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Jérôme Coville

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TRAVELLING FRONTS IN ASYMMETRIC NONLOCAL REACTION DIFFUSION EQUATIONS: THE BISTABLE AND IGNITION CASES

JÉRÔME COVILLE^{1,2}

ABSTRACT. This paper is devoted to the study of the travelling front solutions which appear in a nonlocal reaction-diffusion equations of the form

$$\frac{\partial u}{\partial t} = \mathcal{J} \star u - u + f(u).$$

When the nonlinearity f is of bistable or ignition type, and the dispersion kernel \mathcal{J} is asymmetric, the existence of a travelling wave is proved. The uniqueness of the speed of the front is also established. The construction of the front essentially relies on the vanishing viscosity techniques, some a priori estimates on the speed's front and various comparison principles.

1. Introduction

In 1930, Fisher [26] suggested to model the spatial spread of a mutant in a given population by the following reaction-diffusion equation :

$$(1.1) \quad U_t - \Delta U = U(1 - U),$$

where U represents the gene fraction of the mutant. The dispersion of the genetic characters is assumed to follow a diffusion law while the logistic term $U(1 - U)$ takes into account the saturation of this dispersion process.

Since then, much attention has been devoted to the reaction-diffusion equations, as they have proved to give a robust and accurate description of a wide variety of phenomena, ranging from combustion to bacterial growth, phase transitions, nerve propagation or epidemiology. See for example [2, 5, 24, 28, 31] and their many references.

During the past ten years, the nonlocal versions of (1.1) where the diffusion is modeled by a convolution operator have been introduced to analyze the long range effects of the dispersion. See [4, 16, 21, 22, 32, 34]. The corresponding reaction diffusion equation is then the following:

$$(1.2) \quad U_t - (\mathcal{J} \star U - U) = f(U),$$

where $\mathcal{J} : \mathbb{R}^n \rightarrow \mathbb{R}$ is a nonnegative function of mass one and f a given nonlinearity, see for example [4, 18, 17, 23, 28, 30].

According to the application, three types of nonlinearities are usually considered in the literature: bistable, ignition and monostable. More precisely, for $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(1) < 0$, we say that

- f is of bistable type if there exists $\rho \in (0, 1)$ such that

$$f < 0 \text{ in } (0, \rho), \quad f(\rho) = 0 \quad \text{and} \quad f > 0 \text{ in } (\rho, 1)$$

- f is of ignition type if there exists $\rho \in (0, 1)$ such that

$$f|_{[0, \rho]} \equiv 0, \quad f|_{(\rho, 1)} > 0 \quad \text{and} \quad f(1) = 0.$$

- f is of monostable type if

$$f > 0 \text{ in } (0, 1).$$

In this article we focus on the bistable and ignition nonlinearities, which arise in Ising and combustion models. See [16, 21, 29, 33]. We are particularly interested in the travelling wave solutions of equation (1.2), i.e., the solutions of the form

$$U(x, t) = u(x \cdot e + ct),$$

where $e \in \mathbb{S}^{n-1}$ is a given unit vector, $c \in \mathbb{R}$ is called the wave speed and u the wave profile, which is required to satisfy the equation

$$(1.3) \quad \begin{cases} [J \star u - u] - cu' + f(u) = 0 & \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1, \end{cases}$$

where $u(\pm\infty)$ denotes the limit of $u(z)$ as $z \rightarrow \pm\infty$ and J is the real function defined by

$$J(s) := \int_{\Pi_s} \mathcal{J}(y) dy$$

where $\Pi_s = \{y \in \mathbb{R}^N : \langle y, e \rangle = s\}$.

In the context of an Ising model, the existence of a travelling front solution with unique speed of (1.3) was first investigated by De Masi, Orlandi, Presutti and Triolo [21, 22] for the specific nonlinearity $f(s) := \text{Arctanh}(s) - s$ and a smooth symmetric kernel \mathcal{J} with exponential decay. Then Bates, Fife, Ren and Wang [4] using an homotopy approach investigated the case of a smooth bistable nonlinearity and a smooth symmetric kernel. Later, Alberti and Bellettini [1] using a variational formulation of the problem have extended the results of Bates *et al.* to more general symmetric kernels \mathcal{J} (i.e. $\mathcal{J} \in L^1$) and the general Lipschitz bistable nonlinearities f . Recently in [16] the author has extended the results of Alberti and Bellettini to the case of ignition nonlinearities. The following Theorem summarizes all these results.

Theorem 1.1. [1, 4, 16, 22]

Let $J \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ be such that $J \geq 0$, $J(z) = J(-z)$, $\int_{\mathbb{R}} J = 1$, $\int_{\mathbb{R}} J(z)|z|dz < +\infty$. Let $f \in C^1(\mathbb{R})$ be an ignition or bistable nonlinearity satisfying $f'(0) < 0$. Then there exists an increasing travelling wave u with speed c satisfying (1.3). Furthermore, if v is another travelling wave with speed c' , then $c = c'$ and when u or v is smooth $v(x) = u(x + \tau)$ for some $\tau \in \mathbb{R}$.

When the kernel J is asymmetric the different approaches used in [1, 4, 22] fail and some new ideas are needed. In this direction, assuming that J is smooth, $J \star u - u$ satisfies a strong maximum principle and f is a bistable nonlinearity, Chen [10] has proved the uniqueness of the speed of a travelling wave using a squeezing technique. Moreover, with more assumptions on \mathcal{J} and f , Chen has constructed a travelling wave using a careful analysis of the ω -limit set of the evolution equation (1.2). In particular in his analysis Chen assumes that \mathcal{J} is at least C^2 while f is at least $C^{1,1}$, $f'(0) < 0$ and $f(s) - s$ is monotone decreasing. Recently the author in [14, 15] has extended the uniqueness result of Chen to more general kernel assuming only that $\mathcal{J} \star u - u$ satisfying a strong maximum principle and a nonlinearity f satisfying the following assumption:

There exists $\epsilon > 0$ such that f is non-increasing for $s \leq \epsilon$ (H).

For a L^1 kernel J , a necessary and sufficient condition to ensure that $J \star u - u$ satisfies a strong maximum principle can be expressed as follow:

$J \star u - u$ satisfies a strong maximum principle if and only if the following holds

(H1) There exists $a \leq 0 \leq b$, $a \neq b$ such that $J(a) > 0$ and $J(b) > 0$

A short proof of this well known condition can be found in [15].

The aim of this paper is to construct a travelling wave solution with the most minimal set of assumptions on the nonlinearity f and the kernel J . In particular, we investigate the case when the operator $J \star u - u$ does not satisfy anymore a strong maximum principle and the case of $J := \mu$ a Borel probability measure. We present and develop here an approach briefly discussed by Chen [10] which is inspired by the viscosity solution theory introduced by Crandall - Lions [3, 20, 19].

For simplicity of the presentation of our results, let us assume that J satisfies

(H2) $J \in C^0(\mathbb{R})$, $J \geq 0$, $\int_{\mathbb{R}} J = 1$ and $\int_{\mathbb{R}} J(z)|z|dz < +\infty$.

Our first result concern the existence of a travelling wave. More precisely, we prove the following

Theorem 1.2.

Assume that J satisfies assumption (H2) and let f be of ignition or bistable type and satisfy (H). Then there exists a constant $c \in \mathbb{R}$ and a non-decreasing function u satisfying (1.3). Moreover the speed c is unique.

