq \int_{\Sigma} \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|^2} dm(\xi) \\
&= \int_{\Sigma \cap I_{\lambda_n}^{\alpha}} \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|^2} dm(\xi) + \int_{\Sigma \cap (\mathbb{T} \setminus I_{\lambda_n}^{\alpha})} \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|^2} dm(\xi) \\
&\leq \int_{\Sigma \cap I_{\lambda_n}^{\alpha}} \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|^2} dm(\xi) + \frac{c_1}{2} \\
&\lesssim \frac{m(\Sigma \cap I_{\lambda_n}^{\alpha})}{1 - |\lambda_n|^2} + \frac{c_1}{2},
\end{aligned}$$

which yields the desired conclusion.  $\square$

If the spectrum of the inner function  $\Theta$  is not the whole unit circle, it turns out that dominating sets for  $(\Theta H^2)^{\perp}$  exist – in abundance. Here are two results in this direction.

**Theorem 5.6.** *Let  $\Theta$  be an inner function and assume that  $\sigma(\Theta) \neq \mathbb{T}$ . Then  $(\Theta H^2)^{\perp}$  has dominating sets. More precisely, if  $\Sigma$  is an open subset of  $\mathbb{T}$  such that  $\sigma(\Theta) \subset \Sigma$  and  $m(\Sigma) < 1$ , then  $\Sigma$  is a dominating set for  $(\Theta H^2)^{\perp}$ .*

*Proof.* Since  $\sigma(\Theta)$  is a closed subset of  $\mathbb{T}$ , it turns out that if  $\sigma(\Theta) \neq \mathbb{T}$ , then we can find an open subset  $\Sigma$  of  $\mathbb{T}$  such that  $\sigma(\Theta) \subset \Sigma$  and  $m(\Sigma) < 1$ . We now prove that this open set is a dominating set for  $(\Theta H^2)^{\perp}$ . As already noticed, this is equivalent to saying if  $d\mu = \chi_{\Sigma} dm$  the norms  $\|\cdot\|_{\mu}$  and  $\|\cdot\|_2$  are equivalent on  $(\Theta H^2)^{\perp}$ . By Volberg's result we will show that if  $\lambda_n \in \mathbb{D}$  is such that  $\lim_{n \rightarrow +\infty} \widehat{\chi}_{\Sigma}(\lambda_n) = 0$ , then  $\lim_{n \rightarrow +\infty} |\Theta(\lambda_n)| = 1$ . To do this, we first note that

$$(5.1) \quad \widehat{\chi}_{\Sigma}(\lambda_n) = \int_{\Sigma} \frac{1 - |\lambda_n|^2}{|1 - \bar{\lambda}_n \zeta|^2} dm(\zeta).$$

Let  $\zeta_0 \in \mathbb{T}$  be such that some subsequence  $\lambda_{n_{\ell}} \rightarrow \zeta_0$ . It suffices to show that  $\zeta_0 \in \mathbb{T} \setminus \sigma(\Theta)$ . Indeed since we know that the inner function  $\Theta$  is analytic on  $\mathbb{T} \setminus \sigma(\Theta)$  we will thus get

$$\lim_{\ell \rightarrow +\infty} |\Theta(\lambda_{n_{\ell}})| = |\Theta(\zeta_0)| = 1.$$

In order to prove that  $\zeta_0 \in \mathbb{T} \setminus \sigma(\Theta)$ , we argue, assuming to the contrary, that  $\zeta_0 \in \sigma(\Theta)$ . Then for  $\zeta \in \mathbb{T} \setminus \Sigma$  and for sufficiently large  $\ell$ , we have

$$\begin{aligned}
|1 - \bar{\lambda}_{n_{\ell}} \zeta| &\geq |\zeta_0 - \zeta| - |\lambda_{n_{\ell}} - \zeta_0| \geq d(\zeta_0, \mathbb{T} \setminus \Sigma) - |\lambda_{n_{\ell}} - \zeta_0| \\
&\geq \frac{1}{2} d(\zeta_0, \mathbb{T} \setminus \Sigma).
\end{aligned}$$

Since  $\zeta_0 \in \sigma(\Theta) \subset \Sigma$  and  $\Sigma$  is open, we have  $d(\zeta_0, \mathbb{T} \setminus \Sigma) > 0$ . Hence

$$\int_{\mathbb{T} \setminus \Sigma} \frac{1 - |\lambda_{n_\ell}|^2}{|1 - \bar{\lambda}_{n_\ell} \zeta|^2} dm(\zeta) \leq \frac{4(1 - |\lambda_{n_\ell}|^2)}{d^2(\zeta_0, \mathbb{T} \setminus \Sigma)} \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

On the other hand, by hypothesis, we have  $\widehat{\chi}_\Sigma(\lambda_{n_\ell}) \rightarrow 0$ ,  $\ell \rightarrow +\infty$ . Taking into account (5.1), we get

$$\begin{aligned} \int_{\mathbb{T}} \frac{1 - |\lambda_{n_\ell}|^2}{|1 - \bar{\lambda}_{n_\ell} \zeta|^2} dm(\zeta) &= \\ \int_{\Sigma} \frac{1 - |\lambda_{n_\ell}|^2}{|1 - \bar{\lambda}_{n_\ell} \zeta|^2} dm(\zeta) + \int_{\mathbb{T} \setminus \Sigma} \frac{1 - |\lambda_{n_\ell}|^2}{|1 - \bar{\lambda}_{n_\ell} \zeta|^2} dm(\zeta) &\rightarrow 0 \end{aligned}$$

as  $\ell \rightarrow +\infty$ . But this is a contradiction since the left hand side is always 1.  $\square$

**Corollary 5.7.** *Let  $\Theta$  be an inner function such that  $m(\sigma(\Theta)) = 0$ . Then, for every  $0 < \varepsilon < 1$ , there is a dominating set  $\Sigma$  for  $(\Theta H^2)^\perp$  such that  $m(\Sigma) < \varepsilon$ .*

*Proof.* By outer regularity of Lebesgue measure, we know that for every  $0 < \varepsilon < 1$ , there exists an open subset  $\Sigma$  of  $\mathbb{T}$  such that  $\sigma(\Theta) \subset \Sigma$  and  $m(\Sigma) < \varepsilon$ . Now Theorem 5.6 implies that such a  $\Sigma$  is a dominating set for  $(\Theta H^2)^\perp$ .  $\square$

**Remark 5.8.** Theorems 5.6 and Corollary 5.7 apply in particular when  $\Theta \in (CLS)$ , because in that situation we know by a result of Aleksandrov [2] that  $m(\sigma(\Theta)) = 0$ .

