HAL open science

Reverse Carleson Embeddings for Model Spaces

Alain Blandignères, Emmanuel Fricain, Frederic Gaunard, Andreas Hartmann, William T. Ross

To cite this version:

Alain Blandignères, Emmanuel Fricain, Frederic Gaunard, Andreas Hartmann, William T. Ross. Reverse Carleson Embeddings for Model Spaces. 2012. hal-00697223v1

HAL Id: hal-00697223

https://hal.science/hal-00697223v1

Preprint submitted on 14 May 2012 (v1), last revised 24 Sep 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

REVERSE CARLESON EMBEDDINGS FOR MODEL SPACES

ALAIN BLANDIGNÈRES, EMMANUEL FRICAIN, FRÉDÉRIC GAUNARD, ANDREAS HARTMANN, AND WILLIAM T. ROSS

Abstract. The classical embedding theorem of Carleson deals with finite positive Borel measures μ on the closed unit disk for which there exists a positive constant c such that $\|f\|_{L^2(\mu)} \leq c\|f\|_{H^2}$ for all $f \in H^2$, the Hardy space of the unit disk. Lefèvre et al. examined measures μ for which there exists a positive constant c such that $\|f\|_{L^2(\mu)} \geq c\|f\|_{H^2}$ for all $f \in H^2$. The first type of inequality above was explored with H^2 replaced by one of the model spaces $(\Theta H^2)^\perp$ by Aleksandrov, Baranov, Cohn, Treil, and Volberg. In this paper we continue this discussion.

1. Introduction

Let H^2 be the classical Hardy space of the open unit disk $\mathbb{D}$ [20, 24] with norm $\|\cdot\|_2$, and let H^∞ be the space of bounded analytic functions on $\mathbb{D}$. Let $M_+(\mathbb{D})$ denote the finite positive Borel measures on $\mathbb{D}$, and, for $\mu \in M_+(\mathbb{D})$, let $\|\cdot\|_\mu$ be the norm in $L^2(\mu)$. A beautiful theorem of L. Carleson [24] says that H^2 can be continuously embedded into $L^2(\mu)$, i.e.,

$$\exists\, c > 0 \text{ such that } \|f\|_\mu \leq c\|f\|_2, \qquad \forall f \in H^2 \tag{1.1}$$

if and only if

$$\sup_I \frac{\mu(S(I))}{m(I)} < \infty, \tag{1.2}$$

where the above supremum is taken over all arcs I of the unit circle $\mathbb{T} = \partial\mathbb{D}$, $m := d\theta/2\pi$ is normalized Lebesgue measure on $\mathbb{T}$, and $S(I)$ is the Carleson window

$$S(I) := \left\{ |z| \leq 1 : \ \frac{z}{|z|} \in I, \ 1 - |z| \leq \frac{m(I)}{2} \right\}. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 30J05, 30H10, 46E22.

Key words and phrases. model spaces, embeddings, dominating sets, Carleson measures, Clark measures.

This work is supported by ANR FRAB: ANR-09-BLAN-0058-02.

We will write $H^2 \hookrightarrow L^2(\mu)$ to represent the fact that H^2 can be continuously embedded into $L^2(\mu)$ via the map $f \mapsto f|E_\mu$, where E_μ is a carrier of μ. Measures for which this is true are called *Carleson measures*. We will also write statements such as (1.1) in the more convenient form

$$\|f\|_\mu \lesssim \|f\|_2, \quad \forall f \in H^2.$$

Carleson's result can be extended to $\mu \in M_+(\mathbb{D}^-)$ ($\mathbb{D}^-$ is the closure of $\mathbb{D}$) but, in the initial definition of embedding in (1.1), we need to change the phrase $\forall f \in H^2$ into $\forall f \in H^2 \cap C(\mathbb{D}^-)$, owing to the fact that functions in H^2 are defined m-almost everywhere on $\mathbb{T}$ while functions in $L^2(\mu)$ are defined μ-almost everywhere on $\mathbb{T}$. However, condition (1.2) clearly implies that the restrictions of Carleson measures to the unit circle are absolutely continuous with respect to m, and so the initial concern in examining $\|f\|_\mu$ for all $f \in H^2$, and not just the continuous ones, evaporates.

Lefèvre et al. [29] examined the *reverse embedding problem*, i.e., when is the above embedding injective with closed range, equivalently, when does (1.1) hold as well as the reverse inequality

$$\|f\|_2 \lesssim \|f\|_\mu, \quad \forall f \in H^2? \tag{1.4}$$

They proved that if μ is a Carleson measure, then the reverse embedding happens if and only if

$$\inf_I \frac{\mu(S(I))}{m(I)} > 0. \tag{1.5}$$

In other words, the norms $\|\cdot\|_\mu$ and $\|\cdot\|_2$ are equivalent on H^2 if and only if both (1.2) and (1.5) are satisfied.

We would like to point out that D. Luecking studied the question of reverse embeddings for Bergman spaces in [31, 32] and G. Chacón [11] has some related results for certain Dirichlet type spaces.

The purpose of this paper is to explore reverse embeddings for model spaces $(\Theta H^2)^\perp = H^2 \ominus \Theta H^2$, where Θ is a non-constant inner function, that is $\Theta \in H^\infty$ and Θ admits radial limits of modulus one almost everywhere on $\mathbb{T}$. Aleksandrov [5] showed that if $\mu \in M_+(\mathbb{D}^-)$ satisfies

$$\|f\|_\mu \lesssim \|f\|_2, \quad \forall f \in (\Theta H^2)^\perp \cap C(\mathbb{D}^-), \tag{1.6}$$

then each $f \in (\Theta H^2)^\perp$ has a finite radial limit at μ-almost every point of $\mathbb{T}$ and the inequality in (1.6) holds for every $f \in (\Theta H^2)^\perp$. We point out that, amazingly, $(\Theta H^2)^\perp \cap C(\mathbb{D}^-)$ is dense in $(\Theta H^2)^\perp$ [3] (see also [12, p. 188]). Again we use the notation $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ to denote the embedding $f \mapsto f|E_\mu$, where $E_\mu \subset \mathbb{D}^-$ is a carrier of μ.

Treil and Volberg [45], along with Cohn [14], examined when embedding of model spaces actually occurs. In particular, they showed that $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ as soon as there is an $\varepsilon \in (0,1)$ such that μ satisfies the condition (1.2) but the supremum is taken only over arcs $I \subset \mathbb{T}$ satisfying

$$S(I) \cap L(\Theta, \varepsilon) \neq \varnothing, \tag{1.7}$$

where

$$L(\Theta, \varepsilon) := \{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\} \tag{1.8}$$

is a sub-level set for Θ. In particular, we see that this condition is weaker than Carleson's one since $(\Theta H^2)^\perp$ functions are much more regular than H^2-functions, particularly when we are far from $L(\Theta, \varepsilon)$. Using weighted Bernstein inequalities, Baranov [8] slightly improved the embedding result of Treil–Volberg.

Conversely, assuming that $L(\Theta, \varepsilon)$ is connected for some $\varepsilon > 0$[1], if $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$, then μ satisfies (1.2) for arcs I satisfying (1.7).

Again, as was asked by Lefèvre et al. for Carleson measures in H^2, when is the embedding $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ injective with closed range (equivalently, induces a reverse embedding)? For example, if μ is any one of the *Clark measures* for Θ (we will define these measures in a moment), then by results of Clark [13] and Poltoratskii [38] we have the isometric embedding $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$. The same is true when μ is a Clark measure for an inner multiple of Θ.

For another example, let $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ be a *complete interpolating sequence* for $(\Theta H^2)^\perp$. This means that the sequence of reproducing kernels $(k^\Theta_{\lambda_n})_{n\geq 1}$ for $(\Theta H^2)^\perp$ forms an unconditional basis in $(\Theta H^2)^\perp$ [27, 34]. In this situation it turns out that for the measure

$$\mu = \sum_{n\geq 1} \frac{1}{\|k_{\lambda_n}\|_2^2} \delta_{\lambda_n},$$

the norms $\|\cdot\|_\mu$ and $\|\cdot\|_2$ are equivalent on $(\Theta H^2)^\perp$ and, in particular, we have a reverse embedding.

For a third example, suppose M is a subspace of H^2 satisfying $f/z \in M$ whenever $f \in M$ and $f(0) = 0$. Such subspaces M are called *nearly invariant* and were initially studied by Hitt and Sarason [26, 40]. If $g \in M$ is the unique solution to the extremal problem,

$$\Re(g(0)) = \sup\{\Re(f(0)) : f \in M, \|f\|_2 \leq 1\},$$

[1]Such inner functions are said to satisfy the *connected level set condition* (CLS)

then there exists an inner function Θ such that $M = g(\Theta H^2)^\perp$ and g is an isometric multiplier of $(\Theta H^2)^\perp$. That is to say

$$\|gf\|_2 = \|f\|_2, \quad \forall f \in (\Theta H^2)^\perp.$$

Rephrasing this a bit we see that if $d\mu = |g|^2 dm$ then the embedding $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ is isometric. As a matter of fact, the measures $\mu \in M_+(\mathbb{T})$ which ensure an isometric embedding of $(\Theta H^2)^\perp$ have been characterized by Aleksandrov [4].

Theorem 1.1 (Aleksandrov)**.** *Let $\mu \in M_+(\mathbb{T})$. Then the following assertions are equivalent:*

(i) *$(\Theta H^2)^\perp$ embeds isometrically into $L^2(\mu)$;*
(ii) *the function Θ has non-tangential boundary values μ-almost everywhere on $\mathbb{T}$ and*

$$\int_{\mathbb{T}} \left| \frac{1 - \overline{\Theta(z)}\Theta(\zeta)}{1 - \overline{z}\zeta} \right|^2 d\mu(\zeta) = \frac{1 - |\Theta(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D};$$

(iii) *there exists a $\varphi \in H^\infty$ such that $\|\varphi\|_\infty \leq 1$ and*

$$\int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta) = \Re\left(\frac{1 + \varphi(z)\Theta(z)}{1 - \varphi(z)\Theta(z)}\right), \quad z \in \mathbb{D}. \tag{1.9}$$

Note that in [18], de Branges has proven the result for meromorphic inner functions and in [25], Krein has obtained a characterization of isometric measures for $(\Theta H^2)^\perp$ in a more operator-theoric langage.

In the particular case when $\Theta(z) = \Theta_a(z) = \exp(-a\frac{1+z}{1-z})$, $a > 0$, the question of equivalence of norms in $(\Theta_a H^2)^\perp$ was discussed in [28, 30, 36, 37]. In this case, the question is equivalent to the following one: for which (positive) measures μ on the real line are the norms

$$\left(\int_{\mathbb{R}} |f(x)|^2 \, d\mu(x)\right)^{1/2} \quad \text{and} \quad \left(\int_{\mathbb{R}} |f(x)|^2 \, dx\right)^{1/2}$$

equivalent on the space of entire functions of exponential type not exceeding $a/2$ and which belong to $L^2(\mathbb{R})$?

In [44] Volberg generalized the previous results and gave a complete answer for general model spaces and absolutely continuous measures $d\mu = wdm, w \in L^\infty(\mathbb{T})$,

Theorem 1.2 (Volberg)**.** *Let $d\mu = wdm$, with $w \in L^\infty(\mathbb{T})$, $w \geq 0$, and let Θ be an inner function. Then the following assertions are equivalent:*

(i) *the norms $\|\cdot\|_\mu$ and $\|\cdot\|_2$ are equivalent on $(\Theta H^2)^\perp$;*

(ii) *if* $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$, *then*

$$\lim_{n\to+\infty} \widehat{w}(\lambda_n) = 0 \Longrightarrow \lim_{n\to+\infty} |\Theta(\lambda_n)| = 1;$$

(iii) *we have*

$$\inf_{\lambda\in\mathbb{D}} (\widehat{w}(\lambda) + |\Theta(\lambda)|) > 0.$$

In the above, $\widehat{w}$ represents the harmonic continuation of the function w into the open unit disk, that is,

$$\widehat{w}(z) = \int_{\mathbb{T}} w(\zeta) \frac{1-|z|^2}{|z-\zeta|^2}\, dm(\zeta), \qquad z \in \mathbb{D}.$$

The aim of this paper is to prove a model space version of the Lefèvre et al. reverse Carleson embedding theorem for H^2 where we weaken the condition (1.5) in the spirit of the result of Treil–Volberg for the direct embedding. Along the way, we will also develop a notion of dominating sets for model spaces and use this to state another reverse embedding theorem.

