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ON WEAK*-CONVERGENCE IN $H_L^1(\mathbb{R}^d)$

LUONG DANG KY

ABSTRACT. Let $L = -\Delta + V$ be a Schrödinger operator on \mathbb{R}^d , $d \geq 3$, where V is a nonnegative function, $V \neq 0$, and belongs to the reverse Hölder class $RH_{d/2}$. In this paper, we prove a version of the classical theorem of Jones and Journé on weak*-convergence in the Hardy space $H_L^1(\mathbb{R}^d)$.

1. INTRODUCTION

A famous and classical result of Fefferman [7] states that the John-Nirenberg space $BMO(\mathbb{R}^d)$ is the dual of the Hardy space $H^1(\mathbb{R}^d)$. It is also well-known that $H^1(\mathbb{R}^d)$ is one of the few examples of separable, nonreflexive Banach space which is a dual space. In fact, let $VMO(\mathbb{R}^d)$ denote the closure of the space $C_c^\infty(\mathbb{R}^d)$ in $BMO(\mathbb{R}^d)$, where $C_c^\infty(\mathbb{R}^d)$ is the set of C^∞ -functions with compact support, Coifman and Weiss showed in [1] that $H^1(\mathbb{R}^d)$ is the dual space of $VMO(\mathbb{R}^d)$, which gives to $H^1(\mathbb{R}^d)$ a richer structure than $L^1(\mathbb{R}^d)$. For example, the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are not bounded on $L^1(\mathbb{R}^d)$, but are bounded on $H^1(\mathbb{R}^d)$. In addition, the weak*-convergence is true in $H^1(\mathbb{R}^d)$, which is useful in the application of Hardy spaces to compensated compactness (see [2]). More precisely, in [9], Jones and Journé proved the following.

Theorem J-J. *Suppose that $\{f_j\}_{j \geq 1}$ is a bounded sequence in $H^1(\mathbb{R}^d)$, and that $f_j(x) \rightarrow f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in H^1(\mathbb{R}^d)$ and $\{f_j\}_{j \geq 1}$ weak*-converges to f , that is, for every $\varphi \in VMO(\mathbb{R}^d)$, we have*

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} f_j(x) \varphi(x) dx = \int_{\mathbb{R}^d} f(x) \varphi(x) dx.$$

The aim of this paper is to prove an analogous version of the above theorem in the setting of function spaces associated with Schrödinger operators.

Let $L = -\Delta + V$ be a Schrödinger differential operator on \mathbb{R}^d , $d \geq 3$, where V is a nonnegative potential, $V \neq 0$, and belongs to the reverse Hölder class $RH_{d/2}$. In the recent years, there is an increasing interest on the study of the problems of harmonic analysis associated with these operators, see for example [4, 5, 6, 10, 11, 13, 14]. In [6], Dziubański and Zienkiewicz considered the Hardy space $H_L^1(\mathbb{R}^d)$ as the set of functions $f \in L^1(\mathbb{R}^d)$ such that

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$\|f\|_{H_L^1} := \|\mathcal{M}_L f\|_{L^1} < \infty$, where $\mathcal{M}_L f(x) := \sup_{t>0} |e^{-tL} f(x)|$. There, they characterized $H_L^1(\mathbb{R}^d)$ in terms of atomic decomposition and in terms of the Riesz transforms associated with L . Later, in [5], Dziubański et al. introduced a BMO -type space $BMO_L(\mathbb{R}^d)$ associated with L , and established the duality between $H_L^1(\mathbb{R}^d)$ and $BMO_L(\mathbb{R}^d)$. Recently, Deng et al. [4] introduced and developed new VMO -type function spaces $VMO_A(\mathbb{R}^d)$ associated with some operators A which have a bounded holomorphic functional calculus on $L^2(\mathbb{R}^d)$. When $A \equiv L$, their space $VMO_L(\mathbb{R}^d)$ is just the set of all functions f in $BMO_L(\mathbb{R}^d)$ such that $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\begin{aligned} \gamma_1(f) &= \lim_{r \rightarrow 0} \left(\sup_{x \in \mathbb{R}^d, t \leq r} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-tL} f(y)|^2 dy \right)^{1/2} \right), \\ \gamma_2(f) &= \lim_{R \rightarrow \infty} \left(\sup_{x \in \mathbb{R}^d, t \geq R} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-tL} f(y)|^2 dy \right)^{1/2} \right), \\ \gamma_3(f) &= \lim_{R \rightarrow \infty} \left(\sup_{B(x, t) \cap B(0, R) = \emptyset} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-tL} f(y)|^2 dy \right)^{1/2} \right). \end{aligned}$$

The authors in [4] further showed that $H_L^1(\mathbb{R}^d)$ is in fact the dual of $VMO_L(\mathbb{R}^d)$, which allows us to study the weak*-convergence in $H_L^1(\mathbb{R}^d)$. This is useful in the study of the Hardy estimates for commutators of singular integral operators related to L , see for example Theorem 7.1 and Theorem 7.3 of [10].

Our main result is the following theorem.