In the beginning of this section we have seen a simple argument that a dominating set has to be close to every point of the complement of  $\text{ADC}_\Theta$ . Also recall that  $\sigma(\Theta)$  contains this complement. In order to show that a dominating set has to be close to every point of  $\sigma(\Theta)$  we use Volberg's theorem (note that  $\mathbb{T} \setminus \text{ADC}_\Theta$  can be empty while  $\sigma(\Theta)$  is never empty).

**Proposition 5.9.** *Let  $\Theta$  be an inner function. If  $\Sigma$  is a dominating set for  $(\Theta H^2)^\perp$ , then for every  $\zeta \in \sigma(\Theta)$ , we have  $d(\zeta, \Sigma) = 0$ .*

*Proof.* By Volberg's theorem, we know that there exists a  $\delta > 0$  such that

$$(5.2) \quad \widehat{\chi}_\Sigma(z) + |\Theta(z)| \geq \delta, \quad z \in \mathbb{D}.$$

Assuming there is  $\zeta \in \sigma(\Theta)$  with  $d(\zeta, \Sigma) > 0$ . By (3.1) there exists a sequence  $(z_n)_{n \geq 1} \subset \mathbb{D}$  such that  $z_n \rightarrow \zeta$  and  $\Theta(z_n) \rightarrow 0$ , as  $n \rightarrow +\infty$ .

Now

$$\widehat{\chi}_\Sigma(z_n) = \int_\Sigma \frac{1 - |z_n|^2}{|z_n - \xi|^2} dm(\xi),$$

and for  $\xi \in \Sigma$

$$|\xi - z_n| \geq |\xi - \zeta| - |\zeta - z_n| \geq d(\zeta, \Sigma) - |\zeta - z_n|.$$

Since  $|\zeta - z_n| \rightarrow 0$ , as  $n \rightarrow +\infty$ , we get, for  $n$  sufficiently large,  $|\xi - z_n| \geq d(\zeta, \Sigma^-)/2$ , which yields

$$\widehat{\chi}_\Sigma(z_n) \leq \frac{4}{d(\zeta, \Sigma^-)^2} (1 - |z_n|^2) \rightarrow 0, \quad \text{as } n \rightarrow +\infty,$$

contradicting (5.2).  $\square$

We have seen that if the spectrum of  $\Theta$  is not the whole circle, then there are many dominating sets. On the contrary, if the spectrum of  $\Theta$  is the whole circle, the preceding proposition allows to deduce a topological condition on the size of the dominating set.

**Corollary 5.10.** *Let  $\Theta$  be an inner function such that  $\sigma(\Theta) = \mathbb{T}$ . If  $\Sigma$  is a dominating set for  $(\Theta H^2)^\perp$ , then  $\Sigma$  is dense in  $\mathbb{T}$ .*

**Remark 5.11.** According to Theorem 5.6, we see that if  $\sigma(\Theta) \neq \mathbb{T}$ , then one can construct closed dominating sets for  $(\Theta H^2)^\perp$ . Indeed it is sufficient to choose an open set  $\Sigma$  such that  $\sigma(\Theta) \subset \Sigma \subset \Sigma^-$  and  $m(\Sigma^-) < 1$ . Corollary 5.10 shows that the converse is true. In other words, the space  $(\Theta H^2)^\perp$  has *closed* dominating sets if and only if  $\sigma(\Theta) \neq \mathbb{T}$ . In Proposition 5.16 we will construct a Blaschke product  $B$  with  $\sigma(B) = \mathbb{T}$  and which admits an open dominating set.

We now discuss, with the help of examples, the conditions of our previous results and the “size” of dominating sets.

**Example 5.12.** Let  $\Theta(z) = \exp(-(1+z)/(1-z))$ . In this case,  $\sigma(\Theta) = \{1\}$  and  $\Theta \in (CLS)$ . According to Theorem 5.6, every open arc containing 1 is a dominating set for  $\Theta$ . In this example, the “open” condition seems to be necessary. Indeed, we have the following result.

**Proposition 5.13.** *Let  $\Theta(z) = \exp(-(1+z)/(1-z))$ . Suppose that  $\Sigma \subset \mathbb{T}$  is a closed arc with endpoints 1 and  $e^{i\gamma_0}$ ,  $\gamma_0 \in (0, \pi)$ , then  $\Sigma$  is not dominating for  $(\Theta H^2)^\perp$ .*

*Proof.* Let

$$z_n = r_n e^{-i\theta_n}, \quad 1 - r_n = \theta_n^{3/2},$$

where  $\theta_n > 0$  and  $\theta_n \rightarrow 0$ . The sequence  $z_n$  goes tangentially “from below” to 1, i.e.  $\arg(z_n) \in (-\frac{\pi}{2}, 0)$ . In particular we have

$$|1 - z_n|^2 \asymp (1 - r_n)^2 + \theta_n^2 = \theta_n^3 + \theta_n^2 \asymp \theta_n^2, \quad n \rightarrow \infty.$$