2. Main results

Our first result is a reverse embedding theorem along the lines of Treil-Volberg for which we need the following notation: given an arc $I \subset \mathbb{T}$ and a number $n > 0$ we define the amplified arc nI as the arc with same center as I and of length $n \times m(I)$.

Theorem 2.1. *Let* Θ *be a* (CLS) *inner function,* $L(\Theta, \varepsilon_1)$ *its connected sublevel set, and* $\mu \in M_+(\mathbb{D}^-)$ *such that* $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$. *There exists an* $N = N(\Theta, \varepsilon_1) > 1$ *such that if*

$$\inf_I \frac{\mu(S(I))}{m(I)} > 0, \tag{2.1}$$

where the infimum is taken over all arcs $I \subset \mathbb{T}$ *with*

$$S(NI) \cap L(\Theta, \varepsilon_1) \neq \varnothing,$$

then

$$\|f\|_2 \lesssim \|f\|_\mu, \qquad \forall f \in (\Theta H^2)^\perp. \tag{2.2}$$

The proof of this theorem will show how N depends on Θ and ε_1. There will also be a discussion in Section 6 of why the N is needed.

Our second reverse embedding result involves the notion of a dominating set for $(\Theta H^2)^\perp$. A (Lebesgue measurable) subset $\Sigma \subset \mathbb{T}$, with $m(\Sigma) < 1$, is called a *dominating set* for $(\Theta H^2)^\perp$ if

$$\int_{\mathbb{T}} |f|^2 dm \lesssim \int_\Sigma |f|^2 dm, \quad \forall f \in (\Theta H^2)^\perp.$$

Notice how this is equivalent to saying that the measure $d\mu = \chi_\Sigma dm$ satisfies the reverse embedding property for $(\Theta H^2)^\perp$. We will discuss some elementary properties of dominating sets in Section 5. In particular, we will show that dominating sets always exists for inner functions Θ such that $\sigma(\Theta) \neq \mathbb{T}$ – see (3.1) for the definition of $\sigma(\Theta)$. Moreover, if $m(\sigma(\Theta)) = 0$, then dominating sets can be of arbitrary small Lebesgue measure. This is the case for (CLS) inner functions. With regards to reverse embeddings, we will prove the following result.

Theorem 2.2. *Let Θ be an inner function, Σ be a dominating set for $(\Theta H^2)^\perp$, and $\mu \in M_+(\mathbb{D}^-)$ such that $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$. Suppose that*

$$\inf_I \frac{\mu\left(S\left(I\right)\right)}{m\left(I\right)} > 0,$$

where the above infimum is taken over all arcs $I \subset \mathbb{T}$ such that

$$I \cap \Sigma \neq \varnothing.$$

Then

$$\|f\|_2 \lesssim \|f\|_\mu, \qquad \forall f \in (\Theta H^2)^\perp. \tag{2.3}$$

In both Theorems 2.1 and 2.2, the hypothesis $(\Theta H^2)^\perp \subset L^2(\mu)$ ensures that the reverse inequality holds for every function in $(\Theta H^2)^\perp$. As already mentioned, when μ is carried on $\mathbb{T}$ we might not be able to define $\|f\|_\mu$ properly for certain $f \in (\Theta H^2)^\perp$. Still, it will be clear from the proofs that the embeddings (2.2) and (2.3) still hold on the dense subspaces $(\Theta H^2)^\perp \cap C(\mathbb{D}^-)$ under the reverse Carleson inequality even without the assumption on the embedding.

Finally we would like to discuss an alternate proof of Aleksandrov's isometric embedding theorem. Our proof uses the theory of de Branges–Rovnyak spaces.

Theorem 2.3 (Aleksandrov)**.** *Suppose Θ is an inner function and $\mu \in M_+(\mathbb{T})$. Then the embedding $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ is isometric if and only if there is a function b in the closed unit ball of H^∞ such that*

$$\mu = \sigma^1_{\Theta b},$$

where $\sigma^1_{\Theta b}$ is a so-called Aleksandrov-Clark[2] measure associated with Θb.

In the following section, we recall some useful facts concerning model spaces $(\Theta H^2)^\perp$ and de Branges–Rovnyak spaces $\mathcal{H}(b)$. In Section 4, we prove Theorem 2.3. Section 5 is devoted to dominating sets, and finally in Section 6 we prove Theorems 2.1 and 2.2.

[2]It satisfies (1.9) with φ replaced by b, see also Section 3

Acknowledgement. We would like to thank A. Baranov for some useful comments and discussion.

3. Model spaces and de Branges–Rovnyak spaces

Before getting underway, let us first gather up some well-known facts about the model spaces $(\Theta H^2)^\perp$ that will be needed later. References for much of this can be found in [12]. Model spaces were initially studied as the typical invariant subspaces for the adjoint of the unilateral shift on H^2 [19] but the subject has expanded in many ways since then.

By the Nevenlinna theory [24], an inner function Θ can be factored as $\Theta = e^{i\varphi} B_\Lambda S_\nu$, where φ is a real constant, B_Λ is a Blaschke product with zeros (repeated according to multiplicity) $\Lambda = (z_n)_{n\geq 1}$ and

$$S_\nu(z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\nu(\xi)\right),$$

where $\nu \in M_+(\mathbb{T})$ with $\nu \perp m$, is a singular inner function. The *boundary spectrum* of Θ is the set

$$\sigma(\Theta) := \left\{\xi \in \mathbb{T} : \varliminf_{z\to\xi} |\Theta(z)| = 0\right\}.^3 \tag{3.1}$$

It turns out that $\sigma(\Theta)$ is equal to $(\Lambda^- \cap \mathbb{T}) \cup \operatorname{supt}(\nu)$ – the union of cluster points of Λ on $\mathbb{T}$ together with the support of the measure ν [35, p. 63]. It is well known [33] that Θ, along with every function in $(\Theta H^2)^\perp$, has an analytic continuation to an open neighborhood of $\mathbb{T} \setminus \sigma(\Theta)$.

Several authors [1, 5, 14] have examined the non-tangential limits of functions in $(\Theta H^2)^\perp$ near $\sigma(\Theta)$, where analytic continuation fails. Indeed, let ADC_Θ denote the set of points $\xi \in \mathbb{T}$ where the angular derivative of Θ, in the sense of Carathéodory, exists. More specifically, $\xi \in ADC_\Theta$ if the radial limit of Θ exists and is unimodular and the radial limit of Θ' exists. We use the notation $|\Theta'(\xi)|$ to denote the modulus of the angular derivative of Θ at ξ (when it exists). Note that $|\Theta'(\xi)| > 0$ whenever $\xi \in ADC_\Theta$. A result of Ahern-Clark [1] yields that $\xi \in ADC_\Theta$ if and only if

$$|\Theta'(\xi)| = \sum_{n\geq 1} \frac{1 - |z_n|^2}{|\xi - z_n|^2} + 2\int_{\mathbb{T}} \frac{d\nu(\zeta)}{|\xi - \zeta|^2} < \infty. \tag{3.2}$$

[3] We should emphasize that this definition does not take into account the zeros of Θ inside $\mathbb{D}$.

The model space $(\Theta H^2)^\perp$ is a reproducing kernel Hilbert space with kernel

$$k_\lambda^\Theta(z) = \frac{1-\overline{\Theta(\lambda)}\Theta(z)}{1-\overline{\lambda}z}, \quad \lambda \in \mathbb{D}. \tag{3.3}$$

These kernels satisfy the reproducing property

$$\langle f, k_\lambda^\Theta\rangle_2 = f(\lambda), \quad \forall f \in (\Theta H^2)^\perp, \tag{3.4}$$

and have norm

$$\|k_\lambda^\Theta\|_2 = \sqrt{\frac{1-|\Theta(\lambda)|^2}{1-|\lambda|^2}}. \tag{3.5}$$

It is also possible to define k_ξ^Θ for $\xi \in ADC_\Theta$. In this case $k_\xi^\Theta \in (\Theta H^2)^\perp$, every $f \in (\Theta H^2)^\perp$ has a finite non-tangential limit $f(\xi)$ at ξ, $\langle f, k_\xi^\Theta\rangle_2 = f(\xi)$, and k_ξ^Θ has norm

$$\|k_\xi^\Theta\|_2 = \sqrt{|\Theta'(\xi)|}.$$

For b in the closed unit ball of H^∞, the associated *de Branges–Rovnyak space* $\mathcal{H}(b)$ is defined by

$$\mathcal{H}(b) := (I - T_b T_b^*)^{1/2} H^2,$$

and is equipped with the following scalar product

$$\langle (I - T_b T_b^*)^{1/2} f, (I - T_b T_b^*)^{1/2} g\rangle_b = \langle f, g\rangle_2,$$

for any $f, g \in (\ker(I - T_b T_b^*)^{1/2})^\perp$, making $\mathcal{H}(b)$ a Hilbert space contractively contained in H^2. Here T_b is the Toeplitz operator with symbol b, $T_b f = P_+(bf)$, $f \in H^2$ and $P_+ : L^2 \to H^2$ is the Riesz projection. When $b = \Theta$ is an inner function, $\mathcal{H}(\Theta) = (\Theta H^2)^\perp$ with equality of norms. The space $\mathcal{H}(b)$ is also a reproducing kernel Hilbert space with kernel k_λ^b given by the same formula as (3.3) where Θ is replaced by b. Formulas (3.4) and (3.5) are also valid in this $\mathcal{H}(b)$ setting.

For each $\alpha \in \mathbb{T}$,

$$\Re\left(\frac{\alpha + b(z)}{\alpha - b(z)}\right) > 0, \quad z \in \mathbb{D},$$

and hence, by Herglotz's theorem [20], there is a unique $\sigma_b^\alpha \in M_+(\mathbb{T})$, called the *Aleksandrov-Clark measure* [12, 39, 41] associated with b and α, such that

$$\Re\left(\frac{\alpha + b(z)}{\alpha - b(z)}\right) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\sigma_b^\alpha(\zeta), \quad z \in \mathbb{D}. \tag{3.6}$$

The measure σ_b^α is singular (with respect to the Lebesgue measure m) if and only if b is an inner function. In this case, σ_b^α is often called a *Clark measure* since they were initially studied by Clark [13].

Conversely, if we start with a positive Borel measure μ on $\mathbb{T}$, then we can define the analytic function b by the formula

$$\frac{1+b(z)}{1-b(z)} = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta), \qquad z \in \mathbb{D}. \tag{3.7}$$

Taking the real part of both sides, we see that the resulting function b is an element in the closed unit ball of H^∞, with $-1 < b(0) < 1$, and (3.6) holds with $\mu = \sigma_b^1$. The special case $b(0) = 0$ happens if and only if μ is a probability measure.

Moreover, it follows easily from the uniqueness of the representation (3.6) that

$$\sigma_b^\alpha = \sigma_{\bar{\alpha}b}^1. \tag{3.8}$$

Clark [13], for the inner case, and Ball [6, 7] for the general case (see also [41]), proved that for each fixed $\alpha \in \mathbb{T}$ the operator

$$\omega_b^\alpha : L^2(\sigma_b^\alpha) \to \mathcal{H}(b), \quad \omega_b^\alpha h = (1-\overline{\alpha}b)\, C_{\sigma_b^\alpha} h, \tag{3.9}$$

is an onto partial isometry whose kernel is $(H^2(\sigma_b^\alpha))^\perp$, where

$$(C_{\sigma_b^\alpha} h)(z) := \int_{\mathbb{T}} \frac{h(\xi)}{1-\bar{\xi}z} d\sigma_b^\alpha(\xi), \quad z \in \mathbb{D},$$

denotes the Cauchy transform of the measure $h d\sigma_b^\alpha$ (see [12] for more details) and $H^2(\sigma_b^\alpha)$ is the closure of the polynomials in the $L^2(\sigma_b^\alpha)$-norm.