Theorem 1.1. *Suppose that $\{f_j\}_{j \geq 1}$ is a bounded sequence in $H_L^1(\mathbb{R}^d)$, and that $f_j(x) \rightarrow f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in H_L^1(\mathbb{R}^d)$ and $\{f_j\}_{j \geq 1}$ weak*-converges to f , that is, for every $\varphi \in VMO_L(\mathbb{R}^d)$, we have*

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} f_j(x) \varphi(x) dx = \int_{\mathbb{R}^d} f(x) \varphi(x) dx.$$

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In \mathbb{R}^d , we denote by $B = B(x, r)$ an open ball with center x and radius $r > 0$. For any measurable set E , we denote by $|E|$ its Lebesgue measure.

The paper is organized as follows. In Section 2, we present some notations and preliminary results. Section 3 is devoted to the proof of Theorem 1.1. In the last section, we prove that $C_c^\infty(\mathbb{R}^d)$ is dense in the space $VMO_L(\mathbb{R}^d)$.

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2. SOME PRELIMINARIES AND NOTATIONS

In this paper, we consider the Schrödinger differential operator

$$L = -\Delta + V$$

on \mathbb{R}^d , $d \geq 3$, where V is a nonnegative potential, $V \neq 0$. As in the works of Dziubański et al [5, 6], we always assume that V belongs to the reverse Hölder class $RH_{d/2}$. Recall that a nonnegative locally integrable function V is said to belong to a reverse Hölder class RH_q , $1 < q < \infty$, if there exists a constant $C > 0$ such that for every ball $B \subset \mathbb{R}^d$,

$$\left(\frac{1}{|B|} \int_B (V(x))^q dx \right)^{1/q} \leq \frac{C}{|B|} \int_B V(x) dx.$$

Let $\{T_t\}_{t>0}$ be the semigroup generated by L and $T_t(x, y)$ be their kernels. Namely,

$$T_t f(x) = e^{-tL} f(x) = \int_{\mathbb{R}^d} T_t(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$

Since V is nonnegative, the Feynman-Kac formula implies that

$$(2.1) \quad 0 \leq T_t(x, y) \leq \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}.$$

According to [6], the space $H_L^1(\mathbb{R}^d)$ is defined as the completion of

$$\{f \in L^2(\mathbb{R}^d) : \mathcal{M}_L f \in L^1(\mathbb{R}^d)\}$$

in the norm

$$\|f\|_{H_L^1} := \|\mathcal{M}_L f\|_{L^1},$$

where $\mathcal{M}_L f(x) := \sup_{t>0} |T_t f(x)|$ for all $x \in \mathbb{R}^d$.

In [5] it was shown that the dual space of $H_L^1(\mathbb{R}^d)$ can be identified with the space $BMO_L(\mathbb{R}^d)$ which consists of all functions $f \in BMO(\mathbb{R}^d)$ with

$$(2.2) \quad \|f\|_{BMO_L} := \|f\|_{BMO} + \sup_{\rho(x) \leq r} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy < \infty,$$

where ρ is the auxiliary function defined as in [13], that is,

$$(2.3) \quad \rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\},$$

$x \in \mathbb{R}^d$. Clearly, $0 < \rho(x) < \infty$ for all $x \in \mathbb{R}^d$, and thus $\mathbb{R}^d = \bigcup_{n \in \mathbb{Z}} \mathcal{B}_n$, where the sets \mathcal{B}_n are defined by

$$(2.4) \quad \mathcal{B}_n = \{x \in \mathbb{R}^d : 2^{-(n+1)/2} < \rho(x) \leq 2^{-n/2}\}.$$

The following fundamental property of the function ρ is due to Shen [13].

Proposition 2.1 (see [13], Lemma 1.4). *There exist $C_0 > 1$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^d$,*

$$C_0^{-1} \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C_0 \rho(x) \left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}.$$

Let $VMO_L(\mathbb{R}^d)$ be the subspace of $BMO_L(\mathbb{R}^d)$ consisting of those functions f satisfying $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\begin{aligned} \gamma_1(f) &= \lim_{r \rightarrow 0} \left(\sup_{x \in \mathbb{R}^d, t \leq r} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-tL} f(y)|^2 dy \right)^{1/2} \right), \\ \gamma_2(f) &= \lim_{R \rightarrow \infty} \left(\sup_{x \in \mathbb{R}^d, t \geq R} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-tL} f(y)|^2 dy \right)^{1/2} \right), \\ \gamma_3(f) &= \lim_{R \rightarrow \infty} \left(\sup_{B(x, t) \cap B(0, R) = \emptyset} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - e^{-tL} f(y)|^2 dy \right)^{1/2} \right). \end{aligned}$$

In [4] it was shown that $H_L^1(\mathbb{R}^d)$ is the dual space of $VMO_L(\mathbb{R}^d)$.

In the sequel, we denote by \mathcal{C}_L the L -constant

$$\mathcal{C}_L = 8.9^{k_0} C_0$$

where k_0 and C_0 are defined as in Proposition 2.1.

Following Dziubański and Zienkiewicz [6], we define atoms as follows.

Definition 2.1. *Given $1 < q \leq \infty$. A function a is called a (H_L^1, q) -atom related to the ball $B(x_0, r)$ if $r \leq \mathcal{C}_L \rho(x_0)$ and*

- i) $\text{supp } a \subset B(x_0, r)$,*
- ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,*
- iii) if $r \leq \frac{1}{\mathcal{C}_L} \rho(x_0)$ then $\int_{\mathbb{R}^d} a(x) dx = 0$.*

Then, we have the following atomic characterization of $H_L^1(\mathbb{R}^d)$.

