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ABSTRACT. Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative function, $V \neq 0$, and belongs to the reverse Hölder class $RH_{d/2}$. In this paper, we prove a version of the classical theorem of Jones and Journé on weak*-convergence in the Hardy space $H^1_L(\mathbb{R}^d)$.

1. INTRODUCTION

A famous and classical result of Fefferman [7] states that the John-Nirenberg space $BMO(\mathbb{R}^d)$ is the dual of the Hardy space $H^1(\mathbb{R}^d)$. It is also well-known that $H^1(\mathbb{R}^d)$ is one of the few examples of separable, nonreflexive Banach space which is a dual space. In fact, let $VMO(\mathbb{R}^d)$ denote the closure of the space $C_c^\infty(\mathbb{R}^d)$ in $BMO(\mathbb{R}^d)$, where $C_c^\infty(\mathbb{R}^d)$ is the set of C^∞-functions with compact support, Coifman and Weiss showed in [1] that $H^1(\mathbb{R}^d)$ is the dual space of $VMO(\mathbb{R}^d)$, which gives to $H^1(\mathbb{R}^d)$ a richer structure than $L^1(\mathbb{R}^d)$. For example, the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ are not bounded on $L^1(\mathbb{R}^d)$, but are bounded on $H^1(\mathbb{R}^d)$. In addition, the weak*-convergence is true in $H^1(\mathbb{R}^d)$, which is useful in the application of Hardy spaces to compensated compactness (see [2]). More precisely, in [9], Jones and Journé proved the following.

Theorem J-J. *Suppose that $\{f_j\}_{j\geq 1}$ is a bounded sequence in $H^1(\mathbb{R}^d)$, and that $f_j(x) \to f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in H^1(\mathbb{R}^d)$ and $\{f_j\}_{j\geq 1}$ weak*-converges to f, that is, for every $\varphi \in VMO(\mathbb{R}^d)$, we have*

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} f_j(x)\varphi(x)dx = \int_{\mathbb{R}^d} f(x)\varphi(x)dx.$$

The aim of this paper is to prove an analogous version of the above theorem in the setting of function spaces associated with Schrödinger operators.

Let $L = -\Delta + V$ be a Schrödinger differential operator on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential, $V \neq 0$, and belongs to the reverse Hölder class $RH_{d/2}$. In the recent years, there is an increasing interest on the study of the problems of harmonic analysis associated with these operators, see for example [4, 5, 6, 10, 11, 13, 14]. In [6], Dziubański and Zienkiewicz considered the Hardy space $H^1_L(\mathbb{R}^d)$ as the set of functions $f \in L^1(\mathbb{R}^d)$ such that

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$\|f\|_{H^1_L} := \|\mathcal{M}_L f\|_{L^1} < \infty$, where $\mathcal{M}_L f(x) := \sup_{t>0} |e^{-tL} f(x)|$. There, they characterized $H^1_L(\mathbb{R}^d)$ in terms of atomic decomposition and in terms of the Riesz transforms associated with L. Later, in [5], Dziubański et al. introduced a BMO-type space $BMO_L(\mathbb{R}^d)$ associated with L, and established the duality between $H^1_L(\mathbb{R}^d)$ and $BMO_L(\mathbb{R}^d)$. Recently, Deng et al. [4] introduced and developed new VMO-type function spaces $VMO_A(\mathbb{R}^d)$ associated with some operators A which have a bounded holomorphic functional calculus on $L^2(\mathbb{R}^d)$. When $A \equiv L$, their space $VMO_L(\mathbb{R}^d)$ is just the set of all functions f in $BMO_L(\mathbb{R}^d)$ such that $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{r\to 0} \left(\sup_{x\in\mathbb{R}^d, t\le r} \Big(\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL} f(y)|^2 dy \Big)^{1/2} \right),$$

$$\gamma_2(f) = \lim_{R\to\infty} \left(\sup_{x\in\mathbb{R}^d, t\ge R} \Big(\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL} f(y)|^2 dy \Big)^{1/2} \right),$$

$$\gamma_3(f) = \lim_{R\to\infty} \left(\sup_{B(x,t)\cap B(0,R)=\emptyset} \Big(\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} |f(y) - e^{-tL} f(y)|^2 dy \Big)^{1/2} \right).$$

The authors in [4] further showed that $H^1_L(\mathbb{R}^d)$ is in fact the dual of $VMO_L(\mathbb{R}^d)$, which allows us to study the weak*-convergence in $H^1_L(\mathbb{R}^d)$. This is useful in the study of the Hardy estimates for commutators of singular integral operators related to L, see for example Theorem 7.1 and Theorem 7.3 of [10].

Our main result is the following theorem.

Theorem 1.1. *Suppose that $\{f_j\}_{j\ge1}$ is a bounded sequence in $H^1_L(\mathbb{R}^d)$, and that $f_j(x) \to f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in H^1_L(\mathbb{R}^d)$ and $\{f_j\}_{j\ge1}$ weak*-converges to f, that is, for every $\varphi \in VMO_L(\mathbb{R}^d)$, we have*

$$\lim_{j\to\infty} \int\limits_{\mathbb{R}^d} f_j(x)\varphi(x)dx = \int\limits_{\mathbb{R}^d} f(x)\varphi(x)dx.$$

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In $\mathbb{R}^d$, we denote by $B = B(x,r)$ an open ball with center x and radius $r > 0$. For any measurable set E, we denote by $|E|$ its Lebesgue measure.

