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A UNIFIED APPROACH TO THE THEORY OF NORMED STRUCTURES - PART I: THE SINGLE-SORTED CASE

SALVATORE TRINGALI

ABSTRACT. We introduce the concept of a prenormed model of a particular kind of single-sorted finitary first-order theories, that we refer to as prealgebraic theories. These are characterized by a signature comprising, together with arbitrary function symbols, only relation symbols whose interpretation, in any possible model, is a preorder. The result is an abstract approach to the notion of norm and, consequently, to the theory of normed structures.

1. INTRODUCTION

There is no doubt that norms, along with diverse analog concepts such as valuations and seminorms, occupy a central place in mathematics, not only in relation to the notion of distance and metric spaces, but also in their own right, as for instance in the theory of Banach spaces [6], valuated rings [3] and normed groups [2]. In fact, the present article is intended as one-half of a two-part work in a series of papers devoted to norms and normed structures. Since the “many-sorted case” is an essentially technical complication of the “one-sorted case” and adds no significant insights to the theory (at least in its basic aspects), we will concentrate here on the latter and consider the former only in the second part of this work. The long-term goal, as well as our original motivation, is the development of a framework suitable to carry out computations relevant to the a priori convergence theory of approximation schemes in numerical analysis, with a special focus on reduced basis methods [4] (a standard technique used by several authors in applied mathematics to provide effective solutions of numerical problems depending on a large number of parameters). The link is the spectral theory of linear operators and Banach algebras [10], but we are not really going to dig into this in the sequel.

One of the main achievements here is, instead, the introduction of an abstract notion of norm for models of a special kind of single-sorted finitary first-order theories interpreted over a category \mathbf{C} with finite powers for which there exists a functor \mathcal{F} to the category of sets preserving those same powers. Such theories will be called subalgebraic, due to the fact that their signature includes, together with arbitrary function symbols, only relation symbols whose interpretation, in any possible model “generated” by \mathcal{F} , is a partial order (algebraic structures are a special case). The corresponding models will be referred to as \mathcal{F} -models, to stress the role of \mathcal{F} in the picture. In particular, it will be proved that all the \mathcal{F} -models of a given collection of prealgebraic theories form a category whose morphisms can be eventually understood as “norms”. This will be used, in turn, to build up another category (over a fixed \mathcal{F} -model), whose objects are ultimately an abstraction of normed spaces and whose morphisms are, in a generalized sense, “short maps” between them.

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Starting from this, we show by a number of examples how to recover down-to-earth constructions of common use in the everyday practice, for a large variety of applications ranging from functional analysis to linear algebra and number theory, such as normed groups and valuated rings (Part I), normed spaces and normed algebras (Part II), and variants, generalizations or specializations thereof including seminormed semigroups, non-Archimedean pseudo-semivaluated fields, normed modules over valuated rings, etc.

For this purpose, we will face, in the next pages, the following (intentionally vague) questions: What is abstractly a norm? And what are the fundamental features that one should retain in order to give a purely algebraic definition of norms, to the degree that normed structures can be ultimately identified with the objects of an appropriate category and norms with the arrows between these objects? To the best of our knowledge, the matter has not been considered, in such generality, by previous authors, and the answers provided in the sequel are certainly far from being exhaustive and definite, but we are confident that the subject may be worth the effort and our hope is that it can attract the interest of other researchers in the field.

Many ideas in this paper have been deeply influenced by the prominent work of R. Lowen on approach spaces [9] and F. W. Lawvere on algebraic theories [8] and generalized metric spaces [7]. From a categorical point of view, extended pseudometric spaces and extended pseudoquasimetric spaces, along with their corresponding Lipschitz maps, have the best properties that one can actually expect from a category of metric spaces: It is possible, within them, to form quotients and take arbitrary products and coproducts. Dropping the attribute “extended” implies that, in general, only finite products and coproducts will exist, while curtailing the prefix “pseudo” affects the existence of quotients. Moving from these considerations, it seems quite reasonable, in search of a “good” answer to questions concerning the “real nature” of norms, to focus first on the weaker notions of seminorm and pseudoseminorm. This leads to one more basic insight, which has been central in this research and can be roughly outlined as follows.

Loosely speaking, a homomorphism of two algebraic structures of the same type, as described in the language of model theory, is a function between the underlying sets with the property of “preserving the operations”. Then, one observes that, with a little effort of imagination, a seminorm, say, on a real vector space exhibits almost the same behaviour:

- (i) Its codomain is a special “reference structure”. In the toy case that we are considering, this is the set of non-negative real numbers, herein denoted by \mathbb{R}_0^+ , together with its standard structure of totally ordered semifield.
- (ii) It preserves the additive identity (a nullary operation). This has always been something subtle (and, hence, interesting) to our eyes: In the final analysis, one is basically requiring a seminorm to map a distinguished element a in the domain to a distinguished element b in the codomain, in a context where a and b play the same (algebraic) role, but still in a match lining up essentially different teams (both of them are identities, but in structures marked by significant differences).
- (iii) It transforms a sum (of vectors) to a sum (of scalars) and relates the one to the other by means of an inequality.
- (iv) It equals the product of a scalar by a vector to a product of two scalars, which is informally the same as saying that it preserves the products, except that the one product and the other have very little in common, at least at a first glance.

That said, the next step is to give emphasis to something absolutely obvious, i.e., that equalities and inequalities, appearing in such a fundamental way in the (classical) definition of seminorms, have in common the property of being orders. Some of them are partial, as for the equality

relation, while others are total, like in the case of the standard order on the set of real numbers, but they all are orders, i.e., reflexive, antisymmetric and transitive binary relations. And it is just by using orders and relaxing equalities to inequalities that we can manage to relate structures of different types and “let them play a good game.”

This intuition is strengthened by the inspection of other similar constructions encountered in various fields of the mathematical landscape. E.g., a group seminorm can be abstractly defined, based on common terminology and notation from model theory (cf. Remarks 2 and 4), as a function $\|\cdot\|$ from a group $(G; +, -, 0_G)$ to an ordered monoid $(M; +, 0_M; \leq_M)$ such that $\|a+b\| \leq_M \|a\| + \|b\|$ for all $a, b \in G$ and $\|0_G\| = 0_M$, and this is, indeed, called a group norm if it is symmetric (with respect to the unary operation of negation) and $\|a\| = 0_M$ for some $a \in G$ and only if $a = 0_G$. Likewise, an absolute value is defined, in the context of ring and field theory, as a function $|\cdot|$ from a domain $(D; +, \cdot, -, 0_D)$ to an ordered ring $(R; +, \cdot, -, 0_R; \leq_R)$ such that $|\cdot|$ is a group seminorm from the [Abelian] group $(D; +, -, 0_D)$ to the [Abelian] ordered monoid $(R_0^+; +, 0_R; \leq_R)$ such that $|a \cdot b| = |a| \cdot |b|$ for all $a, b \in D$, where $R_0^+ := \{a \in R : 0_R \leq_R a\}$. Thus, it is naively apparent the existence of a common pattern among these definitions, and the primary goal of the paper is, indeed, to give an explicit formal description of such a pattern.

Basic notation and terminology. We set our foundations in the Neumann-Bernays-Gödel axiomatic class theory (NBG), as presented in [11, Ch. IV]. We use \mathbb{N} for the non-negative integers and \mathbb{Z} , \mathbb{Q} and \mathbb{R} according to their standard meaning. Unless differently stated, each of these sets will be endowed with its ordinary order and operations.

If X, Y are sets, $D \subseteq X$ and $f \subseteq D \times Y$ is such that, for every $x \in D$, there exists only one $y \in Y$ such that $(x, y) \in f$, we say that f is a (total) function (or map, mapping, or similia) $D \rightarrow Y$, but also that f is a partial function from X to Y . In this case, D , X and Y are called, each in turn, the domain, the source and the target of f . In particular, we write $\text{dom}(f)$ for D and use the notation $f : X \dashrightarrow Y$ (an arrow with a vertical stroke) for a partial function f from X to Y . Note that, formally, a partial map from X to Y is an ordered triple (X, Y, f) for which f is a function $D \rightarrow Y$ for some $D \subseteq X$. Yet, we will often identify (X, Y, f) with f when it is convenient to do that and it is clear from the context which sets must be used as source and target. Lastly, if $S \subseteq X$ and g is a function $X \rightarrow Y$, then we denote by $g|_S$, as is customary, the mapping $S \rightarrow Y : x \mapsto f(x)$ and refer to $g|_S$ as the restriction of g to S .

Organization. Section 2 is an essentially preparatory section in which we recall facts from model and category theory and introduce concepts useful to adapt them to our specific needs, while fixing up, once and for all, more notation and terminology used throughout and in future work. In Section 3, we define prealgebraic [resp. subalgebraic] theories and prenorms [resp. subnorms], prove the main (and unique) results of the paper (i.e. Proposition 3.1) and, subsequently, present the category of prealgebraic [resp. subalgebraic] \mathcal{F} -models relative to a given family of prealgebraic [resp. subalgebraic] theories, for \mathcal{F} a functor from a category \mathbf{C} with finite powers to **Set**. Then, Section 4 discusses prenormed [resp. subnormed] models and Section 5 shows how these are ultimately an abstraction of familiar normed structures, such as normed groups and valuated rings, by a number of worked examples.

2. PRELIMINARIES ON MODEL AND CATEGORY THEORY

In the traditional language of model theory and first-order logic [5], a (finitary single-sorted) signature σ is a triple $(\Sigma_f, \Sigma_r, \text{ar})$, where Σ_f and Σ_r are disjoint sets (not containing basic logical symbols of the underlying formal language) and ar is a function $\Sigma_f \cup \Sigma_r \rightarrow \mathbb{N}$. The elements of

Σ_f are called function symbols, those of Σ_r relation symbols. For each symbol $\varsigma \in \Sigma_f \cup \Sigma_r$, $\text{ar}(\varsigma)$ is referred to as the arity of ς . A subsignature of σ is any signature $\sigma_0 = (\Sigma_{f,0}, \Sigma_{r,0}, \text{ar}_0)$ such that $\Sigma_{f,0} \subseteq \Sigma_f$, $\Sigma_{r,0} \subseteq \Sigma_r$ and ar_0 is the restriction of ar to $\Sigma_{f,0} \cup \Sigma_{r,0}$. In addition to this, we say that σ is algebraic if $\Sigma_r = \emptyset$ and *balanced* if there exists a bijection $\phi : \Sigma_f \rightarrow \Sigma_r$.

Remark 1. A balanced signature $\sigma = (\Sigma_f, \Sigma_r, \text{ar})$ will be systematically represented, without loss of generality, as $(\{(\varsigma_r, \varrho_r)\}_{r \in \Sigma_f}; \text{ar})$, where $\{\varsigma_r : r \in \Sigma_f\} = \Sigma_f$ and $\{\varrho_r : r \in \Sigma_f\} = \Sigma_r$.