Poltoratskii [38] (see also [12]) went on further to show that for each $h \in L^2(\sigma_b^\alpha)$,

$$\lim_{r\to 1^-} (\omega_b^\alpha h)(r\xi) = h(\xi), \quad (\sigma_b^\alpha)_s\text{-a.e.},$$

where $(\sigma_b^\alpha)_s$ is the singular part of σ_b^α. In particular, if $b = \Theta$ is an inner function, then σ_Θ^α is singular, $H^2(\sigma_\Theta^\alpha) = L^2(\sigma_\Theta^\alpha)$ and putting this all together, we have the isometric, in fact unitary, embedding

$$(\Theta H^2)^\perp \hookrightarrow L^2(\sigma_\Theta^\alpha),$$

i.e.,

$$\int_{\mathbb{T}} |f|^2\, d\sigma_\Theta^\alpha = \|f\|_2^2, \qquad \forall f \in (\Theta H^2)^\perp. \tag{3.10}$$

For $a \in \mathbb{D}$, let

$$\phi_a(z) = \frac{a-z}{1-\bar{a}z}$$

be the (involutive) conformal automorphism of the disk which sends a to 0. Straightforward computations [2] show that

$$\sigma_{\phi_a\circ b}^{\phi_a(\alpha)} = \frac{1}{|\phi_a'(\alpha)|}\sigma_b^\alpha. \tag{3.11}$$

If $b = \Theta$ is an inner function, the *Crofoot transform*

$$(U_\Theta^a f)(z) := \frac{\sqrt{1-|a|^2}}{1-\overline{a}\Theta(z)} f(z) \tag{3.12}$$

is a unitary operator from $(\Theta H^2)^\perp$ onto $((\phi_a\circ\Theta)H^2)^\perp$ [17].

The Clark measure σ_Θ^α is carried by the set

$$\left\{\xi\in\mathbb{T} : \lim_{r\to 1^-}\Theta(r\xi) = \alpha\right\}. \tag{3.13}$$

By this last statement we mean that the σ_Θ^α-measure of the complement of this set is zero. The carrier is not to be confused with the support of σ_Θ^α – which is different. A Clark measure σ_Θ^α has a point mass at $\xi\in\mathbb{T}$ if and only if $|\Theta'(\xi)|<\infty$ and $\Theta(\xi)=\alpha$. Moreover, in this case $\sigma_\Theta^\alpha(\{\xi\}) = |\Theta'(\xi)|^{-1}$. Thus if σ_Θ^α is discrete then

$$\sigma_\Theta^\alpha = \sum_{\{\Theta(\xi)=\alpha, |\Theta'(\xi)|<\infty\}} \frac{1}{|\Theta'(\xi)|}\delta_\xi.$$

When the carrier of σ_Θ^α in (3.13) is discrete, say $(\xi_n)_{n\geq 1}\subset\mathbb{T}$, Clark [13] showed that the system

$$\left\{\frac{k_{\xi_n}^\Theta}{\sqrt{|\Theta'(\xi_n)|}} : n\geq 1\right\} \tag{3.14}$$

forms an orthonormal basis for $(\Theta H^2)^\perp$ – called the *Clark basis*. In [22] it is shown that this situation cannot occur in the general setting of de Branges–Rovnyak spaces.

An inner function Θ is said to have the *connected level set* property (written $\Theta\in(CLS)$), if there exists $\eta\in(0,1)$ such that the sub-level set

$$L(\Theta,\eta) := \{z\in\mathbb{D} : \ |\Theta(z)|<\eta\}$$

is connected. For example, the atomic inner function

$$\Theta(z) = \exp\left(\frac{z+1}{z-1}\right)$$

satisfies this condition since the sub-level set is a disk tangent to $\mathbb{T}$ at $\zeta = 1$. Though more difficult to see, one can show that the Blaschke

products with zeros

$$\left\{1-\frac{1}{n^2}:n\geq 1\right\} \quad \text{or} \quad \left\{1-\frac{1}{2^n}:n\geq 1\right\}$$

are CLS inner functions. The Blaschke product whose zeros are

$$\left\{1-\frac{1}{2^{n^2}}:n\geq 1\right\}$$

is not a CLS inner function. If $\Theta \in (CLS)$, Aleksandrov [5] showed that $m(\sigma(\Theta))=0$ and moreover, for every $\alpha\in\mathbb{T}$, $\sigma_\Theta^\alpha(\sigma(\Theta))=0$. Since $\mathbb{T}\setminus\sigma(\Theta)$ is a countable union of arcs where Θ continues analytically, the carrier of σ_Θ^α,

$$\{\xi\in\mathbb{T}\setminus\sigma(\Theta):\ \Theta(\xi)=\alpha\}$$

is a discrete set $(\xi_n)_{n\geq 1}$. Hence, as discussed above, every $\Theta\in(CLS)$ has a Clark basis (3.14).

4. Isometric Embeddings

In this section, we propose an alternate proof of Aleksandrov's isometric embedding theorem. The main idea is to first prove the result when Θ is a Blaschke product with simple zeros and then use an approximation argument based on Frostman shifts.

Proof of Theorem 2.3. The first implication is Aleksandrov's. We flesh out more of the details. First assume that $\mu=\sigma^1_{\Theta b}$ for some function b in the closed unit ball of H^∞. Then, by Carathéodory's theorem (see [24, p.6]), there exists a sequence of (finite) Blaschke products $(B_n)_{n\geq 1}$ which converges pointwise to b. In particular, if for $z\in\mathbb{D}$, we denote by

$$P_z(\zeta)=\frac{1-|z|^2}{|\zeta-z|^2},\quad \zeta\in\mathbb{T},$$

the associated Poisson kernel, then according to (3.6), we get that for every $z\in\mathbb{D}$,

$$\int_{\mathbb{T}} P_z(\zeta)\,d\sigma^1_{\Theta B_n}(\zeta)\to\int_{\mathbb{T}} P_z(\zeta)\,d\mu(\zeta),\qquad n\to+\infty.$$

Since $\|\sigma^1_{\Theta B_n}\|$ is bounded (apply (3.6) to $b=\Theta B_n$ and $z=0$ and use the fact that $(\Theta B_n)(0)\to\Theta(0)b(0)$) and since the closed linear span of $\{P_z:z\in\mathbb{D}\}$ is dense in $C(\mathbb{T})$, we get that $\sigma^1_{\Theta B_n}\to\mu$ in the weak$-*$ topology. In others words, for any $f\in C(\mathbb{T})$, we have

$$\int_{\mathbb{T}} f(\zeta)\,d\sigma^1_{\Theta B_n}(\zeta)\to\int_{\mathbb{T}} f(\zeta)\,d\mu(\zeta),\qquad n\to+\infty. \tag{4.1}$$

Moreover, Clark's theorem says that the embedding $(B_n\Theta H^2)^\perp \hookrightarrow L^2(\sigma^1_{\Theta B_n})$ is isometric. Since $(\Theta H^2)^\perp \subset (\Theta B_n H^2)^\perp$, we also have the isometric embedding $(\Theta H^2)^\perp \hookrightarrow L^2(\sigma^1_{\Theta B_n})$. Thus, according to (4.1), for any $f \in (\Theta H^2)^\perp \cap C(\mathbb{T})$, we have

$$\|f\|_2^2 = \int_{\mathbb{T}} |f(\zeta)|^2 \, d\sigma^1_{\Theta B_n}(\zeta) \to \int_{\mathbb{T}} |f(\zeta)|^2 \, d\mu(\zeta).$$

Then, each function $f \in (\Theta H^2)^\perp$ has a finite radial limit at μ-almost every point of $\mathbb{T}$ and we have $\|f\|_2 = \|f\|_\mu$.

Conversely, suppose that $\mu \in M_+(\mathbb{T})$ and suppose that the embedding $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ is isometric. In view of (3.7), we know that there exists a function b_0 in the closed unit ball of H^∞ such that

$$\mu = \sigma^1_{b_0}. \tag{4.2}$$

We will now show that Θ divides b_0. We first do this when $\Theta = B$ is a Blaschke product with simple zeros $\Lambda = (\lambda_n)_{n\geq 1}$. By our discussion of the Clark theory (generalized by Ball) we have the onto partial isometry

$$\omega_{b_0} : L^2(\mu) \to \mathcal{H}(b_0), \quad \omega_{b_0} h = (1 - b_0) C_\mu h.$$

Note that for any $n \geq 1$,

$$k^B_{\lambda_n}(z) = k_{\lambda_n}(z) := \frac{1}{1 - \overline{\lambda_n} z} \in (BH^2)^\perp \cap C(\mathbb{D}^-),$$

which yields, since $k_{\lambda_n} \in H^2(\mu) = (\ker \omega_{b_0})^\perp$,

$$\langle \omega_{b_0} k_{\lambda_n}, \omega_{b_0} k_{\lambda_\ell} \rangle_{b_0} = \langle k_{\lambda_n}, k_{\lambda_\ell} \rangle_\mu = \langle k_{\lambda_n}, k_{\lambda_\ell} \rangle_2 , \quad n, \ell \geq 1. \tag{4.3}$$

A standard computation (see for instance [42, III.6]) shows that

$$\omega_{b_0} k_{\lambda_n} = (1 - \overline{b_0(\lambda_n)})^{-1} k^{b_0}_{\lambda_n}, \qquad n \geq 1.$$

Apply formula (4.3) to the above identity and use the reproducing property to get

$$\begin{aligned}
\langle k_{\lambda_n}, k_{\lambda_\ell} \rangle_2 &= (1 - \overline{b_0(\lambda_n)})^{-1} (1 - b_0(\lambda_\ell))^{-1} \langle k^{b_0}_{\lambda_n}, k^{b_0}_{\lambda_\ell} \rangle_{b_0} \\
&= (1 - \overline{b_0(\lambda_n)})^{-1} (1 - b_0(\lambda_\ell))^{-1} k^{b_0}_{\lambda_n}(\lambda_\ell) \\
&= (1 - \overline{b_0(\lambda_n)})^{-1} (1 - b_0(\lambda_\ell))^{-1} (1 - \overline{b_0(\lambda_n)} b_0(\lambda_\ell)) k_{\lambda_n}(\lambda_\ell).
\end{aligned}$$

Since

$$\langle k_{\lambda_n}, k_{\lambda_\ell} \rangle_2 = k_{\lambda_n}(\lambda_\ell) = (1 - \overline{\lambda_n} \lambda_\ell)^{-1} \neq 0,$$

we obtain

$$(1 - \overline{b_0(\lambda_n)})(1 - b_0(\lambda_\ell)) = 1 - \overline{b_0(\lambda_n)} b_0(\lambda_\ell),$$

which, after a little algebra, gives us

$$\overline{b_0(\lambda_n)} - 2\overline{b_0(\lambda_n)}b_0(\lambda_\ell) + b_0(\lambda_\ell) = 0.$$

The above can be re–arranged as

$$\overline{b_0(\lambda_n)}(1 - b_0(\lambda_\ell)) = -b_0(\lambda_\ell)(1 - \overline{b_0(\lambda_n)}), \qquad (n, \ell \geq 1).$$

Setting $f := b_0/(1 - b_0)$, the last equality implies that, for every $n \geq 1$,

$$f(\lambda_n) = -\overline{f(\lambda_1)} = c, \tag{4.4}$$

and so

$$b_0(\lambda_n) = \frac{c}{1+c} =: \delta, \qquad n \geq 1. \tag{4.5}$$

Setting $\phi_\delta(z) := (\delta - z)/\left(1 - \overline{\delta} z\right)$, we see that $\phi_\delta \circ b_0$ vanishes on Λ and so B, since it has *simple* zeros, divides $\phi_\delta \circ b_0$. This implies the existence of a function ϑ in the closed unit ball of H^∞ such that $B\vartheta = \phi_\delta \circ b_0$. Since $\phi_\delta \circ \phi_\delta$ is the identity we get $b_0 = \phi_\delta \circ (B\vartheta)$, which shows that

$$\mu = \sigma^1_{\phi_{\delta \circ (B\vartheta)}}.$$

To finish this off, we use (3.11) and (3.8) to get

$$\mu = \frac{1}{|\phi_\delta'(\phi_\delta(1))|}\sigma^{\phi_\delta(1)}_{B\vartheta} = \frac{1-|\delta|^2}{|1-\delta|^2}\sigma^{\phi_\delta(1)}_{B\vartheta} = \frac{1-|\delta|^2}{|1-\delta|^2}\sigma^1_{Bb},$$

where b is the function in the closed unit ball of H^∞ defined by $b = \overline{\phi_\delta(1)}\vartheta$ (note that $|\phi_\delta(1)| = 1$). Finally, using (4.4), we easily see that c is imaginary. Thus from the definition of δ in (4.5) we have

$$\frac{1-|\delta|^2}{|1-\delta|^2} = 1,$$

which concludes the proof in the case when $\Theta = B$ is a Blaschke product with simple zeros.