Observe that the profile u is not *a priori* unique up to translation when J is asymmetric. In particular, when the speed $c = 0$, the solution u may be discontinuous and infinite many profile can exists. There is a very nice example of this phenomena in [4, 10].

With an extra assumption on the speed c , we can still establish the monotone behavior of any bounded solutions of (1.3). Namely, we show the following

Theorem 1.3.

Assume that J satisfies assumption (H2) and let f be of ignition or bistable type which satisfies (H). Let (u, c) be a solution of (1.3) and assume that $c \neq 0$, then u is monotone non decreasing.

1.1. Method and plan. The proof of Theorem 1.2 is achieved in three steps using the vanishing viscosity technique which was briefly suggested in [10]. This approach is inspired by the viscosity solution theory introduced by Crandall and Lions [20] in the early eighty's. In a first step, we study the following problem:

$$(1.4) \quad \begin{cases} \mathcal{M}[u] + f(u) &= 0 & \text{in } \mathbb{R} \\ u(-\infty) &= 0 \\ u(+\infty) &= 1, \end{cases}$$

where

$$(1.5) \quad \mathcal{M}[u] = \mathcal{M}(\epsilon, J, c)[u] = \epsilon u'' + [J \star u - u] - cu',$$

J satisfies (H1&H2), $\epsilon > 0$ and $c \in \mathbb{R}$.

For this problem we show the following:

Theorem 1.4.

Let f be of ignition or bistable type satisfying (H) and assume that J has a compact support, satisfies (H2) and there exists $a \in \mathbb{R}$ such that $J(a) > 0$ and $J(-a) > 0$. Then there exists a constant $c_\epsilon \in \mathbb{R}$ and an increasing function u_ϵ such that (u_ϵ, c_ϵ) is a solution of (1.4). Moreover the solution (u_ϵ, c_ϵ) is unique in following sense: If (v, c') is another solution of (1.4) then $c_\epsilon = c'$ and $v(x) = u_\epsilon(x + \tau)$ for some $\tau \in \mathbb{R}$.

In a second step, using a standard approximation of the kernel, we extend the above theorem to non-compact kernels J .

Theorem 1.5.

Assume that J satisfies (H2) and let f be of ignition or bistable type satisfying (H). Then there exists a constant $c_\epsilon \in \mathbb{R}$ and an increasing function u_ϵ such that (u_ϵ, c_ϵ) is a solution of (1.4). Moreover the solution (u_ϵ, c_ϵ) is unique in following sense: If (v, c') is another solution of (1.4) then $c_\epsilon = c'$ and $v(x) = u_\epsilon(x + \tau)$ for some $\tau \in \mathbb{R}$.

Theorem 1.2 is proved in the final step, where a careful study of the singular limit of (1.4) as $\epsilon \rightarrow 0$ is performed.

Though elementary in nature, the proofs require a number of lemmas which are listed and prove in the Appendix. In Section 2, we establish some *a priori* estimates that we use throughout this paper. Theorems 1.4 and 1.5 are proved in Section 3 and 4 respectively. In the last section, we proved Theorems 1.2 and 1.3.

1.2. Remarks and general comments. Uniqueness of the speed c and the front profile u of 1.4 and 1.5 have already been proved in [14, 15]. We do not repeat the proofs here.

The existence of travelling waves with unique speed is not surprising. Indeed, let us recall a connection between the nonlocal problem (1.2) and a local version, which arises by considering a family of kernels J_ϵ that approach the Dirac mass δ_0 , that is, $J_\epsilon(x) = \frac{1}{\epsilon} J(\frac{x}{\epsilon})$ with $\epsilon > 0$. Assuming that u is smooth and J decays fast enough, by expanding $J_\epsilon \star u - u$ in powers of ϵ we see that

$$\begin{aligned} J_\epsilon \star u(x) - u(x) &= \frac{1}{\epsilon} \int_{\mathbb{R}} J\left(\frac{x-y}{\epsilon}\right) (u(y) - u(x)) dy = \int_{\mathbb{R}} J(-z) (u(x + \epsilon z) - u(x)) dz \\ (1.6) \quad &= \epsilon \beta u'(x) + \epsilon^2 \alpha u''(x) + o(\epsilon^2) \end{aligned}$$

as $\epsilon \rightarrow 0$, where

$$\alpha = \frac{1}{2} \int_{\mathbb{R}} J(z) z^2 dz \quad \text{and} \quad \beta = \int_{\mathbb{R}} J(-z) z dz.$$

Thus there is a formal analogy between $J \star u - u$ and $\beta u'(x) + \epsilon \alpha u''(x)$. When J is symmetric then $\beta = 0$ and the results for travelling waves of (1.3) are similar to those for travelling wave solutions of

$$(1.7) \quad u'' - cu' + f(u) = 0 \quad \text{in } \mathbb{R}, \quad u(-\infty) = 0, \quad u(+\infty) = 1.$$

For (1.7) there exists a unique speed c such that a travelling front solution exists. For general asymmetric J we see from (1.6) that a better analogue than (1.7) for (1.3) is the problem

$$\tilde{u}'' - (c - \tilde{\beta})u' + f(u) = 0 \quad \text{in } \mathbb{R}, \quad u(-\infty) = 0, \quad u(+\infty) = 1$$

for some $\tilde{\beta}$ and $\tilde{\alpha} \geq 0$. This equation is the same as (1.7) with a shift in the speed, that is, the unique speed is $c + \tilde{\beta}$ where c is the unique speed in (1.7). This new speed can be either positive or negative depending on the size and sign of $\tilde{\beta}$ and $\tilde{\alpha}$, which are related to the asymmetry of J .

Using the techniques developed in this paper, Theorem 1.2 can be easily extend to operators of the form

$$\mathcal{E}[u] + \mathcal{J} \star u - u,$$

where \mathcal{E} is a translation invariant elliptic operator which satisfies the positive maximum principle of Courrège [9, 12]:

Positive maximum Principle : *Let A be a continuous linear map from $C^0(\mathbb{R}^n) \rightarrow C(\mathbb{R})$. Then A satisfies the positive maximum principle if and only if*

$$\text{For all } f \in D(A) \text{ and } x \in \mathbb{R}^n, \text{ such that } f(x) = \sup(f) \geq 0, \text{ then } A[f](x) \leq 0.$$

Another consequence of the vanishing viscosity techniques and the estimates proved here is that the existence results in Theorem 1.2 is still true when f is only assume to be Lipschitz and $J := \mu$ is a Borel probability measure with a finite first moment. In this context the convolution is defined by $\mu \star u := \int_{\mathbb{R}} u(x-y) d\mu(y)$. In particular, we can construct a travelling wave solution for the following discrete reaction diffusion equations of the form

$$(1.8) \quad \frac{\partial U}{\partial t} = \frac{1}{2}[U(x+1) + U(x-1) - 2U(x)] + f(U) \text{ in } \mathbb{R}^+ \times \mathbb{R}.$$

Indeed, let ψ be a mollifier with compact support, i.e., a positive even function of mass one with a support on the unit ball. Let us define a kernel J_ϵ the following way :

$$J_\epsilon(x) := \frac{1}{2\epsilon} \left(\psi\left(\frac{x+1}{\epsilon}\right) + \psi\left(\frac{x-1}{\epsilon}\right) \right).$$

We easily check that for any continuous function u

$$\lim_{\epsilon \rightarrow 0} [J_\epsilon \star u(x) - u(x)] \rightarrow \frac{1}{2}[u(x+1) + u(x-1) - 2u(x)].$$

Therefore for any ϵ , according to Theorem 1.2 there exists a travelling wave (u_ϵ, c_ϵ) to the equation (1.3) with J_ϵ . Pick a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ which converges to 0 and let (u_n, c_n) be the corresponding sequence of travelling wave solution. Using now Helly's Theorem and a priori estimate on the speed see Remark 4.1, we can extract a subsequence of the normalized sequence $(u_n, c_n)_{n \in \mathbb{N}}$ which converges to a non trivial travelling wave solution (u, c) of the equation (1.8). The boundary conditions are obtained following the same argument as in the regular case, see Remark 5.3.