Provided that $\sigma_i = (\Sigma_{f,i}, \Sigma_{r,i}, \text{ar}_i)$ is a signature ($i = 1, 2$), we define a signature homomorphism from σ_1 to σ_2 to be a function $\alpha : \Sigma_{f,1} \cup \Sigma_{r,1} \rightarrow \Sigma_{f,2} \cup \Sigma_{r,2}$ such that $\alpha(\Sigma_{f,1}) \subseteq \Sigma_{f,2}$, $\alpha(\Sigma_{r,1}) \subseteq \Sigma_{r,2}$ and $\text{ar}_2(\alpha(\varsigma)) = \text{ar}_1(\varsigma)$ for every $\varsigma \in \Sigma_{f,1} \cup \Sigma_{r,1}$. If so, we write that $\alpha : \sigma_1 \rightarrow \sigma_2$ is a signature homomorphism. In addition to this, for $\sigma_0 = (\Sigma_{f,0}, \Sigma_{r,0}, \text{ar}_0)$ a subsignature of σ_1 , we say that a signature homomorphism $\alpha_0 : \sigma_0 \rightarrow \sigma_2$ is the restriction of α to σ_0 if, regarded as a function, it is the restriction of α to $\Sigma_{f,0} \cup \Sigma_{r,0}$.

Remark 2. Pick $\sigma = (\Sigma_f, \Sigma_r, \text{ar})$ to be a signature. For $k, \ell \in \mathbb{N}$, suppose that Σ_f and Σ_r can be respectively partitioned into k families of function symbols $\{\varsigma_{1,r}\}_{r \in R_1}, \dots, \{\varsigma_{k,r}\}_{r \in R_k}$ and ℓ families of relation symbols $\{\varrho_{1,s}\}_{s \in S_1}, \dots, \{\varrho_{\ell,s}\}_{s \in S_\ell}$. Then, σ is possibly denoted by

$$(\{\varsigma_{1,r}\}_{r \in R_1}, \dots, \{\varsigma_{k,r}\}_{r \in R_k}; \{\varrho_{1,s}\}_{s \in S_1}, \dots, \{\varrho_{\ell,s}\}_{s \in S_\ell}; \text{ar}). \quad (1)$$

On another hand, assume that σ is balanced and let $\sigma = (\{(\varsigma_r, \varrho_r)\}_{r \in \Sigma_f}; \text{ar})$ (see Remark 1). Admit that there exists $k \in \mathbb{N}$ such that $\{(\varsigma_r, \varrho_r)\}_{r \in \Sigma_f}$ can be partitioned into k collections of the form $\{(\varsigma_{1,r}, \varrho_{1,r})\}_{r \in R_1}, \dots, \{(\varsigma_{k,r}, \varrho_{k,r})\}_{r \in R_k}$. Then, we possibly write σ as

$$(\{(\varsigma_{1,r}, \varrho_{1,r})\}_{r \in R_1}; \dots; \{(\varsigma_{k,r}, \varrho_{k,r})\}_{r \in R_k}; \text{ar}). \quad (2)$$

These notations are further simplified, in the most obvious way, whenever a family of symbols consists of one element (i.e. is a singleton), to the extent of writing, for instance, $(+, \star, 1; \leq, \sim; \text{ar})$ in place of $\sigma = (\{+, \star, 1\}, \{\leq, \sim\}, \text{ar})$ or $(-, \preceq; \star, \simeq; \text{ar})$ instead of $(\{-, \star\}, \{\preceq, \simeq\}, \text{ar})$.

We will use extensively the language of categories. Following [12], these are henceforth defined as 6-uples of type $(C_o, C_h, s, t, \text{id}, \circ)$, where C_o and C_h are classes, the former referred to as the collection of objects, the latter as the collection of morphisms (or arrows); s and t are functions $C_h \rightarrow C_o$ which assign, to every arrow, its source and target; id is a further function $C_o \rightarrow C_h$ which sends each object X to a distinguished morphism $X \rightarrow X$, called the identity on X ; \circ is a partial operation $C_h \times C_h \rightarrow C_h$; and all is accompanied by the usual axioms that s , t , id and \circ are required to satisfy (cf. [1, p. 21, Definition 3.1]). If \mathbf{C} is a category, we denote by $\text{obj}(\mathbf{C})$ the class of its objects and by $\text{hom}(\mathbf{C})$ the one of its arrows. We write $\text{src}_{\mathbf{C}}$ and $\text{trg}_{\mathbf{C}}$ for the functions mapping a morphism to its source and target, respectively, and $\text{id}_{\mathbf{C}}$ for that sending an object X to the identity on X . Subsequently, for $X, Y \in \text{obj}(\mathbf{C})$, we adopt the notation $f : X \rightarrow Y$ to mean that f is a morphism of $\text{hom}(\mathbf{C})$ with $\text{src}_{\mathbf{C}}(f) = X$ and $\text{trg}_{\mathbf{C}}(f) = Y$. Lastly, we use $\circ_{\mathbf{C}}$ for the composition law of \mathbf{C} , or simply \circ when there is no danger of ambiguity, and $\text{hom}_{\mathbf{C}}(X, Y)$ for the collection of arrows $f : X \rightarrow Y$ in \mathbf{C} , or simply $\text{hom}_{\mathbf{C}}(X)$ when $X = Y$, and we write $g \circ_{\mathbf{C}} f$ in place of $\circ_{\mathbf{C}}(f, g)$ for $(f, g) \in \text{dom}(\circ_{\mathbf{C}})$ (cf. [1, p. 21, Remarks 3.2]). As is customary, we refer to $g \circ_{\mathbf{C}} f$ as the composition of g with f . In particular, we will consider the following basic categories (cf. [1, p. 22, Example 3.3]):

- (i) **Rel**, the category having the class of sets as objects and all triples \mathfrak{R} of type (X, Y, R) as morphisms, where X and Y are sets and R is a subset of the Cartesian product $X \times Y$, i.e., a binary relation between X and Y . When this does not lead to confusion, we identify \mathfrak{R} with R . The composition of two relations $R : X \rightarrow Y_1$ and $S : Y_2 \rightarrow Z$

is defined if and only if $Y_1 = Y_2$, and then it is equal to the triple (X, Z, T) , where $T \subseteq X \times Z$ and $(x, z) \in T$ for $x \in X$ and $z \in Z$ if and only if $(x, y) \in R$ and $(y, z) \in S$ for some $y \in Y_i$. The identities and the maps of source and target are the obvious ones.

- (ii) **Set**, the subcategory of **Rel** whose morphisms are functions.
- (iii) **Pre**, the category having the class of presets as objects and all triples \mathfrak{f} of type $(\mathcal{P}, \mathcal{Q}, f)$ as morphisms, where $\mathcal{P} = (P, \leq_P)$ and $\mathcal{Q} = (Q, \leq_Q)$ are presets, i.e., ordered pairs consisting of a set and a preorder on it, and f is a function $P \rightarrow Q$ such that $f(x) \leq_Q f(y)$ if $x, y \in P$ and $x \leq_P y$ (also referred to as a monotonic function $\mathcal{P} \rightarrow \mathcal{Q}$). When there is no danger of confusion, we will use f as a shorthand of \mathfrak{f} . The composition of two morphisms $(\mathcal{P}_1, \mathcal{P}_2, f)$ and $(\mathcal{Q}_1, \mathcal{Q}_2, g)$ in **Pos** is defined if and only if $\mathcal{P}_2 = \mathcal{Q}_1$, and then it is set equal to the triple $(\mathcal{P}_1, \mathcal{Q}_2, g \circ_{\mathbf{Set}} f)$. The identities and the maps of source and target are defined in the most straightforward way.
- (iv) **Pos**, the full subcategory of **Pre** whose objects are posets, i.e., partially ordered sets.
- (v) **Sgn**, the category having the class of (one-sorted finitary) signatures as objects and all triples of the form (σ, τ, α) as morphisms, where σ and τ are (one-sorted finitary) signatures and $\alpha : \sigma \rightarrow \tau$ is a signature homomorphism. As in other cases, when there is no likelihood of ambiguity, we will use α as a shorthand of (σ, τ, α) . The composition of two morphisms $(\sigma_1, \sigma_2, \alpha)$ and (τ_1, τ_2, β) is defined if and only if $\sigma_2 = \tau_1$, and then it is given by the triple $(\sigma_1, \tau_2, \beta \circ_{\mathbf{Set}} \alpha)$. The identities and the maps of source and target are once more specified in the obvious way.

If \mathbf{C} is a category, $n \in \mathbb{N}$ and $f_i : X_i \rightarrow Y_i$ is a morphism of \mathbf{C} ($i = 1, 2, \dots, n$), we write $\text{prod}_{\mathbf{C}}(X_1, X_2, \dots, X_n)$ for the product of (X_1, X_2, \dots, X_n) in \mathbf{C} (unique up to isomorphism, if it exists), or simply $\prod_{i=1}^n X_i$ when \mathbf{C} is clear from the context, and $f_1 \times_{\mathbf{C}} f_2 \times_{\mathbf{C}} \dots \times_{\mathbf{C}} f_n$ for the universal map in the \mathbf{C} -product of $(f_1 \circ_{\mathbf{C}} \pi_1, f_2 \circ_{\mathbf{C}} \pi_2, \dots, f_n \circ_{\mathbf{C}} \pi_n)$, where π_i is the canonical projection $\text{prod}_{\mathbf{C}}(X_1, X_2, \dots, X_n) \rightarrow X_i$. In particular, we use $\text{pwr}_{\mathbf{C}}(X, n)$ instead of $\text{prod}_{\mathbf{C}}(X_1, X_2, \dots, X_n)$ if $X_1 = X_2 = \dots = X_n =: X$ (this is called the n -th power of X in \mathbf{C}) and $\text{pwr}_{\mathbf{C}}(f, n)$ in place of $f_1 \times_{\mathbf{C}} f_2 \times_{\mathbf{C}} \dots \times_{\mathbf{C}} f_n$ if $f_1 = f_2 = \dots = f_n =: f$. Indeed, we will write X^n for $\text{pwr}_{\mathbf{C}}(X, n)$ and f^n for $\text{pwr}_{\mathbf{C}}(f, n)$ if there exists no danger of ambiguity.

That said, assume henceforth that \mathbf{C} is a category with all finite powers and \mathcal{F} a functor $\mathbf{C} \rightarrow \mathbf{Set}$ preserving these powers. In our understanding, a (finitary single-sorted) \mathcal{F} -structure is then any 4-uple $\mathfrak{A} = (A, \chi, \sigma, I)$ consisting of

- (i) an object A of \mathbf{C} , referred to as the carrier of the structure and denoted by $|\mathfrak{A}|$;
- (ii) a choice function $\chi : \mathbb{N} \rightarrow \text{obj}(\mathbf{C})$ sending each $n \in \mathbb{N}$ to a distinguished element $\chi(n)$ in the isomorphism class of A^n , where A^0 is, in particular, the terminal object of \mathbf{C} ;
- (iii) a signature $\sigma = (\Sigma_f, \Sigma_r, \text{ar})$;
- (iv) an interpretation function $I : \Sigma_f \cup \Sigma_r \rightarrow \text{obj}(\mathbf{C}) \cup \text{obj}(\mathbf{Set})$.