In the general case, the nice little fact that will contribute here is Frostman's result [23] (see also [16]) which says that for an arbitrary inner function Θ, the *Frostman shift* $\phi_\lambda \circ \Theta$ is a Blaschke product for every $\lambda \in \mathbb{D}$ with the possible exception of a set of logarithmic capacity zero. This result can be refined to fit to our situation: $\phi_\lambda \circ \Theta$ is a Blaschke product with *simple* zeros for all $\lambda \in \mathbb{D}$ with the possible exception of a set of Lebesgue area measure zero [21, p. 677]. So let $(\lambda_n)_{n\geq 1}$ be a sequence in $\mathbb{D}$, $\lambda_n \to 0$, such that $B_n := \phi_{\lambda_n} \circ (-\Theta)$ is a Blaschke product with simple zeros. A trivial estimate shows that

$$\|B_n - \Theta\|_\infty \leq \frac{2|\lambda_n|}{1-|\lambda_n|} \to 0, \quad n \to +\infty. \tag{4.6}$$

Fix $n \geq 1$. Since $\phi_{\lambda_n} \circ \phi_{\lambda_n}$ is the identity, remember from (3.12) the Crofoot transform $U_{B_n}^{\lambda_n}$ which is a unitary operator from $(B_n H^2)^\perp$ onto $(\Theta H^2)^\perp$. If we define $\mu_n' \in M_+(\mathbb{T})$ by

$$d\mu_n' := \frac{1-|\lambda_n|^2}{\left|1-\overline{\lambda_n}B_n\right|^2} d\mu, \tag{4.7}$$

then $(B_n H^2)^\perp$ embeds isometrically into $L^2(\mu_n')$. Indeed, for $g \in (B_n H^2)^\perp$, note that $U_{B_n}^{\lambda_n} g \in (\Theta H^2)^\perp$ and $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$ isometrically. Thus,

$$\int_{\mathbb{T}} |g|^2 dm = \int_{\mathbb{T}} |U_{B_n}^{\lambda_n} g|^2 dm = \int_{\mathbb{T}} |U_{B_n}^{\lambda_n} g|^2 d\mu = \int_{\mathbb{T}} |g|^2 d\mu_n'.$$

By the first case of the proof, there is a function b_n in the closed unit ball of H^∞ such that $\mu_n' = \sigma^1_{B_n b_n}$, that is

$$\mu = \frac{|1-\overline{\lambda_n}B_n|^2}{1-|\lambda_n|^2} \sigma^1_{B_n b_n}, \quad n \geq 1. \tag{4.8}$$

By the Banach-Alaoglu theorem, there is a subsequence $(b_{n_\ell})_{\ell \geq 1}$ and a function b in the closed unit ball of H^∞ such that b_{n_ℓ} converges to b in the weak-$*$ topology, which means that

$$\int_{\mathbb{T}} f b_{n_\ell}\, dm \to \int_{\mathbb{T}} f b\, dm, \quad \ell \to +\infty,$$

for any $f \in L^1(\mathbb{T})$. In particular, for any $z \in \mathbb{D}$, $b_{n_\ell}(z) \to b(z)$ as $\ell \to +\infty$. Now we argue that $\mu = \sigma^1_{\Theta b}$. Indeed according to (4.8), we have

$$\begin{aligned} \int_{\mathbb{T}} P_z(\zeta)\, d\mu(\zeta) &= \int_{\mathbb{T}} P_z(\zeta) \left(\frac{|1-\overline{\lambda_{n_\ell}}B_{n_\ell}(\zeta)|^2}{1-|\lambda_{n_\ell}|^2} - 1 \right) d\sigma^1_{B_{n_\ell} b_{n_\ell}}(\zeta) + \\ &+ \int_{\mathbb{T}} P_z(\zeta)\, d\sigma^1_{B_{n_\ell} b_{n_\ell}}(\zeta). \end{aligned}$$

Observe that

$$\frac{|1-\overline{\lambda_n}B_n(\zeta)|^2}{1-|\lambda_n|^2} - 1 \to 0, \quad n \to +\infty,$$

uniformly on $\mathbb{T}$ and, according to (4.6), B_{n_ℓ} tends to Θ uniformly on $\mathbb{D}^-$ and $b_{n_\ell}(z) \to b(z)$, $z \in \mathbb{D}$, as $\ell \to +\infty$. Hence,

$$\int_{\mathbb{T}} P_z(\zeta)\, d\sigma^1_{B_{n_\ell} b_{n_\ell}}(\zeta) = \frac{1-|B_{n_\ell}(z) b_{n_\ell}(z)|^2}{|1-B_{n_\ell}(z) b_{n_\ell}(z)|^2}$$

tends to

$$\frac{1-|\Theta(z)b(z)|^2}{|1-\Theta(z)b(z)|^2} = \int_{\mathbb{T}} P_z(\zeta)\, d\sigma^1_{\Theta b}(\zeta).$$

Thus, we obtain that for any $z \in \mathbb{D}$,

$$\int_{\mathbb{T}} P_z(\zeta)\, d\mu(\zeta) = \int_{\mathbb{T}} P_z(\zeta)\, d\sigma^1_{\Theta b}(\zeta).$$

But the closed linear span of $\{P_z : z \in \mathbb{D}\}$ is dense in $C(\mathbb{T})$, which yields that

$$\mu = \sigma^1_{\Theta b},$$

and thus concludes the proof. □

5. Dominating Sets

In this section, we introduce and discuss the notion of dominating sets for $(\Theta H^2)^\perp$ where Θ is a non-constant inner function. This terminology is perhaps reminiscent of the concept of a dominating sequence for H^∞ [10].

Definition 5.1. A (Lebesgue) measurable subset $\Sigma \subset \mathbb{T}$, with $m(\Sigma) < 1$, is called a *dominating set* for $(\Theta H^2)^\perp$ if

$$\|f\|_2^2 \lesssim \int_\Sigma |f|^2 dm, \quad \forall f \in (\Theta H^2)^\perp.$$

Necessarily, in the above definition, $m(\Sigma) > 0$. Also notice that the hypothesis $m(\Sigma) < 1$ is crucial since otherwise the whole matter becomes trivial. Another observation is that if Σ is a measurable subset of $\mathbb{T}$ such that $m(\Sigma) < 1$, then Σ is a dominating set if and only if the measure $d\mu = \chi_\Sigma dm$ yields a reverse embedding for $(\Theta H^2)^\perp$ (and hence equivalent norms). Therefore, by Volberg's theorem (Theorem 1.2), we obtain a criterion for dominating sets Σ in terms of the harmonic continuation of χ_Σ into the open unit disc. However, this criterion is not so easy to deal with. In particular, it is not clear that dominating sets even exist – for *any* model space.

The aim of this section is to give simpler necessary or sufficient conditions for dominating sets. More precisely, we would like to highlight some interesting relationship between dominating sets for $(\Theta H^2)^\perp$ and the spectrum of the inner function Θ. Indeed, a simple look at the definition of dominating set tells us that such a set cannot be too 'far' from the spectrum or from the points where the modulus of the angular derivative is infinite. We make this more precise with the next two results. For sets A and B and a point x, let

$$\begin{aligned} d(A,B) &:= \inf\{|a-b| : a \in A, b \in B\}, \\ d(x,A) &:= d(\{x\}, A). \end{aligned}$$

Proposition 5.2. *If Σ is a dominating set for $(\Theta H^2)^\perp$, and $\zeta \in \mathbb{T} \setminus ADC_\Theta$ then $d(\zeta, \Sigma) = 0$.*

Proof. Let $\zeta \in \mathbb{T}$ such that $|\Theta'(\zeta)| = \infty$ and $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ with radial limit ζ. Let us suppose, towards a contradiction, that $d(\zeta, \Sigma) > 0$. Then, for n large enough, we see that $d(\lambda_n, \Sigma) \gtrsim 1$. If we consider the reproducing kernels $k^\Theta_{\lambda_n} \in (\Theta H^2)^\perp$, the defining property of Σ along with (3.5) gives us

$$\begin{aligned}
\frac{1-|\Theta(\lambda_n)|^2}{1-|\lambda_n|^2} &= \int_{\mathbb{T}} |k^\Theta_{\lambda_n}|^2 dm \lesssim \int_\Sigma |k^\Theta_{\lambda_n}|^2 dm \\
&\lesssim \int_\Sigma \left| \frac{\Theta(\xi) - \Theta(\lambda_n)}{\lambda_n - \xi} \right|^2 dm(\xi) \\
&\lesssim \frac{1}{\mathrm{d}(\lambda_n, \Sigma)} \\
&\lesssim 1,
\end{aligned}$$

whereas, by Ahern-Clark [1],

$$\frac{1-|\Theta(\lambda_n)|^2}{1-|\lambda_n|^2} \longrightarrow \infty, \quad n \to \infty,$$

This yields the required contradiction. □

The following consequence is obvious.

Corollary 5.3. *If Σ is a dominating set for $(\Theta H^2)^\perp$, then*

$$d(\Sigma, \mathbb{T} \setminus \mathrm{ADC}_\Theta) = 0.$$

Now, since $\mathbb{T} \setminus ADC_\Theta$ is a subset of the spectrum $\sigma(\Theta)$, the previous corollary directly implies the following one.

Proposition 5.4. *If Σ is a dominating set for $(\Theta H^2)^\perp$, then*

$$d(\Sigma, \sigma(\Theta)) = 0.$$

We will see in Corollary 5.10 that if the spectrum of the inner function is the whole circle, then a dominating set — if it exists — must be dense. However, dense sets can be of measure zero. Here is a little fact that states that any neighborhood of a point in the spectrum has to contain portions of the dominating set Σ with non negligible Lebesgue measure. First let us introduce the *Privalov shadow*: for $\lambda \in \mathbb{D}$ and $\alpha > 0$, let I^α_λ be the arc in $\mathbb{T}$ centered at $\lambda/|\lambda|$ with length $\alpha(1-|\lambda|)$.

For $\alpha = 1$ we simply write $I_\lambda := I^1_\lambda$, (see Figure 1). Observe that I^α_λ is the α-amplification of I_λ.

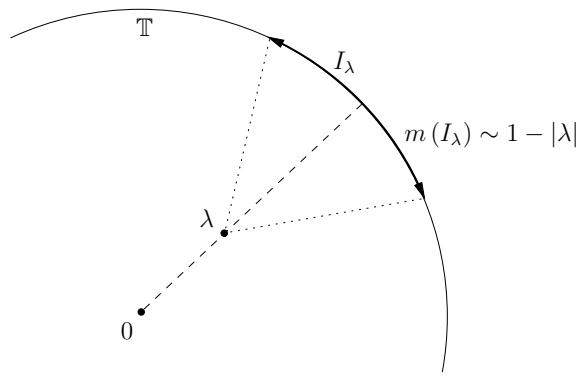

FIGURE 1. Privalov shadow

Proposition 5.5. *Let $\zeta \in \sigma(\Theta)$ and Σ dominating. Then there exists an $\alpha > 0$ such that for every sequence $\lambda_n \to \zeta$ with $\Theta(\lambda_n) \to 0$, there is an integer N with*

$$m(\Sigma \cap I^\alpha_{\lambda_n}) \gtrsim m(I^\alpha_{\lambda_n}), \quad n \geq N.$$

Proof. Since Σ is dominating we have

$$c \leq \frac{\int_\Sigma |k^\Theta_{\lambda_n}|^2 dm}{\|k^\Theta_{\lambda_n}\|_2^2}.$$

Now, since $\Theta(\lambda_n) \to 0$, there exists an N such that $|\Theta(\lambda_n)| \leq 1/2$ for $n \geq N$, so that in the above inequality we can replace $k^\Theta_{\lambda_n}$ by k_{λ_n}, $n \geq N$. Hence (with a change of constant from c to c_1) for $n \geq N$,

$$c_1 \leq \int_\Sigma \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|^2} dm(\xi).$$

Now with an appropriate choice of α we have

$$\int_{\mathbb{T} \setminus I^\alpha_{\lambda_n}} \frac{1 - |\lambda_n|^2}{|\xi - \lambda_n|^2} dm(\xi) < \frac{c_1}{2}.$$

Hence

$$\begin{aligned} c_1 &\leq \int_\Sigma \frac{1-|\lambda_n|^2}{|\xi-\lambda_n|^2} dm(\xi) \\ &= \int_{\Sigma\cap I^\alpha_{\lambda_n}} \frac{1-|\lambda_n|^2}{|\xi-\lambda_n|^2} dm(\xi) + \int_{\Sigma\cap(\mathbb{T}\setminus I^\alpha_{\lambda_n})} \frac{1-|\lambda_n|^2}{|\xi-\lambda_n|^2} dm(\xi) \\ &\leq \int_{\Sigma\cap I^\alpha_{\lambda_n}} \frac{1-|\lambda_n|^2}{|\xi-\lambda_n|^2} dm(\xi) + \frac{c_1}{2} \\ &\lesssim \frac{m(\Sigma\cap I^\alpha_{\lambda_n})}{1-|\lambda_n|^2} + \frac{c_1}{2}, \end{aligned}$$

which yields the desired conclusion. □

If the spectrum of the inner function Θ is not the whole unit circle, it turns out that dominating sets for $(\Theta H^2)^\perp$ exist – in abundance. Here are two results in this direction.