To get the results for any Borel probability measure μ , we just observe that μ can be approximated by $\rho_\epsilon \star \mu$ where ρ_ϵ is a regular mollifier. This construction extends known results on the existence of travelling wave in discrete system, see for example [11, 13, 27].

The monotonicity of the front, Theorems 1.3, holds as well for a general Borel probability measure μ provided that μ satisfies the conditions $\text{supp}(\mu) \cap \mathbb{R}^+ \neq \emptyset$ and $\text{supp}(\mu) \cap \mathbb{R}^- \neq \emptyset$, see remarks 5.6. To obtain the uniqueness of the speed of the front we require further that μ satisfies that there exists two real $a, b \in \text{supp}(\mu)$ such that $\frac{a}{b} \notin \mathbb{Q}$, see Remarks 5.8. The last condition ensures that the operator $\mu \star u - u$ satisfies a strong maximum principle.

2. USEFUL ESTIMATES ON THE SOLUTION OF (1.4)

In this section we prove some estimates on solutions (u, c) of (1.4). We start by showing some useful *a priori* estimates on u . More precisely, we have

Lemma 2.1.

Let $\epsilon > 0$ and (u, c) be a bounded smooth non-decreasing solution of the following problem

$$(2.1) \quad \mathcal{M}[u] + f(u) = 0 \quad \text{in } \mathbb{R},$$

$$(2.2) \quad u(0) = \rho,$$

where $\mathcal{M} = \mathcal{M}(\epsilon, J, c)$ is defined in (1.5). Let us denote $\Sigma := \{z \in \mathbb{R} \mid J(-z) > 0\}$. Then the following hold:

- (i) $u'(x), u''(x) \rightarrow 0$ as $x \rightarrow \pm\infty$ and $f(l^\pm) = 0$ where l^\pm denotes the corresponding limits of u at $\pm\infty$.
- (ii) If $l^+ \leq \rho$, then $u \equiv \rho$ and $u' \equiv 0$ in $\Sigma \cup \mathbb{R}^+$.
- (iii) If $l^- \geq \rho$, then $u \equiv \rho$ and $u' \equiv 0$ in $\Sigma \cup \mathbb{R}^-$.
- (iv) $\int_{\mathbb{R}} (u')^2 < \infty$, $\int_{\mathbb{R}} f(u) = (c - \int_{\mathbb{R}} J(-z)z \, dz)[l^+ - l^-] < \infty$.
- (v) When $c \neq 0$, there exists $K > 0$ independent of u and c such that $|c| \|u'\|_{L^\infty(\mathbb{R})} < K$ and $c^2 \|u''\|_{L^\infty(\mathbb{R})} < K|c| \|u'\|_{L^\infty(\mathbb{R})}$.
- (vi) There exists $K' > 0$ such that $\epsilon \|u'\|_{L^\infty(\mathbb{R})} < K' + 2|c|(l^+ - l^-) + \epsilon u'(0)$.

Remark 2.1. Note that when J satisfies (H1), $u \equiv \rho$ in all \mathbb{R} .

The proofs of (i),(iv),(v) follow standard ideas that we can find in [2, 7, 8, 24, 25] in the elliptic case and in [4] in the case of a symmetric kernel. For the sake of completeness of this paper, we give a detailed proof of them.

Proof:

First observe that since u is a bounded non-decreasing function, u achieves its limits at $\pm\infty$. Let l^+ and l^- denote these limits. Observe also that it is a trivial matter to prove (ii)-(iii). Indeed, since u is a non-decreasing smooth function, in both cases u achieves a global extrema at zero. At this point we have:

$$\begin{aligned} 0 &= \mathcal{M}[u](0) + f(u(0)) \\ &= \epsilon u''(0) - cu'(0) + J \star u(0) - \rho + f(\rho) \\ &= J \star u(0) - \rho \\ &= \int_{\mathbb{R}} J(-y)[u(y) - \rho] dy. \end{aligned}$$

Therefore, for all $-y \in \text{supp}(J)$, $u(y) = \rho$. Hence (ii)-(iii) hold.

Now, let us prove the behavior of the derivatives of u at $\pm\infty$.

We only show that $u' \rightarrow 0$ as $x \rightarrow +\infty$. The other cases $u' \rightarrow 0$ as $x \rightarrow -\infty$ and $u'' \rightarrow 0$ as $x \rightarrow \pm\infty$ are obtained similarly.

To obtain $u' \rightarrow 0$ as $x \rightarrow +\infty$, we argue by contradiction. If not, there exists a positive constant α and a sequence $(x_p)_{p \in \mathbb{N}}$ such that

$$(2.3) \quad \forall p \in \mathbb{N} \quad u'(x_p) \geq \alpha > 0.$$

Let $(\gamma_p)_{p \in \mathbb{N}}$ be the following sequence $(\gamma_p)_{p \in \mathbb{N}} = \liminf_{p \in \mathbb{N}} (x_p)_{p \in \mathbb{N}}$. Now let us define the following sequence of functions $(u_p)_{p \in \mathbb{N}} = (u(\cdot + \gamma_p))_{p \in \mathbb{N}}$. Then $(u_p)_{p \in \mathbb{N}}$ is a monotonic

sequence and for each p , u_p is a uniformly bounded non-decreasing function and satisfies

$$\mathcal{M}[u_p] + f(u_p) = 0 \quad \text{in} \quad \mathbb{R}.$$

Using now local $C^{2,\alpha}$ estimates, Helly's Theorem, and diagonal extraction, we deduce that there exists a subsequence still denoted u_p which converges pointwise to some nondecreasing function \tilde{u} . Since $(u_p)_{p \in \mathbb{N}}$ is a monotonic sequence, the all sequence converge pointwise. Moreover, using standard elliptic theory, and Lebesgue's Theorem, then the sequence $(u_p)_{p \in \mathbb{N}}$ converges in the $C_{loc}^{2,\alpha}$ topology to \tilde{u} and \tilde{u} satisfies

$$(2.4) \quad \mathcal{M}[\tilde{u}] + f(\tilde{u}) = 0 \quad \text{in} \quad \mathbb{R}.$$

By definition, u_p converges pointwise to l^+ . Therefore by uniqueness of the limit, $\tilde{u} \equiv l^+$. Hence $u'_p \rightarrow 0$ uniformly on every compact set.

In particular, we have $u'(\gamma_p) = u'_p(0) \rightarrow 0$, which contradicts (2.3).