Given $n \in \mathbb{N}$, the interpretation of an n -ary (function) symbol $f \in \Sigma_f$ in \mathfrak{A} is a morphism $I(f) \in \text{hom}_{\mathbf{C}}(\chi(n), A)$, and the interpretation of an n -ary (relation) symbol $R \in \Sigma_r$ is a relation $I(R) \subseteq \mathcal{F}(\chi(n))$. As is customary, a nullary function symbol ζ is called a constant symbol, because its interpretation $I(\zeta)$ can be identified with a distinguished element of $\mathcal{F}(A)$.

Remark 3. In the sequel, dealing with a (finitary single-sorted) \mathcal{F} -structure $\mathfrak{A} = (A, \chi, \sigma, I)$, we will systematically forget about χ and write simply A^n in place of $\chi(n)$, with the implicit understanding that this is only a shorthand for a chosen representative of the isomorphism class of A^n in \mathbf{C} . As a result, we will use a 3-uple, instead of a 4-uple, to describe an \mathcal{F} -structure and omit any further explicit reference to any choice function. On the other hand, in the cases

explicitly considered below to work out the basics of the abstract theory of normed structures, we will restrict ourselves to \mathcal{F} -structures of type (A, σ, I) , where each relation symbol ϱ of σ , if any is present, is a binary symbol whose interpretation is a preorder [resp. a partial order] on $\mathcal{F}(A)$, in such a way that $(\mathcal{F}(A), I(\varrho))$ is an object of **Pre** [resp. **Pos**]. When this occurs, \mathfrak{A} will be referred to as a prealgebraic [resp. subalgebraic] \mathcal{F} -structure, and indeed as an algebraic \mathcal{F} -structure if $I(\varrho)$ is the equality relation on $\mathcal{F}(A)$ for every $\varrho \in \Sigma_r$. Algebraic $\mathbf{1}_{\mathbf{Set}}$ -structures are precisely the (finitary single-sorted) structures traditionally studied by universal algebra (here and later, $\mathbf{1}_{\mathbf{Set}}$ denotes the identity functor $\mathbf{Set} \rightarrow \mathbf{Set}$).

Remark 4. As far as there is no likelihood of confusion, given an \mathcal{F} -structure $\mathfrak{A} = (A, \sigma, I)$, with $\sigma = (\Sigma_f, \Sigma_r, \text{ar})$, we do not make any notational distinction between a symbol ς of σ and its interpretation under I , i.e., we use ς for $I(\varsigma)$. This will be especially the case when $\mathcal{F} = \mathbf{1}_{\mathbf{Set}}$ and ς is the function symbol of a binary operation, such as $+$, \cdot or \star , or the relation symbol of a preorder, such as \leq or \preceq : In these circumstances, for $a, b \in \mathcal{A}$, we will write, e.g., $a + b$ in place of $\mathcal{I}(+)(a, b)$ and $a \leq b$ instead of $(a, b) \in I(\leq)$, without additional explanation.

Upon these premises, assume that, for some $k, \ell \in \mathbb{N}$, it is possible to partition Σ_f and Σ_r , each in turn, into k families of function symbols $\{\varsigma_{1,r}\}_{r \in R_1}, \dots, \{\varsigma_{k,r}\}_{r \in R_k}$ and ℓ families of relation symbols $\{\varrho_{1,s}\}_{s \in S_1}, \dots, \{\varrho_{\ell,s}\}_{s \in S_\ell}$ (see Remark 2). In this case, \mathfrak{A} is possibly represented by

$$(A; \{\varsigma_{1,r}\}_{r \in R_1}, \dots, \{\varsigma_{k,r}\}_{r \in R_k}; \{\varrho_{1,s}\}_{s \in S_1}, \dots, \{\varrho_{\ell,s}\}_{s \in S_\ell}). \quad (3)$$

On another hand, admit that σ is balanced and let $\sigma = (\{\varsigma_r, \varrho_r\}_{r \in \Sigma_f}; \text{ar})$ (see Remark 1). Suppose that there exists $k \in \mathbb{N}$ such that $\{\varsigma_r, \varrho_r\}_{r \in \Sigma_f}$ can be partitioned into k collections of the form $\{\varsigma_{1,r}, \varrho_{1,r}\}_{r \in R_1}, \dots, \{\varsigma_{k,r}, \varrho_{k,r}\}_{r \in R_k}$. Then, we possibly denote \mathfrak{A} by

$$(A; \{\varsigma_{1,r}, \varrho_{1,r}\}_{r \in R_1}; \dots; \{\varsigma_{k,r}, \varrho_{k,r}\}_{r \in R_k}). \quad (4)$$

Moreover, these notations are further simplified, in the most obvious way, if a family of symbols is a singleton, to the degree of writing, e.g., $(A; +, \star, 1; \leq, \sim)$ in place of (A, σ, I) provided $\sigma = (\{+, \star, 1\}, \{\leq, \sim\}, \text{ar})$ or $(A; -, \preceq; \star, \simeq)$ instead of (A, σ, I) for $\sigma = (\{-, \star\}, \{\preceq, \simeq\}, \text{ar})$.

3. PREALGEBRAIC THEORIES AND PRENORMS

Assume henceforth that \mathbf{C} is a category with finite powers and \mathcal{F} a functor $\mathbf{C} \rightarrow \mathbf{Set}$. Let σ be a signature and V an infinite set of variables. We denote by $\langle V; \sigma \rangle$ the collection of all well-formed formulas in the variables V generated by σ . An \mathcal{F} -structure $\mathfrak{A} = (A, \sigma, I)$ is said to satisfy a formula $\phi \in \langle V; \sigma \rangle$ of n arguments if the interpretation $\phi(\mathfrak{A})$ of ϕ in \mathfrak{A} is a true statement, where $\phi(\mathfrak{A})$ is obtained by replacing each variables with an element of $\mathcal{F}(A)$, each function symbol $\varsigma \in \Sigma_f$ with $\mathcal{F}(I(\varsigma))$, and each relation symbol $\varrho \in \Sigma_r$ with $I(\varrho)$. In that case, one writes $\mathcal{F}(\mathfrak{A}) \models \phi$. A σ -theory, or a theory of type σ , in the variables V is then any triple $T = (V, \sigma, \Xi)$ such that Ξ is a (possibly empty) subset of $\langle V; \sigma \rangle$, while a model \mathfrak{A} of T under \mathcal{F} , or equivalently an \mathcal{F} -model of T , is an \mathcal{F} -structure (A, σ, I) that satisfies every axiom $\phi \in \Xi$. Such a condition is equivalently expressed by writing $\mathcal{F}(\mathfrak{A}) \models T$ and saying that \mathfrak{A} satisfies T under \mathcal{F} : in this respect, σ will be also referred to as the signature of (T, \mathfrak{A}) . If $T = (V, \sigma, \Xi)$ is a σ -theory, a subtheory of T is any theory $T_S = (V, \sigma_S, \Xi_S)$ such that σ_S is a subsignature of σ and $\Xi_S = \Xi \cap \langle V; \sigma_S \rangle$, while an \mathcal{F} -submodel of T is an \mathcal{F} -model of a subtheory of T . If T_S is a subtheory of T , we can as well say that T is an extension, or a supertheory, of T_S .

Remark 5. Let σ be a signature including two binary symbols \vee and \wedge , a unary symbol u and a nullary symbol e . Then, consider the following well-formed formulas from $\langle V; \sigma \rangle$:

- (A.1) $\forall x, y, z \in V : (x \vee y) \vee z = x \vee (y \vee z)$.
(A.2) $\forall x \in V : x \vee e = e \vee x = x$.
(A.3) $\forall x \in V : x \vee u(x) = u(x) \vee x = e$.
(A.4) $\forall x, y, z \in V : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
(A.5) $\forall x, y, z \in V : (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

We refer, as is usual, to (A.1) as the axiom of associativity for the symbol \vee ; to (A.2) as the axiom of neutrality for the pair (\vee, e) ; to (A.3) as the axiom of inverses for the triple (\vee, u, e) ; to (A.4) and (A.5), respectively, as the axioms of left and right distributiveness of \wedge over \vee . We list them here for future reference: They will be used later to deal with examples in Section 5.

Definition 1. We say that a theory $T = (V, \sigma, \Xi)$ is *prealgebraic* if σ is a balanced signature of type $(\Sigma_f, \Sigma_r, \text{ar})$, where all relation symbols are binary and, for each $\varrho \in \Sigma_r$, the axioms of T include at least the axiom of reflexivity: $\forall x \in V : (x, x) \in \varrho$, and the axiom of transitivity:

$$\forall x, y, z \in V : (((x, y) \in \varrho) \text{ AND } ((y, z) \in \varrho)) \implies ((x, z) \in \varrho).$$

In addition to this, T will be called *subalgebraic* if it is prealgebraic and, for every $\varrho \in \Sigma_r$, Ξ contains also the axiom of antisymmetry:

$$\forall x, y \in V : (((x, y) \in \varrho) \text{ AND } ((y, x) \in \varrho)) \implies (x = y).$$

Lastly, we convey that T is algebraic if $\Sigma_r = \emptyset$ and algebraic under \mathcal{F} if, for each \mathcal{F} -model $\mathfrak{A} = (A, \sigma, I)$ of T , there exists one more \mathcal{F} -model $\mathfrak{A}_a = (A, \sigma, I_a)$ of T such that $I_a(\varsigma) = I(\varsigma)$ for each $\varsigma \in \Sigma_f$ and $I_a(\varrho)$ is the equality relation on $\mathcal{F}(A)$ for each $\varrho \in \Sigma_r$, so that \mathfrak{A}_a is an algebraic \mathcal{F} -structure (see Remark 3), referred to as an algebraization of \mathfrak{A} .

Definition 2. An \mathcal{F} -model $\mathfrak{A} = (A, \sigma, I)$ of a theory $T = (V, \sigma, \Xi)$, with $\sigma = (\Sigma_f, \Sigma_r, \text{ar})$, will be said *pivotal* if the symbols of Σ_r have all the same arity and there exists $\varrho_0 \in \Sigma_r$ such that $I(\varrho) \subseteq I(\varrho_0)$ for every $\varrho \in \Sigma_r$, in which case we refer to $I(\varrho_0)$ as the pivot of \mathfrak{A} .

These definitions stem from the fact that, if T is prealgebraic [resp. subalgebraic], any relation symbol of its signature will be interpreted, in any possible \mathcal{F} -model of T , as a preorder [resp. a partial order]. In particular, pivotal models will be crucial, later in this section, for the definition of the categories of prenormed and subnormed structures over a fixed “target” (see Section 4).