Hence

$$\frac{1 - |z_n|^2}{|1 - z_n|^2} \asymp \frac{1 - r_n}{\theta_n^2} = \frac{\theta_n^{3/2}}{\theta_n^2} = \frac{1}{\theta_n^{1/2}} \longrightarrow \infty,$$

and thus

$$|\Theta(z_n)| = \exp\left(-\frac{1 - |\lambda_n|^2}{|1 - \lambda_n|^2}\right) \longrightarrow 0.$$

Observe that

$$\begin{aligned} \widehat{\chi}_\Sigma(z_n) &= \int_\Sigma \frac{1 - r_n^2}{|e^{it} - z_n|^2} \frac{dt}{2\pi} \lesssim \int_0^{\gamma_0} \frac{\theta_n^{3/2}}{|e^{it} - e^{-i\theta_n}|^2} dt \\ &= \int_0^{\gamma_0} \frac{\theta_n^{3/2}}{2(1 - \cos(t + \theta_n))} dt \asymp \int_0^{\gamma_0} \frac{\theta_n^{3/2}}{(t + \theta_n)^2} dt \\ &= \theta_n^{3/2} \left[ \frac{-1}{t + \theta_n} \right]_0^{\gamma_0} \\ &\asymp \theta_n^{1/2} \longrightarrow 0. \end{aligned}$$

So  $\inf_{\lambda \in \mathbb{D}} (\widehat{\chi}_\Sigma(\lambda) + |\Theta(\lambda)|) = 0$ , and Volberg's theorem allows us to conclude that  $\Sigma$  is not dominating.  $\square$

The situation is, of course, the same if we replace  $\gamma_0 \in (0, \pi)$  by  $\gamma_0 \in (-\pi, 0)$ .

Compare this with the next situation.

**Example 5.14.** Let  $\Theta = B$  be a Blaschke product with simple zeros  $\lambda_n = r_n e^{i\theta_n}$ ,  $\theta_n = 2^{-n}$  and  $1 - r_n = 16^{-n}$ . This is an inner function such that  $\sigma(B) = \{1\}$  and  $\text{ADC}_B = \mathbb{T}$  (Ahern-Clark). We have the following result.

**Proposition 5.15.** *Let  $B$  be as above. Suppose that  $\Sigma \subset \mathbb{T}$  is a closed arc with endpoints 1 and  $e^{i\gamma_0}$ ,  $\gamma_0 \in (0, \pi)$ , then  $\Sigma$  is dominating for  $(BH^2)^\perp$ .*

*Proof.* Again we use Volberg's theorem. Let  $(z_n)_{n \geq 1} \subset \mathbb{D}$  be such that  $\widehat{\chi}_\Sigma(z_n) \rightarrow 0$ . We have to show that  $|B(z_n)| \rightarrow 1$ . Pick a convergent subsequence  $z_{n_k} \rightarrow \zeta \in \mathbb{T}$ . Since  $\sigma(B) = \{1\}$ , the only critical situation is when  $\zeta = 1$ . First we argue that  $z_{n_k} \rightarrow 1$  "from below". Indeed, assume on the contrary that there is a subsequence, also denoted by  $z_{n_k}$ , such that  $\arg(z_{n_k}) \in (0, \pi/2)$ . For a point  $\lambda \in \mathbb{D}$ , recall that  $I_\lambda$  denotes the Privalov shadow associated to  $\lambda$ , that is the arc in  $\mathbb{T}$  centered at  $\lambda/|\lambda|$  with length  $(1 - |\lambda|)$  – see Figure 1. It is easy to see that for  $\zeta \in I_\lambda$ , we have

$$\frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} \asymp \frac{1}{1 - |\lambda|^2}.$$

Now, since  $z_{n_k} \rightarrow 1$ , with  $\arg(z_{n_k}) \in (0, \pi/2)$ , there exists an integer  $N$  such that for any  $k \geq N$ , we have  $I_{z_{n_k}} \cap \mathbb{T}_+ \subset \Sigma$  and  $m(I_{z_{n_k}} \cap \mathbb{T}_+) \geq (1 - |z_{n_k}|)/2$ , where  $\mathbb{T}_+ = \{z \in \mathbb{T} : \arg(z) \in (0, \pi)\}$ . Thus

$$\begin{aligned} \widehat{\chi}_\Sigma(z_{n_k}) &= \int_\Sigma \frac{1 - |z_{n_k}|^2}{|\zeta - z_{n_k}|^2} dm(\zeta) \geq \int_{I_{z_{n_k}} \cap \mathbb{T}_+} \frac{1 - |z_{n_k}|^2}{|\zeta - z_{n_k}|^2} dm(\zeta) \\ &\gtrsim \frac{1}{1 - |z_{n_k}|^2} \frac{1 - |z_{n_k}|}{2} \gtrsim 1, \end{aligned}$$

which contradicts the fact that  $\widehat{\chi}_\Sigma(z_n) \rightarrow 0$ . Hence we can assume that  $z_{n_k} \rightarrow 1$  with  $\arg(z_{n_k}) \in (-\pi/2, 0)$ . Taking the logarithmic derivative and using (3.2), it is easy to see that

$$|B'(z)| \leq \sum_{n \geq 1} \frac{1 - |\lambda_n|^2}{|1 - \bar{\lambda}_n z|^2}, \quad z \in \mathbb{D}^-,$$

and standard estimates show that if  $z$  belongs to the closed lower half-disc  $\Upsilon := \{z : |z| \leq 1, \operatorname{Im}(z) \leq 0\}$ , then