Theorem A (see [6], Theorem 1.5). *Let $1 < q \leq \infty$. A function f is in $H_L^1(\mathbb{R}^d)$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are (H_L^1, q) -atoms and $\sum_j |\lambda_j| < \infty$. Moreover, there exists a constant $C > 0$ such that*

$$\|f\|_{H_L^1} \leq \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\} \leq C \|f\|_{H_L^1}.$$

Let $P(x) = (4\pi)^{-d/2} e^{-|x|^2/4}$ be the Gauss function. According to [6], the space $h_n^1(\mathbb{R}^d)$, $n \in \mathbb{Z}$, denotes the space of all integrable functions f such that

$$\mathcal{M}_n f(x) = \sup_{0 < t < 2^{-n/2}} |P_t * f(x)| \in L^1(\mathbb{R}^d),$$

where $P_t(\cdot) := t^{-d}P(t^{-1}\cdot)$. The norm on $h_n^1(\mathbb{R}^d)$ is then defined by

$$\|f\|_{h_n^1} := \|\mathcal{M}_n f\|_{L^1}.$$

It was shown in [8] that the dual space of $h_n^1(\mathbb{R}^d)$ can be identified with $bmo_n(\mathbb{R}^d)$ the space of all locally integrable functions f such that

$$\|f\|_{bmo_n} = \|f\|_{BMO} + \sup_{x \in \mathbb{R}^d, 2^{-n/2} \leq r} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy < \infty.$$

Here and in what follows, for a ball B and a locally integrable function f , we denote by f_B the average of f on B . Following Dafni [3], we define $vmo_n(\mathbb{R}^d)$ as the subspace of $bmo_n(\mathbb{R}^d)$ consisting of those f such that

$$\lim_{\sigma \rightarrow 0} \left(\sup_{x \in \mathbb{R}^d, r < \sigma} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f_{B(x, r)}| dy \right) = 0$$

and

$$\lim_{R \rightarrow \infty} \left(\sup_{B(x, r) \cap B(0, R) = \emptyset, r \geq 2^{-n/2}} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy \right) = 0.$$

Recall that $C_c^\infty(\mathbb{R}^d)$ is the space of all C^∞ -functions with compact support. Then, the following was established by Dafni [3].

Theorem B (see [3], Theorem 6 and Theorem 9). *Let $n \in \mathbb{Z}$. Then,*

- i) The space $vmo_n(\mathbb{R}^d)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ in $bmo_n(\mathbb{R}^d)$.*
- ii) The dual of $vmo_n(\mathbb{R}^d)$ is the space $h_n^1(\mathbb{R}^d)$.*

Furthermore, the weak*-convergence is true in $h_n^1(\mathbb{R}^d)$.

Theorem C (see [3], Theorem 11). *Let $n \in \mathbb{Z}$. Suppose that $\{f_j\}_{j \geq 1}$ is a bounded sequence in $h_n^1(\mathbb{R}^d)$, and that $f_j(x) \rightarrow f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in h_n^1(\mathbb{R}^d)$ and $\{f_j\}_{j \geq 1}$ weak*-converges to f , that is, for every $\varphi \in vmo_n(\mathbb{R}^d)$, we have*

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} f_j(x) \varphi(x) dx = \int_{\mathbb{R}^d} f(x) \varphi(x) dx.$$

3. PROOF OF THEOREM 1.1

We begin by recalling the following two lemmas due to [6]. These two lemmas together with Proposition 2.1 play an important role in our study.

Lemma 3.1 (see [6], Lemma 2.3). *There exists a constant $C > 0$ and a collection of balls $B_{n,k} = B(x_{n,k}, 2^{-n/2})$, $n \in \mathbb{Z}, k = 1, 2, \dots$, such that $x_{n,k} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \bigcup_k B_{n,k}$, and*

$$\text{card} \{(n', k') : B(x_{n,k}, R2^{-n/2}) \cap B(x_{n',k'}, R2^{-n/2}) \neq \emptyset\} \leq R^C$$

for all n, k and $R \geq 2$.

Lemma 3.2 (see [6], Lemma 2.5). *There are nonnegative C^∞ -functions $\psi_{n,k}$, $n \in \mathbb{Z}, k = 1, 2, \dots$, supported in the balls $B(x_{n,k}, 2^{1-n/2})$ such that*

$$\sum_{n,k} \psi_{n,k} = 1 \quad \text{and} \quad \|\nabla \psi_{n,k}\|_{L^\infty} \leq C2^{n/2}.$$

The following corollary is useful, its proof follows directly from Lemma 3.1. We omit the details here (see also Corollary 1 of [5]).