Remark 6. A subalgebraic theory is always a prealgebraic theory. More interestingly, there is a canonical way to identify an \mathcal{F} -model $\mathfrak{A} = (A, \sigma, I)$ of an algebraic theory $T = (V, \sigma, \Xi)$ with an \mathcal{F} -model of a subalgebraic theory (in the same variables), so that any algebraic theory can be definitely identified with an algebraic theory under \mathcal{F} . To see how, let $\sigma = (\Sigma_f, \emptyset, \text{ar})$. For each $\varsigma \in \Sigma_f$, consider a binary relation symbol $\varrho_\varsigma \notin \Sigma_f$, not already comprised among the basic symbols of the underlying logic, and set $\Sigma_r := \{\varrho_\varsigma\}_{\varsigma \in \Sigma_f}$. Then, extend ar to the function $\text{ar}_e : \Sigma_f \cup \Sigma_r \rightarrow \mathbb{N}$ by taking $\text{ar}_e(\varrho) := 2$ for every $\varrho \in \Sigma_r$ and define $\sigma_e := (\Sigma_f, \Sigma_r, \text{ar}_e)$. Lastly, expand Ξ to a larger set of axioms, namely Ξ_e , in such a way as to include all and only the axioms of reflexivity, symmetry and transitivity relative to every relation symbol $\varrho \in \Sigma_r$. Then, $T_e = (V, \sigma_e, \Xi_e)$ is a subalgebraic (and hence prealgebraic) theory and \mathfrak{A} can be identified with the model $\mathfrak{A}_e = (A, \sigma_e, I_e)$ of T_e defined by assuming that $I_e = I$ on Σ_f and $I_e(\varrho)$ is the equality relation on A for each $\varrho \in \Sigma_r$. On another hand, say that T is an *arbitrary* theory. Then, there is always a smallest prealgebraic [resp. subalgebraic] theory in the same variables and with the same signature as T , where “smallest” means “with the fewest possible axioms”. This will be denoted by $\sharp_p T$ [resp. $\sharp_s T$] and called the prealgebraic [resp. subalgebraic] embodiment of T .

Remark 7. When the set of variables V is well understood from the context, we will use simply (σ, Ξ) in place of (V, σ, Ξ) to indicate a theory in the variables V .

If P is a set and $\mathcal{Q} = (Q, \leq_Q)$ a preset, we agree to denote by $\wp(P, \mathcal{Q})$ the preorder induced on $\text{hom}_{\mathbf{Set}}(P, Q)$ by \leq_Q as follows: if f and g are functions $P \rightarrow Q$, we let $(f, g) \in \wp(P, \mathcal{Q})$ if and only if $f(x) \leq_Q g(x)$ for all $x \in P$. Observe that, up to an isomorphism, $\wp(P, \mathcal{Q})$ is, indeed, the preorder of the exponential object \mathcal{Q}^P of $(\mathcal{P}, \mathcal{Q})$ in \mathbf{Pre} when \mathcal{P} is the trivial poset on P . Note also that $\wp(P, \mathcal{Q})$ is, in fact, a partial order when \mathcal{Q} is a poset.

Lemma 3.1. *Given that P is a set and $\mathcal{Q} = (Q, \leq_Q)$ and $\mathcal{R} = (R, \leq_R)$ are presets, let $f, f_1, f_2 \in \text{hom}_{\mathbf{Set}}(P, Q)$ and $g, g_1, g_2 \in \text{hom}_{\mathbf{Set}}(Q, R)$. Then the following holds:*

- (i) *If $(g_1, g_2) \in \wp(Q, \mathcal{R})$, then $(g_1 \circ_{\mathbf{Set}} f, g_2 \circ_{\mathbf{Set}} f) \in \wp(P, \mathcal{R})$.*
- (ii) *If $(f_1, f_2) \in \wp(P, \mathcal{Q})$ and $g \in \text{hom}_{\mathbf{Pre}}(\mathcal{Q}, \mathcal{R})$, then $(g \circ_{\mathbf{Set}} f_1, g \circ_{\mathbf{Set}} f_2) \in \wp(P, \mathcal{R})$.*

Proof. The first claim is obvious. As for the second, pick $x \in P$. Since $(f_1, f_2) \in \wp(P, \mathcal{Q})$, it is $f_1(x) \leq_Q f_2(x)$, from which $g(f_1(x)) \leq_R g(f_2(x))$, for g is a monotonic function $\mathcal{Q} \rightarrow \mathcal{R}$. By the arbitrariness of $x \in P$, this completes the proof. \blacksquare

Now, in what follows, let $T_i = (V, \sigma_i, \Xi_i)$ be a prealgebraic [resp. subalgebraic] theory and $\mathfrak{A}_i = (A_i, \sigma_i, I_i)$ an \mathcal{F} -model of T_i ($i = 1, 2$), with $\sigma_i = (\{\varsigma_{r,i}, \varrho_{r,i}\}_{r \in R_i}; \text{ar}_i)$. It is then possible to regard a signature homomorphism $\alpha : \sigma_1 \rightarrow \sigma_2$ as a pair (α_1, α_2) of maps $\alpha_1, \alpha_2 : R_1 \rightarrow R_2$ by imposing that $\varsigma_{\alpha_1(r),2} = \alpha(\varsigma_{r,1})$ and $\varrho_{\alpha_2(r),2} = \alpha(\varrho_{r,1})$ for each $r \in R_1$. In similar circumstances, we will systematically abuse notation and identify α with its ‘‘components’’ α_1 and α_2 , to the extent of writing $\varsigma_{\alpha(r),2}$ in place of $\varsigma_{\alpha_1(r),2}$ and $\varrho_{\alpha(s),2}$ for $\varrho_{\alpha_2(s),2}$.

Definition 3. We say that \mathfrak{A}_1 is prehomomorphic [resp. subhomomorphic] to \mathfrak{A}_2 if there exist $\alpha \in \text{hom}_{\mathbf{Sgn}}(\sigma_1, \sigma_2)$ and $\varphi \in \text{hom}_{\mathbf{C}}(A_1, A_2)$ such that, for each $r \in R_1$,

- (i) $(\mathcal{F}(\varphi \circ_{\mathbf{C}} \varsigma_{r,1}), \mathcal{F}(\varsigma_{\alpha(r),2} \circ_{\mathbf{C}} \varphi^n)) \in \wp(\mathcal{F}(A_1^n), \mathcal{A}_{\alpha(r),2})$;
- (ii) $\mathcal{F}(\varphi) \in \text{hom}_{\mathbf{Pre}}(\mathcal{A}_{r,1}, \mathcal{A}_{\alpha(r),2})$, i.e., $\mathcal{F}(\varphi)$ is a monotonic function $\mathcal{A}_{r,1} \rightarrow \mathcal{A}_{\alpha(r),2}$,

where $n := \text{ar}_1(\varsigma_{r,1})$ and $\mathcal{A}_{s,i} := (\mathcal{F}(A_i), \varrho_{s,i})$ for every $s \in R_i$ (take in mind Remark 4). In that case, we refer to $\Phi := (\alpha, \varphi)$ as an \mathcal{F} -prenorm [resp. -subnorm] or write that $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ is a prenorm [resp. subnorm] of \mathcal{F} -models. In particular, an \mathcal{F} -prenorm $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ will be said an \mathcal{F} -homomorphism if $I_2(\varrho_{\alpha(r),2})$ is the equality on $\mathcal{F}(A_2)$ for each $r \in R_1$.

Remark 8. Definition 3 returns exactly the standard notion of a homomorphism of algebraic structures, as given in the framework of universal algebra, in the special case when T_i is an algebraic theory under $1_{\mathbf{Set}}$ (and thus $\mathbf{C} = \mathbf{Set}$), $1_{\mathbf{Set}}(\mathfrak{A}_i) \models T_i$, and $T_1 = T_2$.

Remark 9. Except for the axioms required to turn T_1 and T_2 into prealgebraic [resp. subalgebraic] theories, the definition of a prenorm [resp. subnorm] $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ does not depend at all on the axioms that \mathfrak{A}_1 and \mathfrak{A}_2 are required to satisfy as models of T_1 and T_2 , respectively. This ultimately means that other axioms, if any is present, do not play an active role in the foundations of the abstract theory so far developed. Rather, they (can) contribute to determining ‘‘extrinsic’’ properties of prenorms (and, later, prenormed structures), i.e., properties complementary to the inherent ones stemming directly from their very definition.

Remark 10. Clearly enough, for an \mathcal{F} -prenorm $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ to exist, it is necessary that any n -ary function symbol of σ_1 has a corresponding n -ary function symbol in σ_2 , though it is not necessary that σ_1 is smaller than σ_2 (in the sense that the former contains less symbols than the latter). Also, if $R_1 = \emptyset$, the pair (α, φ) is an \mathcal{F} -prenorm $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ if and only if $\alpha = (\sigma_1, \sigma_2, \emptyset)$ and $\varphi \in \text{hom}_{\mathbf{C}}(A_1, A_2)$ is monotonic in the sense of condition (ii) of Definition 3. Lastly, for \mathfrak{A}_1 an algebraic \mathcal{F} -model of T_1 , (α, φ) is an \mathcal{F} -prenorm $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ if and only if it satisfies condition (i) in the aforementioned definition (since the other, in this case, is automatically fulfilled).

Remark 11. Suppose that (α, φ) is an \mathcal{F} -prenorm $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ and pick an index $r \in R_1$, if any exists, such that $\varsigma_{r,1}$ is (interpreted as) a nullary operation of \mathfrak{A}_1 . It then follows from Definition 3 that $(\mathcal{F}(\varphi)(c_{r,1}), c_{\alpha(r),2}) \in \varrho_{\alpha(r),2}$, where $c_{r,1} := \mathcal{F}(\varsigma_{r,1})(\mathcal{F}(A_1^0))$ is a distinguished element of $\mathcal{F}(A_1)$ and $c_{r,2} := \mathcal{F}(\varsigma_{\alpha(r),2})(\mathcal{F}(A_2^0))$ a distinguished element of $\mathcal{F}(A_2)$. In particular, if $\varrho_{\alpha(r),2}$ is antisymmetric, thus a partial order on $\mathcal{F}(A_2)$, and $c_{\alpha(r),2}$ is the least element of $(\mathcal{F}(A_2), \varrho_{\alpha(r),2})$, in the sense that $(c_{\alpha(r),2}, a) \in \varrho_{\alpha(r),2}$ for every $a \in \mathcal{F}(A_2)$, this implies $\mathcal{F}(\varphi)(c_{r,1}) = c_{\alpha(r),2}$. On account of the worked examples examined in Section 5, such a result represents a minor but attractive byproduct of the framework set up in this work. In the ultimate analysis, it shows that there is no need to *assume*, say, that a group norm or a ring seminorm, as defined in the traditional setting by taking them to be valued in \mathbb{R}_0^+ (cf. Section 5, Examples E.3 and E.5), preserve the additive identities. Actually, this is a *consequence* of the inherent properties of subnorms, which can help to better understand the intimate nature of the concept itself of norm.

Definition 4. Assume that \mathfrak{A}_2 is pivotal and denote its pivot by \leq . Take ς_2 to be a nullary function symbol in σ_2 (if any exists) and $\Phi = (\alpha, \varphi)$ an \mathcal{F} -prenorm $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$. We say that Φ is

- (i) upward [resp. downward] semidefinite with respect to ς_2 if $\mathcal{F}(\varsigma_2 \circ_{\mathbf{C}} \varphi^0)(a) \leq \mathcal{F}(\varphi)(a)$ [resp. $\mathcal{F}(\varphi)(a) \leq \mathcal{F}(\varsigma_2 \circ_{\mathbf{C}} \varphi^0)$] for every $a \in \mathcal{F}(A_1)$;
- (ii) upward [resp. downward] definite (with respect to ς_2) if it is upward [resp. downward] semidefinite and $\mathcal{F}(\varsigma_2 \circ_{\mathbf{C}} \varphi^0)(a) \neq \mathcal{F}(\varphi)(a)$ for all $a \in \mathcal{F}(A_1) \setminus \bigcup_{\varsigma_1 \in \alpha^{-1}(\varsigma_2)} \mathcal{F}(\varphi \circ_{\mathbf{C}} \varsigma_1)(A_1^0)$, which will be equivalently expressed by writing that $\mathcal{F}(\varsigma_2 \circ_{\mathbf{C}} \varphi^0)(a) \lesssim \mathcal{F}(\varphi)(a)$ [resp. $\mathcal{F}(\varphi)(a) \lesssim \mathcal{F}(\varsigma_2 \circ_{\mathbf{C}} \varphi^0)$] for every $a \in \mathcal{F}(A_1) \setminus \bigcup_{\varsigma_1 \in \alpha^{-1}(\varsigma_2)} \mathcal{F}(\varphi \circ_{\mathbf{C}} \varsigma_1)(A_1^0)$.
- (iii) indefinite (with respect to ς_2) if it is neither upward nor downward semidefinite.