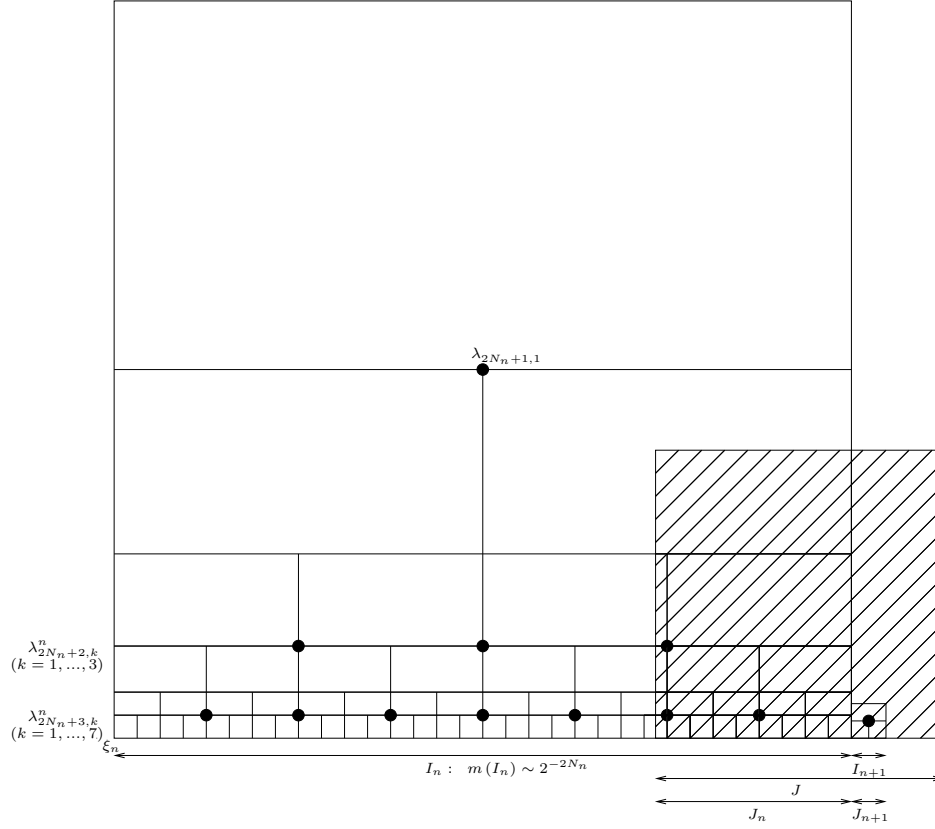
$$|B'(z)| \lesssim \sum_{n \geq 1} \frac{1}{4^n} < +\infty,$$

which means that  $B'$  is uniformly bounded on  $\Upsilon$ . Hence  $B$  is continuous on  $\Upsilon$ . In particular, we get that  $|B(z_{n_k})| \rightarrow 1$ . We conclude from Volberg's theorem that  $\Sigma$  is dominating for  $(BH^2)^\perp$ .  $\square$

We finish this section with an example of a Blaschke product whose boundary spectrum is the whole circle and which admits a dominating set  $\Sigma$ .

**Proposition 5.16.** *There exists a Blaschke product  $B$  with  $\sigma(B) = \mathbb{T}$  and an open subset  $\Sigma \subsetneq \mathbb{T}$  dominating for  $(BH^2)^\perp$ .*

*Proof.* Let  $C$  be a Smith-Volterra-Cantor set of  $\mathbb{T}$  (i.e., a closed subset of  $\mathbb{T}$ , nowhere dense and with positive measure, constructed in a similar way as the usual Cantor set by removing the middle *fourth* instead of the middle third). We define  $\Sigma := \mathbb{T} \setminus C$  which is clearly a dense open subset of  $\mathbb{T}$  with  $0 < m(\Sigma) < 1$ . Since  $\Sigma$  is open, there is a sequence of open arcs  $(I_n)_{n \geq 1}$  such that  $\Sigma = \bigcup_{n \geq 1} I_n$ . We denote by  $\xi_n$  the first endpoint (moving counterclockwise) of  $I_n$  and  $N_n$  the integer such that  $2^{-(2N_n+2)} \leq \theta_n := m(I_n) < 2^{-2N_n}$  and  $\alpha_n := \theta_n 2^{2N_n} \in [1/4, 1)$ . Observe that the second endpoint of  $I_n$  then corresponds to  $\xi_n e^{i2\pi\theta_n}$ . We now fix  $n$ . For each  $l > 2N_n$  and each  $k = 1, \dots, 2^{l-2N_n} - 1$ , we set


 FIGURE 2.  $\Lambda_n$ ,  $J_n$  and  $J_{n+1}$ 

(see Figure 2)

$$\lambda_{l,k}^n := \left(1 - \frac{1}{2^{2l}}\right) \xi_n e^{i \frac{k}{2^l} 2\pi \alpha_n}, \quad \Lambda_n := \{\lambda_{l,k}^n : l, k\}, \quad \text{and } \Lambda := \bigcup_{n \geq 1} \Lambda_n.$$

Observe that every  $\lambda_{l,k}^n \in \Lambda_n$  lies in the Carleson window  $S(I_n)$ . For each  $l \geq N_n$ , every arc  $\{|z| = 1 - 2^{-2l}\} \cap S(I_n)$  contains  $2^{l-2N_n} - 1$  points of  $\Lambda$ .

Thus, the sequence  $\Lambda$  satisfies the Blaschke condition:

$$\begin{aligned} \sum_{\lambda \in \Lambda} (1 - |\lambda|) &= \sum_{n \geq 1} \sum_{l > 2N_n} \frac{1}{2^{2l}} (2^{l-2N_n} - 1) \\ &\leq \sum_{n \geq 1} \frac{1}{2^{4N_n}} \\ &\asymp \sum_{n \geq 1} m(I_n)^2 < \infty. \end{aligned}$$



Let  $B$  be the corresponding Blaschke product. It is clear that every point of  $I_n$  is a cluster point of  $\Lambda$ . Since  $\Sigma = \bigcup_n I_n$  is dense in  $\mathbb{T}$ , it easily follows that  $\sigma(B) = \mathbb{T}$ . It remains to show that  $\Sigma$  is a dominating set for  $(BH^2)^\perp$ . We will use Volberg's theorem and show that

$$\inf_{z \in \mathbb{D}} (|B(z)| + \widehat{\chi}_\Sigma(z)) > 0.$$

The idea is the following. We will show that  $\Lambda$  is an interpolating sequence, which implies that  $|B|$  is big outside a pseudohyperbolic neighborhood of  $\Lambda$  (see below for definition). Inside a pseudohyperbolic neighborhood of a point  $\lambda \in \Lambda$ , an easy computation will show that  $\widehat{\chi}_\Sigma$  is big.