We can now show that $f(l^\pm) = 0$. Since u is a solution of (2.1), we have

$$(2.5) \quad -f(u) = \mathcal{M}[u] = \epsilon u'' - cu' + J \star u - u.$$

Taking $x \rightarrow +\infty$ in the above equation (2.5) yields

$$-f(l^+) = J \star l^+ - l^+ = l^+ - l^+ = 0.$$

We show that $f(l^-) = 0$ by taking $x \rightarrow -\infty$ in (2.5).

Next, we obtain (iv). Let us first integrate (2.1) over $(-r, r)$.

This leads to

$$\begin{aligned} -\int_{-r}^r f(u) &= \epsilon \int_{-r}^r u'' + \int_{-r}^r (J \star u - u) - c \int_{-r}^r u' \\ &= \epsilon [u'(r) - u'(-r)] + \int_{-r}^r \left(\int_{\mathbb{R}} J(x-y)(u(y) - u(x)) dy \right) dx - c[u(r) - u(-r)] \\ &= \epsilon [u'(r) - u'(-r)] + \int_{-r}^r \left(\int_{\mathbb{R}} J(-z)z \int_0^1 u'(x+tz) dt dz \right) dx - c[u(r) - u(-r)] \\ &= \epsilon [u'(r) - u'(-r)] + \int_0^1 \left(\int_{\mathbb{R}} J(-z)z [u(r+tz) - u(-r+tz)] dz \right) dt - c[u(r) - u(-r)]. \end{aligned}$$

Letting $r \rightarrow \infty$ we find

$$(2.6) \quad \int_{\mathbb{R}} f(u) = (c - \int_{\mathbb{R}} J(-z)z dz) [l^+ - l^-].$$

We denote $M_1 := \int_{\mathbb{R}} J(-z)z dz$.

We obtain the second inequality in a similar way. Multiplying equation (2.1) by u and integrating over $(-r, r)$ leads to

$$\begin{aligned} \int_{-r}^r f(u)u &= -\epsilon \int_{-r}^r u''u - \int_{-r}^r (J \star u - u)u + c \int_{-r}^r u'u \\ &= \epsilon \int_{-r}^{+R} (u'(x))^2 dx - \int_{-r}^r (J \star u(x) - u(x))u(x) dx + \frac{c[u^2(r) - u^2(-r)]}{2} + o(1). \end{aligned}$$

By letting $r \rightarrow \infty$ we find

$$(2.7) \quad \epsilon \int_{\mathbb{R}} (u'(x))^2 dx = -c \frac{[(l^+)^2 - (l^-)^2]}{2} + \int_{\mathbb{R}} (J \star u(x) - u(x))u(x) dx + \int_{\mathbb{R}} f(u(x))u(x) dx$$

and (iv) is proved.

Let us now show (v). From (i) u' vanishes at infinity. Since u is smooth and non-decreasing, u' achieves a global maximum at some point $x_0 \in \mathbb{R}$. At x_0 , since u is uniformly bounded and f globally Lipschitz, we have

$$\begin{aligned} cu'(x_0) &= (J \star u - u)(x_0) + f(u) \\ |cu'(x_0)| &\leq |J \star u(x_0)| + |u|(x_0) + K_1 \|u\|_{L^\infty(\mathbb{R})} \\ |c| \|u'\|_{L^\infty(\mathbb{R})} &\leq K \|u\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

with $K = K_1 + 2$. Differentiating (2.1), it follows that

$$cu'' = f'(u(x))u' - \epsilon u''' + u' - J \star u'.$$

Using that u is bounded and (i), u'' achieves a global extrema at some points x_1, x_2 . Assume that u'' achieves global maximum at x_1 . At this point, we have

$$\begin{aligned} |c| |cu''(x_1)| &= |c| |f'(u(x_1))u'(x_1) + u'(x_1) - J \star u'(x_1)| \\ &\leq (K_1 + 2) |c| \|u'\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Similarly, if u'' achieves global minimum at x_1 we have

$$\begin{aligned} |c| |cu''(x_2)| &= |c| |f'(u(x_2))u'(x_2) + u'(x_2) - J \star u'(x_2)| \\ &\leq (K_1 + 2) |c| \|u'\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Hence (v) holds.

Finally we obtain (vi). Let us integrate equation (2.1) over $(-\infty, x)$. It follows that

$$\begin{aligned} \epsilon u'(x) - c(u(x) - l^-) + \int_{-\infty}^x \int_0^1 \int_{\mathbb{R}} J(-z) z u'(s + tz) dt dz dx &= - \int_{-\infty}^x f(u) \\ \epsilon u'(x) - c(u(x) - l^-) + \int_{-\infty}^x \int_0^1 \int_{\mathbb{R}} J(-z) z u'(s + tz) dt dz dx &\leq - \int_{-\infty}^0 f(u) \end{aligned}$$

Therefore we have

$$(2.8) \quad \epsilon |u'(x)| \leq |c|(l^+ - l^-) + \int_{-\infty}^x \int_0^1 \int_{\mathbb{R}} J(-z) |z| u'(s + tz) dt dz dx + \left| \int_{-\infty}^0 f(u) \right|$$

$$(2.9) \quad \leq \left(\int_{\mathbb{R}} J(z) |z| dz + |c| \right) (l^+ - l^-) + \left| \int_{-\infty}^0 f(u) \right|.$$

Observe that in the case of an ignition nonlinearity, we are done since $f(u) \equiv 0$ on \mathbb{R}^- . Now, let us compute $\int_{-\infty}^0 f(u)$. By integrating (2.1) over \mathbb{R}^- and using Fubini's Theorem, it follows that

$$(2.10) \quad \epsilon u'(0) - c(\rho - l^-) + \int_0^1 \int_{\mathbb{R}} J(-z) z [u(tz) - l^-] dt dz = - \int_{-\infty}^0 f(u).$$

Substituting (2.10) into (2.9) yields

$$\epsilon |u'(x)| \leq 2 \left(\int_{\mathbb{R}} J(z) |z| dz + |c| \right) (l^+ - l^-) + \epsilon u'(0).$$

Hence

$$\epsilon \|u'\|_{L^\infty(\mathbb{R})} \leq K' + 2|c|(l^+ - l^-) + \epsilon u'(0),$$

with $K' := 2 \left(\int_{\mathbb{R}} J(z) |z| dz \right) (l^+ - l^-)$. Thus, (vi) holds.

□

Remark 2.2. Observe that even for a non-smooth function u that has limits at infinity, it yields $\lim_{x \rightarrow \pm\infty} (J \star u - u) = 0$. Therefore, if u satisfies

$$J \star u - u + f(u) = 0 \quad \text{in } \mathbb{R},$$

then l^\pm must satisfy $f(l^\pm) = 0$. For bounded solutions u satisfying $J \star u - u - cu' + f(u) = 0$, we can easily see that, when $c \neq 0$, (i)-(v) hold as well.

We now obtain some *a priori* estimates on the speed c of solutions (u, c) of (1.4). Namely we have

Lemma 2.2.

Let (u, c) be a bounded non-decreasing solution of (1.4). Then there exists positive constants μ, κ such that

$$\mu c^2 - \kappa |c| \leq \epsilon K,$$

where K is defined in the above Lemma 2.1.