Corollary 3.1. *i) Let \mathbb{K} be a compact set. Then, there exists a finite set $\Gamma \subset \mathbb{Z} \times \mathbb{Z}^+$ such that $\mathbb{K} \cap B(x_{n,k}, 2^{1-n/2}) = \emptyset$ whenever $(n, k) \notin \Gamma$.*

ii) There exists a constant $C > 0$ such that for every $x \in \mathbb{R}^d$,

$$\text{card} \{(n, k) \in \mathbb{Z} \times \mathbb{Z}^+ : B(x_{n,k}, 2^{1-n/2}) \cap B(x, 2\rho(x)) \neq \emptyset\} \leq C.$$

iii) There exists a constant $C > 0$ such that for every ball $B(x, r)$ with $\rho(x) \leq r$, we have

$$|B(x, r)| \leq \sum_{B(x_{n,k}, 2^{-n/2}) \cap B(x, r) \neq \emptyset} |B(x_{n,k}, 2^{-n/2})| \leq C|B(x, r)|.$$

The key point in the proof of Theorem 1.1 is the following result that we will prove in the last section.

Theorem 3.1. *The space $C_c^\infty(\mathbb{R}^d)$ is dense in the space $VMO_L(\mathbb{R}^d)$.*

To prove Theorem 1.1, we need also the following two lemmas.

Lemma 3.3 (see [10], Lemma 6.5). *Let $1 < q \leq \infty$, $n \in \mathbb{Z}$ and $x \in \mathcal{B}_n$. Suppose that $f \in h_n^1(\mathbb{R}^d)$ with $\text{supp } f \subset B(x, 2^{1-n/2})$. Then, there are (H_L^1, q) -atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \sum_{j=1}^{\infty} |\lambda_j| \leq C\|f\|_{h_n^1}$$

with a positive constant C independent of n and f .

Lemma 3.4 (see (4.7) in [6]). *For every $f \in H_L^1(\mathbb{R}^d)$, we have*

$$\sum_{n,k} \|\psi_{n,k} f\|_{h_n^1} \leq C\|f\|_{H_L^1}.$$

Now, we are ready to give the proof of the main theorem.

Proof of Theorem 1.1. By assumption, there exists $\mathcal{M} > 0$ such that

$$\|f_j\|_{H_L^1} \leq \mathcal{M}, \quad \text{for all } j \geq 1.$$

Let $(n, k) \in \mathbb{Z} \times \mathbb{Z}^+$. Then, for almost every $x \in \mathbb{R}^d$, $\psi_{n,k}(x)f_j(x) \rightarrow \psi_{n,k}(x)f(x)$ since $f_j(x) \rightarrow f(x)$. By Theorem C, this yields that $\psi_{n,k}f$ belongs to $h_n^1(\mathbb{R}^d)$ and $\{\psi_{n,k}f_j\}_{j \geq 1}$ weak*-converges to $\psi_{n,k}f$ in $h_n^1(\mathbb{R}^d)$, that is,

$$(3.1) \quad \lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} \psi_{n,k}(x)f_j(x)\phi(x)dx = \int_{\mathbb{R}^d} \psi_{n,k}(x)f(x)\phi(x)dx,$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$. Furthermore,

$$(3.2) \quad \|\psi_{n,k}f\|_{h_n^1} \leq \liminf_{j \rightarrow \infty} \|\psi_{n,k}f_j\|_{h_n^1}.$$

As $x_{n,k} \in \mathcal{B}_n$ and $\text{supp } \psi_{n,k}f \subset B(x_{n,k}, 2^{1-n/2})$, by Lemma 3.3, there are $(H_L^1, 2)$ -atoms $a_j^{n,k}$ related to the balls $B(x_j^{n,k}, r_j^{n,k}) \subset B(x_{n,k}, 2^{2-n/2})$ such that

$$\psi_{n,k}f = \sum_j \lambda_j^{n,k} a_j^{n,k}, \quad \sum_j |\lambda_j^{n,k}| \leq C \|\psi_{n,k}f\|_{h_n^1}.$$

Let $N, K \in \mathbb{Z}^+$ be arbitrary. Then, the above together with (3.2) and Lemma 3.4 imply that there exists $m_{N,K} \in \mathbb{Z}^+$ such that

$$\begin{aligned} \sum_{n=-N}^N \sum_{k=1}^K \sum_j |\lambda_j^{n,k}| &\leq \sum_{n=-N}^N \sum_{k=1}^K C \left(\frac{\mathcal{M}}{(1+n^2)(1+k^2)} + \|\psi_{n,k}f_{m_{N,K}}\|_{h_n^1} \right) \\ &\leq C \sum_{n,k} \frac{\mathcal{M}}{(1+n^2)(1+k^2)} + C \|f_{m_{N,K}}\|_{H_L^1} \\ &\leq C\mathcal{M}, \end{aligned}$$

where the constants C are independent of N, K . By Theorem A, this allows to conclude that

$$f = \sum_{n,k} \psi_{n,k}f \in H_L^1(\mathbb{R}^d) \quad \text{and} \quad \|f\|_{H_L^1} \leq \sum_{n,k} \sum_j |\lambda_j^{n,k}| \leq C\mathcal{M}.$$

Finally, we need to show that for every $\phi \in VMO_L(\mathbb{R}^d)$,

$$(3.3) \quad \lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} f_j(x)\phi(x)dx = \int_{\mathbb{R}^d} f(x)\phi(x)dx.$$

By Theorem 3.1, we only need to prove (3.3) for $\phi \in C_c^\infty(\mathbb{R}^d)$. In fact, by (i) of Corollary 3.1, there exists a finite set $\Gamma_\phi \subset \mathbb{Z} \times \mathbb{Z}^+$ such that

$$f\phi = \sum_{(n,k) \in \Gamma_\phi} \psi_{n,k}f\phi \quad \text{and} \quad f_j\phi = \sum_{(n,k) \in \Gamma_\phi} \psi_{n,k}f_j\phi$$

since $\text{supp } \psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$. This together with (3.1) give