The paper is organized as follows. In Section 2, we present some notations and preliminary results. Section 3 is devoted to the proof of Theorem 1.1. In the last section, we prove that $C^\infty_c(\mathbb{R}^d)$ is dense in the space $VMO_L(\mathbb{R}^d)$.

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2. Some preliminaries and notations

In this paper, we consider the Schrödinger differential operator

$$L = -\Delta + V$$

on $\mathbb{R}^d$, $d \geq 3$, where V is a nonnegative potential, $V \neq 0$. As in the works of Dziubański et al [5, 6], we always assume that V belongs to the reverse Hölder class $RH_{d/2}$. Recall that a nonnegative locally integrable function V is said to belong to a reverse Hölder class RH_q, $1 < q < \infty$, if there exists a constant $C > 0$ such that for every ball $B \subset \mathbb{R}^d$,

$$\Big(\frac{1}{|B|}\int\limits_B (V(x))^q dx\Big)^{1/q} \leq \frac{C}{|B|}\int\limits_B V(x)dx.$$

Let $\{T_t\}_{t>0}$ be the semigroup generated by L and $T_t(x,y)$ be their kernels. Namely,

$$T_t f(x) = e^{-tL}f(x) = \int\limits_{\mathbb{R}^d} T_t(x,y)f(y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$

Since V is nonnegative, the Feynman-Kac formula implies that

$$0 \leq T_t(x,y) \leq \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}. \tag{2.1}$$

According to [6], the space $H^1_L(\mathbb{R}^d)$ is defined as the completion of

$$\{f \in L^2(\mathbb{R}^d) : \mathcal{M}_L f \in L^1(\mathbb{R}^d)\}$$

in the norm

$$\|f\|_{H^1_L} := \|\mathcal{M}_L f\|_{L^1},$$

where $\mathcal{M}_L f(x) := \sup_{t>0} |T_t f(x)|$ for all $x \in \mathbb{R}^d$.

In [5] it was shown that the dual space of $H^1_L(\mathbb{R}^d)$ can be identified with the space $BMO_L(\mathbb{R}^d)$ which consists of all functions $f \in BMO(\mathbb{R}^d)$ with

$$\|f\|_{BMO_L} := \|f\|_{BMO} + \sup_{\rho(x)\leq r} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|dy < \infty, \tag{2.2}$$

where ρ is the auxiliary function defined as in [13], that is,

$$\rho(x) = \sup\Big\{r > 0 : \frac{1}{r^{d-2}}\int\limits_{B(x,r)} V(y)dy \leq 1\Big\}, \tag{2.3}$$

$x \in \mathbb{R}^d$. Clearly, $0 < \rho(x) < \infty$ for all $x \in \mathbb{R}^d$, and thus $\mathbb{R}^d = \bigcup_{n\in\mathbb{Z}} \mathcal{B}_n$, where the sets $\mathcal{B}_n$ are defined by

$$\mathcal{B}_n = \{x \in \mathbb{R}^d : 2^{-(n+1)/2} < \rho(x) \leq 2^{-n/2}\}. \tag{2.4}$$

The following fundamental property of the function ρ is due to Shen [13].

Proposition 2.1 (see [13], Lemma 1.4)**.** *There exist $C_0 > 1$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^d$,*

$$C_0^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq C_0\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}.$$

Let $VMO_L(\mathbb{R}^d)$ be the subspace of $BMO_L(\mathbb{R}^d)$ consisting of those functions f satisfying $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d, t\leq r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)-e^{-tL}f(y)|^2dy\Big)^{1/2}\right),$$

$$\gamma_2(f) = \lim_{R\to\infty}\left(\sup_{x\in\mathbb{R}^d, t\geq R}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)-e^{-tL}f(y)|^2dy\Big)^{1/2}\right),$$

$$\gamma_3(f) = \lim_{R\to\infty}\left(\sup_{B(x,t)\cap B(0,R)=\emptyset}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)-e^{-tL}f(y)|^2dy\Big)^{1/2}\right).$$

In [4] it was shown that $H^1_L(\mathbb{R}^d)$ is the dual space of $VMO_L(\mathbb{R}^d)$.

In the sequel, we denote by $\mathcal{C}_L$ the L-constant

$$\mathcal{C}_L = 8.9^{k_0}C_0$$

where k_0 and C_0 are defined as in Proposition 2.1.

Following Dziubański and Zienkiewicz [6], we define atoms as follows.

Definition 2.1. *Given $1 < q \leq \infty$. A function a is called a (H^1_L, q)-atom related to the ball $B(x_0, r)$ if $r \leq \mathcal{C}_L\rho(x_0)$ and*

i) supp $a \subset B(x_0, r)$,

ii) $\|a\|_{L^q} \leq |B(x_0, r)|^{1/q-1}$,

iii) if $r \leq \frac{1}{\mathcal{C}_L}\rho(x_0)$ then $\int_{\mathbb{R}^d} a(x)dx = 0$.

Then, we have the following atomic characterization of $H^1_L(\mathbb{R}^d)$.