Proof:

Without loss of generality, we may assume that $u(0) = \rho$. Observe that since u is non-decreasing and bounded, we have $\|u\|_{L^\infty(\mathbb{R})} \leq u(+\infty) = 1$. We study two cases.

- Case 1 $c \geq 0$:

In this case, let us now integrate (1.4) over \mathbb{R}^- . This leads to

$$\underbrace{\int_{\mathbb{R}^-} (J \star u - u) dy}_{I_1} + \underbrace{\epsilon \int_{\mathbb{R}^-} u'' dy}_{I_2} - \underbrace{c \int_{\mathbb{R}^-} u' dy}_{I_3} = - \underbrace{\int_{\mathbb{R}^-} f(u) dy}_{I_4}.$$

A quick computation shows that:

- $I_2 = \epsilon u'(0)$,
- $I_3 = c\rho$,
- $I_4 \leq 0$.

So we have,

$$c\rho \leq \epsilon u'(0) + |I_1|.$$

Since $c \geq 0$ multiplying the above equation by c leads to

$$c^2 \rho \leq \epsilon c u'(0) + c |I_1|.$$

Using the Fundamental Theorem of Calculus and Fubini's Theorem, we have

$$\begin{aligned} |I_1| &= \left| \int_{\mathbb{R}^-} (J \star u - u) \right| \leq \int_{\mathbb{R}^-} \left(\int_{\mathbb{R}} J(-z) |z| \int_0^1 u'(x + tz) dt dz \right) dx \\ &\leq \int_{\mathbb{R}^-} \left(\int_{\mathbb{R}} J(-z) |z| \int_0^1 u'(x + tz) dt dz \right) dx \\ &\leq \int_{\mathbb{R}} J(-z) |z| \int_0^1 u(tz) dt dz \\ &\leq \kappa, \end{aligned}$$

where $\kappa = \int_{\mathbb{R}} J(z)|z| dz$. Therefore we have

$$\begin{aligned} c^2 \rho &\leq \epsilon c \|u'\|_{L^\infty(\mathbb{R})} + |c| \kappa, \\ c^2 \rho - |c| \kappa &\leq \epsilon K \|u\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Let $\mu := \inf\{\rho, 1 - \rho\}$. Then we have the desired conclusion

$$\mu c^2 - c \kappa \leq \epsilon K.$$

- Case 2 $c \leq 0$:

In this case, we argue similarly. Let us now integrate (1.4) over \mathbb{R}^+ . This leads to

$$\underbrace{\int_{\mathbb{R}^+} (J \star u - u) dy}_{I_5} + \underbrace{\epsilon \int_{\mathbb{R}^+} u'' dy}_{I_6} - \underbrace{c \int_{\mathbb{R}^+} u' dy}_{I_7} = - \underbrace{\int_{\mathbb{R}^+} f(u) dy}_{I_8}.$$

A quick computation shows that:

- $I_6 = -\epsilon u'(0)$,
- $I_7 = c(1 - \rho)$,
- $I_8 \geq 0$.

So we have

$$(1 - \rho)c \geq -\epsilon u'(0) + I_5.$$

Since $c \leq 0$, multiplying the above equation by c leads to

$$(1 - \rho)c^2 \leq -\epsilon c u'(0) + c I_5.$$

Hence

$$(1 - \rho)c^2 \leq -\epsilon c u'(0) + |c| |I_5|.$$

Using a similar computation as for $|I_1|$, we can show that $|I_5| \leq \kappa$. Therefore we obtain

$$(1 - \rho)c^2 - |c| \kappa \leq \epsilon K.$$

Hence

$$\mu c^2 - |c| \kappa \leq \epsilon K.$$

□

Remark 2.3. Observe that the constants K, μ depend only on f and κ depends only on J .

3. Construction of a solution of (3.1) for $\epsilon > 0$ and J with compact support

In this section, assuming that J has compact support, we show that there exists a unique positive increasing solution (u_ϵ, c_ϵ) of the following problem

$$(3.1) \quad \begin{cases} \mathcal{M}[u] + f(u) = 0 & \text{in } \mathbb{R}, \\ u(x) \rightarrow 0 & \text{as } x \rightarrow -\infty, \\ u(0) = \rho, \\ u(x) \rightarrow 1 & \text{as } x \rightarrow +\infty, \end{cases}$$

where $\mathcal{M} = \mathcal{M}(\epsilon, J, c)$ is defined by (1.5) and f is a bistable or an ignition nonlinearity. We prove the following

Theorem 3.1.

Let $\epsilon > 0$ and assume that J has compact support and satisfies (H2). Assume further that there exists $a \in \mathbb{R}$ such that $\{-a, a\} \subset \text{supp}(J)$. Then there exists a unique positive increasing solution (u_ϵ, c_ϵ) of (3.1), i.e. The uniqueness is state as follow, if $(v_\epsilon, c'_\epsilon)$ is a solution of (3.1), then $c_\epsilon = c'_\epsilon$ and $u_\epsilon \equiv v_\epsilon$.

The proof of Theorem 3.1 uses ideas developed by Dupaigne and the present author in [17] combined with ideas of Berestycki–Larroutourou–Lions in [6] and Berestycki–Nirenberg in [8]. We prove Theorem 3.1 in two steps. In the first step, subsections 3.1 and 3.2, using sub- and super-solutions techniques and standard limiting procedure we construct normalised solutions for an approximate problem on a semi-infinite interval. Then in a second step, using a standard limiting procedure and some *a priori* estimates, we extract a solution of (3.1). The uniqueness of the solution has already been established in [14, 15], so we omit it.

3.1. Existence of a normalized solution in an interval. In this subsection we construct an increasing solution (u, c) of the following problem :

$$(3.2) \quad \begin{cases} \mathcal{L}[u] + f(u) + h_R = 0 & \text{for } x \in \Omega, \\ u(-r) = 0, \\ u(+R) = 1, \end{cases}$$

where $\Omega = (-r, +R)$ and where $\mathcal{L} = \mathcal{L}(\epsilon, J, r, R, c)$ and h_R are defined by

$$(3.3) \quad \begin{aligned} \mathcal{L}[u] &= \mathcal{L}(\epsilon, J, r, R, c)[u] = \epsilon u'' + \left[\int_{-r}^{+R} J(x-y)u(y)dy - u \right] - cu', \\ h_R(x) &= \int_{+R}^{+\infty} J(x-y)dy. \end{aligned}$$

Namely, we have for $\epsilon > 0$,

Theorem 3.2.

Assume that J has compact support, satisfies (H2) and there exists $a \in \mathbb{R}$ such that $J(a) > 0$ and $J(-a) > 0$. For any $\epsilon > 0$, $1 \leq r < R$ so that $\text{supp}J \subset (-r, R)$ and $c \in \mathbb{R}$, there exists a unique positive increasing solution u_c of (3.2). Moreover there exists a unique $c := c_\epsilon^{r,R}$ such that $u_{c_\epsilon^{r,R}}(0) = \rho$.

Proof of Theorem 3.2:

Observe that, since the constant functions $\underline{u} = 0$ and $\bar{u} = 1$ are sub- and super-solutions of (3.2) respectively, using a basic iterative scheme (see the Appendix), there exists a solution u_c of (3.2) for any values of $c \in \mathbb{R}$. The uniqueness and the monotonicity of this solution is obtained in [15], so we omit the proof.