Theorem 5.6. *Let Θ be an inner function and assume that $\sigma(\Theta) \neq \mathbb{T}$. Then $(\Theta H^2)^\perp$ has dominating sets. More precisely, if Σ is an open subset of $\mathbb{T}$ such that $\sigma(\Theta) \subset \Sigma$ and $m(\Sigma) < 1$, then Σ is a dominating set for $(\Theta H^2)^\perp$.*

Proof. Since $\sigma(\Theta)$ is a closed subset of $\mathbb{T}$, it turns out that if $\sigma(\Theta) \neq \mathbb{T}$, then we can find an open subset Σ of $\mathbb{T}$ such that $\sigma(\Theta) \subset \Sigma$ and $m(\Sigma) < 1$. We now prove that this open set is a dominating set for $(\Theta H^2)^\perp$. As already noticed, this is equivalent to saying if $d\mu = \chi_\Sigma dm$ the norms $\|\cdot\|_\mu$ and $\|\cdot\|_2$ are equivalent on $(\Theta H^2)^\perp$. By Volberg's result we will show that if $\lambda_n \in \mathbb{D}$ is such that $\lim_{n\to+\infty} \widehat{\chi_\Sigma}(\lambda_n) = 0$, then $\lim_{n\to+\infty} |\Theta(\lambda_n)| = 1$. To do this, we first note that

$$\widehat{\chi_\Sigma}(\lambda_n) = \int_\Sigma \frac{1-|\lambda_n|^2}{|1-\bar{\lambda}_n\zeta|^2}\, dm(\zeta). \tag{5.1}$$

Let $\zeta_0 \in \mathbb{T}$ be such that some subsequence $\lambda_{n_\ell} \to \zeta_0$. It suffices to show that $\zeta_0 \in \mathbb{T}\setminus\sigma(\Theta)$. Indeed since we know that the inner function Θ is analytic on $\mathbb{T}\setminus\sigma(\Theta)$ we will thus get

$$\lim_{\ell\to+\infty} |\Theta(\lambda_{n_\ell})| = |\Theta(\zeta_0)| = 1.$$

In order to prove that $\zeta_0 \in \mathbb{T}\setminus\sigma(\Theta)$, we argue, assuming to the contrary, that $\zeta_0 \in \sigma(\Theta)$. Then for $\zeta \in \mathbb{T}\setminus\Sigma$ and for sufficiently large ℓ, we have

$$\begin{aligned} |1-\bar{\lambda}_{n_\ell}\zeta| &\geq |\zeta_0-\zeta| - |\lambda_{n\ell}-\zeta_0| \geq d(\zeta_0, \mathbb{T}\setminus\Sigma) - |\lambda_{n\ell}-\zeta_0| \\ &\geq \frac{1}{2} d(\zeta_0, \mathbb{T}\setminus\Sigma). \end{aligned}$$

Since $\zeta_0 \in \sigma(\Theta) \subset \Sigma$ and Σ is open, we have $d(\zeta_0, \mathbb{T} \setminus \Sigma) > 0$. Hence

$$\int_{\mathbb{T}\setminus\Sigma} \frac{1-|\lambda_{n_\ell}|^2}{|1-\bar{\lambda}_n\zeta|^2}\, dm(\zeta) \le \frac{4(1-|\lambda_{n_\ell}|^2)}{d^2(\zeta_0, \mathbb{T}\setminus\Sigma)} \to 0 \qquad \text{as } \ell \to +\infty.$$

On the other hand, by hypothesis, we have $\widehat{\chi_\Sigma}(\lambda_{n_\ell}) \to 0$, $\ell \to +\infty$. Taking into account (5.1), we get

$$\int_{\mathbb{T}} \frac{1-|\lambda_{n_\ell}|^2}{|1-\bar{\lambda}_{n_\ell}\zeta|^2}\, dm(\zeta) = \int_{\Sigma} \frac{1-|\lambda_{n_\ell}|^2}{|1-\bar{\lambda}_{n_\ell}\zeta|^2}\, dm(\zeta) + \int_{\mathbb{T}\setminus\Sigma} \frac{1-|\lambda_{n_\ell}|^2}{|1-\bar{\lambda}_{n_\ell}\zeta|^2}\, dm(\zeta) \to 0$$

as $\ell \to +\infty$. But this is a contradiction since the left hand side is always 1. □

Corollary 5.7. *Let Θ be an inner function such that $m(\sigma(\Theta)) = 0$. Then, for every $0 < \varepsilon < 1$, there is a dominating set Σ for $(\Theta H^2)^\perp$ such that $m(\Sigma) < \varepsilon$.*

Proof. By outer regularity of Lebesgue measure, we know that for every $0 < \varepsilon < 1$, there exists an open subset Σ of $\mathbb{T}$ such that $\sigma(\Theta) \subset \Sigma$ and $m(\Sigma) < \varepsilon$. Now Theorem 5.6 implies that such a Σ is a dominating set for $(\Theta H^2)^\perp$.

□

Remark 5.8. Theorems 5.6 and Corollary 5.7 apply in particular when $\Theta \in (CLS)$, because in that situation we know by a result of Aleksandrov [2] that $m(\sigma(\Theta)) = 0$.

In the beginning of this section we have seen a simple argument that a dominating set has to be close to every point of the complement of ADC_Θ. Also recall that $\sigma(\Theta)$ contains this complement. In order to show that a dominating set has to be close to every point of $\sigma(\Theta)$ we use Volberg's theorem (note that $\mathbb{T} \setminus \mathrm{ADC}_\Theta$ can be empty while $\sigma(\Theta)$ is never empty).

Proposition 5.9. *Let Θ be an inner function. If Σ is a dominating set for $(\Theta H^2)^\perp$, then for every $\zeta \in \sigma(\Theta)$, we have $d(\zeta, \Sigma) = 0$.*

Proof. By Volberg's theorem, we know that there exists a $\delta > 0$ such that

$$\widehat{\chi}_\Sigma(z) + |\Theta(z)| \ge \delta, \qquad z \in \mathbb{D}. \tag{5.2}$$

Assuming there is $\zeta \in \sigma(\Theta)$ with $d(\zeta, \Sigma) > 0$. By (3.1) there exists a sequence $(z_n)_{n\ge1} \subset \mathbb{D}$ such that $z_n \to \zeta$ and $\Theta(z_n) \to 0$, as $n \to +\infty$.

Now

$$\widehat{\chi}_\Sigma(z_n) = \int_\Sigma \frac{1-|z_n|^2}{|z_n-\xi|^2}\, dm(\xi),$$

and for $\xi \in \Sigma$

$$|\xi - z_n| \geq |\xi - \zeta| - |\zeta - z_n| \geq d(\zeta, \Sigma) - |\zeta - z_n|.$$

Since $|\zeta - z_n| \to 0$, as $n \to +\infty$, we get, for n sufficiently large, $|\xi - z_n| \geq d(\zeta, \Sigma^-)/2$, which yields

$$\widehat{\chi}_\Sigma(z_n) \leq \frac{4}{d(\zeta, \Sigma^-)^2}(1-|z_n|^2) \to 0, \quad \text{as } n \to +\infty,$$

contradicting (5.2). □

We have seen that if the spectrum of Θ is not the whole circle, then there are many dominating sets. On the contrary, if the spectrum of Θ is the whole circle, the preceding proposition allows to deduce a topological condition on the size of the dominating set.

Corollary 5.10. *Let Θ be an inner function such that $\sigma(\Theta) = \mathbb{T}$. If Σ is a dominating set for $(\Theta H^2)^\perp$, then Σ is dense in $\mathbb{T}$.*

Remark 5.11. According to Theorem 5.6, we see that if $\sigma(\Theta) \neq \mathbb{T}$, then one can construct closed dominating sets for $(\Theta H^2)^\perp$. Indeed it is sufficient to choose an open set Σ such that $\sigma(\Theta) \subset \Sigma \subset \Sigma^-$ and $m(\Sigma^-) < 1$. Corollary 5.10 shows that the converse is true. In other words, the space $(\Theta H^2)^\perp$ has *closed* dominating sets if and only if $\sigma(\Theta) \neq \mathbb{T}$. In Proposition 5.16 we will construct a Blaschke product B with $\sigma(B) = \mathbb{T}$ and which admits an open dominating set.

We now discuss, with the help of examples, the conditions of our previous results and the "size" of dominating sets.

Example 5.12. Let $\Theta(z) = \exp(-(1+z)/(1-z))$. In this case, $\sigma(\Theta) = \{1\}$ and $\Theta \in (CLS)$. According to Theorem 5.6, every open arc containing 1 is a dominating set for Θ. In this example, the "open" condition seems to be necessary. Indeed, we have the following result.

Proposition 5.13. *Let $\Theta(z) = \exp(-(1+z)/(1-z))$. Suppose that $\Sigma \subset \mathbb{T}$ is a closed arc with endpoints 1 and $e^{i\gamma_0}$, $\gamma_0 \in (0, \pi)$, then Σ is not dominating for $(\Theta H^2)^\perp$.*

Proof. Let

$$z_n = r_n e^{-i\theta_n}, \quad 1 - r_n = \theta_n^{3/2},$$

where $\theta_n > 0$ and $\theta_n \longrightarrow 0$. The sequence z_n goes tangentially "from below" to 1, i.e. $\arg(z_n) \in (-\frac{\pi}{2}, 0)$. In particular we have

$$|1 - z_n|^2 \asymp (1 - r_n)^2 + \theta_n^2 = \theta_n^3 + \theta_n^2 \asymp \theta_n^2, \quad n \to \infty.$$

Hence

$$\frac{1-|z_n|^2}{|1-z_n|^2} \asymp \frac{1-r_n}{\theta_n^2} = \frac{\theta_n^{3/2}}{\theta_n^2} = \frac{1}{\theta_n^{1/2}} \longrightarrow \infty,$$

and thus

$$|\Theta(z_n)| = \exp\left(-\frac{1-|\lambda_n|^2}{|1-\lambda_n|^2}\right) \longrightarrow 0.$$

Observe that

$$\begin{aligned}
\widehat{\chi}_\Sigma(z_n) &= \int_\Sigma \frac{1-r_n^2}{|e^{it}-z_n|^2}\frac{dt}{2\pi} \lesssim \int_0^{\gamma_0} \frac{\theta_n^{3/2}}{|e^{it}-e^{-i\theta_n}|^2}dt \\
&= \int_0^{\gamma_0} \frac{\theta_n^{3/2}}{2(1-\cos(t+\theta_n))}dt \asymp \int_0^{\gamma_0} \frac{\theta_n^{3/2}}{(t+\theta_n)^2}dt \\
&= \theta_n^{3/2}\left[\frac{-1}{t+\theta_n}\right]_0^{\gamma_0} \\
&\asymp \theta_n^{1/2} \longrightarrow 0.
\end{aligned}$$

So $\inf_{\lambda\in\mathbb{D}}(\widehat{\chi}_\Sigma(\lambda)+|\Theta(\lambda)|) = 0$, and Volberg's theorem allows us to conclude that Σ is not dominating. □

The situation is, of course, the same if we replace $\gamma_0 \in (0,\pi)$ by $\gamma_0 \in (-\pi, 0)$.

Compare this with the next situation.

Example 5.14. Let $\Theta = B$ be a Blaschke product with simple zeros $\lambda_n = r_n e^{i\theta_n}$, $\theta_n = 2^{-n}$ and $1-r_n = 16^{-n}$. This is an inner function such that $\sigma(B) = \{1\}$ and $\mathrm{ADC}_B = \mathbb{T}$ (Ahern-Clark). We have the following result.