In order to prove that  $\Lambda$  is an interpolating sequence, it suffices to prove that  $\nu := \sum_{\lambda \in \Lambda} (1 - |\lambda|^2) \delta_\lambda$  is a Carleson measure [24] since the sequence  $\Lambda$  is separated by construction. Let  $J$  be an arc of  $\mathbb{T}$ . It is possible to write  $J = \bigcup_{n \geq 1} J_n$ , where  $J_n := J \cap I_n$  are disjoint arcs (observe that we have three configurations: either  $J_n = \emptyset$  or  $I_n \subset J$  or  $I_n$  meets  $J$  without being contained in  $J$ , this latter situation occurs at most two times at the endpoints of  $J$  - see Figure 2).

Let us introduce the sector:

$$\alpha(I) := \left\{ z \in \mathbb{D}^- : \frac{z}{|z|} \in I \right\}.$$

In the following computation, we want to count the number of points of  $\Lambda_n$  at level  $1 - 1/2^{2l}$ ,  $l \geq 2N_n + 1$ , that fall in the sector  $\alpha(K)$  where  $K$  is any arc in  $\mathbb{T}$ . Since the arguments of those points are separated by  $2\pi\alpha_n/2^l$  (recall that  $\alpha_n \in [1/4, 1)$ ) we get

$$\# (\Lambda_n \cap \{|z| = 1 - 2^{-2l}\} \cap \alpha(K)) \leq \frac{m(K)}{2\pi\alpha 2^{-l}} \asymp 2^l m(K).$$

Hence

$$\begin{aligned} \nu(S(J)) &= \sum_{\lambda \in S(J)} (1 - |\lambda|^2) \lesssim \sum_{\lambda \in S(J)} (1 - |\lambda|) \\ &\leq \sum_{n \geq 1} \sum_{l > 2N_n} 2^{-2l} \cdot \# (\Lambda_n \cap \{|z| = 1 - 2^{-2l}\} \cap \alpha(J_n)) \\ &\leq \sum_{n \geq 1} m(J_n) \sum_{l > 2N_n} 2^{-l} \\ &\leq \sum_{n \geq 1} m(J_n) \\ &\leq m(J), \end{aligned}$$

so that  $\nu$  is a Carleson measure and thus  $\Lambda$  is an interpolating sequence. So, for arbitrarily fixed  $\eta \in (0, 1)$ , if

$$z \in \mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} \Omega(\lambda, \eta), \text{ with } \Omega(\lambda, \eta) := \left\{ z \in \mathbb{D} : \left| \frac{\lambda - z}{1 - \bar{\lambda}z} \right| < \eta \right\},$$

then  $|B(z)| \asymp 1$  (see for instance [35, p. 218]). On the other hand, if  $z \in \Omega(\lambda, \eta)$ , then

$$\begin{aligned} \widehat{\chi}_\Sigma(z) &\asymp \int_\Sigma \frac{1 - |\lambda|^2}{|\lambda - \xi|^2} dm(\xi) \\ &\geq \int_{I_\lambda \cap \Sigma} \frac{1 - |\lambda|^2}{|\lambda - \xi|^2} dm(\xi) \\ &\gtrsim \frac{m(I_\lambda \cap \Sigma)}{1 - |\lambda|}, \end{aligned}$$

where  $I_\lambda$  is the Privalov shadow of  $\lambda$ . Let  $n$  be such that  $\lambda \in \Lambda_n$ . Then  $\lambda \in S(I_n)$  and thus  $m(I_\lambda \cap \Sigma) \geq m(I_\lambda \cap I_n) \geq m(I_\lambda)/2 \asymp 1 - |\lambda|$ , so that  $\widehat{\chi}_\Sigma(z) \gtrsim 1$ ,  $z \in \Omega(\lambda, \eta)$ . Finally, we obtain that

$$\inf_{z \in \mathbb{D}} (|B(z)| + \widehat{\chi}_\Sigma(z)) > 0,$$

which ends the proof.  $\square$

We have seen that if  $\sigma(\Theta) \neq \mathbb{T}$  then  $(\Theta H^2)^\perp$  admits a dominating set. In Proposition 5.16, we have constructed an example of inner functions  $\Theta$  such that  $\sigma(\Theta) = \mathbb{T}$  and the corresponding model space admits also a dominating set. It is natural to formulate the following:

**Open Question.** Does every model space  $(\Theta H^2)^\perp$  admit a dominating set?

## 6. REVERSE EMBEDDINGS FOR $(\Theta H^2)^\perp$ – PROOFS OF THEOREM 2.1 AND THEOREM 2.2

The proof of Theorem 2.1 requires a few preliminaries on perturbation of bases. Recall that a sequence  $(x_n)_{n \geq 1} \subset \mathcal{H}$  is a *Riesz basis* for a separable Hilbert space  $\mathcal{H}$  if the closed linear span of  $(x_n)_{n \geq 1}$  is  $\mathcal{H}$  and

$$\left\| \sum_{n \geq 1} a_n x_n \right\|_{\mathcal{H}}^2 \asymp \sum_{n \geq 1} |a_n|^2, \quad \forall (a_n)_{n \geq 1} \in \ell^2(\mathbb{N}).$$

Also recall that if  $\Theta \in (CLS)$ , then a carrier of the Clark measure  $\sigma_\Theta^\alpha$ ,

$$\{\xi \in \mathbb{T} \setminus \sigma(\Theta) : \Theta(\xi) = \alpha\},$$

is a discrete set  $(\xi_n)_{n \geq 1}$  and so  $\{k_{\xi_n}^\Theta / \|k_{\xi_n}^\Theta\|_2 : n \geq 1\}$  is an orthonormal basis, a Clark basis, for  $(\Theta H^2)^\perp$ . Recall that  $\|k_{\xi_n}^\Theta\|_2 = \sqrt{|\Theta'(\xi_n)|}$ .