Knowing the existence and uniqueness of solution of (3.2), the main difficulty now remains on finding a real c which achieves the normalisation $u_c(0) = \rho$.

Define now the following map:

$$\begin{aligned} T : \mathbb{R} &\rightarrow \mathbb{R} \\ c &\mapsto u_c(0). \end{aligned}$$

Observe that T is continuous by standard elliptic *a priori* estimates and the uniqueness of u_c . Furthermore from the monotonicity and uniqueness of solution of (3.2), T is decreasing.

Indeed, choose any $c' \geq c$. Then u_c is a super-solution of the following problem

$$(3.4) \quad \begin{cases} \mathcal{L}(\epsilon, J, r, R, c')[u_c] + f(u_c) + h_R = (c - c')(u_c)' \leq 0 & \text{in } \Omega \\ u(-r) = 0, \\ u(+R) = 1. \end{cases}$$

Since 0 is a sub-solution and $0 \leq u_c$, it follows that $0 < u_{c'} < u_c$.

Hence $T(c') < T(c)$.

Next we show that there exists c_1 and c_2 such that $T(c_1) < \rho$ and $T(c_2) > \rho$.

Lemma 3.1.

Let ϵ, r, R and J be as in Theorem 3.2. Then there exists c_1 and c_2 such that

$$T(c_1) < \rho < T(c_2).$$

Assume for a moment that the Lemma is proved, then using a basic continuity argument, we achieve $T(c_\epsilon^{r,R}) = \rho$, for some unique real $c_\epsilon^{r,R}$. This ends the proof of Theorem 3.2. \square

Now let us prove the Lemma.

Proof of the Lemma 3.1:

We find c_1 and c_2 using a comparison argument with solutions of (3.2) and adequate sub- and super-solutions. We first extend f continuously by 0 outside $[0, 1]$.

Let us consider the following function $\bar{u} := \rho e^{\lambda_0 x}$ with λ_0 such that $\rho e^{\lambda_0} = 1$. Choose c_1 such that

$$(3.5) \quad \epsilon \lambda_0^2 - c_1 \lambda_0 - 1 + k + \int_{-\infty}^{+\infty} J(-z) e^{\lambda_0 z} dz \leq 0,$$

where k is the Lipschitz constant of f . First observe that, since $\bar{u} \geq 1$ in $[1, +\infty)$ and $\bar{u} \geq 0$ in $(-\infty, -1]$, for any $R > r \geq 1$, we have the following

$$\int_{-r}^R J(x-y) \bar{u}(y) dy + h_R < J \star \bar{u}.$$

Now, let us compute $\mathcal{L}[\bar{u}] + f(\bar{u}) + h_R$ with $c = c_1$. Since J has compact support, f is globally Lipschitz, $\bar{u} \geq 1$ in $[R, +\infty)$ and $\bar{u} \geq 0$ in $(-\infty, -r]$, we have

$$\begin{aligned} \mathcal{L}[\bar{u}] + f(\bar{u}) + h_R &= \epsilon \bar{u}'' - c_1 \bar{u}' + \int_{-r}^{+R} J(x-y) \bar{u}(y) dy - \bar{u} + h_R + f(\bar{u}), \\ &\leq \epsilon \bar{u}'' - c_1 \bar{u}' + \int_{-\infty}^{+\infty} J(x-y) \bar{u}(y) dy - \bar{u} + k \bar{u}, \\ &\leq \rho e^{\lambda_0 x} \left(\epsilon \lambda_0^2 - c_1 \lambda_0 - 1 + k + \int_{-\infty}^{+\infty} J(x-y) e^{\lambda_0(y-x)} dy \right), \\ &\leq \rho e^{\lambda_0 x} \left(\epsilon \lambda_0^2 - c_1 \lambda_0 - 1 + k + \int_{-\infty}^{+\infty} J(-z) e^{\lambda_0 z} dz \right). \end{aligned}$$

Using (3.5), it follows that

$$\mathcal{L}[\bar{u}] + f(\bar{u}) + h_R \leq 0 \quad \text{in } (r, R).$$

Since $\bar{u} > 0$, we get $u_{c_1} < \bar{u}$ and in particular $T(c_1) = u_{c_1}(0) < \bar{u}(0) = \rho$.

To obtain c_2 , we argue similarly. Let us take $\underline{u} := 1 - e^{-\delta_0(x+1)}$ with δ_0 such that $1 - e^{-\delta_0} = \rho$. Choose c_2 so that,

$$(3.6) \quad -\epsilon\delta_0^2 - c_2\delta_0 + 1 - \int_{-\infty}^{+\infty} J(-z)e^{-\delta_0 z} dz \geq M e^{\delta_0},$$

where $M = \max_{[0,1]} f$. For convenience, let us define the constant γ by

$$\gamma := -\epsilon\delta_0^2 - c_2\delta_0 + 1 - \int_{-\infty}^{+\infty} J(-z)e^{-\delta_0 z} dz.$$

From (3.6) it follows that

$$\gamma e^{-\delta_0} - M \geq 0.$$

Observe now that, for any $R, r > 1$, \underline{u} is a sub-solution of (3.2) with $c = c_2$. Indeed, since $\underline{u} \leq 1$ in $[1, +\infty)$ and $\underline{u} \leq 0$ in $(-\infty, -1]$, we have for any $R, r > 1$,

$$\int_{-r}^R J(x-y)\bar{u}(y) + h_R > J \star \underline{u}.$$

Computing $\mathcal{L}[\underline{u}] + f(\underline{u}) + h_R$ in $(-r, R)$ with $c = c_2$ yields

$$\begin{aligned} \mathcal{L}[\underline{u}] + f(\underline{u}) + h_R &= \epsilon \underline{u}'' - c_2 \underline{u}' + \int_{-r}^{+R} J(x-y)\underline{u}(y)dy - \underline{u} + h_R + f(\underline{u}), \\ &\geq \epsilon \underline{u}'' - c_2 \underline{u}' + J \star \underline{u} - \underline{u} + f(\underline{u}), \\ &\geq \gamma e^{-\delta_0(x+1)} + f(\underline{u}). \end{aligned}$$

Observe that in $[0, R)$, since $\underline{u} \geq \rho$, we have $f(\underline{u}) \geq 0$. Therefore we have

$$(3.7) \quad \gamma e^{-\delta_0(x+1)} + f(\underline{u}) \geq 0,$$

since $\gamma \geq 0$. Observe now that in $(-r, 0)$, since $e^{-\delta_0(x+1)} \geq e^{-\delta_0}$ and $f(s) \geq -M$, we have

$$(3.8) \quad \gamma e^{-\delta_0(x+1)} + f(\underline{u}) \geq \gamma e^{-\delta_0} - M \geq 0.$$

By combining (3.7) and (3.8) it follows that

$$\mathcal{L}[\underline{u}] + f(\underline{u}) + h_R \geq 0 \quad \text{in } (r, R).$$

By construction, $\underline{u} \leq 1$. Thus we have $\underline{u} < u_{c_2}$. In particular, we have $T(c_2) = u_{c_2}(0) > \underline{u}(0) = \rho$.

□

An easy consequence of the above proof is the boundedness of the speed $c_\epsilon^{r,R}$. Namely, we have the following corollary:

Corollary 3.1.