Proposition 5.15. *Let B be as above. Suppose that $\Sigma \subset \mathbb{T}$ is a closed arc with endpoints 1 and $e^{i\gamma_0}$, $\gamma_0 \in (0,\pi)$, then Σ is dominating for $(BH^2)^\perp$.*

Proof. Again we use Volberg's theorem. Let $(z_n)_{n\geq 1} \subset \mathbb{D}$ be such that $\widehat{\chi}_\Sigma(z_n) \to 0$. We have to show that $|B(z_n)| \to 1$. Pick a convergent subsequence $z_{n_k} \to \zeta \in \mathbb{T}$. Since $\sigma(B) = \{1\}$, the only critical situation is when $\zeta = 1$. First we argue that $z_{n_k} \to 1$ "from below". Indeed, assume on the contrary that there is a subsequence, also denoted by z_{n_k}, such that $\arg(z_{n_k}) \in (0,\pi/2)$. For a point $\lambda \in \mathbb{D}$, recall that I_λ denotes the Privalov shadow associated to λ, that is the arc in $\mathbb{T}$ centered at $\lambda/|\lambda|$ with length $(1-|\lambda|)$ – see Figure 1. It is easy to see that for $\zeta \in I_\lambda$, we have

$$\frac{1-|\lambda|^2}{|\zeta-\lambda|^2} \asymp \frac{1}{1-|\lambda|^2}.$$

Now, since $z_{n_k} \to 1$, with $\arg(z_{n_k}) \in (0, \pi/2)$, there exists an integer N such that for any $k \geq N$, we have $I_{z_{n_k}} \cap \mathbb{T}_+ \subset \Sigma$ and $m(I_{z_{n_k}} \cap \mathbb{T}_+) \geq (1 - |z_{n_k}|)/2$, where $\mathbb{T}_+ = \{z \in \mathbb{T} : \arg(z) \in (0, \pi)\}$. Thus

$$\begin{aligned} \widehat{\chi}_\Sigma(z_{n_k}) &= \int_\Sigma \frac{1 - |z_{n_k}|^2}{|\zeta - z_{n_k}|^2}\, dm(\zeta) \geq \int_{I_{z_{n_k}} \cap \mathbb{T}_+} \frac{1 - |z_{n_k}|^2}{|\zeta - z_{n_k}|^2}\, dm(\zeta) \\ &\gtrsim \frac{1}{1 - |z_{n_k}|^2} \frac{1 - |z_{n_k}|}{2} \gtrsim 1, \end{aligned}$$

which contradicts the fact that $\widehat{\chi}_\Sigma(z_n) \to 0$. Hence we can assume that $z_{n_k} \to 1$ with $\arg(z_{n_k}) \in (-\pi/2, 0)$. Taking the logarithmic derivative and using (3.2), it is easy to see that

$$|B'(z)| \leq \sum_{n \geq 1} \frac{1 - |\lambda_n|^2}{|1 - \bar{\lambda}_n z|^2}, \qquad z \in \mathbb{D}^-,$$

and standard estimates show that if z belongs to the closed lower half-disc $\Upsilon := \{z : \ |z| \leq 1, \mathrm{Im}(z) \leq 0\}$, then

$$|B'(z)| \lesssim \sum_{n \geq 1} \frac{1}{4^n} < +\infty,$$

which means that B' is uniformly bounded on Υ. Hence B is continuous on Υ. In particular, we get that $|B(z_{n_k})| \to 1$. We conclude from Volberg's theorem that Σ is dominating for $(BH^2)^\perp$. □

We finish this section with an example of a Blaschke product whose boundary spectrum is the whole circle and which admits a dominating set Σ.

Proposition 5.16. *There exists a Blaschke product B with $\sigma(B) = \mathbb{T}$ and an open subset $\Sigma \subsetneq \mathbb{T}$ dominating for $(BH^2)^\perp$.*

Proof. Let C be a Smith-Volterra-Cantor set of $\mathbb{T}$ (i.e., a closed subset of $\mathbb{T}$, nowhere dense and with positive measure, constructed in a similar way as the usual Cantor set by removing the middle *fourth* instead of the middle third). We define $\Sigma := \mathbb{T} \setminus C$ which is clearly a dense open subset of $\mathbb{T}$ with $0 < m(\Sigma) < 1$. Since Σ is open, there is a sequence of open arcs $(I_n)_{n \geq 1}$ such that $\Sigma = \bigcup_{n \geq 1} I_n$. We denote by ξ_n the first endpoint (moving counterclockwise) of I_n and N_n the integer such that $2^{-(2N_n+2)} \leq \theta_n := m(I_n) < 2^{-2N_n}$ and $\alpha_n := \theta_n 2^{2N_n} \in [1/4, 1)$. Observe that the second endpoint of I_n then corresponds to $\xi_n e^{i2\pi\theta_n}$. We now fix n. For each $l > 2N_n$ and each $k = 1, ..., 2^{l-2N_n} - 1$, we set

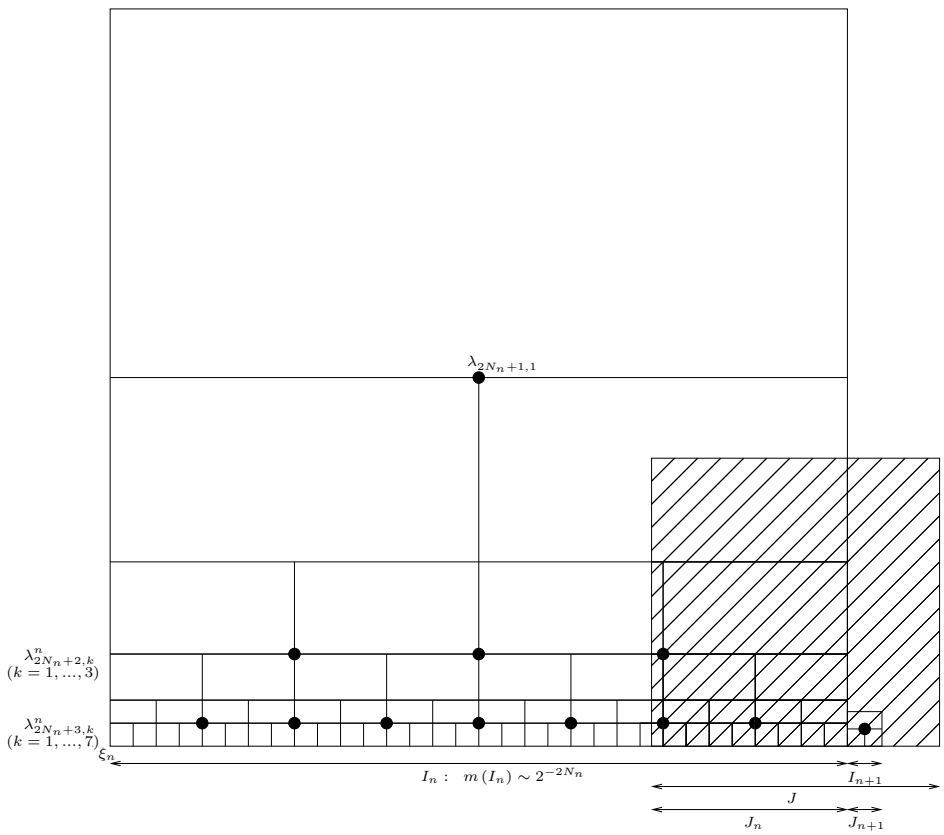

FIGURE 2. Λ_n, J_n and J_{n+1}

(see Figure 2)

$$\lambda^n_{l,k} := \left(1-\frac{1}{2^{2l}}\right)\xi_n e^{i\frac{k}{2^l}2\pi\alpha_n},\ \Lambda_n := \left\{\lambda^n_{l,k}:\ l,k\right\},\ \text{and}\ \Lambda := \bigcup_{n\geq 1}\Lambda_n.$$

Observe that every $\lambda^n_{\ell,k}\in\Lambda_n$ lies in the Carleson window $S(I_n)$. For each $l\geq N_n$, every arc $\left\{|z|=1-2^{-2l}\right\}\cap S(I_n)$ contains $2^{l-2N_n}-1$ points of Λ.

Thus, the sequence Λ satisfies the Blaschke condition:

$$\begin{aligned}\sum_{\lambda\in\Lambda}(1-|\lambda|) &= \sum_{n\geq 1}\sum_{l>2N_n}\frac{1}{2^{2l}}\left(2^{l-2N_n}-1\right)\\ &\leq \sum_{n\geq 1}\frac{1}{2^{4N_n}}\\ &\asymp \sum_{n\geq 1}m(I_n)^2<\infty.\end{aligned}$$

Let B be the corresponding Blaschke product. It is clear that every point of I_n is a cluster point of Λ. Since $\Sigma = \bigcup_n I_n$ is dense in $\mathbb{T}$, it easily follows that $\sigma(B) = \mathbb{T}$. It remains to show that Σ is a dominating set for $(BH^2)^\perp$. We will use Volberg's theorem and show that

$$\inf_{z\in\mathbb{D}} \left(|B(z)| + \widehat{\chi_\Sigma}(z)\right) > 0.$$

The idea is the following. We will show that Λ is an interpolating sequence, which implies that $|B|$ is big outside a pseudohyperbolic neighborhood of Λ (see below for definition). Inside a pseudohyperbolic neighborhood of a point $\lambda \in \Lambda$, an easy computation will show that $\widehat{\chi_\Sigma}$ is big.

In order to prove that Λ is an interpolating sequence, it suffices to prove that $\nu := \sum_{\lambda\in\Lambda} \left(1 - |\lambda|^2\right) \delta_\lambda$ is a Carleson measure [24] since the sequence Λ is separated by construction. Let J be an arc of $\mathbb{T}$. It is possible to write $J = \bigcup_{n\geq 1} J_n$, where $J_n := J \cap I_n$ are disjoint arcs (observe that we have three configurations: either $J_n = \emptyset$ or $I_n \subset J$ or I_n meets J without being contained in J, this latter situation occurs at most two times at the endpoints of J - see Figure 2).

Let us introduce the sector:

$$\alpha(I) := \left\{ z \in \mathbb{D}^- : \frac{z}{|z|} \in I \right\}.$$

In the following computation, we want to count the number of points of Λ_n at level $1 - 1/2^{2l}$, $l \geq 2N_n + 1$, that fall in the sector $\alpha(K)$ where K is any arc in $\mathbb{T}$. Since the arguments of those points are separated by $2\pi\alpha_n/2^l$ (recall that $\alpha_n \in [1/4, 1)$) we get

$$\#\left(\Lambda_n \cap \left\{|z| = 1 - 2^{-2l}\right\} \cap \alpha(K)\right) \leq \frac{m(K)}{2\pi\alpha 2^{-l}} \asymp 2^l m(K).$$

Hence

$$\begin{aligned}
\nu(S(J)) &= \sum_{\lambda\in S(J)} \left(1 - |\lambda|^2\right) \lesssim \sum_{\lambda\in S(J)} (1 - |\lambda|) \\
&\leq \sum_{n\geq 1} \sum_{l>2N_n} 2^{-2l} \cdot \#\left(\Lambda_n \cap \left\{|z| = 1 - 2^{-2l}\right\} \cap \alpha(J_n)\right) \\
&\leq \sum_{n\geq 1} m(J_n) \sum_{l>2N_n} 2^{-l} \\
&\leq \sum_{n\geq 1} m(J_n) \\
&\leq m(J),
\end{aligned}$$

so that ν is a Carleson measure and thus Λ is an interpolating sequence. So, for arbitrarily fixed $\eta \in (0,1)$, if

$$z \in \mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} \Omega(\lambda, \eta), \text{ with } \Omega(\lambda, \eta) := \left\{ z \in \mathbb{D} : \left| \frac{\lambda - z}{1 - \overline{\lambda} z} \right| < \eta \right\},$$

then $|B(z)| \asymp 1$ (see for instance [35, p. 218]). On the other hand, if $z \in \Omega(\lambda, \eta)$, then

$$\begin{aligned} \widehat{\chi_\Sigma}(z) &\asymp \int_\Sigma \frac{1 - |\lambda|^2}{|\lambda - \xi|^2} dm(\xi) \\ &\geq \int_{I_\lambda \cap \Sigma} \frac{1 - |\lambda|^2}{|\lambda - \xi|^2} dm(\xi) \\ &\gtrsim \frac{m(I_\lambda \cap \Sigma)}{1 - |\lambda|}, \end{aligned}$$

where I_λ is the Privalov shadow of λ. Let n be such that $\lambda \in \Lambda_n$. Then $\lambda \in S(I_n)$ and thus $m(I_\lambda \cap \Sigma) \geq m(I_\lambda \cap I_n) \geq m(I_\lambda)/2 \asymp 1 - |\lambda|$, so that $\widehat{\chi_\Sigma}(z) \gtrsim 1$, $z \in \Omega(\lambda, \eta)$. Finally, we obtain that

$$\inf_{z \in \mathbb{D}} \left(|B(z)| + \widehat{\chi_\Sigma}(z) \right) > 0,$$

which ends the proof. □

We have seen that if $\sigma(\Theta) \neq \mathbb{T}$ then $(\Theta H^2)^\perp$ admits a dominating set. In Proposition 5.16, we have constructed an example of inner functions Θ such that $\sigma(\Theta) = \mathbb{T}$ and the corresponding model space admits also a dominating set. It is natural to formulate the following:

Open Question. Does every model space $(\Theta H^2)^\perp$ admit a dominating set?