The following result is due to Baranov ([9, Corollary 1.3 and proof of Theorem 1.1]).

**Theorem 6.1** (Baranov). *Let  $\Theta \in (CLS)$ . There exists  $\varepsilon_0 = \varepsilon_0(\Theta) \in (0, 1)$  making the following true: if  $(k_{\xi_n}^\Theta / \|k_{\xi_n}^\Theta\|_2)_{n \geq 1}$ , with  $\xi_n \in ADC_\Theta$ , is a Riesz basis for  $(\Theta H^2)^\perp$  and  $\lambda_n \in \mathbb{D}^-$  satisfy*

$$|\lambda_n - \xi_n| < \varepsilon_0 |\Theta'(\xi_n)|^{-1},$$

*then  $(k_{\lambda_n}^\Theta / \|k_{\lambda_n}^\Theta\|_2)_{n \geq 1}$  is also a Riesz basis for  $(\Theta H^2)^\perp$ . Moreover, there is a positive constant  $C$  such that for every  $f \in (\Theta H^2)^\perp$ , we have*

$$(6.1) \quad \sum_{n \geq 1} \frac{|f(\xi_n) - f(\lambda_n)|^2}{|\Theta'(\xi_n)|} \leq \varepsilon_0 C \|f\|_2^2.$$

In particular one can choose  $(k_{\xi_n}^\Theta / \|k_{\xi_n}^\Theta\|_2)_{n \geq 1}$  to be a Clark basis. Let us mention that Cohn [15] also established an interesting result about the stability of Clark basis for one-component inner functions.

*Proof of Theorem 2.1.* Let  $\varepsilon_0 \in (0, 1)$  be the constant given in Theorem 6.1 and let  $(k_{\xi_n}^\Theta / \|k_{\xi_n}^\Theta\|_2)_{n \geq 1}$  be a Clark basis (we know that such a basis exists because  $\Theta \in (CLS)$ ). For  $\eta = \varepsilon_0 \sqrt{2}/2$ , we define  $\sigma_n \subset \mathbb{T}$  to be the arc centered at  $\xi_n$  of length  $\eta |\Theta'(\xi_n)|^{-1}$ . Since  $\eta < \varepsilon_0$ , we easily check that

$$S(\sigma_n) \subset D(\xi_n, \varepsilon_0 |\Theta'(\xi_n)|^{-1}).$$

Recall from (1.3) that  $S(\sigma_n)$  is the Carleson window over  $\sigma_n$ . Moreover, we argue that

$$(6.2) \quad |\xi_n - \xi_m| \geq \varepsilon_0 \min(|\Theta'(\xi_n)|^{-1}, |\Theta'(\xi_m)|^{-1}), \quad n \neq m.$$

Indeed, for a point  $\xi_n$ , if for some  $m \neq n$ , there exists a point  $\xi_m \in D(\xi_n, \varepsilon_0 |\Theta'(\xi_n)|^{-1})$ , then we could define

$$\lambda_\ell = \begin{cases} \xi_\ell & \text{for } \ell \neq m \\ \xi_n & \text{for } \ell = m. \end{cases}$$

While this sequence satisfies the perturbation condition of Theorem 6.1, it is certainly no longer a basis (the same vector appears two times).

Now it follows from (6.2) that  $\sigma_n \cap \sigma_m = \emptyset$ ,  $n \neq m$ , and thus the Carleson windows  $S(\sigma_n)$ ,  $n \geq 1$ , are disjoint. Moreover, since  $\Theta \in (CLS)$ , we know [9] that

$$\min(|\Theta'(\xi_n)|^{-1}, d(\xi_n, \sigma(\Theta))) \asymp d(\xi_n, L(\Theta, \varepsilon_1)),$$

where  $L(\Theta, \varepsilon_1)$  is the connected sub-level set for  $\Theta$  defined in (1.8). In particular,

$$|\Theta'(\xi_n)|^{-1} \geq cd(\xi_n, L(\Theta, \varepsilon_1))$$

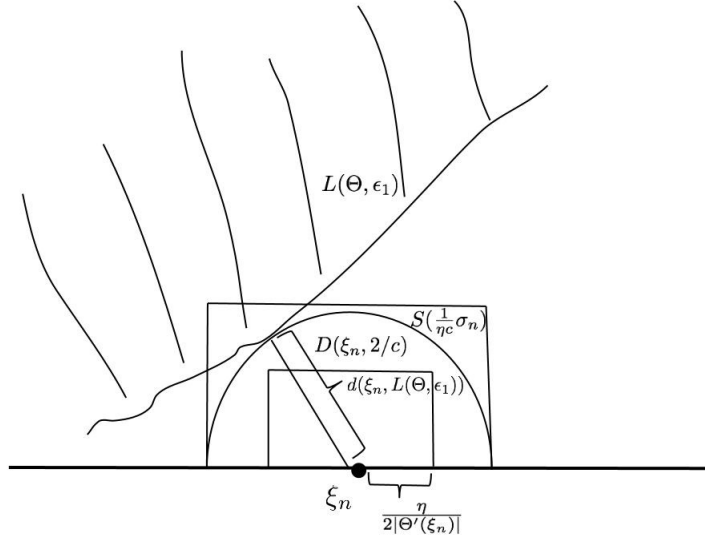


FIGURE 3

for a suitable constant  $c$ . The definition of  $\sigma_n$  yields that

$$m(\sigma_n) \geq \eta c d(\xi_n, L(\Theta, \varepsilon_1)),$$

or more explicitly,

$$d(\xi_n, L(\Theta, \varepsilon_1)) \leq \frac{m(\sigma_n)}{\eta c} = \frac{|\Theta'(\xi_n)|^{-1}}{c},$$