Theorem A (see [6], Theorem 1.5)**.** *Let $1 < q \leq \infty$. A function f is in $H^1_L(\mathbb{R}^d)$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are (H^1_L, q)-atoms and $\sum_j |\lambda_j| < \infty$. Moreover, there exists a constant $C > 0$ such that*

$$\|f\|_{H^1_L} \leq \inf\left\{\sum_j|\lambda_j| : f = \sum_j \lambda_j a_j\right\} \leq C\|f\|_{H^1_L}.$$

Let $P(x) = (4\pi)^{-d/2}e^{-|x|^2/4}$ be the Gauss function. According to [6], the space $h^1_n(\mathbb{R}^d)$, $n \in \mathbb{Z}$, denotes the space of all integrable functions f such that

$$\mathcal{M}_n f(x) = \sup_{0<t<2^{-n/2}}|P_t * f(x)| \in L^1(\mathbb{R}^d),$$

where $P_t(\cdot) := t^{-d}P(t^{-1}\cdot)$. The norm on $h^1_n(\mathbb{R}^d)$ is then defined by

$$\|f\|_{h^1_n} := \|\mathcal{M}_n f\|_{L^1}.$$

It was shown in [8] that the dual space of $h^1_n(\mathbb{R}^d)$ can be identified with $bmo_n(\mathbb{R}^d)$ the space of all locally integrable functions f such that

$$\|f\|_{bmo_n} = \|f\|_{BMO} + \sup_{x\in\mathbb{R}^d, 2^{-n/2}\le r} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|dy < \infty.$$

Here and in what follows, for a ball B and a locally integrable function f, we denote by f_B the average of f on B. Following Dafni [3], we define $vmo_n(\mathbb{R}^d)$ as the subspace of $bmo_n(\mathbb{R}^d)$ consisting of those f such that

$$\lim_{\sigma\to 0}\left(\sup_{x\in\mathbb{R}^d, r<\sigma} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|dy\right) = 0$$

and

$$\lim_{R\to\infty}\left(\sup_{B(x,r)\cap B(0,R)=\emptyset, r\ge 2^{-n/2}} \frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|dy\right) = 0.$$

Recall that $C^\infty_c(\mathbb{R}^d)$ is the space of all C^∞-functions with compact support. Then, the following was established by Dafni [3].

Theorem B (see [3], Theorem 6 and Theorem 9)**.** *Let $n \in \mathbb{Z}$. Then,*
i) The space $vmo_n(\mathbb{R}^d)$ is the closure of $C^\infty_c(\mathbb{R}^d)$ in $bmo_n(\mathbb{R}^d)$.
ii) The dual of $vmo_n(\mathbb{R}^d)$ is the space $h^1_n(\mathbb{R}^d)$.

Furthermore, the weak*-convergence is true in $h^1_n(\mathbb{R}^d)$.

Theorem C (see [3], Theorem 11)**.** *Let $n \in \mathbb{Z}$. Suppose that $\{f_j\}_{j\ge1}$ is a bounded sequence in $h^1_n(\mathbb{R}^d)$, and that $f_j(x) \to f(x)$ for almost every $x \in \mathbb{R}^d$. Then, $f \in h^1_n(\mathbb{R}^d)$ and $\{f_j\}_{j\ge1}$ weak*-converges to f, that is, for every $\varphi \in vmo_n(\mathbb{R}^d)$, we have*

$$\lim_{j\to\infty}\int\limits_{\mathbb{R}^d} f_j(x)\varphi(x)dx = \int\limits_{\mathbb{R}^d} f(x)\varphi(x)dx.$$

3. Proof of Theorem 1.1

We begin by recalling the following two lemmas due to [6]. These two lemmas together with Proposition 2.1 play an important role in our study.

Lemma 3.1 (see [6], Lemma 2.3). *There exists a constant $C > 0$ and a collection of balls $B_{n,k} = B(x_{n,k}, 2^{-n/2})$, $n \in \mathbb{Z}, k = 1, 2, ...$, such that $x_{n,k} \in \mathcal{B}_n$, $\mathcal{B}_n \subset \bigcup_k B_{n,k}$, and*

$$card\,\{(n', k') : B(x_{n,k}, R2^{-n/2}) \cap B(x_{n',k'}, R2^{-n/2}) \neq \emptyset\} \leq R^C$$

for all n, k and $R \geq 2$.

Lemma 3.2 (see [6], Lemma 2.5). *There are nonnegative C^∞-functions $\psi_{n,k}$, $n \in \mathbb{Z}, k = 1, 2, ...$, supported in the balls $B(x_{n,k}, 2^{1-n/2})$ such that*

$$\sum_{n,k} \psi_{n,k} = 1 \quad and \quad \|\nabla \psi_{n,k}\|_{L^\infty} \leq C 2^{n/2}.$$

The following corollary is useful, its proof follows directly from Lemma 3.1. We omit the details here (see also Corollary 1 of [5]).

Corollary 3.1. *i) Let $\mathbb{K}$ be a compact set. Then, there exists a finite set $\Gamma \subset \mathbb{Z} \times \mathbb{Z}^+$ such that $\mathbb{K} \cap B(x_{n,k}, 2^{1-n/2}) = \emptyset$ whenever $(n, k) \notin \Gamma$.*

ii) There exists a constant $C > 0$ such that for every $x \in \mathbb{R}^d$,

$$card\,\{(n, k) \in \mathbb{Z} \times \mathbb{Z}^+ : B(x_{n,k}, 2^{1-n/2}) \cap B(x, 2\rho(x)) \neq \emptyset\} \leq C.$$

iii) There exists a constant $C > 0$ such that for every ball $B(x, r)$ with $\rho(x) \leq r$, we have

$$|B(x, r)| \leq \sum_{B(x_{n,k}, 2^{-n/2}) \cap B(x,r) \neq \emptyset} |B(x_{n,k}, 2^{-n/2})| \leq C|B(x, r)|.$$

The key point in the proof of Theorem 1.1 is the following result that we will prove in the last section.