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} f_j(x) \phi(x) dx &= \lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} \sum_{(n,k) \in \Gamma_\phi} \psi_{n,k}(x) f_j(x) \phi(x) dx \\ &= \sum_{(n,k) \in \Gamma_\phi} \lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} \psi_{n,k}(x) f_j(x) \phi(x) dx \\ &= \sum_{(n,k) \in \Gamma_\phi} \int_{\mathbb{R}^d} \psi_{n,k}(x) f(x) \phi(x) dx \\ &= \int_{\mathbb{R}^d} f(x) \phi(x) dx, \end{aligned}$$

which ends the proof of Theorem 1.1. \square

4. PROOF OF THEOREM 3.1

The main point in the proof of Theorem 3.1 is the theorem.

Theorem 4.1. *Let $CMO_L(\mathbb{R}^d)$ be the closure of $C_c^\infty(\mathbb{R}^d)$ in $BMO_L(\mathbb{R}^d)$. Then, $H_L^1(\mathbb{R}^d)$ is the dual space of $CMO_L(\mathbb{R}^d)$.*

To prove Theorem 4.1, we need the following three lemmas.

Lemma 4.1. *There exists a constant $C > 0$ such that*

$$2^{-n/2} \leq Cr$$

whenever $B(x_{n,k}, 2^{1-n/2}) \cap B(x, r) \neq \emptyset$ and $\rho(x) \leq r$.

The proof of Lemma 4.1 follows directly from Proposition 2.1. We leave the details to the reader.

Lemma 4.2. *Let $\psi_{n,k}$, $(n, k) \in \mathbb{Z} \times \mathbb{Z}^+$, be as in Lemma 3.2. Then, there exists a constant C independent of $n, k, \psi_{n,k}$, such that*

$$(4.1) \quad \|\psi_{n,k} f\|_{bmo_n} \leq C \|f\|_{bmo_n}$$

for all $f \in bmo_n(\mathbb{R}^d)$, and

$$(4.2) \quad \|\psi_{n,k} \phi\|_{BMO_L} \leq C \|\phi\|_{bmo_n}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

Lemma 4.3. *For every $f \in BMO_L(\mathbb{R}^d)$, we have*

$$\|f\|_{BMO_L} \approx \sup_{r \leq \rho(x)} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f_{B(x, r)}| dy + \sup_{\rho(x) \leq r \leq 2\rho(x)} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

Proof of Lemma 4.2. Noting that $\psi_{n,k}$ is a multiplier of $bmo_n(\mathbb{R}^d)$ and $\|\psi_{n,k}\|_{L^\infty} \leq 1$, Theorem 2 of [12] allows us to reduce (4.1) to showing that

$$(4.3) \quad \frac{\log\left(e + \frac{2^{-n/2}}{r}\right)}{|B(x,r)|} \int_{B(x,r)} \left| \psi_{n,k}(y) - \frac{1}{|B(x,r)|} \int_{B(x,r)} \psi_{n,k}(z) dz \right| dy \leq C$$

holds for every ball $B(x,r)$ which satisfies $r \leq 2^{-n/2}$. In fact, from $\|\nabla\psi_{n,k}\|_{L^\infty} \leq C2^{n/2}$ and the estimate $\frac{r}{2^{-n/2}} \log\left(e + \frac{2^{-n/2}}{r}\right) \leq \sup_{0 < t \leq 1} t \log(e + 1/t) < \infty$,

$$\begin{aligned} & \frac{\log\left(e + \frac{2^{-n/2}}{r}\right)}{|B(x,r)|} \int_{B(x,r)} \left| \psi_{n,k}(y) - \frac{1}{|B(x,r)|} \int_{B(x,r)} \psi_{n,k}(z) dz \right| dy \\ & \leq \frac{\log\left(e + \frac{2^{-n/2}}{r}\right)}{|B(x,r)|^2} \int_{B(x,r)} \int_{B(x,r)} |\psi_{n,k}(y) - \psi_{n,k}(z)| dz dy \\ & \leq \log\left(e + \frac{2^{-n/2}}{r}\right) \|\nabla\psi_{n,k}\|_{L^\infty} 2r \\ & \leq C \frac{r}{2^{-n/2}} \log\left(e + \frac{2^{-n/2}}{r}\right) \leq C, \end{aligned}$$

which proves (4.3), and thus (4.1) holds.