Upward (semi)definiteness abstracts and generalizes one of the most basic properties of standard norms, to wit, positive (semi)definiteness. More than this, Definition 4 suggests that, at least in principle, (semi)definiteness of norms has nothing really special to do with the additive identities in group-like, ring-like or module-like structures, as one might naively conclude from the classical perspective. Rather, it is an issue related to constants, all constants: Which one of these is more significant than the others strongly depends on the case at hand.

Proposition 3.1. *Let $T_i = (V, \sigma_i, \Xi_i)$ be a prealgebraic [resp. subalgebraic] theory and $\mathfrak{A}_i = (A_i, \sigma_i, I_i)$ an \mathcal{F} -model of T_i ($i = 1, 2, 3$). Suppose $\Phi = (\alpha, \varphi)$ is an \mathcal{F} -prenorm [resp. -subnorm] $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ and $\Psi = (\beta, \psi)$ a prenorm [resp. subnorm] $\mathfrak{A}_2 \rightarrow \mathfrak{A}_3$. Then, $\Theta = (\gamma, \vartheta)$, where $\gamma := \beta \circ_{\mathbf{Sgn}} \alpha$ and $\vartheta := \psi \circ_{\mathbf{C}} \varphi$, is an \mathcal{F} -prenorm [resp. -subnorm] $\mathfrak{A}_1 \rightarrow \mathfrak{A}_3$.*

Proof. Assume $\sigma_i = (\{\varsigma_{r,i}, \varrho_{r,i}\}_{r \in R_i}; \text{ar}_i)$ and, for every $r \in R_i$, take $\mathcal{A}_{r,i}$ to be the preset [resp. poset] $(\mathcal{F}(A_i), \varrho_{r,i})$. Since **Sgn** and **Pre** [resp. **Pos**] are categories, γ is obviously a signature homomorphism $\sigma_1 \rightarrow \sigma_2$ and $\mathcal{F}(\vartheta)$, i.e., $\mathcal{F}(\psi) \circ_{\mathbf{Set}} \mathcal{F}(\varphi)$, a monotonic function $\mathcal{A}_{r,1} \rightarrow \mathcal{A}_{\gamma(r),3}$ for every $r \in R_1$. Thus, it is left to prove that $(\mathcal{F}(\vartheta \circ_{\mathbf{C}} \varsigma_{r,1}), \mathcal{F}(\varsigma_{\gamma(r),3} \circ_{\mathbf{C}} \vartheta^n)) \in \wp(\mathcal{F}(A_1^n), \mathcal{A}_{\gamma(r),3})$ for each n -ary function symbol $\varsigma_{r,1} \in \sigma_1$. To this end, pick $r \in R_1$ and set $n := \text{ar}_1(\varsigma_{r,1})$. Since, by hypothesis, $(\mathcal{F}(\varphi \circ_{\mathbf{C}} \varsigma_{r,1}), \mathcal{F}(\varsigma_{\alpha(r),2} \circ_{\mathbf{C}} \varphi^n)) \in \wp(\mathcal{F}(A_1^n), \mathcal{A}_{\alpha(r),2})$ and $\mathcal{F}(\psi)$ is a monotonic function $\mathcal{A}_{\alpha(r),2} \rightarrow \mathcal{A}_{\gamma(r),3}$, it follows from the second point of Lemma 3.1 that

$$(\mathcal{F}(\psi) \circ_{\mathbf{Set}} \mathcal{F}(\varphi \circ_{\mathbf{C}} \varsigma_{r,1}), \mathcal{F}(\psi) \circ_{\mathbf{Set}} \mathcal{F}(\varsigma_{\alpha(r),2} \circ_{\mathbf{C}} \varphi^n)) \in \wp(\mathcal{F}(A_1^n), \mathcal{A}_{\gamma(r),3}). \quad (5)$$

By the functoriality of \mathcal{F} (and the associativity of $\circ_{\mathbf{C}}$), this in turn is equivalent to

$$(\mathcal{F}(\vartheta \circ_{\mathbf{C}} \varsigma_{r,1}), \mathcal{F}(\psi \circ_{\mathbf{C}} \varsigma_{\alpha(r),2}) \circ_{\mathbf{Set}} \mathcal{F}(\varphi^n)) \in \wp(\mathcal{F}(A_1^n), \mathcal{A}_{\gamma(r),3}). \quad (6)$$

On the other hand, again by hypothesis, $(\mathcal{F}(\psi \circ_{\mathbf{C}} \varsigma_{\alpha(r),2}), \mathcal{F}(\varsigma_{\gamma(r),3} \circ_{\mathbf{C}} \psi^n)) \in \wp(\mathcal{F}(A_2^n), \mathcal{A}_{\gamma(r),3})$. Also, $\vartheta^n = \psi^n \circ_{\mathbf{C}} \varphi^n$. Hence, the first point of Lemma 3.1 implies that

$$(\mathcal{F}(\psi \circ_{\mathbf{C}} \varsigma_{\alpha(r),2}) \circ_{\mathbf{Set}} \mathcal{F}(\varphi^n), \mathcal{F}(\varsigma_{\gamma(r),3} \circ_{\mathbf{C}} \psi^n) \circ_{\mathbf{Set}} \mathcal{F}(\varphi^n)) \in \wp(\mathcal{F}(A_1^n), \mathcal{A}_{\gamma(r),3}). \quad (7)$$

Using once more the functoriality of \mathcal{F} (and the associativity of $\circ_{\mathbf{C}}$), along with the fact that $\psi^n \circ_{\mathbf{C}} \varphi^n = (\psi \circ_{\mathbf{C}} \varphi)^n = \vartheta^n$, this equation can be rearranged in the form:

$$(\mathcal{F}(\psi \circ_{\mathbf{C}} \varsigma_{\alpha(r),2}) \circ_{\mathbf{Set}} \mathcal{F}(\varphi^n), \mathcal{F}(\varsigma_{\gamma(r),3} \circ_{\mathbf{C}} \vartheta^n)) \in \wp(\mathcal{F}(A_1^n), \mathcal{A}_{\gamma(r),3}). \quad (8)$$

In the light of Equation (6), it then follows that

$$(\mathcal{F}(\vartheta \circ_{\mathbf{C}} \varsigma_{r,1}), \mathcal{F}(\varsigma_{\gamma(r),3} \circ_{\mathbf{C}} \vartheta^n)) \in \wp(\mathcal{F}(A_1^n), \mathcal{A}_{\gamma(r),3}), \quad (9)$$

since $\wp(\mathcal{F}(A_1^n), \mathcal{A}_{\gamma(r),3})$ is a preorder on $\text{hom}_{\mathbf{Set}}(\mathcal{F}(A_1^n), \mathcal{F}(A_3))$. And this ultimately proves, by the arbitrariness of $r \in R_1$, that (γ, ϑ) is an \mathcal{F} -prenorm [resp. -subnorm] $\mathfrak{A}_1 \rightarrow \mathfrak{A}_3$. \blacksquare

With this in hand, take J to be a set and $\mathfrak{T} = \{(\sigma_j, \Xi_j)\}_{j \in J}$ a collection of prealgebraic [resp. subalgebraic] theories in the variables V . We define C_o , on the one hand, as the class of all pairs (T, \mathfrak{A}) such that $T \in \mathfrak{T}$ and $\mathcal{F}(\mathfrak{A}) \models T$, and C_h , on the other, as the class of all triples $(\mathcal{M}_1, \mathcal{M}_2, \Phi)$ such that $\mathcal{M}_i = (T_i, \mathfrak{A}_i)$, with $T_i \in \mathfrak{T}$ and $\mathcal{F}(\mathfrak{A}_i) \models T_i$, and Φ is a prenorm [resp. subnorm] of \mathcal{F} -models $\mathfrak{A}_1 \rightarrow \mathfrak{A}_2$. We let s and t be, each in turn, the maps $C_h \rightarrow C_o : (\mathcal{M}_1, \mathcal{M}_2, \Phi) \mapsto \mathcal{M}_1$ and $C_h \rightarrow C_o : (\mathcal{M}_1, \mathcal{M}_2, \Phi) \mapsto \mathcal{M}_2$, while denoting by id the mapping $C_o \rightarrow C_h$ that sends a pair $\mathcal{M} = (T, \mathfrak{A})$ of C_o , with $T = (\sigma, \Xi)$ and $\mathfrak{A} = (A, \sigma, I)$, to the triple $(\mathcal{M}, \mathcal{M}, \varepsilon)$ of C_h with $\varepsilon := (\text{id}_{\mathbf{Sgn}}(\sigma), \text{id}_{\mathbf{C}}(A))$. Last but not least, we specify a partial function $\circ : C_h \times C_h \rightarrow C_h$ as follows: Pick $\mathbf{m} = (\mathcal{M}_1, \mathcal{M}_2, \Phi)$ and $\mathbf{n} = (\mathcal{N}_1, \mathcal{N}_2, \Psi)$ in C_h . If $\mathcal{M}_2 \neq \mathcal{N}_1$, then $\circ(\mathbf{m}, \mathbf{n})$ is not defined. Otherwise, in the light of Proposition 3.1, assume $\Phi = (\alpha, \varphi)$ and $\Psi = (\beta, \psi)$ and set $\circ(\mathbf{m}, \mathbf{n}) := (\mathcal{M}_1, \mathcal{N}_2, \Theta)$, where $\Theta := (\beta \circ_{\mathbf{Sgn}} \alpha, \psi \circ_{\mathbf{C}} \varphi)$.

It is then routine to check that $(C_o, C_h, s, t, \text{id}, \circ)$ is a category. We call it the category of prealgebraic [resp. subalgebraic] \mathcal{F} -models of \mathfrak{T} . It will be denoted, in general, by $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$ [resp. $\mathbf{Snr}_{\mathcal{F}}(\mathfrak{T})$], and especially written as $\mathbf{Pnr}_{\mathcal{F}}(T)$ [resp. $\mathbf{Snr}_{\mathcal{F}}(T)$] in the case where \mathfrak{T} consists of a unique prealgebraic [resp. subalgebraic] theory T (in the variables V). In the latter occurrence, whenever T is implied by the context, we use \mathfrak{A} in place of (T, \mathfrak{A}) to mean an object of $\mathbf{Pnr}_{\mathcal{F}}(T)$ [resp. $\mathbf{Snr}_{\mathcal{F}}(T)$]. A thorough investigation of the properties of these categories is behind the scope of the present paper: it will, in fact, be the subject of a subsequent article. For the moment, we restrict ourselves to a few trivial remarks and observations.