Theorem 3.1. *The space $C^\infty_c(\mathbb{R}^d)$ is dense in the space $VMO_L(\mathbb{R}^d)$.*

To prove Theorem 1.1, we need also the following two lemmas.

Lemma 3.3 (see [10], Lemma 6.5). *Let $1 < q \leq \infty$, $n \in \mathbb{Z}$ and $x \in \mathcal{B}_n$. Suppose that $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x, 2^{1-n/2})$. Then, there are (H^1_L, q)-atoms a_j related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and*

$$f = \sum_{j=1}^\infty \lambda_j a_j, \quad \sum_{j=1}^\infty |\lambda_j| \leq C\|f\|_{h^1_n}$$

with a positive constant C independent of n and f.

Lemma 3.4 (see (4.7) in [6]). *For every $f \in H^1_L(\mathbb{R}^d)$, we have*

$$\sum_{n,k} \|\psi_{n,k} f\|_{h^1_n} \leq C\|f\|_{H^1_L}.$$

Now, we are ready to give the proof of the main theorem.

Proof of Theorem 1.1. By assumption, there exists $\mathscr{M} > 0$ such that

$$\|f_j\|_{H^1_L} \le \mathscr{M}\ , \quad \text{for all } j \ge 1.$$

Let $(n,k) \in \mathbb{Z} \times \mathbb{Z}^+$. Then, for almost every $x \in \mathbb{R}^d$, $\psi_{n,k}(x)f_j(x) \to \psi_{n,k}(x)f(x)$ since $f_j(x) \to f(x)$. By Theorem C, this yields that $\psi_{n,k}f$ belongs to $h^1_n(\mathbb{R}^d)$ and $\{\psi_{n,k}f_j\}_{j\ge1}$ weak*-converges to $\psi_{n,k}f$ in $h^1_n(\mathbb{R}^d)$, that is,

$$\lim_{j\to\infty}\int_{\mathbb{R}^d} \psi_{n,k}(x)f_j(x)\phi(x)dx = \int_{\mathbb{R}^d} \psi_{n,k}(x)f(x)\phi(x)dx, \tag{3.1}$$

for all $\phi \in C^\infty_c(\mathbb{R}^d)$. Furthermore,

$$\|\psi_{n,k}f\|_{h^1_n} \le \varliminf_{j\to\infty} \|\psi_{n,k}f_j\|_{h^1_n}. \tag{3.2}$$

As $x_{n,k} \in \mathcal{B}_n$ and supp $\psi_{n,k}f \subset B(x_{n,k}, 2^{1-n/2})$, by Lemma 3.3, there are $(H^1_L, 2)$-atoms $a^{n,k}_j$ related to the balls $B(x^{n,k}_j, r^{n,k}_j) \subset B(x_{n,k}, 2^{2-n/2})$ such that

$$\psi_{n,k}f = \sum_j \lambda^{n,k}_j a^{n,k}_j, \quad \sum_j |\lambda^{n,k}_j| \le C\|\psi_{n,k}f\|_{h^1_n}.$$

Let $N, K \in \mathbb{Z}^+$ be arbitrary. Then, the above together with (3.2) and Lemma 3.4 imply that there exists $m_{N,K} \in \mathbb{Z}^+$ such that

$$\begin{aligned}
\sum_{n=-N}^{N}\sum_{k=1}^{K}\sum_j |\lambda^{n,k}_j| &\le \sum_{n=-N}^{N}\sum_{k=1}^{K} C\Big(\frac{\mathscr{M}}{(1+n^2)(1+k^2)} + \|\psi_{n,k}f_{m_{N,K}}\|_{h^1_n}\Big)\\
&\le C\sum_{n,k}\frac{\mathscr{M}}{(1+n^2)(1+k^2)} + C\|f_{m_{N,K}}\|_{H^1_L}\\
&\le C\mathscr{M},
\end{aligned}$$

where the constants C are independent of N, K. By Theorem A, this allows to conclude that

$$f = \sum_{n,k}\psi_{n,k}f \in H^1_L(\mathbb{R}^d) \quad \text{and} \quad \|f\|_{H^1_L} \le \sum_{n,k}\sum_j |\lambda^{n,k}_j| \le C\mathscr{M}.$$

Finally, we need to show that for every $\phi \in VMO_L(\mathbb{R}^d)$,

$$\lim_{j\to\infty}\int_{\mathbb{R}^d} f_j(x)\phi(x)dx = \int_{\mathbb{R}^d} f(x)\phi(x)dx. \tag{3.3}$$