meaning that  $D(\xi_n, (c|\Theta'(\xi_n)|)^{-1})$  meets  $L(\Theta, \varepsilon_1)$  (see Figure 1). Since  $D(\xi_n, (c|\Theta'(\xi_n)|)^{-1}) \subset S(\frac{2}{\eta c}\sigma_n)$  it is enough to pick  $N = 2/(\eta c)$ . So

$$S(N\sigma_n) \cap L(\Theta, \varepsilon_1) \neq \emptyset,$$

which implies, by hypothesis, that for every  $n$

$$(6.3) \quad \mu(S(\sigma_n)) \geq c_N m(\sigma_n).$$

Let us now take  $f \in (\Theta H^2)^\perp$ . Since, as mentioned in the introduction,  $(\Theta H^2)^\perp \cap C(\mathbb{D}^-)$  is dense in  $(\Theta H^2)^\perp$  and since  $(\Theta H^2)^\perp$  embeds continuously into  $L^2(\mu)$ , we may assume that  $f$  is continuous on  $\mathbb{D}^-$ . Define  $\mu_n$  as the point of  $S(\sigma_n)$  satisfying

$$|f(\mu_n)| = \inf_{\zeta \in \sigma_n} |f(\zeta)|.$$

Since  $(k_{\xi_n}/\sqrt{|\Theta'(\xi_n)|})_{n \geq 1}$  is an orthonormal basis of  $(\Theta H^2)^\perp$ , we have

$$\|f\|_2^2 = \sum_{n \geq 1} \frac{|f(\xi_n)|^2}{|\Theta'(\xi_n)|} \leq 2 \left( \sum_{n \geq 1} \frac{|f(\xi_n) - f(\mu_n)|^2}{|\Theta'(\xi_n)|} + \sum_{n \geq 1} \frac{|f(\mu_n)|^2}{|\Theta'(\xi_n)|} \right).$$

Using Theorem 6.1, along with (6.3), and remembering that the Carleson windows  $S(\sigma_n)$  are disjoint, we get

$$\begin{aligned} \|f\|_2^2 &\leq 2\varepsilon_0 C \|f\|_2^2 + \frac{2}{\eta} \sum_{n \geq 1} m(\sigma_n) |f(\mu_n)|^2 \\ &\leq 2\varepsilon_0 C \|f\|_2^2 + \frac{2}{\eta c_N} \sum_{n \geq 1} \mu(S(\sigma_n)) |f(\mu_n)|^2 \\ &\leq 2\varepsilon_0 C \|f\|_2^2 + \frac{2}{\eta c_N} \sum_{n \geq 1} \int_{S(\sigma_n)} |f|^2 d\mu \\ (6.4) \quad &\leq 2\varepsilon_0 C \|f\|_2^2 + \frac{2}{\eta c_N} \int_{\mathbb{D}} |f|^2 d\mu. \end{aligned}$$

Finally, for sufficiently small  $\varepsilon_0$  we obtain

$$\|f\|_2^2 \leq \frac{2}{(1 - 2\varepsilon_0 C)\eta c_N} \int_{\mathbb{D}} |f|^2 d\mu,$$

which yields the required estimate.  $\square$

**Remark 6.2.** Note that from the first line of (6.4) we obtain

$$\|f\|_2^2 \leq \frac{2}{(1 - 2\varepsilon_0 C)\eta} \int_{\Sigma} |f|^2 dm,$$

where  $\Sigma = \bigcup \sigma_n$ . This gives another way of constructing dominating sets (at least when the Clark measure is discrete): here we take neighborhoods of the support points of the Clark basis instead of an open neighborhood of the spectrum as in Theorem 5.6.

**Remark 6.3.** Here is a simple example showing that it is not sufficient that condition (2.1) of Theorem 2.1 is satisfied if  $N$  is not suitably chosen. Let us discuss this example in the upper half plane and consider the special case of  $\Theta(z) = e^{2iz}$ . It is known that with this  $\Theta$ , the model space  $(\Theta H^2)^\perp$  is isomorphic to the Paley-Wiener space  $PW_\pi^2$  (the space of entire functions of exponential type at most  $\pi$ , whose restriction to  $\mathbb{R}$  belongs to  $L^2$ ). Given  $\varepsilon > 0$ , the sub-level set is exactly

$$L(\Theta; \varepsilon) = \left\{ z \in \mathbb{C} : \operatorname{Im}(z) > \theta := \frac{1}{2} \ln \frac{1}{\varepsilon} \right\}.$$

In what follows we will choose  $\varepsilon = e^{-6}$  so that  $\theta = 3$ .

Note that for the sets  $(\sigma_n)_n$  appearing in the proof of Theorem 2.1, by the Kadets-Ingham theorem we can take  $\sigma_n = (n - \delta, n + \delta)$ ,  $n \in \mathbb{Z}$ , where  $0 < \delta < 1/4$  (see for instance [34, Theorem D4.1.2]).