The first one is the existence, substantially by construction, of an obvious “forgetful” functor $\mathcal{D} : \mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T}) \rightarrow \mathbf{Set}$, defined by sending an object (T, \mathfrak{A}) of $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$ to $\mathcal{F}(|\mathfrak{A}|)$ and a morphism $(\mathcal{M}_1, \mathcal{M}_2, \Phi)$, with $\mathcal{M}_i = (T, \mathfrak{A}_i)$ and $\Phi = (\alpha, \varphi)$, to the function $\mathcal{F}(\varphi) : \mathcal{F}(|\mathfrak{A}_1|) \rightarrow \mathcal{F}(|\mathfrak{A}_2|)$. In fact, \mathcal{D} is full if \mathcal{F} is full, and [essentially] surjective on objects whenever \mathcal{F} is such. On another hand, $\mathbf{Snr}_{\mathcal{F}}(\mathfrak{T})$ is obviously contained in $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$ as a full subcategory, so that we can partially reduce the study of the former to the study of the latter. Moreover, suppose $T = (V, \sigma, \Xi)$ is a prealgebraic [resp. subalgebraic] theory and $T_0 = (V, \sigma_0, \Xi_0)$ a prealgebraic [resp. subalgebraic] subtheory of T . Then, there exists one more “forgetful” functor $\mathcal{C}_{T_0} : \mathbf{Pnr}_{\mathcal{F}}(T) \rightarrow \mathbf{Pnr}_{\mathcal{F}}(T_0)$ [resp. $\mathcal{C}_{T_0} : \mathbf{Snr}_{\mathcal{F}}(T) \rightarrow \mathbf{Snr}_{\mathcal{F}}(T_0)$]. This is defined by mapping

- (i) a prealgebraic [resp. subalgebraic] \mathcal{F} -model $\mathcal{M} = (T, \mathfrak{A})$ of T , with $\mathfrak{A} = (A, \sigma, I)$, to the pair (T_0, \mathfrak{A}_0) , where $\mathfrak{A}_0 := (A, \sigma_0, I|_{\sigma_0})$;
- (ii) a morphism $(\alpha, \varphi) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ of $\mathbf{Pnr}_{\mathcal{F}}(T)$ [resp. $\mathbf{Snr}_{\mathcal{F}}(T)$] to the \mathcal{F} -prenorm [resp. -subnorm] $(\alpha_0, \varphi) : \mathcal{C}_{T_0}(\mathcal{M}_1) \rightarrow \mathcal{C}_{T_0}(\mathcal{M}_2)$, where α_0 is the restriction of α to σ_0 .

In particular, \mathcal{C}_{T_0} returns a “forgetful” functor to \mathbf{C} in the extreme case where T_0 is the “empty theory” $(V, \emptyset, \emptyset)$, so that \mathcal{D} , as given above, is simply the functor composition of \mathcal{F} with \mathcal{C}_{T_0} (in this latter case). One interesting question is, then, to establish under which conditions \mathcal{C}_{T_0} admits a left or right adjoint. Nevertheless, this question, as well as other properties of $\mathbf{Pnr}_{\mathcal{F}}(T)$ and $\mathbf{Snr}_{\mathcal{F}}(T)$, appears to be strongly dependent on the actual characteristics of \mathbf{C} and \mathcal{F} and will be considered in future work.

4. PRENORMED MODELS OVER A FIXED TARGET

Continuing with the notation of the previous section (unless explicitly overridden), assume henceforth that $\mathcal{M}_0 = (T_0, \mathfrak{A}_0)$ is a fixed “target” in $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$, with $\mathfrak{A}_0 = (A_0, \sigma_0, I_0)$, and suppose that \mathfrak{A}_0 is pivotal; let us denote its pivot by \leq and set $\mathcal{A}_0 := (\mathcal{F}(A_0), \leq)$. A prenormed \mathcal{F} -model of \mathfrak{T} over \mathcal{M}_0 is, then, any pair $\mathbf{M} = (\mathcal{M}, \Phi)$ such that \mathcal{M} is another object of $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$ and Φ an \mathcal{F} -prenorm $\mathcal{M} \rightarrow \mathcal{M}_0$. We will, in this case, refer to Φ as an \mathcal{M}_0 -valued \mathcal{F} -prenorm on \mathcal{M} (or an \mathcal{F} -prenorm on \mathcal{M} with values in \mathcal{M}_0), and indeed as an \mathcal{M}_0 -valued \mathcal{F} -subnorm on \mathcal{M} (or an \mathcal{F} -subnorm on \mathcal{M} with values in \mathcal{M}_0) if \mathcal{M} and \mathcal{M}_0 are both prenormed \mathcal{F} -models of \mathfrak{T} . Observe that \mathbf{M} can be well identified with the morphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}_0$ of $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$.

Given a prenormed \mathcal{F} -model $\mathbf{M}_i = (\mathcal{M}_i, \Phi_i)$ of \mathfrak{T} over \mathcal{M}_0 ($i = 1, 2$), with $\Phi_i = (\alpha_i, \varphi_i)$, we define an \mathcal{F} -short morphism $\mathbf{M}_1 \rightarrow \mathbf{M}_2$ to be any morphism $(\beta, \psi) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ in $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$ such that $\beta \circ_{\mathbf{Sgn}} \alpha_2 = \alpha_1$ and $(\mathcal{F}(\varphi_2 \circ_{\mathbf{C}} \psi), \mathcal{F}(\varphi_1)) \in \wp(\mathcal{F}(A_0), \mathcal{A}_0)$. The terminology is prompted by the fact that the latter condition is ultimately equivalent to saying, in a much more familiar notation, that $\|\mathcal{F}(\psi)(a)\|_2 \leq \|a\|_1$ for all $a \in \mathcal{F}(A)$, with $\|\cdot\|_i := \mathcal{F}(\varphi_i)$. In particular, we write that Ψ is an \mathcal{F} -isometry $\mathbf{M}_1 \rightarrow \mathbf{M}_2$ if $\|\mathcal{F}(\psi)(a)\|_2 = \|a\|_1$ for all $a \in \mathcal{F}(A)$.

Remark 12. As shown shortly, prenormed \mathcal{F} -models of \mathfrak{T} over \mathcal{M}_0 and \mathcal{F} -short morphisms thereof give rise to a further category, besides $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$. In the many-sorted case (discussed in the second part of the present work), this provides a full abstraction of the usual category of left modules over a fixed valuated ring (normed spaces over a fixed valuated field can be viewed as a special case of these), with morphisms given by weakly contractive linear transformations between the underlying vector spaces. This sounds intriguing, for it seems to suggest that, from the perspective of the framework set up in this paper, the “right choice” about the kind of morphisms to be considered in relation to normed structures, based only on abstract nonsense reasoning (and especially regardless of any further considerations relevant to applications), should “naturally” fall on short maps. Simply for the fact that common alternatives available in the “localized” context of normed spaces, such as bounded transformations or continuous functions between the standard topologies induced by the norms on the underlying sets, are ruled out, for they are not even possible, in the setting where we are planting the foundations of the general theory of normed structures.

Lemma 4.1. *Let $\mathbf{M}_i = (\mathcal{M}_i, \Phi_i)$ be a prenormed \mathcal{F} -model of \mathfrak{T} over \mathcal{M}_0 ($i = 1, 2, 3$) and suppose that Ψ and Θ are, respectively, \mathcal{F} -short maps $\mathbf{M}_1 \rightarrow \mathbf{M}_2$ and $\mathbf{M}_2 \rightarrow \mathbf{M}_3$. The composition of Θ with Ψ in $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$ is then an \mathcal{F} -short map $\mathbf{M}_1 \rightarrow \mathbf{M}_3$.*

Proof. Let $\Psi = (\beta, \psi)$ and $\Theta = (\gamma, \vartheta)$ and set $\mathcal{M}_i = (T_i, \mathfrak{A}_i)$, $\Phi_i = (\alpha_i, \varphi_i)$ and $\|\cdot\|_i := \mathcal{F}(\varphi_i)$. It suffices to show that $\|\mathcal{F}(\vartheta \circ_{\mathbf{C}} \psi)(a)\|_3 \leq \|a\|_1$ for all $a \in \mathcal{F}(|\mathfrak{A}_1|)$. But this is straightforward since, by hypothesis, $\|\mathcal{F}(\vartheta)(\mathcal{F}(\psi)(a))\|_3 \leq \|\mathcal{F}(\psi)(a)\|_2$ on the one hand, and on the other hand $\|\mathcal{F}(\psi)(a)\|_2 \leq \|a\|_1$, so that the conclusion follows from the transitivity of \leq . \blacksquare

Define C_o as the class of prenormed \mathcal{F} -models over \mathcal{M}_0 and C_h as that of triples (M_1, M_2, Ψ) such that M_i is a prenormed \mathcal{F} -model of \mathfrak{T} over \mathcal{M}_0 and Ψ an \mathcal{F} -short morphism $M_1 \rightarrow M_2$. Then, take s and t to be the maps $C_h \rightarrow C_o : (M_1, M_2, \Phi) \mapsto M_1$ and $C_h \rightarrow C_o : (M_1, M_2, \Phi) \mapsto M_2$, respectively, and denote by id the function $C_o \rightarrow C_h$ sending a prenormed \mathcal{F} -model $M = (\mathcal{M}, \Phi)$ to the triple (M, M, ε) , where ε is the identity $\mathcal{M} \rightarrow \mathcal{M}$ in $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T})$. Lastly, let \circ be the partial function $C_h \times C_h \rightarrow C_h$ specified as follows: Take $\mathbf{m} = (M_1, M_2, \Psi)$ and $\mathbf{n} = (N_1, N_2, \Theta)$ in C_h . If $M_2 \neq N_1$, then $\circ(\mathbf{m}, \mathbf{n})$ is not defined. Otherwise, based on Lemma 4.1, assume $\Psi = (\beta, \psi)$ and $\Theta = (\gamma, \vartheta)$ and set $\circ(\mathbf{m}, \mathbf{n})$ equal to the triple (M_1, N_2, Π) , where $\Pi := (\gamma \circ_{\mathbf{Sgn}} \beta, \vartheta \circ_{\mathbf{C}} \psi)$.

It is easy to verify that the 6-uple $(C_o, C_h, s, t, \text{id}, \circ)$ gives a category. We call it the category of prenormed \mathcal{F} -models of \mathfrak{T} over \mathcal{M}_0 and refer to its objects as prenormed \mathcal{F} -models of \mathfrak{T} over \mathcal{M}_0 . This category will be denoted, in general, by $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0)$, and especially by $\mathbf{Pnr}_{\mathcal{F}}(T; \mathcal{M}_0)$ in the case where \mathfrak{T} consists of one prealgebraic theory T . When \mathcal{M}_0 is a seminormed \mathcal{F} -model of \mathfrak{T} , then the objects in $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0)$ that are, indeed, seminormed \mathcal{F} -models of \mathfrak{T} over \mathcal{M}_0 , form a full subcategory of $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0)$. This will be written as $\mathbf{Snr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0)$, or as $\mathbf{Snr}_{\mathcal{F}}(T; \mathcal{M}_0)$ for $\mathfrak{T} = \{T\}$, and called the category of seminormed \mathcal{F} -models over \mathcal{M}_0 .