As (4.1) holds, we get

$$\|\psi_{n,k}\phi\|_{BMO} \leq \|\psi_{n,k}\phi\|_{bmo_n} \leq C\|\phi\|_{bmo_n}.$$

Therefore, to prove (4.2), we only need to show that

$$(4.4) \quad \frac{1}{|B(x,r)|} \int_{B(x,r)} |\psi_{n,k}(y)\phi(y)| dy \leq C\|\phi\|_{bmo_n}$$

holds for every $x \in \mathbb{R}^d$ and $r \geq \rho(x)$. Since $\text{supp } \psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$, (4.4) is obvious if $B(x,r) \cap B(x_{n,k}, 2^{1-n/2}) = \emptyset$. Otherwise, as $\rho(x) \leq r$, Lemma 4.1 gives $2^{-n/2} \leq Cr$. As a consequence, we get

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\psi_{n,k}(y)\phi(y)| dy & \leq C \sup_{2^{-n/2} \leq r} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\phi(y)| dy \\ & \leq C\|\phi\|_{bmo_n}, \end{aligned}$$

which proves (4.4), and hence (4.2) holds. \square

Proof of Lemma 4.3. Clearly, it is sufficient to prove that

$$(4.5) \quad \sup_{\rho(x) \leq r} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy \leq C \sup_{\rho(x) \leq r \leq 2\rho(x)} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

In fact, for every ball $B(x, r)$ which satisfies $\rho(x) \leq r$, setting

$$G = \{(n, k) \in \mathbb{Z} \times \mathbb{Z}^+ : B(x_{n,k}, 2^{-n/2}) \cap B(x, r) \neq \emptyset\},$$

one has

$$B(x, r) \subset \cup_{(n,k) \in G} B(x_{n,k}, 2^{-n/2}) \quad \text{and} \quad \sum_{(n,k) \in G} |B(x_{n,k}, 2^{-n/2})| \leq C|B(x, r)|$$

since $\mathbb{R}^d = \cup_{n \in \mathbb{Z}} \mathcal{B}_n \subset \cup_{n,k} B(x_{n,k}, 2^{-n/2})$ and (iii) of Corollary 3.1. Therefore,

$$\begin{aligned} & \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy \leq \frac{1}{|B(x, r)|} \int_{\cup_{(n,k) \in G} B(x_{n,k}, 2^{-n/2})} |f(y)| dy \\ & \leq \frac{1}{|B(x, r)|} \sum_{(n,k) \in G} |B(x_{n,k}, 2^{-n/2})| \sup_{\rho(z) \leq s \leq 2\rho(z)} \frac{1}{|B(z, s)|} \int_{B(z, s)} |f(y)| dy \\ & \leq C \sup_{\rho(z) \leq s \leq 2\rho(z)} \frac{1}{|B(z, s)|} \int_{B(z, s)} |f(y)| dy, \end{aligned}$$

which implies that (4.5) holds. \square

Proof of Theorem 4.1. Since $CMO_L(\mathbb{R}^d)$ is a subspace of $BMO_L(\mathbb{R}^d)$, which is the dual of $H_L^1(\mathbb{R}^d)$, every function f in $H_L^1(\mathbb{R}^d)$ determines a bounded linear functional on $CMO_L(\mathbb{R}^d)$ of norm bounded by $\|f\|_{H_L^1}$.

Conversely, given a bounded linear functional \mathcal{T} on $CMO_L(\mathbb{R}^d)$. Then, for every $(n, k) \in \mathbb{Z} \times \mathbb{Z}^+$, from (4.2) and density of $C_c^\infty(\mathbb{R}^d)$ in $vmO_n(\mathbb{R}^d)$, the linear functional $\mathcal{T}_{n,k}(g) \mapsto \mathcal{T}(\psi_{n,k}g)$ is continuous on $vmO_n(\mathbb{R}^d)$. Consequently, by Theorem B, there exists $f_{n,k} \in h_n^1(\mathbb{R}^d)$ such that for every $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$(4.6) \quad \mathcal{T}(\psi_{n,k}\phi) = \mathcal{T}_{n,k}(\phi) = \int_{\mathbb{R}^d} f_{n,k}(y)\phi(y)dy,$$

moreover,

$$(4.7) \quad \|f_{n,k}\|_{h_n^1} \leq C\|\mathcal{T}_{n,k}\|,$$

where C is a positive constant independent of $n, k, \psi_{n,k}$ and \mathcal{T} .

Noting that $\text{supp } \psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$, (4.6) implies that $\text{supp } f_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$. Consequently, as $x_{n,k} \in \mathcal{B}_n$, Lemma 3.3 yields that there are

$(H_L^1, 2)$ -atoms $a_j^{n,k}$ related to the balls $B(x_j^{n,k}, r_j^{n,k})$ such that

$$(4.8) \quad f_{n,k} = \sum_{j=1}^{\infty} \lambda_j^{n,k} a_j^{n,k}, \quad \sum_{j=1}^{\infty} |\lambda_j^{n,k}| \leq C \|f_{n,k}\|_{h_n^1}$$

with a positive constant C independent of $\psi_{n,k}$ and $f_{n,k}$.

Since $\text{supp } f_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$, by Lemma 3.1, the function

$$x \mapsto f(x) = \sum_{n,k} f_{n,k}(x)$$

is well defined, and belongs to $L_{\text{loc}}^1(\mathbb{R}^d)$. Moreover, for every $\phi \in C_c^\infty(\mathbb{R}^d)$, by (i) of Corollary 3.1, there exists a finite set $\Gamma_\phi \subset \mathbb{Z} \times \mathbb{Z}^+$ such that

$$\mathcal{T}(\phi) = \sum_{(n,k) \in \Gamma_\phi} \mathcal{T}(\psi_{n,k}\phi) = \sum_{(n,k) \in \Gamma_\phi} \int_{\mathbb{R}^d} f_{n,k}(y)\phi(y)dy = \int_{\mathbb{R}^d} f(y)\phi(y)dy.$$

Next, we need to show that $f \in H_L^1(\mathbb{R}^d)$.