Let us now consider  $\Lambda = \{\lambda_n; n \in \mathbb{Z} \setminus \{0\}\}$  with  $\lambda_n := n + \frac{1}{8}i$  and

$$\mu_\Lambda := \sum_{n \geq 1} \delta_{\lambda_n}.$$

Note that  $\mathbb{Z}$  is a complete interpolating sequence for  $PW_\pi^2$  (see e.g. [43]) as will be  $\Lambda \cup \{\lambda_0\}$ . Hence, we can find  $f \in PW_\pi^2$ , such that  $f(\lambda_n) = 0$ ,  $n \in \mathbb{Z} \setminus \{0\}$  and  $f(\lambda_0) = 1$ . In particular,  $\int |f|^2 d\mu_\Lambda = 0$  and of course  $\|f\|_2 \neq 0$ . So, the reverse embedding fails, while (2.1) is valid for  $N = 1$ . Indeed, if  $I$  is an interval such that  $S(I) \cap L(\Theta, \varepsilon) \neq \emptyset$ , then  $m(I) \geq 3$  and since  $0 \leq \lambda_{n+1} - \lambda_n \leq 2$  we have  $S(I) \cap \Lambda \neq \emptyset$ . It actually turns out that  $\mu_\Lambda(S(I)) = \#(S(I) \cap \Lambda) \asymp m(I)$ .

An appropriate choice for  $N$  here is  $N \geq 6$ . In particular  $S(6\sigma_0)$  meets  $L(\Theta, \varepsilon)$ . But  $S(\sigma_0)$  does not contain any point of  $\Lambda$  so that  $\mu(S(\sigma_0)) = 0 \not\asymp m(\sigma_0)$  and (2.1) fails as it should.

The above example is also instructive in that it allows to observe that Theorem 2.1 does not apply when the sequence  $\Lambda$  is “too far” from  $\mathbb{R}$ . Consider for instance  $\Lambda_1 = \{n+i; n \in \mathbb{Z} \setminus \{0\}\}$  and  $\Lambda_2 = \{n+i; n \in \mathbb{Z}\}$ . For  $\Lambda_2$  we get reverse embedding while for  $\Lambda_1$  we won't. Observe that for both sequences we will have  $\mu(S(\sigma_n)) = 0$  for every  $n$ , so that the reverse Carleson inequality fails.

Now we give the proof of our reverse embeddings result which involves dominating sets. We will need the following lemma, for which we omit the proof.

**Lemma 6.4.** *Let  $\Sigma \subsetneq \mathbb{T}$  and  $\delta > 0$ . It is possible to find a (finite) sequence  $(\sigma_n)_{n \geq 1}$  of disjoint semi-open arcs of  $\mathbb{T}$  such that  $\Sigma \subset \bigcup_n \sigma_n$ ,  $\Sigma \cap \sigma_n \neq \emptyset$  and  $m(\sigma_n) < \delta$ .*

*Proof of Theorem 2.2.* Since  $(\Theta H^2)^\perp \cap C(\mathbb{D}^-)$  is dense in  $(\Theta H^2)^\perp$  and since  $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ , it suffices to show the inequality  $\|f\|_2 \lesssim \|f\|_\mu$ , for every function  $f \in (\Theta H^2)^\perp$  continuous on  $\mathbb{D}^-$ . Fix  $\varepsilon > 0$ . Since  $f$  is uniformly continuous, it is possible to find a  $\delta > 0$  such that, for every arc  $I$  of length less than  $\delta$ , we have

$$|f(z) - f(\mu)| < \varepsilon, \quad \forall z, \mu \in S(I).$$

According to Lemma 6.4, we can construct a sequence of disjoint semi-open arcs of length less than  $\delta$ , covering and intersecting  $\Sigma$ . In particular,  $m(\sigma_n) \leq k\mu(S(\sigma_n))$  for some  $k > 0$  independent of  $n$ . We now

introduce the points  $\xi_n$  and  $z_n$  such that

$$|f(\xi_n)| = \max\{|f(\xi)| : \xi \in \sigma_n^-\}, \quad |f(z_n)| = \min\{|f(z)| : z \in S(\sigma_n^-)\}$$

These points satisfy

$$|f(\xi_n)| \leq \varepsilon + |f(z_n)|.$$

Since  $\Sigma$  is dominating, there is some constant  $c > 0$  such that

$$\begin{aligned} \|f\|_2^2 &\leq c \int_{\Sigma} |f|^2 dm \\ &\leq c \sum_{n \geq 1} m(\sigma_n) |f(\xi_n)|^2 \\ &\leq 2c \left( \varepsilon^2 \sum_{n \geq 1} m(\sigma_n) + \sum_{n \geq 1} m(\sigma_n) |f(z_n)|^2 \right) \\ &\leq 2c \left( \varepsilon^2 + k \sum_{n \geq 1} \mu(S(\sigma_n)) |f(z_n)|^2 \right) \\ &\leq 2c\varepsilon^2 + 2ck \sum_{n \geq 1} \int_{S(\sigma_n)} |f|^2 d\mu \\ &\leq 2c\varepsilon^2 + 2ck \int_{\mathbb{D}^-} |f|^2 d\mu. \end{aligned}$$

Since  $\varepsilon$  can be arbitrarily small, we have the desired estimate  $\|f\|_2 \lesssim \|f\|_{\mu}$ .  $\square$

**Remark 6.5.** If  $\Theta$  is inner and  $\sigma(\Theta) = \mathbb{T}$ , then, according to Corollary 5.10, any dominating set  $\Sigma$  for  $(\Theta H^2)^\perp$  is dense in  $\mathbb{T}$ . Hence requiring the reverse Carleson inequality  $\mu(S(I)) \gtrsim m(I)$  for any arc  $I \subset \mathbb{T}$  meeting  $\Sigma$  is the same as requiring this inequality for *every* arc  $I \subset \mathbb{T}$ . This situation is completely described by Lefèvre et al. Hence Theorem 2.2 is interesting for inner functions  $\Theta$  for which  $\sigma(\Theta) \neq \mathbb{T}$ . In that case Theorem 5.6 states that dominating sets always exist.

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