Let ϵ, r, R and J be as in Theorem 3.2. Then $c_2 < c_\epsilon^{r,R} < c_1$.

Remark 3.1. Note that the speed c_2 can be negative.

Remark 3.2. The bound c_1 and c_2 found in the Corollary 3.1 are independent of r and R .

Proof:

From Lemma 3.1, we have for any $R > r \geq 1$, $T(c_1) < T(c_\epsilon^{r,R}) < T(c_2)$. Since T is decreasing we have $c_2 < c_\epsilon^{r,R} < c_1$. \square

3.2. Existence of a normalized solution in semi-infinite intervals. In this subsection we construct a normalized solution of the following problem :

$$(3.9) \quad \begin{cases} \mathcal{S}[u] + f(u) = 0 & \text{in } \omega, \\ u(-r) = 0, \\ u(+\infty) = 1, \end{cases}$$

where $\omega := (-r, +\infty)$ and $\mathcal{S}(\epsilon, J, r, c)$ is defined by

$$(3.10) \quad \mathcal{S}[u] = \mathcal{S}(\epsilon, J, r, c)[u] = \epsilon u'' + \left[\int_{-r}^{\infty} J(x-y)u(y)dy - u \right] - cu'.$$

More precisely, we show:

Theorem 3.3.

Let $\epsilon > 0$, J be as in Theorem 3.2 and $r > 1$ be such that $\text{supp} J \subset \omega$. Then there exists a sequence of solutions $(u_\epsilon^{r,R_n}, c_\epsilon^{r,R_n})$ of (3.2) which converges locally uniformly to $(u_\epsilon^r, c_\epsilon^r)$. Moreover, $(u_\epsilon^r, c_\epsilon^r)$ satisfies (3.9) and $u_\epsilon^r(0) = \rho$. Furthermore u_ϵ^r is increasing and satisfies $(u_\epsilon^r)' \geq 0$.

Proof of Theorem 3.3:

Let us fix $\epsilon > 0$ and $r > 1$ such that $\text{supp}(J) \subset \omega$. Let $(R_n)_{n \in \mathbb{N}}$ be a sequence of real numbers which converges to $+\infty$. Since J has compact support, without loss of generality we may assume that $\text{supp}(J) \subset (-r, R_n)$ for all $n \in \mathbb{N}$. Let us denote by (u_n, c_n) the corresponding normalized solution given by theorem 3.2 (i.e., $c_n = c_\epsilon^{r,R_n}$, $u_n := u_{c_\epsilon^{r,R_n}}$).

Using Corollary 3.1 and Remark 3.2, we have $c_2 \leq c_n \leq c_1$ for all $n \in \mathbb{N}$. So we may choose a subsequence of R_n such that c_n converges to some number c with $c_2 \leq c \leq c_1$. Clearly, $h_{R_n} \rightarrow 0$ pointwise as $n \rightarrow \infty$. Observe now that $(u_n)_{n \in \mathbb{N}}$ is a uniformly bounded sequence of increasing functions. Therefore using Helly's Theorem there exists a subsequence which converges pointwise to a non-decreasing function u . Since $\epsilon > 0$, using local $C^{2,\alpha}$ estimates, up to extraction, the subsequence converges in $C_{loc}^{2,\alpha}$. Therefore $u \in C^{2,\alpha}$ and satisfies

$$(3.11) \quad \begin{cases} \epsilon u'' + \int_{-r}^{+\infty} J(x-y)u(y)dy - u - cu' + f(u) = 0 & \text{in } \omega, \\ u(-r) = 0, \\ u(0) = \rho. \end{cases}$$

To end the construction, it remains to show that u tends to 1 as $x \rightarrow +\infty$. Since u is a bounded smooth increasing function, we have

$$l^+ := \lim_{x \rightarrow +\infty} u(x)$$

for some nonnegative l^+ .

Observe that the arguments developed in the proof of Lemma 2.1 (i-ii) can easily be adapted to the solution of (3.11), whence we have

$$f(l^+) = 0.$$

Moreover, $l^+ \in \{\rho, 1\}$ since $f|_{(\rho,1)} > 0$ and u is non-decreasing.

Observe that if $l^+ = \rho$, from the monotonicity of u it follows that $u \equiv \rho$ in \mathbb{R}^+ . Therefore, u achieves a global maximum at $x_0 = 0$ and by the maximum principle, it follows that $u \equiv \rho$ in ω , which is impossible since $u(-r) = 0$. Hence, $l^+ = 1$ and $(u, c) := (u_\epsilon^r, c_\epsilon^r)$ is the desired solution.

□

Next we prove some useful estimates on the solution $(u_\epsilon^r, c_\epsilon^r)$ of (3.9). More precisely we prove:

Lemma 3.2.

Let $\epsilon > 0$, J be as in Theorem 3.2 and $r > 1$ be such that $\text{supp} J \subset \omega$. Then there exists positive constants K and τ independent of r such that the following hold:

$$(3.12) \quad \|(u_\epsilon^r)'\|_{L^\infty(\omega)} < K,$$

$$(3.13) \quad \int_0^{+\infty} f(u_\epsilon^r) \geq \tau > 0,$$

$$(3.14) \quad \epsilon(u_\epsilon^r)'(0) + \int_0^1 \int_{-r}^{+\infty} J(-z) z u_\epsilon^r(tz) dz dt - M_1 + (1 - \rho) c_\epsilon^r \geq \tau,$$

where $M_1 = \int_{-\infty}^{+\infty} J(-z) z dz$ and $(u_\epsilon^r, c_\epsilon^r)$ is the solution obtained in Theorem 3.3.

Proof of Lemma 3.2:

Let us start with (3.12). Choose any $x \in \omega$ and integrate (3.9) over $(x, +\infty)$. It follows that

$$(3.15) \quad \epsilon(u_\epsilon^r)'(x) + (1 - u_\epsilon^r(x)) c_\epsilon^r - \int_x^{+\infty} \left(\int_{-r}^{+\infty} J(s - y) u_\epsilon^r(y) dy - u_\epsilon^r \right) ds = \int_x^{+\infty} f(u_\epsilon^r).$$

Observe that we can rewrite the above equation in the following way:

$$\epsilon(u_\epsilon^r)'(x) = -(1 - u_\epsilon^r(x)) c_\epsilon^r + \int_x^{+\infty} \int_{-r}^{+\infty} J(s - y) [u_\epsilon^r(y) - u_\epsilon^r(s)] dy ds - \int_x^\infty g_r(s) u_\epsilon^r + \int_x^{+\infty} f(u_\epsilon^r),$$

where $g_r(s) := \int_{-\infty}^{-r-s} J(-z) dz$.

Since u_ϵ^r is non-decreasing and $f(u_\epsilon^r) \leq 0$ in $(-r, 0)$, using the normalisation, we have for any $x \in (-r, 0)$,

$$\int_x^0 f(u_\epsilon^r) \leq 0.$$

Using now the Fundamental Theorem of Calculus, we see that

$$\epsilon(u_\epsilon^r)'(x) \leq |c_\epsilon^r| + \underbrace{\int_0^1 \int_x^{+\infty} \int_{-r-s}^{+\infty} J(-z) |z| (u_\epsilon^r)'(s + tz) dz ds dt}_{I_1} + \underbrace{\int_{-r}^\infty g_r(s) ds}_{I_2} + \underbrace{\int_0^{+\infty} f(u_\epsilon^r)}_{I_3}.$$

Now it remains to find bounds for the I_i to complete the proof.