An extended study of the properties of these categories is beyond the scope of the present paper. Here, we just observe that there exists an obvious “forgetful” functor $\mathcal{D} : \mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0) \rightarrow \mathbf{Set}$, defined by sending a prenormed \mathcal{F} -model (\mathcal{M}, Φ) of \mathfrak{T} over \mathcal{M}_0 , with $\mathcal{M} = (T, \mathfrak{A})$, to $\mathcal{F}(|\mathfrak{A}|)$, and an \mathcal{F} -short morphism $(\alpha, \psi) : (\mathcal{M}_1, \Phi_1) \rightarrow (\mathcal{M}_2, \Phi_2)$ of $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0)$, with $\mathcal{M}_i = (T_i, \mathfrak{A}_i)$, to the function $\mathcal{F}(\psi) : \mathcal{F}(|\mathfrak{A}_1|) \rightarrow \mathcal{F}(|\mathfrak{A}_2|)$. This is full if \mathcal{F} is full, and [essentially] surjective on objects if \mathcal{F} is such. The same \mathcal{D} restricts to a functor $\mathbf{Snr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0) \rightarrow \mathbf{Set}$ for \mathcal{M}_0 being a seminormed \mathcal{F} -model of \mathfrak{T} , in which case $\mathbf{Snr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0)$ is, indeed, a full subcategory of $\mathbf{Pnr}_{\mathcal{F}}(\mathfrak{T}; \mathcal{M}_0)$. On another hand, assume that $T = (V, \sigma, \Xi)$ is a prealgebraic [resp. subalgebraic] theory and $T_0 = (V, \sigma_0, \Xi_0)$ a prealgebraic subtheory of T . Then, we know from the previous section that there exists a “forgetful” functor $\mathcal{C}_{T_0} : \mathbf{Pnr}_{\mathcal{F}}(T) \rightarrow \mathbf{Pnr}_{\mathcal{F}}(T_0)$ [resp. $\mathcal{C}_{T_0} : \mathbf{Pnr}_{\mathcal{F}}(T) \rightarrow \mathbf{Pnr}_{\mathcal{F}}(T_0)$]. This in turn induces a “forgetful” functor $\mathcal{E}_{T_0} : \mathbf{Pnr}_{\mathcal{F}}(T; \mathcal{M}_0) \rightarrow \mathbf{Pnr}_{\mathcal{F}}(T_0; \mathcal{C}_{T_0}(\mathcal{M}_0))$ [resp. $\mathcal{E}_{T_0} : \mathbf{Snr}_{\mathcal{F}}(T; \mathcal{M}_0) \rightarrow \mathbf{Snr}_{\mathcal{F}}(T_0; \mathcal{C}_{T_0}(\mathcal{M}_0))$], defined by sending

- (i) a prenormed [resp. subnormed] \mathcal{F} -model (\mathcal{M}, Φ) of T over \mathcal{M}_0 to $(\mathcal{C}_{T_0}(\mathcal{M}), \mathcal{C}_{T_0}(\Phi))$;
- (ii) an arrow $\Psi : (\mathcal{M}_1, \Phi_1) \rightarrow (\mathcal{M}_2, \Phi_2)$ of $\mathbf{Pnr}_{\mathcal{F}}(T; \mathcal{M}_0)$ [resp. $\mathbf{Snr}_{\mathcal{F}}(T; \mathcal{M}_0)$] to the morphism $\mathcal{C}_{T_0}(\Psi) : (\mathcal{C}_{T_0}(\mathcal{M}_1), \mathcal{C}_{T_0}(\Phi_1)) \rightarrow (\mathcal{C}_{T_0}(\mathcal{M}_2), \mathcal{C}_{T_0}(\Phi_2))$ of $\mathbf{Pnr}_{\mathcal{F}}(T_0; \mathcal{C}_{T_0}(\mathcal{M}_0))$ [resp. $\mathbf{Snr}_{\mathcal{F}}(T_0; \mathcal{C}_{T_0}(\mathcal{M}_0))$].

In particular, \mathcal{E}_{T_0} returns a “forgetful” functor to \mathbf{C} in the extreme case where T_0 is the “empty theory” $(V, \emptyset, \emptyset)$, so that \mathcal{D} , as given above, is simply the functor composition of \mathcal{F} with \mathcal{E}_{T_0} (in this latter case). It is then interesting to ask when \mathcal{E}_{T_0} admits adjoints. However, the question, along with other properties of $\mathbf{Pnr}_{\mathcal{F}}(T)$ and $\mathbf{Snr}_{\mathcal{F}}(T)$, is seen to critically depend on the specificity of \mathbf{C} and \mathcal{F} and will be investigated in a separate paper.

5. SOME WORKED EXAMPLES

Unless explicitly overridden, the notation throughout is based on that of the previous section. Here, we show how the framework developed so far succeeds to capture all the essential features of the notion itself of norm as this is traditionally defined in the classical approach to the theory of normed groups, valuated rings, normed (vector) spaces, etc. In each of the examples examined, we assume $\mathbf{C} = \mathbf{Set}$ and $\mathcal{F} = \mathbf{1}_{\mathbf{Set}}$. Therefore, we omit any further reference to \mathcal{F} and use, e.g., “model” in place of “ \mathcal{F} -model”, “prenorm” [resp. “subnorm”] instead of “ \mathcal{F} -prenorm” [resp. “ \mathcal{F} -subnorm”], and so on. In fact, it is enough to focus on a family \mathfrak{T} consisting of two

theories $T = (\sigma, \Xi)$ and $T_0 = (\sigma_0, \Xi_0)$, possibly equal to each other: The former is required to be algebraic under $\mathbf{1}_{\mathbf{Set}}$, the latter prealgebraic [resp. subalgebraic].

Upon these premises, we take $\mathfrak{A} = (A, \sigma, I)$ to be an algebraic model of T and $\mathfrak{A}_0 = (A_0, \sigma_0, I_0)$ a pivotal prealgebraic [resp. subalgebraic] model of T_0 . We set $\mathcal{M} := (T, \mathfrak{A})$ and $\mathcal{M}_0 := (T_0, \mathfrak{A}_0)$ and denote the pivot of \mathfrak{A}_0 by \leq . Furthermore, in the light of Remark 10, we assume that σ is a subsignature of σ_0 and concentrate only on \mathcal{M}_0 -valued prenorms [resp. subnorms] on \mathcal{M} of the form $\Phi = (\alpha, \|\cdot\|)$ such that α is the canonical injection $\sigma \rightarrow \sigma_0$, hence identifying Φ with $\|\cdot\|$ by a convenient abuse of notation. Lastly, for T_s a subtheory of T_0 , we use \mathcal{C}_{T_s} for the “forgetful” functor $\mathbf{Pnr}_{\mathbf{1Set}}(T_0) \rightarrow \mathbf{Pnr}_{\mathbf{1Set}}(T_s)$ defined by the end of Section 3.

Now, we pick a distinguished set of (non-logical) function symbols, $\Sigma_f = \{+, \star, u, 0, 1\}$, and a distinguished set of (non-logical) relation symbols, $\Sigma_r = \{\leq_+, \leq_\star, \leq_u, \leq_0, \leq_1\}$, and introduce a “reference signature” $\sigma_{\text{ref}} = (\Sigma_f, \Sigma_r, \text{ar}_{\text{ref}})$, where ar_{ref} is defined in such a way that $+, \star$ and every member of Σ_r are binary, u is unary, and 0 and 1 are nullary. Then, as is usual, we call

- (s.1) $\sigma_{\text{sgrp}} := (+, \leq_+, \text{ar}_{\text{sgrp}})$ the signature of semigroups;
- (s.2) $\sigma_{\text{mon}} := (+, \leq_+, 0, \leq_0; \text{ar}_{\text{mon}})$ the signature of monoids;
- (s.3) $\sigma_{\text{grp}} := (+, \leq_+, u, \leq_u; 0, \leq_0; \text{ar}_{\text{grp}})$ the signature of groups;
- (s.4) $\sigma_{\text{rg}} := (+, \leq_+, \star, \leq_\star; 0, \leq_0; \text{ar}_{\text{rg}})$ the signature of semirings;
- (s.5) $\sigma_{\text{rig}} := (+, \leq_+, \star, \leq_\star; 0, \leq_0; 1, \leq_1; \text{ar}_{\text{rig}})$ the signature of unital semirings;
- (s.6) $\sigma_{\text{rng}} := (+, \leq_+, \star, \leq_\star; u, \leq_u; 0, \leq_0; \text{ar}_{\text{rng}})$ the signature of rings;
- (s.7) $\sigma_{\text{ring}} := (+, \leq_+, \star, \leq_\star; u, \leq_u; 0, \leq_0; 1, \leq_1; \text{ar}_{\text{ring}})$ the signature of unital rings.

Here, ar_{sgrp} , ar_{mon} , etc are the appropriate restrictions of ar_{ref} to $\{+, \leq_+\}$, $\{+, \leq_+, 0, \leq_0\}$, etc. We then say that a subnorm $\|\cdot\| : \mathcal{M} \rightarrow \mathcal{M}_0$, if any exists, is an \mathcal{M}_0 -valued semigroup [resp. group] subnorm (on \mathcal{M}) if T is the smallest subalgebraic theory of signature σ_{sgrp} [resp. σ_{grp}], an \mathcal{M}_0 -valued monoid subnorm if T is the smallest subalgebraic theory of signature σ_{mon} , an \mathcal{M}_0 -valued semiring [resp. ring] subnorm if T is the smallest subalgebraic theory of signature σ_{rg} [resp. σ_{rng}], and an \mathcal{M}_0 -valued subnorm of unital semirings [resp. unital rings] if T is the smallest subalgebraic theory of signature σ_{rig} [resp. σ_{ring}] (cf. Remark 6). When $T_0 = T$, this implies that a subnorm $\|\cdot\| : \mathcal{M} \rightarrow \mathcal{M}_0$ can be characterized by saying that:

- (E.1) an \mathcal{M}_0 -valued semigroup subnorm (on \mathcal{M}) if and only if

$$\|a + b\| \leq_+ \|a\| + \|b\| \text{ for all } a, b \in A. \quad (10)$$

In the most usual case, \mathfrak{A} and \mathfrak{A}_0 are models of the subalgebraic theory T_{sgrp} of semigroups, which is the smallest subalgebraic theory of signature σ_{sgrp} including the axiom of associativity (see Remark 5) for $+$. In common applications, it is assumed that \mathcal{M}_0 is the ordered additive semigroup of the non-negative real numbers, with the (semigroup and order) structure inherited from \mathbb{R} . As a specialization of the general terminology, we then refer to an \mathcal{M}_0 -valued semigroup subnorm $\|\cdot\|$ (on \mathcal{M}) as an \mathcal{M}_0 -valued semigroup seminorm (on \mathcal{M}), and call \mathcal{M} a seminormed semigroup over \mathcal{M}_0 .