By Theorem 3.1, we only need to prove (3.3) for $\phi \in C^\infty_c(\mathbb{R}^d)$. In fact, by (i) of Corollary 3.1, there exists a finite set $\Gamma_\phi \subset \mathbb{Z} \times \mathbb{Z}^+$ such that

$$f\phi = \sum_{(n,k)\in\Gamma_\phi}\psi_{n,k}f\phi \quad \text{and} \quad f_j\phi = \sum_{(n,k)\in\Gamma_\phi}\psi_{n,k}f_j\phi$$

since supp $\psi_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$. This together with (3.1) give

$$\begin{aligned}\lim_{j\to\infty}\int_{\mathbb{R}^d} f_j(x)\phi(x)dx &= \lim_{j\to\infty}\int_{\mathbb{R}^d}\sum_{(n,k)\in\Gamma_\phi}\psi_{n,k}(x)f_j(x)\phi(x)dx\\ &= \sum_{(n,k)\in\Gamma_\phi}\lim_{j\to\infty}\int_{\mathbb{R}^d}\psi_{n,k}(x)f_j(x)\phi(x)dx\\ &= \sum_{(n,k)\in\Gamma_\phi}\int_{\mathbb{R}^d}\psi_{n,k}(x)f(x)\phi(x)dx\\ &= \int_{\mathbb{R}^d} f(x)\phi(x)dx,\end{aligned}$$

which ends the proof of Theorem 1.1.

□

4. Proof of Theorem 3.1

The main point in the proof of Theorem 3.1 is the theorem.

Theorem 4.1. *Let $CMO_L(\mathbb{R}^d)$ be the closure of $C_c^\infty(\mathbb{R}^d)$ in $BMO_L(\mathbb{R}^d)$. Then, $H^1_L(\mathbb{R}^d)$ is the dual space of $CMO_L(\mathbb{R}^d)$.*

To prove Theorem 4.1, we need the following three lemmas.

Lemma 4.1. *There exists a constant $C > 0$ such that*

$$2^{-n/2} \le Cr$$

whenever $B(x_{n,k}, 2^{1-n/2}) \cap B(x, r) \neq \emptyset$ and $\rho(x) \le r$.

The proof of Lemma 4.1 follows directly from Proposition 2.1. We leave the details to the reader.

Lemma 4.2. *Let $\psi_{n,k}$, $(n,k) \in \mathbb{Z}\times\mathbb{Z}^+$, be as in Lemma 3.2. Then, there exists a constant C independent of $n, k, \psi_{n,k}$, such that*

$$\|\psi_{n,k} f\|_{bmo_n} \le C\|f\|_{bmo_n} \tag{4.1}$$

for all $f \in bmo_n(\mathbb{R}^d)$, and

$$\|\psi_{n,k}\phi\|_{BMO_L} \le C\|\phi\|_{bmo_n} \tag{4.2}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

Lemma 4.3. *For every $f \in BMO_L(\mathbb{R}^d)$, we have*

$$\|f\|_{BMO_L} \approx \sup_{r\le\rho(x)}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f_{B(x,r)}|dy + \sup_{\rho(x)\le r\le 2\rho(x)}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy.$$

Proof of Lemma 4.2. Noting that $\psi_{n,k}$ is a multiplier of $bmo_n(\mathbb{R}^d)$ and $\|\psi_{n,k}\|_{L^\infty} \le 1$, Theorem 2 of [12] allows us to reduce (4.1) to showing that

$$\frac{\log\left(e+\frac{2^{-n/2}}{r}\right)}{|B(x,r)|}\int\limits_{B(x,r)}\Big|\psi_{n,k}(y)-\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}\psi_{n,k}(z)dz\Big|dy\le C \tag{4.3}$$

holds for every ball $B(x,r)$ which satisfies $r\le 2^{-n/2}$. In fact, from $\|\nabla\psi_{n,k}\|_{L^\infty}\le C2^{n/2}$ and the estimate $\frac{r}{2^{-n/2}}\log\left(e+\frac{2^{-n/2}}{r}\right)\le \sup_{0<t\le 1} t\log(e+1/t)<\infty$,

$$\begin{aligned}
&\frac{\log\left(e+\frac{2^{-n/2}}{r}\right)}{|B(x,r)|}\int\limits_{B(x,r)}\Big|\psi_{n,k}(y)-\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}\psi_{n,k}(z)dz\Big|dy\\
\le\ &\frac{\log\left(e+\frac{2^{-n/2}}{r}\right)}{|B(x,r)|^2}\int\limits_{B(x,r)}\int\limits_{B(x,r)}|\psi_{n,k}(y)-\psi_{n,k}(z)|dzdy\\
\le\ &\log\left(e+\frac{2^{-n/2}}{r}\right)\|\nabla\psi_{n,k}\|_{L^\infty}2r\\
\le\ &C\frac{r}{2^{-n/2}}\log\left(e+\frac{2^{-n/2}}{r}\right)\le C,
\end{aligned}$$

which proves (4.3), and thus (4.1) holds.

As (4.1) holds, we get

$$\|\psi_{n,k}\phi\|_{BMO}\le\|\psi_{n,k}\phi\|_{bmo_n}\le C\|\phi\|_{bmo_n}.$$

Therefore, to prove (4.2), we only need to show that

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\psi_{n,k}(y)\phi(y)|dy\le C\|\phi\|_{bmo_n} \tag{4.4}$$

holds for every $x\in\mathbb{R}^d$ and $r\ge\rho(x)$. Since supp $\psi_{n,k}\subset B(x_{n,k},2^{1-n/2})$, (4.4) is obvious if $B(x,r)\cap B(x_{n,k},2^{1-n/2})=\emptyset$. Otherwise, as $\rho(x)\le r$, Lemma 4.1 gives $2^{-n/2}\le Cr$. As a consequence, we get

$$\begin{aligned}
\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\psi_{n,k}(y)\phi(y)|dy &\le C\sup_{2^{-n/2}\le r}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\phi(y)|dy\\
&\le C\|\phi\|_{bmo_n},
\end{aligned}$$

which proves (4.4), and hence (4.2) holds.