We first claim that there exists $C > 0$ such that

$$(4.9) \quad \sum_{n,k} \|f_{n,k}\|_{h_n^1} \leq C \|\mathcal{T}\|.$$

Assume that (4.9) holds for a moment. Then, from (4.8), there are $(H_L^1, 2)$ -atoms $a_j^{n,k}$ and complex numbers $\lambda_j^{n,k}$ such that

$$f = \sum_{n,k} \sum_j \lambda_j^{n,k} a_j^{n,k} \quad \text{and} \quad \sum_{n,k} \sum_j |\lambda_j^{n,k}| \leq C \sum_{n,k} \|f_{n,k}\|_{h_n^1} \leq C \|\mathcal{T}\|.$$

By Theorem A, this proves that $f \in H_L^1(\mathbb{R}^d)$, moreover, $\|f\|_{H_L^1} \leq C \|\mathcal{T}\|$.

Now, we return to prove (4.9).

Without loss of generality, we can assume that \mathcal{T} is a real-valued functional. By (4.7), for each $(n,k) \in \mathbb{Z} \times \mathbb{Z}^+$, there exists $\phi_{n,k} \in C_c^\infty(\mathbb{R}^d)$ such that

$$(4.10) \quad \|\phi_{n,k}\|_{vmo_n} \leq 1 \quad \text{and} \quad \|f_{n,k}\|_{h_n^1} \leq C \mathcal{T}(\psi_{n,k}\phi_{n,k}).$$

For any $\Gamma \subset \mathbb{Z} \times \mathbb{Z}^+$ a finite set, let $\phi = \sum_{(n,k) \in \Gamma} \psi_{n,k}\phi_{n,k} \in C_c^\infty(\mathbb{R}^d)$. We prove that $\|\phi\|_{BMO_L} \leq C$. Indeed, let $B(x, r)$ be an arbitrary ball satisfying $r \leq 2\rho(x)$. Then, by (ii) of Corollary 3.1, we get

$$\text{card} \{(n,k) \in \mathbb{Z} \times \mathbb{Z}^+ : B(x_{n,k}, 2^{1-n/2}) \cap B(x, r) \neq \emptyset\} \leq C.$$

This together with (4.1) and (4.10) give

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\phi(y) - \phi_{B(x,r)}| dy &\leq C \sup_{(n,k) \in \Gamma} \|\psi_{n,k} \phi_{n,k}\|_{BMO} \\ &\leq C \sup_{(n,k) \in \Gamma} \|\psi_{n,k} \phi_{n,k}\|_{bmo_n} \\ &\leq C \sup_{(n,k) \in \Gamma} \|\phi_{n,k}\|_{bmo_n} \leq C \end{aligned}$$

if $r \leq \rho(x)$, and as (4.4),

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\phi(y)| dy &\leq C \sup_{(n,k) \in \Gamma} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\psi_{n,k}(y) \phi_{n,k}(y)| dy \\ &\leq C \sup_{(n,k) \in \Gamma} \|\phi_{n,k}\|_{bmo_n} \leq C \end{aligned}$$

if $\rho(x) \leq r \leq 2\rho(x)$. Therefore, Lemma 4.3 yields

$$\begin{aligned} \|\phi\|_{BMO_L} &\leq C \left\{ \sup_{r \leq \rho(x)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\phi(y) - \phi_{B(x,r)}| dy + \right. \\ &\quad \left. + \sup_{\rho(x) \leq r \leq 2\rho(x)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\phi(y)| dy \right\} \\ &\leq C \end{aligned}$$

since $B(x,r)$ is an arbitrary ball satisfying $r \leq 2\rho(x)$. This implies that

$$\begin{aligned} \sum_{(n,k) \in \Gamma} \|f_{n,k}\|_{h_n^1} &\leq C \sum_{(n,k) \in \Gamma} \mathcal{T}(\psi_{n,k} \phi_{n,k}) = C\mathcal{T}(\phi) \\ &\leq C\|\mathcal{T}\| \|\phi\|_{BMO_L} \leq C\|\mathcal{T}\|. \end{aligned}$$

Consequently, (4.9) holds since $\Gamma \subset \mathbb{Z} \times \mathbb{Z}^+$ is an arbitrary finite set and the constants C are dependent of Γ . This ends the proof of Theorem 4.1. \square

To prove Theorem 3.1, we need to recall the following lemma.

Lemma 4.4 (see [6], Lemma 3.0). *There is a constant $\varepsilon > 0$ such that for every C' there exists $C > 0$ such that for every $t > 0$ and $|x - y| \leq C'\rho(x)$,*

$$\left| \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} - T_t(x,y) \right| \leq C \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(x)} \right)^\varepsilon.$$

Proof of Theorem 3.1. As $H_L^1(\mathbb{R}^d)$ is the dual space of $VMO_L(\mathbb{R}^d)$ (see Theorem 4.1 of [4]), by Theorem 4.1 and the Hahn-Banach theorem, it suffices to show that $C_c^\infty(\mathbb{R}^d) \subset VMO_L(\mathbb{R}^d)$. In fact, for every $f \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp } f \subset B(0, R_0)$, one only needs to establish the following three steps:

Step 1. By (2.1), one has $\|e^{-tL}f\|_{L^2} \leq \|f\|_{L^2}$ for all $t > 0$. Therefore,

$$\frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy \leq \frac{1}{|B(x,t)|} 4\|f\|_{L^2}^2$$

for all $x \in \mathbb{R}^d$ and $t > 0$. This implies that

$$\gamma_2(f) = \lim_{R \rightarrow \infty} \left(\sup_{x \in \mathbb{R}^d, t \geq R} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy \right)^{1/2} \right) = 0.$$

Step 2. For every $R > 2R_0$ and $B(x,t) \cap B(0,R) = \emptyset$, by (2.1) again,

$$\begin{aligned} & \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy \\ & \leq \frac{1}{|B(x,t)|} \int_{B(x,t)} \left(\frac{1}{(4\pi t)^{d/2}} \int_{B(0,R_0)} e^{-\frac{(R-R_0)^2}{4t}} |f(z)| dz \right)^2 dy \\ & \leq (4\pi)^{-d} \|f\|_{L^1}^2 \frac{1}{t^d} e^{-\frac{R^2}{8t}} \leq (4\pi)^{-d} \|f\|_{L^1}^2 \left(\frac{8d}{R^2} \right)^d e^{-d}. \end{aligned}$$

Therefore,

$$\gamma_3(f) = \lim_{R \rightarrow \infty} \left(\sup_{B(x,t) \cap B(0,R) = \emptyset} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy \right)^{1/2} \right) = 0.$$

Step 3. Finally, we need to show that

$$(4.11) \quad \gamma_1(f) = \lim_{r \rightarrow 0} \left(\sup_{x \in \mathbb{R}^d, t \leq r} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y) - e^{-tL}f(y)|^2 dy \right)^{1/2} \right) = 0.$$

For every $x \in \mathbb{R}^d$ and $t > 0$, we have

$$\begin{aligned} & \left\{ \frac{1}{|B(x,t)|} \int_{B(x,t)} \left| f(y) - \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-z|^2}{4t}} f(z) dz \right|^2 dy \right\}^{1/2} \\ & \leq \sup_{|y-z| < t^{1/4}} |f(y) - f(z)| + 2\|f\|_{L^\infty} \frac{1}{(4\pi t)^{d/2}} \int_{|z| \geq t^{1/4}} e^{-\frac{|z|^2}{4t}} dz. \end{aligned}$$

By the uniform continuity of f , the above implies that

$$\lim_{r \rightarrow 0} \left(\sup_{x \in \mathbb{R}^d, t \leq r} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} \left| f(y) - \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-z|^2}{4t}} f(z) dz \right|^2 dy \right)^{1/2} \right) = 0.$$

Therefore, we can reduce (4.11) to showing that

$$(4.12) \quad \lim_{r \rightarrow 0} \left(\sup_{x \in \mathbb{R}^d, t \leq r} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} \left[\int_{\mathbb{R}^d} \left| \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|y-z|^2}{4t}} - T_t(y, z) \right| |f(z)| dz \right]^2 dy \right)^{1/2} \right) = 0.$$

From $\text{supp } f \subset B(0, R_0)$ and $\mathbb{R}^d \equiv \cup_{n,k} B(x_{n,k}, 2^{-n/2})$, there exists a finite set $\Gamma_f \subset \mathbb{Z} \times \mathbb{Z}^+$ such that $\text{supp } f \subset \cup_{(n,k) \in \Gamma_f} B(x_{n,k}, 2^{-n/2})$. As a consequence, (4.12) holds if we can prove that for each $(n, k) \in \Gamma_f$,

$$(4.13) \quad \lim_{r \rightarrow 0} \left(\sup_{x \in \mathbb{R}^d, t \leq r} \left(\frac{1}{|B(x, t)|} \int_{B(x, t)} \left[\int_{B(x_{n,k}, 2^{-n/2})} \left| \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|y-z|^2}{4t}} - T_t(y, z) \right| |f(z)| dz \right]^2 dy \right)^{1/2} \right) = 0.$$

We now prove (4.13). Let $x \in \mathbb{R}^d$ and $0 < t < 2^{-2n}$. As $x_{n,k} \in \mathcal{B}_n$, by Proposition 2.1, there is a constant $C > 1$ such that $C^{-1}2^{-n/2} \leq \rho(z) \leq C2^{-n/2}$ for all $z \in B(x_{n,k}, 2^{-n/2})$. This together with (2.1) and Lemma 4.4, give

$$\begin{aligned} & \left\{ \frac{1}{|B(x, t)|} \int_{B(x, t)} \left[\int_{B(x_{n,k}, 2^{-n/2})} \left| \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|y-z|^2}{4t}} - T_t(y, z) \right| |f(z)| dz \right]^2 dy \right\}^{1/2} \\ & \leq 2\|f\|_{L^\infty} \frac{1}{(4\pi t)^{d/2}} \int_{|z| \geq t^{1/4}} e^{-\frac{|z|^2}{4t}} dz + C2^{n\varepsilon/2} \|f\|_{L^\infty} \int_{|z| < t^{1/4}} \frac{1}{|z|^{d-\varepsilon}} dz, \end{aligned}$$

which implies that (4.13) holds. The proof of Theorem 3.1 is thus completed. \square

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