- (E.2) an \mathcal{M}_0 -valued monoid subnorm if and only if $\mathcal{C}_{\#_s T_{\text{sgrp}}}(\|\cdot\|)$ is a $\mathcal{C}_{\#_s T_{\text{sgrp}}}(\mathcal{M}_0)$ -valued semigroup subnorm on $\mathcal{C}_{\#_s T_{\text{sgrp}}}(\mathcal{M})$ and $\|0\| \leq_0 0$. Typically, \mathfrak{A} and \mathfrak{A}_0 are models of the subalgebraic theory T_{mon} of monoids, to wit, the smallest subalgebraic extension of T_{sgrp} comprising the axiom of neutrality (see Remark 5) for the symbol 0 . Also, one usually takes \mathcal{M}_0 to be the ordered additive monoid of the non-negative real numbers, with the structure inherited from \mathbb{R} . By Remark 11, it then follows that $\|0\| = 0$. Thus, motivated by the “classical theory”, we refer to an \mathcal{M}_0 -valued monoid subnorm $\|\cdot\|$ (on \mathcal{M}) which is upward semidefinite [resp. definite] with respect to 0 as an \mathcal{M}_0 -valued

monoid seminorm [resp. norm] (on \mathcal{M}), and call \mathcal{M} a seminormed [resp. normed] monoid over \mathcal{M}_0 .

(E.3) an \mathcal{M}_0 -valued group subnorm if and only if

$$\|u(a)\| \leq_u u(\|a\|) \text{ for all } a \in A \quad (11)$$

and $\mathcal{C}_{\#sT_{\text{mon}}}(\|\cdot\|)$ is a $\mathcal{C}_{\#sT_{\text{mon}}}(\mathcal{M}_0)$ -valued monoid subnorm on $\mathcal{C}_{\#sT_{\text{mon}}}(\mathcal{M})$. In familiar instances, \mathfrak{A} is a model of the subalgebraic theory T_{grp} of groups, the smallest subalgebraic extension of T_{mon} containing the axiom of inverses (see Remark 5) for the triple $(+, u, 0)$. In these cases, u is usually represented by the symbol $-$, so that the above Equation (11) reads as:

$$\| - a \| \leq_u u(\|a\|) \text{ for all } a \in A \quad (12)$$

and indeed as: $\| - a \| \leq -\|a\|$ for all $a \in A$ if $T_0 = T_{\text{grp}}$. This seems to suggest that group prenorms are “naturally negative” (for an appropriate choice of the target model \mathfrak{A}_0 such as the additive group of the real numbers with its usual order) as far as we look at them as morphisms between (prenormed) structures of the very same algebraic type, i.e., groups. This is intriguing but not completely satisfactory, for it does not cover the relevant case of standard (positive definite) group norms [2, p. 5]. However, similar cases can be brought within the scope of our framework in the light of one trivial consideration: That the target of a standard group norm is taken equal to \mathbb{R}_0^+ , which is everything but a group. In particular, the most obvious solution is to assume that T_0 is not the same as T , but instead the smallest subalgebraic theory of its same signature. Then, u can be interpreted as the identity function on A_0 and Equation (12) becomes: $\| - a \| \leq_u \|a\|$ for all $a \in A$. If \leq is a partial order, it follows from here that $\|\cdot\|$ is *necessarily* symmetric. This looks like another interesting byproduct of our approach, as it suggests that “asymmetric group norms” (cf. [2, Remark 2]) do not really exist as such. They can as monoid norms, which is absolutely reasonable if we think of the fact that an “asymmetric group norm” is ultimately defined without any specific requirement about inverses. Starting from these considerations, we then refer to an \mathcal{M}_0 -valued group subnorm $\|\cdot\|$ (on \mathcal{M}), which is upward semidefinite [resp. definite] with respect to 0, as an \mathcal{M}_0 -valued group seminorm [resp. norm] (on \mathcal{M}), and call \mathcal{M} a seminormed [resp. normed] group over \mathcal{M}_0 .

(E.4) an \mathcal{M}_0 -valued semiring subnorm if and only if

$$\|a \star b\| \leq_u \|a\| \star \|b\| \text{ for all } a, b \in A \quad (13)$$

and $\mathcal{C}_{\#sT_{\text{mon}}}(\|\cdot\|)$ is a $\mathcal{C}_{\#sT_{\text{mon}}}(\mathcal{M}_0)$ -valued monoid subnorm on $\mathcal{C}_{\#sT_{\text{mon}}}(\mathcal{M})$. Note how this approach suggests that norms on ring-like structures are “inherently submultiplicative”: “Multiplicativeness” is covered by assuming that \leq_\star is (interpreted as) the equality relation on A_0 , and the same applies to different [binary] operations, to the extent that, from an abstract point of view, there is no apparent reason to focus on the one rather than the others. For the rest, \mathfrak{A} and \mathfrak{A}_0 will be typically models of the subalgebraic theory T_{rg} of semirings, i.e., the smallest subalgebraic extension of T_{mon} containing the axioms of left and right distributiveness of \star over $+$ and the axiom of associativity for \star . E.g., this is the case with the semiring of non-negative real numbers (with the usual structure inherited from \mathbb{R}). Motivated by the terminology of the theory of valuated

- rings¹, we then refer to an \mathcal{M}_0 -valued semiring subnorm $\|\cdot\|$ (on \mathcal{M}), which is upward semidefinite [resp. definite] with respect to 0, as an \mathcal{M}_0 -valued semiring semivaluation [resp. valuation] (on \mathcal{M}), and call \mathcal{M} a semivaluated [resp. valuated] semiring over \mathcal{M}_0 .
- (E.5) an \mathcal{M}_0 -valued ring subnorm if and only if it is both a $\mathcal{C}_{\sharp_s T_{\text{grp}}}(\|\cdot\|)$ -valued group subnorm and a $\mathcal{C}_{\sharp_s T_{\text{rig}}}(\|\cdot\|)$ -valued semiring subnorm. Thus, all the considerations previously made on group and semiring subnorms also apply to ring subnorms. In particular, we refer to an \mathcal{M}_0 -valued ring subnorm $\|\cdot\|$ (on \mathcal{M}), which is upward semidefinite [resp. definite] with respect to 0, as an \mathcal{M}_0 -valued ring semivaluation [resp. valuation] (on \mathcal{M}), and then call \mathcal{M} a semivaluated [resp. valuated] semiring over \mathcal{M}_0 .
- (E.6) an \mathcal{M}_0 -valued subnorm of unital semirings if and only if it is a $\mathcal{C}_{\sharp_s T_{\text{rig}}}(\|\cdot\|)$ -valued semiring subnorm and $\|1\| \leq_1 1$. In many relevant applications, \mathfrak{A} and \mathfrak{A}_0 are models of the subalgebraic theory T_{rig} of unital semirings, i.e., the smallest subalgebraic extension of T_{rig} including the axiom of neutrality for 1: The non-negative real numbers form a model of T_{rig} with the usual (algebraic and order) structure inherited from \mathbb{R} . The same considerations previously made on the symbol 0 in the case of group subnorms apply to 1. Furthermore, mimicking the case of semiring subnorms, we refer to an \mathcal{M}_0 -valued subnorm $\|\cdot\|$ of unital semirings (on \mathcal{M}), which is upward semidefinite [resp. definite] with respect to 0, as an \mathcal{M}_0 -valued semivaluation [resp. valuation] of unital semirings (on \mathcal{M}), and then call \mathcal{M} a semivaluated [resp. valuated] unital semiring over \mathcal{M}_0 .
- (E.7) an \mathcal{M}_0 -valued subnorm of unital rings if and only if it is both a $\mathcal{C}_{\sharp_s T_{\text{ring}}}(\|\cdot\|)$ -valued subnorm of unital semirings and $\mathcal{C}_{\sharp_s T_{\text{grp}}}(\|\cdot\|)$ -valued group subnorm. In many relevant applications, \mathfrak{A} and \mathfrak{A}_0 are models of the subalgebraic theory T_{ring} of unital rings, the smallest subalgebraic extension of T_{ring} including the axiom of inverses for the triple $(+, u, 0)$. Of course, the same considerations previously made in the case of group subnorms and subnorms of unital semirings apply to ring subnorms. Also, we refer to an \mathcal{M}_0 -valued subnorm $\|\cdot\|$ of unital rings (on \mathcal{M}), which is upward semidefinite [resp. definite] with respect to 0, as an \mathcal{M}_0 -valued semivaluation [resp. valuation] of unital rings (on \mathcal{M}), and then call \mathcal{M} a semivaluated [resp. valuated] unital ring over \mathcal{M}_0 .

\mathcal{M}_0 -valued semigroup subnorms, monoid subnorms, etc are defined and characterized in the very same way, by replacing “subalgebraic” with “prealgebraic” and \sharp_s with \sharp_p in all of their occurrences in the above discussion. Furthermore, most of the considerations made in the subalgebraic case still hold in the prealgebraic one, except for those based on Remark 11.

Field valuations and norms of vector spaces over a fixed valuated field, together with variants thereof, will be discussed in Part II as special instances of many-sorted subnormed structures.

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¹: differently from other authors, we use here the term “valuation” in the meaning of “absolute value”.

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REFERENCES

- [1] J. Adámek, H. Herrlich, and G. Strecker, *Abstract and concrete categories: The joy of cats*, Reprints in Theory and Applications of Categories, No. 17, 2006.
- [2] N. H. Bingham and A. J. Ostaszewski, *Normed groups: dichotomy and duality*, CDAM Research Report Series, LSE-CDAM-2008-10, London School of Economics, 2008.
- [3] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis*, Springer-Verlag, 1984.
- [4] A. Buffa, Y. Maday, A. T. Patera, C. Prud'homme, and G. Turinici, *A Priori Convergence of the Greedy Algorithm for the Parametrized Reduced Basis*, Math. Mod. Num. Anal., Vol. 46, No. 3, pp. 595-603, 2012.
- [5] C. C. Chang and H. J. Keisler, *Model Theory*, Studies in Logic and the Foundations of Mathematics, Elsevier, 1990 (3rd ed.).
- [6] *Handbook of the Geometry of Banach Spaces - Volumes I and II*, ed. by W. B. Johnson and J. Lindenstrauss, North-Holland, 2001 (Vol. I) and 2003 (Vol. II).
- [7] F. W. Lawvere, *Metric spaces, generalised logic, and closed categories*, Reprints in Theory and Applications of Categories, No. 1, pp. 1-37, 2002.
- [8] F. W. Lawvere, *Functorial Semantics of algebraic theories*, Reprints in Theory and Applications of Categories, No. 5, pp 1-121, 2005 (originally published as: Ph.D. thesis, Columbia University, 1963).
- [9] R. Lowen, *Approach spaces. The missing link in the topology-uniformity-metric triad*, Oxford University Press, 1997.
- [10] Y. Maday, A. T. Patera, and G. Turinici, *A priori convergence theory for reduced-basis approximations of single-parametric elliptic partial differential equations*, J. Scientific Computing, Vol. 17, pp. 437-446, 2002.
- [11] E. Mendelson, *Introduction to mathematical logic*, Chapman & Hall, 1997 (4th ed.).
- [12] nLab, *Category*, online at <http://nlab.mathforge.org/nlab/show/category> (version: Dec 16, 2011).

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