□

Proof of Lemma 4.3. Clearly, it is sufficient to prove that

$$\sup_{\rho(x)\le r}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)|dy\le C\sup_{\rho(x)\le r\le 2\rho(x)}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)|dy. \tag{4.5}$$

In fact, for every ball $B(x,r)$ which satisfies $\rho(x)\le r$, setting

$$G=\{(n,k)\in\mathbb{Z}\times\mathbb{Z}^+ : B(x_{n,k},2^{-n/2})\cap B(x,r)\neq\emptyset\},$$

one has

$$B(x,r)\subset\cup_{(n,k)\in G}B(x_{n,k},2^{-n/2})\quad\text{and}\quad\sum_{(n,k)\in G}|B(x_{n,k},2^{-n/2})|\le C|B(x,r)|$$

since $\mathbb{R}^d=\cup_{n\in\mathbb{Z}}\mathcal{B}_n\subset\cup_{n,k}B(x_{n,k},2^{-n/2})$ and (iii) of Corollary 3.1. Therefore,

$$\begin{aligned}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|f(y)|dy&\le\frac{1}{|B(x,r)|}\int\limits_{\cup_{(n,k)\in G}B(x_{n,k},2^{-n/2})}|f(y)|dy\\ &\le\frac{1}{|B(x,r)|}\sum_{(n,k)\in G}|B(x_{n,k},2^{-n/2})|\sup_{\rho(z)\le s\le 2\rho(z)}\frac{1}{|B(z,s)|}\int\limits_{B(z,s)}|f(y)|dy\\ &\le C\sup_{\rho(z)\le s\le 2\rho(z)}\frac{1}{|B(z,s)|}\int\limits_{B(z,s)}|f(y)|dy,\end{aligned}$$

which implies that (4.5) holds.

□

Proof of Theorem 4.1. Since $CMO_L(\mathbb{R}^d)$ is a subspace of $BMO_L(\mathbb{R}^d)$, which is the dual of $H^1_L(\mathbb{R}^d)$, every function f in $H^1_L(\mathbb{R}^d)$ determines a bounded linear functional on $CMO_L(\mathbb{R}^d)$ of norm bounded by $\|f\|_{H^1_L}$.

Conversely, given a bounded linear functional $\mathcal{T}$ on $CMO_L(\mathbb{R}^d)$. Then, for every $(n,k)\in\mathbb{Z}\times\mathbb{Z}^+$, from (4.2) and density of $C^\infty_c(\mathbb{R}^d)$ in $vmo_n(\mathbb{R}^d)$, the linear functional $\mathcal{T}_{n,k}(g)\mapsto\mathcal{T}(\psi_{n,k}g)$ is continuous on $vmo_n(\mathbb{R}^d)$. Consequently, by Theorem B, there exists $f_{n,k}\in h^1_n(\mathbb{R}^d)$ such that for every $\phi\in C^\infty_c(\mathbb{R}^d)$,

$$\mathcal{T}(\psi_{n,k}\phi)=\mathcal{T}_{n,k}(\phi)=\int\limits_{\mathbb{R}^d}f_{n,k}(y)\phi(y)dy, \tag{4.6}$$

moreover,

$$\|f_{n,k}\|_{h^1_n}\le C\|\mathcal{T}_{n,k}\|, \tag{4.7}$$

where C is a positive constant independent of $n,k,\psi_{n,k}$ and $\mathcal{T}$.

Noting that supp $\psi_{n,k}\subset B(x_{n,k},2^{1-n/2})$, (4.6) implies that supp $f_{n,k}\subset B(x_{n,k},2^{1-n/2})$. Consequently, as $x_{n,k}\in\mathcal{B}_n$, Lemma 3.3 yields that there are

$(H^1_L, 2)$-atoms $a_j^{n,k}$ related to the balls $B(x_j^{n,k}, r_j^{n,k})$ such that

$$f_{n,k} = \sum_{j=1}^{\infty} \lambda_j^{n,k} a_j^{n,k}, \quad \sum_{j=1}^{\infty} |\lambda_j^{n,k}| \le C \|f_{n,k}\|_{h^1_n} \tag{4.8}$$

with a positive constant C independent of $\psi_{n,k}$ and $f_{n,k}$.

Since supp $f_{n,k} \subset B(x_{n,k}, 2^{1-n/2})$, by Lemma 3.1, the function

$$x \mapsto f(x) = \sum_{n,k} f_{n,k}(x)$$

is well defined, and belongs to $L^1_{\rm loc}(\mathbb{R}^d)$. Moreover, for every $\phi \in C_c^\infty(\mathbb{R}^d)$, by (i) of Corollary 3.1, there exists a finite set $\Gamma_\phi \subset \mathbb{Z} \times \mathbb{Z}^+$ such that

$$\mathcal{T}(\phi) = \sum_{(n,k)\in\Gamma_\phi} \mathcal{T}(\psi_{n,k}\phi) = \sum_{(n,k)\in\Gamma_\phi} \int_{\mathbb{R}^d} f_{n,k}(y)\phi(y)dy = \int_{\mathbb{R}^d} f(y)\phi(y)dy.$$

Next, we need to show that $f \in H^1_L(\mathbb{R}^d)$.