6. Reverse Embeddings for $(\Theta H^2)^\perp$ – proofs of Theorem 2.1 and Theorem 2.2

The proof of Theorem 2.1 requires a few preliminaries on perturbation of bases. Recall that a sequence $(x_n)_{n \geq 1} \subset \mathcal{H}$ is a *Riesz basis* for a separable Hilbert space $\mathcal{H}$ if the closed linear span of $(x_n)_{n \geq 1}$ is $\mathcal{H}$ and

$$\| \sum_{n \geq 1} a_n x_n \|_{\mathcal{H}}^2 \asymp \sum_{n \geq 1} |a_n|^2, \quad \forall (a_n)_{n \geq 1} \in \ell^2(\mathbb{N}).$$

Also recall that if $\Theta \in (CLS)$, then a carrier of the Clark measure σ_Θ^α,

$$\{\xi \in \mathbb{T} \setminus \sigma(\Theta) : \Theta(\xi) = \alpha\},$$

is a discrete set $(\xi_n)_{n \geq 1}$ and so $\{k_{\xi_n}^\Theta / \|k_{\xi_n}^\Theta\|_2 : n \geq 1\}$ is an orthonormal basis, a Clark basis, for $(\Theta H^2)^\perp$. Recall that $\|k_{\xi_n}^\Theta\|_2 = \sqrt{|\Theta'(\xi_n)|}$.

The following result is due to Baranov ([9, Corollary 1.3 and proof of Theorem 1.1]).

Theorem 6.1 (Baranov)**.** *Let $\Theta \in (CLS)$. There exists $\varepsilon_0 = \varepsilon_0(\Theta) \in (0,1)$ making the following true: if $(k^\Theta_{\xi_n}/\|k^\Theta_{\xi_n}\|_2)_{n\geq 1}$, with $\xi_n \in ADC_\Theta$, is a Riesz basis for $(\Theta H^2)^\perp$ and $\lambda_n \in \mathbb{D}^-$ satisfy*

$$|\lambda_n - \xi_n| < \varepsilon_0 |\Theta'(\xi_n)|^{-1},$$

then $(k^\Theta_{\lambda_n}/\|k^\Theta_{\lambda_n}\|_2)_{n\geq 1}$ is also a Riesz basis for $(\Theta H^2)^\perp$. Moreover, there is a positive constant C such that for every $f \in (\Theta H^2)^\perp$, we have

$$\sum_{n\geq 1} \frac{|f(\xi_n) - f(\lambda_n)|^2}{|\Theta'(\xi_n)|} \leq \varepsilon_0 C \|f\|_2^2. \tag{6.1}$$

In particular one can choose $(k^\Theta_{\xi_n}/\|k^\Theta_{\xi_n}\|_2)_{n\geq 1}$ to be a Clark basis. Let us mention that Cohn [15] also established an interesting result about the stability of Clark basis for one-component inner functions.

Proof of Theorem 2.1. Let $\varepsilon_0 \in (0,1)$ be the constant given in Theorem 6.1 and let $(k^\Theta_{\xi_n}/\|k^\Theta_{\xi_n}\|_2)_{n\geq 1}$ be a Clark basis (we know that such a basis exists because $\Theta \in (CLS)$). For $\eta = \varepsilon_0\sqrt{2}/2$, we define $\sigma_n \subset \mathbb{T}$ to be the arc centered at ξ_n of length $\eta|\Theta'(\xi_n)|^{-1}$. Since $\eta < \varepsilon_0$, we easily check that

$$S(\sigma_n) \subset D(\xi_n, \varepsilon_0|\Theta'(\xi_n)|^{-1}).$$

Recall from (1.3) that $S(\sigma_n)$ is the Carleson window over σ_n. Moreover, we argue that

$$|\xi_n - \xi_m| \geq \varepsilon_0 \min(|\Theta'(\xi_n)|^{-1}, |\Theta'(\xi_m)|^{-1}), \qquad n \neq m. \tag{6.2}$$

Indeed, for a point ξ_n, if for some $m \neq n$, there exists a point $\xi_m \in D(\xi_n, \varepsilon_0|\Theta'(\xi_n)|^{-1})$, then we could define

$$\lambda_\ell = \begin{cases} \xi_\ell & \text{for } \ell \neq m \\ \xi_n & \text{for } \ell = m. \end{cases}$$

While this sequence satisfies the perturbation condition of Theorem 6.1, it is certainly no longer a basis (the same vector appears two times).

Now it follows from (6.2) that $\sigma_n \cap \sigma_m = \varnothing$, $n \neq m$, and thus the Carleson windows $S(\sigma_n)$, $n \geq 1$, are disjoint. Moreover, since $\Theta \in (CLS)$, we know [9] that

$$\min\left(|\Theta'(\xi_n)|^{-1}, d(\xi_n, \sigma(\Theta))\right) \asymp d(\xi_n, L(\Theta, \varepsilon_1)),$$

where $L(\Theta, \varepsilon_1)$ is the connected sub-level set for Θ defined in (1.8). In particular,

$$|\Theta'(\xi_n)|^{-1} \geq c d(\xi_n, L(\Theta, \varepsilon_1))$$

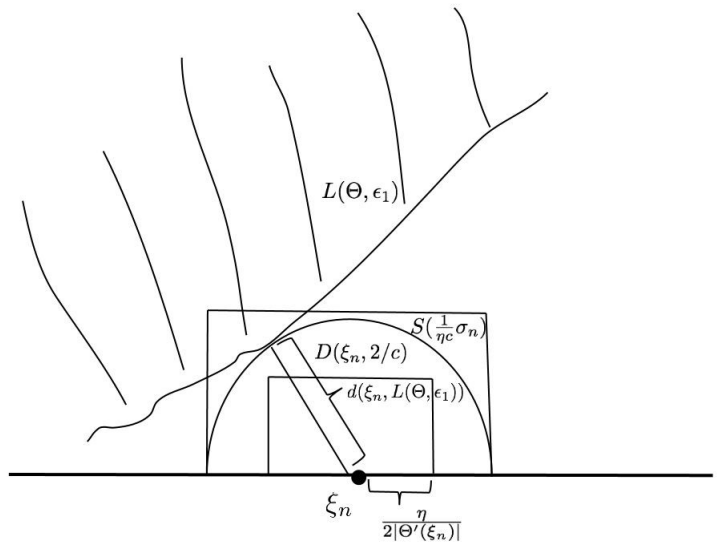

FIGURE 3

for a suitable constant c. The definition of σ_n yields that

$$m(\sigma_n) \geq \eta c d(\xi_n, L(\Theta, \varepsilon_1)),$$

or more explicitely,

$$d(\xi_n, L(\Theta, \varepsilon_1)) \leq \frac{m(\sigma_n)}{\eta c} = \frac{|\Theta'(\xi_n)|^{-1}}{c},$$

meaning that $D(\xi_n, (c|\Theta'(\xi_n)|)^{-1})$ meets $L(\Theta, \varepsilon_1)$ (see Figure 1). Since $D(\xi_n, (c|\Theta'(\xi_n)|)^{-1}) \subset S(\frac{2}{\eta c}\sigma_n)$ it is enough to pick $N = 2/(\eta c)$. So

$$S(N\sigma_n) \cap L(\Theta, \varepsilon_1) \neq \varnothing,$$

which implies, by hypothesis, that for every n

$$\mu(S(\sigma_n)) \geq c_N m(\sigma_n). \tag{6.3}$$

Let us now take $f \in (\Theta H^2)^\perp$. Since, as mentioned in the introduction, $(\Theta H^2)^\perp \cap C(\mathbb{D}^-)$ is dense in $(\Theta H^2)^\perp$ and since $(\Theta H^2)^\perp$ embeds continuously into $L^2(\mu)$, we may assume that f is continuous on $\mathbb{D}^-$. Define μ_n as the point of $S(\sigma_n)$ satisfying

$$|f(\mu_n)| = \inf_{\zeta \in \sigma_n} |f(\zeta)|.$$

Since $(k_{\xi_n}/\sqrt{|\Theta'(\xi_n)|})_{n\geq 1}$ is an orthonormal basis of $(\Theta H^2)^\perp$, we have

$$\|f\|_2^2 = \sum_{n\geq 1} \frac{|f(\xi_n)|^2}{|\Theta'(\xi_n)|} \leq 2\left(\sum_{n\geq 1} \frac{|f(\xi_n)-f(\mu_n)|^2}{|\Theta'(\xi_n)|} + \sum_{n\geq 1} \frac{|f(\mu_n)|^2}{|\Theta'(\xi_n)|}\right).$$

Using Theorem 6.1, along with (6.3), and remembering that the Carleson windows $S(\sigma_n)$ are disjoint, we get

$$\begin{aligned}
\|f\|_2^2 &\leq 2\varepsilon_0 C\|f\|_2^2 + \frac{2}{\eta}\sum_{n\geq 1} m(\sigma_n)|f(\mu_n)|^2 \\
&\leq 2\varepsilon_0 C\|f\|_2^2 + \frac{2}{\eta c_N}\sum_{n\geq 1} \mu(S(\sigma_n))|f(\mu_n)|^2 \\
&\leq 2\varepsilon_0 C\|f\|_2^2 + \frac{2}{\eta c_N}\sum_{n\geq 1} \int_{S(\sigma_n)} |f|^2 d\mu \\
&\leq 2\varepsilon_0 C\|f\|_2^2 + \frac{2}{\eta c_N}\int_{\overline{\mathbb{D}}} |f|^2\, d\mu. \qquad (6.4)
\end{aligned}$$

Finally, for sufficiently small ε_0 we obtain

$$\|f\|_2^2 \leq \frac{2}{(1-2\varepsilon_0 C)\eta c_N}\int_{\overline{\mathbb{D}}} |f|^2\, d\mu,$$

which yields the required estimate. □

Remark 6.2. Note that from the first line of (6.4) we obtain

$$\|f\|_2^2 \leq \frac{2}{(1-2\varepsilon_0 C)\eta}\int_{\Sigma} |f|^2 dm,$$

where $\Sigma = \bigcup \sigma_n$. This gives another way of constructing dominating sets (at least when the Clark measure is discrete): here we take neighborhoods of the support points of the Clark basis instead of an open neighborhood of the spectrum as in Theorem 5.6.

Remark 6.3. Here is a simple example showing that it is not sufficient that condition (2.1) of Theorem 2.1 is satisfied if N is not suitably chosen. Let us discuss this example in the upper half plane and consider the special case of $\Theta(z) = e^{2iz}$. It is known that with this Θ, the model space $(\Theta H^2)^\perp$ is isomorphic to the Paley-Wiener space PW_π^2 (the space of entire functions of exponential type at most π, whose restriction to $\mathbb{R}$ belongs to L^2). Given $\varepsilon > 0$, the sub-level set is exactly

$$L(\Theta;\varepsilon) = \left\{z \in \mathbb{C} :\ \operatorname{Im}(z) > \theta := \frac{1}{2}\ln\frac{1}{\varepsilon}\right\}.$$

In what follows we will choose $\varepsilon = e^{-6}$ so that $\theta = 3$.

Note that for the sets $(\sigma_n)_n$ appearing in the proof of Theorem 2.1, by the Kadets-Ingham theorem we can take $\sigma_n = (n-\delta, n+\delta)$, $n \in \mathbb{Z}$, where $0 < \delta < 1/4$ (see for instance [34, Theorem D4.1.2]).

Let us now consider $\Lambda = \{\lambda_n;\ n \in \mathbb{Z} \setminus \{0\}\}$ with $\lambda_n := n + \frac{1}{8}i$ and

$$\mu_\Lambda := \sum_{n\geq 1} \delta_{\lambda_n}.$$

Note that $\mathbb{Z}$ is a complete interpolating sequence for PW^2_π (see e.g. [43]) as will be $\Lambda \cup \{\lambda_0\}$. Hence, we can find $f \in PW^2_\pi$, such that $f(\lambda_n) = 0$, $n \in \mathbb{Z} \setminus \{0\}$ and $f(\lambda_0) = 1$. In particular, $\int |f|^2 \, d\mu_\Lambda = 0$ and of course $\|f\|_2 \neq 0$. So, the reverse embedding fails, while (2.1) is valid for $N = 1$. Indeed, if I is an interval such that $S(I) \cap L(\Theta, \varepsilon) \neq \varnothing$, then $m(I) \geq 3$ and since $0 \leq \lambda_{n+1} - \lambda_n \leq 2$ we have $S(I) \cap \Lambda \neq \varnothing$. It actually turns out that $\mu_\Lambda(S(I)) = \#(S(I) \cap \Lambda) \asymp m(I)$.

An appropriate choice for N here is $N \geq 6$. In particular $S(6\sigma_0)$ meets $L(\Theta, \varepsilon)$. But $S(\sigma_0)$ does not contain any point of Λ so that $\mu(S(\sigma_0)) = 0 \not\gtrsim m(\sigma_0)$ and (2.1) fails as it should.

The above example is also instructive in that it allows to observe that Theorem 2.1 does not apply when the sequence Λ is "too far" from $\mathbb{R}$. Consider for instance $\Lambda_1 = \{n+i; n \in \mathbb{Z}\setminus\{0\}\}$ and $\Lambda_2 = \{n+i; n \in \mathbb{Z}\}$. For Λ_2 we get reverse embedding while for Λ_1 we won't. Observe that for both sequences we will have $\mu(S(\sigma_n)) = 0$ for every n, so that the reverse Carleson inequality fails.

Now we give the proof of our reverse embeddings result which involves dominating sets. We will need the following lemma, for which we omit the proof.

Lemma 6.4. *Let $\Sigma \subsetneq \mathbb{T}$ and $\delta > 0$. It is possible to find a (finite) sequence $(\sigma_n)_{n\geq 1}$ of disjoint semi-open arcs of $\mathbb{T}$ such that $\Sigma \subset \bigcup_n \sigma_n$, $\Sigma \cap \sigma_n \neq \emptyset$ and $m(\sigma_n) < \delta$.*

Proof of Theorem 2.2. Since $(\Theta H^2)^\perp \cap C(\mathbb{D}^-)$ is dense in $(\Theta H^2)^\perp$ and since $(\Theta H^2)^\perp \hookrightarrow L^2(\mu)$, it suffices to show the inequality $\|f\|_2 \lesssim \|f\|_\mu$, for every function $f \in (\Theta H^2)^\perp$ continuous on $\mathbb{D}^-$. Fix $\varepsilon > 0$. Since f is uniformly continuous, it is possible to find a $\delta > 0$ such that, for every arc I of length less than δ, we have

$$|f(z) - f(\mu)| < \varepsilon, \quad \forall z, \mu \in S(I).$$

According to Lemma 6.4, we can construct a sequence of disjoint semi-open arcs of length less than δ, covering and intersecting Σ. In particular, $m(\sigma_n) \leq k\mu(S(\sigma_n))$ for some $k > 0$ independent of n. We now

introduce the points ξ_n and z_n such that

$$|f(\xi_n)| = \max\{|f(\xi)| : \xi \in \sigma_n^-\}, \quad |f(z_n)| = \min\{|f(z)| : z \in S(\sigma_n^-)\}$$

These points satisfy

$$|f(\xi_n)| \leq \varepsilon + |f(z_n)|.$$

Since Σ is dominating, there is some constant $c > 0$ such that

$$\begin{aligned}
\|f\|_2^2 &\leq c \int_\Sigma |f|^2 dm \\
&\leq c \sum_{n\geq 1} m(\sigma_n)|f(\xi_n)|^2 \\
&\leq 2c \left(\varepsilon^2 \sum_{n\geq 1} m(\sigma_n) + \sum_{n\geq 1} m(\sigma_n)|f(z_n)|^2 \right) \\
&\leq 2c \left(\varepsilon^2 + k \sum_{n\geq 1} \mu(S(\sigma_n))|f(z_n)|^2 \right) \\
&\leq 2c\varepsilon^2 + 2ck \sum_{n\geq 1} \int_{S(\sigma_n)} |f|^2 d\mu \\
&\leq 2c\varepsilon^2 + 2ck \int_{\mathbb{D}^-} |f|^2 d\mu.
\end{aligned}$$

Since ε can be arbitrarily small, we have the desired estimate $\|f\|_2 \lesssim \|f\|_\mu$. □

Remark 6.5. If Θ is inner and $\sigma(\Theta) = \mathbb{T}$, then, according to Corollary 5.10, any dominating set Σ for $(\Theta H^2)^\perp$ is dense in $\mathbb{T}$. Hence requiring the reverse Carleson inequality $\mu(S(I)) \gtrsim m(I)$ for any arc $I \subset \mathbb{T}$ meeting Σ is the same as requiring this inequality for *every* arc $I \subset \mathbb{T}$. This situation is completely described by Lefèvre et al. Hence Theorem 2.2 is interesting for inner functions Θ for which $\sigma(\Theta) \neq \mathbb{T}$. In that case Theorem 5.6 states that dominating sets always exist.

References

1. P. R. Ahern and D. N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. **92** (1970), 332–342. MR 0262511 (41 #7117)
2. A. B. Aleksandrov, *Inner functions and related spaces of pseudocontinuable functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **170** (1989), no. Issled. Linein. Oper. Teorii Funktsii. 17, 7–33, 321. MR 1039571 (91c:30063)
3. ———, *On the existence of angular boundary values of pseudocontinuable functions*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **222**

(1995), no. Issled. po Linein. Oper. i Teor. Funktsii. 23, 5–17, 307. MR 1359992 (97a:30046)

4. ———, *Isometric embeddings of co-invariant subspaces of the shift operator*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **232** (1996), no. Issled. po Linein. Oper. i Teor. Funktsii. 24, 5–15, 213. MR 1464420 (98k:30050)
5. ———, *Embedding theorems for coinvariant subspaces of the shift operator. II*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **262** (1999), no. Issled. po Linein. Oper. i Teor. Funkts. 27, 5–48, 231. MR 1734326 (2001g:46047)
6. J. Ball, *Unitary perturbations of contractions*, Ph.D. thesis, Univ. of Virginia, 1973.
7. Joseph A. Ball and Arthur Lubin, *On a class of contractive perturbations of restricted shifts*, Pacific J. Math. **63** (1976), no. 2, 309–323. MR 0417824 (54 #5872)
8. A. D. Baranov, *Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings*, J. Funct. Anal. **223** (2005), no. 1, 116–146.
9. ———, *Stability of bases and frames of reproducing kernels in model spaces*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 7, 2399–2422. MR 2207388 (2006j:30064)
10. L. Brown, A. Shields, and K. Zeller, *On absolutely convergent exponential sums*, Trans. Amer. Math. Soc. **96** (1960), 162–183. MR 0142763 (26 #332)
11. G. Chacon P., *Carleson-type inequalitites in harmonically weighted Dirichlet spaces*, Ph.D. thesis, University of Tennessee, 2010.
12. J. Cima, A. Matheson, and W. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006. MR 2215991 (2006m:30003)
13. D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. **25** (1972), 169–191.
14. B. Cohn, *Carleson measures for functions orthogonal to invariant subspaces*, Pacific J. Math. **103** (1982), no. 2, 347–364. MR 705235 (84m:30054)
15. William S. Cohn, *Carleson measures and operators on star-invariant subspaces*, J. Operator Theory **15** (1986), no. 1, 181–202.
16. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 56, Cambridge University Press, Cambridge, 1966. MR 0231999 (38 #325)
17. R. B. Crofoot, *Multipliers between invariant subspaces of the backward shift*, Pacific J. Math. **166** (1994), no. 2, 225–246. MR 1313454 (95k:47009)
18. Louis de Branges, *Hilbert spaces of entire functions*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1968. MR 0229011 (37 #4590)
19. R. G. Douglas, H. S. Shapiro, and A. L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator.*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. fasc. 1, 37–76. MR 0270196 (42 #5088)
20. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
21. E. Fricain, *Complétude des noyaux reproduisants dans les espaces modèles*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 2, 661–686. MR 1906486 (2003g:46029)

22. _____, *Bases of reproducing kernels in de Branges spaces*, J. Funct. Anal. **226** (2005), no. 2, 373–405.
23. O. Frostman, *Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Ph.D. thesis, Lund, 1935.
24. J. Garnett, *Bounded analytic functions*, first ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007. MR 2261424 (2007e:30049)
25. M. L. Gorbachuk and V. I. Gorbachuk, *M. G. Krein's lectures on entire operators*, Operator Theory: Advances and Applications, vol. 97, Birkhäuser Verlag, Basel, 1997. MR 1466698 (99f:47001)
26. D. Hitt, *Invariant subspaces of H^2 of an annulus*, Pacific J. Math. **134** (1988), no. 1, 101–120.
27. S. V. Hruščëv, N. K. Nikol′skiĭ, and B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels*, Complex analysis and spectral theory (Leningrad, 1979/1980), Lecture Notes in Math., vol. 864, Springer, Berlin, 1981, pp. 214–335. MR 643384 (84k:46019)
28. V. È. Kacnel′son, *Equivalent norms in spaces of entire functions*, Mat. Sb. (N.S.) **92(134)** (1973), 34–54, 165.
29. P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, *Some revisited results about composition operators on Hardy spaces*, to appear, Revista Mat. Iberoamericana.
30. V. N. Logvinenko and Ju. F. Sereda, *Equivalent norms in spaces of entire functions of exponential type*, Teor. Funkciĭ Funkcional. Anal. i Priložen. (1974), no. Vyp. 20, 102–111, 175.
31. D. H. Luecking, *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math. **107** (1985), no. 1, 85–111.
32. _____, *Dominating measures for spaces of analytic functions*, Illinois J. Math. **32** (1988), no. 1, 23–39.
33. J. W. Moeller, *On the spectra of some translation invariant spaces*, J. Math. Anal. Appl. **4** (1962), 276–296. MR 0150592 (27 #588)
34. N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 2*, Mathematical Surveys and Monographs, vol. 93, American Mathematical Society, Providence, RI, 2002, Model operators and systems, Translated from the French by Andreas Hartmann and revised by the author.
35. N. K. Nikol′skiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986, Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.
36. B. P. Panejah, *On some problems in harmonic analysis*, Dokl. Akad. Nauk SSSR **142** (1962), 1026–1029.
37. _____, *Certain inequalities for functions of exponential type and a priori estimates for general differential operators*, Uspehi Mat. Nauk **21** (1966), no. 3 (129), 75–114.
38. A. Poltoratski, *Boundary behavior of pseudocontinuable functions*, Algebra i Analiz **5** (1993), no. 2, 189–210. MR 94k:30090

39. A. Poltoratski and D. Sarason, *Aleksandrov-Clark measures*, Recent advances in operator-related function theory, Contemp. Math., vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 1–14.
40. D. Sarason, *Nearly invariant subspaces of the backward shift*, Contributions to operator theory and its applications (Mesa, AZ, 1987), Oper. Theory Adv. Appl., vol. 35, Birkhäuser, Basel, 1988, pp. 481–493. MR 1017680 (90m:47012)
41. _____, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, 10, John Wiley & Sons Inc., New York, 1994, A Wiley-Interscience Publication. MR 1289670 (96k:46039)
42. _____, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, 10, John Wiley & Sons Inc., New York, 1994, A Wiley-Interscience Publication.
43. Kristian Seip, *Interpolation and sampling in spaces of analytic functions*, University Lecture Series, vol. 33, American Mathematical Society, Providence, RI, 2004. MR 2040080 (2005c:30038)
44. A. L. Vol′berg, *Thin and thick families of rational fractions*, Complex analysis and spectral theory (Leningrad, 1979/1980), Lecture Notes in Math., vol. 864, Springer, Berlin, 1981, pp. 440–480. MR 643388 (83j:30038)
45. A. L. Vol′berg and S. R. Treil′, *Embedding theorems for invariant subspaces of the inverse shift operator*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **149** (1986), no. Issled. Linein. Teor. Funktsii. XV, 38–51, 186–187. MR 849293 (88f:47028)

Institut Camille Jordan, Université Claude Bernard Lyon 1, 69622 Villeurbanne Cédex, France
E-mail address: blandigneres@math.univ-lyon1.fr

Institut Camille Jordan, Université Claude Bernard Lyon 1, 69622 Villeurbanne Cédex, France
E-mail address: fricain@math.univ-lyon1.fr

Institut de Mathématiques de Bordeaux, Université Bordeaux 1, 351 cours de la Libération 33405 Talence Cédex, France
E-mail address: frederic.gaunard@math.u-bordeaux1.fr

Institut de Mathématiques de Bordeaux, Université Bordeaux 1, 351 cours de la Libération 33405 Talence Cédex, France
E-mail address: Andreas.Hartmann@math.u-bordeaux1.fr

Department of Mathematics and Computer Science, University of Richmond, Richmond, VA 23173, USA
E-mail address: wross@richmond.edu