We first claim that there exists $C > 0$ such that

$$\sum_{n,k} \|f_{n,k}\|_{h^1_n} \le C\|\mathcal{T}\|. \tag{4.9}$$

Assume that (4.9) holds for a moment. Then, from (4.8), there are $(H^1_L, 2)$-atoms $a_j^{n,k}$ and complex numbers $\lambda_j^{n,k}$ such that

$$f = \sum_{n,k}\sum_j \lambda_j^{n,k} a_j^{n,k} \quad \text{and} \quad \sum_{n,k}\sum_j |\lambda_j^{n,k}| \le C \sum_{n,k} \|f_{n,k}\|_{h^1_n} \le C\|\mathcal{T}\|.$$

By Theorem A, this proves that $f \in H^1_L(\mathbb{R}^d)$, moreover, $\|f\|_{H^1_L} \le C\|\mathcal{T}\|$.

Now, we return to prove (4.9).

Without loss of generality, we can assume that $\mathcal{T}$ is a real-valued functional. By (4.7), for each $(n,k) \in \mathbb{Z} \times \mathbb{Z}^+$, there exists $\phi_{n,k} \in C_c^\infty(\mathbb{R}^d)$ such that

$$\|\phi_{n,k}\|_{vmo_n} \le 1 \quad \text{and} \quad \|f_{n,k}\|_{h^1_n} \le C\mathcal{T}(\psi_{n,k}\phi_{n,k}). \tag{4.10}$$

For any $\Gamma \subset \mathbb{Z} \times \mathbb{Z}^+$ a finite set, let $\phi = \sum_{(n,k)\in\Gamma} \psi_{n,k}\phi_{n,k} \in C_c^\infty(\mathbb{R}^d)$. We prove that $\|\phi\|_{BMO_L} \le C$. Indeed, let $B(x,r)$ be an arbitrary ball satisfying $r \le 2\rho(x)$. Then, by (ii) of Corollary 3.1, we get

$$card\,\{(n,k) \in \mathbb{Z} \times \mathbb{Z}^+ : B(x_{n,k}, 2^{1-n/2}) \cap B(x,r) \neq \emptyset\} \le C.$$

This together with (4.1) and (4.10) give

$$\begin{aligned}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\phi(y)-\phi_{B(x,r)}|dy &\le C\sup_{(n,k)\in\Gamma}\|\psi_{n,k}\phi_{n,k}\|_{BMO}\\ &\le C\sup_{(n,k)\in\Gamma}\|\psi_{n,k}\phi_{n,k}\|_{bmo_n}\\ &\le C\sup_{(n,k)\in\Gamma}\|\phi_{n,k}\|_{bmo_n}\le C\end{aligned}$$

if $r\le\rho(x)$, and as (4.4),

$$\begin{aligned}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\phi(y)|dy &\le C\sup_{(n,k)\in\Gamma}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\psi_{n,k}(y)\phi_{n,k}(y)|dy\\ &\le C\sup_{(n,k)\in\Gamma}\|\phi_{n,k}\|_{bmo_n}\le C\end{aligned}$$

if $\rho(x)\le r\le 2\rho(x)$. Therefore, Lemma 4.3 yields

$$\begin{aligned}\|\phi\|_{BMO_L} &\le C\Big\{\sup_{r\le\rho(x)}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\phi(y)-\phi_{B(x,r)}|dy+\\ &\qquad+\sup_{\rho(x)\le r\le 2\rho(x)}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|\phi(y)|dy\Big\}\\ &\le C\end{aligned}$$

since $B(x,r)$ is an arbitrary ball satisfying $r\le 2\rho(x)$. This implies that

$$\begin{aligned}\sum_{(n,k)\in\Gamma}\|f_{n,k}\|_{h^1_n} &\le C\sum_{(n,k)\in\Gamma}\mathcal{T}(\psi_{n,k}\phi_{n,k})=C\mathcal{T}(\phi)\\ &\le C\|\mathcal{T}\|\|\phi\|_{BMO_L}\le C\|\mathcal{T}\|.\end{aligned}$$

Consequently, (4.9) holds since $\Gamma\subset\mathbb{Z}\times\mathbb{Z}^+$ is an arbitrary finite set and the constants C are dependent of Γ. This ends the proof of Theorem 4.1.

□

To prove Theorem 3.1, we need to recall the following lemma.

Lemma 4.4 (see [6], Lemma 3.0)**.** *There is a constant $\varepsilon>0$ such that for every C' there exists $C>0$ such that for every $t>0$ and $|x-y|\le C'\rho(x)$,*

$$\Big|\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|x-y|^2}{4t}}-T_t(x,y)\Big|\le C\frac{1}{|x-y|^d}\Big(\frac{|x-y|}{\rho(x)}\Big)^{\varepsilon}.$$

Proof of Theorem 3.1. As $H^1_L(\mathbb{R}^d)$ is the dual space of $VMO_L(\mathbb{R}^d)$ (see Theorem 4.1 of [4]), by Theorem 4.1 and the Hahn-Banach theorem, it suffices to show that $C^\infty_c(\mathbb{R}^d)\subset VMO_L(\mathbb{R}^d)$. In fact, for every $f\in C^\infty_c(\mathbb{R}^d)$ with supp $f\subset B(0,R_0)$, one only needs to establish the following three steps:

Step 1. By (2.1), one has $\|e^{-tL}f\|_{L^2} \le \|f\|_{L^2}$ for all $t>0$. Therefore,

$$\frac{1}{|B(x,t)|}\int\limits_{B(x,t)} |f(y)-e^{-tL}f(y)|^2dy \le \frac{1}{|B(x,t)|}4\|f\|^2_{L^2}$$

for all $x\in\mathbb{R}^d$ and $t>0$. This implies that

$$\gamma_2(f)=\lim_{R\to\infty}\left(\sup_{x\in\mathbb{R}^d,t\ge R}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)-e^{-tL}f(y)|^2dy\Big)^{1/2}\right)=0.$$

Step 2. For every $R>2R_0$ and $B(x,t)\cap B(0,R)=\emptyset$, by (2.1) again,

$$\begin{aligned}
&\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)-e^{-tL}f(y)|^2dy\\
\le\ &\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big(\frac{1}{(4\pi t)^{d/2}}\int\limits_{B(0,R_0)}e^{-\frac{(R-R_0)^2}{4t}}|f(z)|dz\Big)^2dy\\
\le\ &(4\pi)^{-d}\|f\|^2_{L^1}\frac{1}{t^d}e^{-\frac{R^2}{8t}}\le(4\pi)^{-d}\|f\|^2_{L^1}\Big(\frac{8d}{R^2}\Big)^d e^{-d}.
\end{aligned}$$

Therefore,

$$\gamma_3(f)=\lim_{R\to\infty}\left(\sup_{B(x,t)\cap B(0,R)=\emptyset}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)-e^{-tL}f(y)|^2dy\Big)^{1/2}\right)=0.$$

Step 3. Finally, we need to show that

$$\gamma_1(f)=\lim_{r\to0}\left(\sup_{x\in\mathbb{R}^d,t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)-e^{-tL}f(y)|^2dy\Big)^{1/2}\right)=0. \tag{4.11}$$

For every $x\in\mathbb{R}^d$ and $t>0$, we have

$$\begin{aligned}
&\left\{\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big|f(y)-\frac{1}{(4\pi t)^{d/2}}\int\limits_{\mathbb{R}^d}e^{-\frac{|y-z|^2}{4t}}f(z)dz\Big|^2dy\right\}^{1/2}\\
\le\ &\sup_{|y-z|<t^{1/4}}|f(y)-f(z)|+2\|f\|_{L^\infty}\frac{1}{(4\pi t)^{d/2}}\int\limits_{|z|\ge t^{1/4}}e^{-\frac{|z|^2}{4t}}dz.
\end{aligned}$$

By the uniformly continuity of f, the above implies that

$$\lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d,t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big|f(y)-\frac{1}{(4\pi t)^{d/2}}\int\limits_{\mathbb{R}^d}e^{-\frac{|y-z|^2}{4t}}f(z)dz\Big|^2dy\Big)^{1/2}\right)=0.$$

Therefore, we can reduce (4.11) to showing that

(4.12)
$$\lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d,t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big[\int\limits_{\mathbb{R}^d}\Big|\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|y-z|^2}{4t}}-T_t(y,z)\Big||f(z)|dz\Big]^2dy\Big)^{1/2}\right)=0.$$

From supp $f\subset B(0,R_0)$ and $\mathbb{R}^d\equiv\cup_{n,k}B(x_{n,k},2^{-n/2})$, there exists a finite set $\Gamma_f\subset\mathbb{Z}\times\mathbb{Z}^+$ such that supp $f\subset\cup_{(n,k)\in\Gamma_f}B(x_{n,k},2^{-n/2})$. As a consequence, (4.12) holds if we can prove that for each $(n,k)\in\Gamma_f$,

(4.13)
$$\lim_{r\to 0}\left(\sup_{x\in\mathbb{R}^d,t\le r}\Big(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big[\int\limits_{B(x_{n,k},2^{-n/2})}\Big|\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|y-z|^2}{4t}}-T_t(y,z)\Big||f(z)|dz\Big]^2dy\Big)^{1/2}\right)=0.$$

We now prove (4.13). Let $x\in\mathbb{R}^d$ and $0<t<2^{-2n}$. As $x_{n,k}\in\mathcal{B}_n$, by Proposition 2.1, there is a constant $C>1$ such that $C^{-1}2^{-n/2}\le\rho(z)\le C2^{-n/2}$ for all $z\in B(x_{n,k},2^{-n/2})$. This together with (2.1) and Lemma 4.4, give

$$\begin{aligned}&\left\{\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}\Big[\int\limits_{B(x_{n,k},2^{-n/2})}\Big|\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|y-z|^2}{4t}}-T_t(y,z)\Big||f(z)|dz\Big]^2dy\right\}^{1/2}\\ \le\ & 2\|f\|_{L^\infty}\frac{1}{(4\pi t)^{d/2}}\int\limits_{|z|\ge t^{1/4}}e^{-\frac{|z|^2}{4t}}dz+C2^{n\varepsilon/2}\|f\|_{L^\infty}\int\limits_{|z|<t^{1/4}}\frac{1}{|z|^{d-\varepsilon}}dz,\end{aligned}$$

which implies that (4.13) holds. The proof of Theorem 3.1 is thus completed. □

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