Choose $R_0 > 0$ such that $\text{supp}(J) \subset (-R_0, R_0)$ and observe that for $s \geq 0$ we have $g_r(s) \equiv 0$.

Therefore

$$\begin{aligned}
I_2 &= \int_{-r}^0 g_r(s) ds \\
&= \int_{-r}^0 \int_{-\infty}^{-r-s} J(-z) e^z e^{-z} dz \\
&\leq \int_{-r}^0 e^{-r-s} \int_{-R_0}^{R_0} J(-z) e^{-z} dz \\
&\leq \int_{-R_0}^{R_0} J(z) e^z dz = K_1.
\end{aligned}$$

Now, using that $0 \leq u_\epsilon^r \leq 1$, the Chasles relation and Fubini's Theorem in I_1 , we see that

$$\begin{aligned}
I_1 &= \int_0^1 \int_x^{+\infty} \int_{-r-s}^{+\infty} J(-z) |z| (u_\epsilon^r)'(s + tz) dz ds dt \\
&= \int_0^1 \int_x^{+\infty} \int_{-\infty}^{r+s} J(z) |z| (u_\epsilon^r)'(s - tz) dz ds dt \\
&= \int_0^1 \int_x^{+\infty} \int_{-\infty}^{r+x} J(z) |z| (u_\epsilon^r)'(s - tz) dz ds dt + \int_0^1 \int_x^{+\infty} \int_{r+x}^{r+s} J(z) |z| (u_\epsilon^r)'(s - tz) dz ds dt \\
&= \int_0^1 \int_{-\infty}^{r+x} J(z) |z| (1 - u_\epsilon^r(x - tz)) dz dt + \int_0^1 \int_{r+x}^{+\infty} J(z) |z| \int_{z-r}^{+\infty} (u_\epsilon^r)'(s - tz) ds dz dt \\
&\leq \int_{-\infty}^{r+x} J(z) |z| dz + \int_0^1 \int_{r+x}^{+\infty} J(z) |z| (1 - u_\epsilon^r(z - r - tz)) dz dt \\
&\leq \int_{-\infty}^{+\infty} J(z) |z| dz = K_2.
\end{aligned}$$

From (3.15) with $x = 0$ and the previous computation it follows that

$$I_3 \leq K_2 + |c_\epsilon^r| + \epsilon (u_\epsilon^r)'(0).$$

Now since $-f(u_\epsilon^r) + u - \int_{-r}^{+\infty} J(x-y) u_\epsilon^r(y) dy$ and c_ϵ^r are bounded uniformly independently of r , using standard elliptic estimates in $(-1, 1)$ we get $(u_\epsilon^r)'(0) \leq K_3$. Therefore

$$I_3 \leq K_2 + |c_\epsilon^r| + K_3.$$

Thus

$$\epsilon (u_\epsilon^r)'(x) \leq 2(K_2 + |c_\epsilon^r|) + K_1 + K_3$$

and (3.12) holds.

Now let us show (3.13). Fix a number $\lambda \in (\rho, 1)$. Let $x_0 > 0$ be such that $u_\epsilon^r(x_0) = \lambda$. By (3.12), we have $\|(u_\epsilon^r)'\|_\infty \leq K$. Therefore we see that

$$\frac{1-\lambda}{K} \leq r - x_0 \quad \text{and} \quad \frac{\lambda - \rho}{K} \leq x_0.$$

Thus, we can find $\sigma > 0$ and $\delta > 0$ such that $f(u_\epsilon^r) \geq \sigma$ on $(x_0 - \delta, x_0 + \delta)$. Hence

$$\int_0^{+\infty} f(u_\epsilon^r) \geq \int_{x_0-\delta}^{x_0+\delta} f(u_\epsilon^r) \geq 2\delta\sigma = \tau,$$

which ends the proof of (3.13).

We are now in position to obtain (3.14). Using (3.15) with $x = 0$, it follows that

$$-\epsilon(u_\epsilon^r)'(0) - c_\epsilon^r(1 - \rho) + \int_0^{+\infty} \int_{-r}^{+\infty} J(s - y)[u_\epsilon^r(y) - u_\epsilon^r(s)] dy ds = - \int_0^\infty f(u_\epsilon^r) ds.$$

Now, using (3.13), the Fundamental Theorem of Calculus and that $\text{supp}(J) \subset (-R_0, R_0)$, we have

$$\epsilon(u_\epsilon^r)'(0) + c_\epsilon^r(1 - \rho) - \int_0^{+\infty} \int_{-R_0}^{+\infty} J(-z)z \int_0^1 (u_\epsilon^r)'(s + tz) dt dz ds \geq \tau.$$

Hence, using Fubini's Theorem, we have

$$\begin{aligned} \epsilon(u_\epsilon^r)'(0) + c_\epsilon^r(1 - \rho) - \int_{-R_0}^\infty \int_0^1 J(-z)z[1 - u_\epsilon^r(tz)] dz dt &\geq \tau \\ \epsilon(u_\epsilon^r)'(0) + c_\epsilon^r(1 - \rho) + \int_{-R_0}^\infty \int_0^1 J(-z)z u_\epsilon^r(tz) dz dt - \int_{-R_0}^\infty J(-z)z dz &\geq \tau. \end{aligned}$$

□

3.3. Limit procedure, a priori estimates and construction of a solution of (3.1). Finally, we let $r \rightarrow \infty$ and obtain a solution satisfying almost all the conditions of (3.1), which we summarize in the following proposition:

Proposition 3.1.

Let $\epsilon > 0$ and let J be as in Theorem 3.2. Then there exists a sequence $(r_n)_{n \in \mathbb{N}}$ which goes to $+\infty$ such that the sequence of solutions $(u_\epsilon^{r_n}, c_\epsilon^{r_n})_{n \in \mathbb{N}}$ given by Theorem 3.3 satisfies $u_\epsilon^{r_n}$ converges in C_{loc}^2 to a smooth function u and c_n converges to a constant c . Moreover (u, c) satisfies

$$(3.16) \quad \begin{cases} \mathcal{M}[u] + f(u) = 0 & \text{in } \mathbb{R}, \\ u(0) = \rho, \end{cases}$$

with $\mathcal{M} := \mathcal{M}(\epsilon, J, c)$. Furthermore, $u' \geq 0$.

Proof:

Let $\epsilon > 0$ fixed. Choose a sequence of real numbers $(r_n)_{n \in \mathbb{N}}$ which converges to $+\infty$, such that $\forall n \in \mathbb{N}$, $\text{supp}(J) \subset (-r_n, +\infty)$. This is always possible since J has compact support. Let (u_n, c_n) be the corresponding normalized solution given by Theorem 3.3. Using again Corollary 3.1 and Remark 3.2, $\forall n \in \mathbb{N}$, $c_2 \leq c_n \leq c_1$ and u_n is uniformly bounded. We may now argue as in the proof of Theorem 3.3. Choose a subsequence of r_n such that c_n converges to some number c . Since $(u_n)_{n \in \mathbb{N}}$ is a uniformly bounded sequence of increasing functions, using Helly's Theorem and local $C^{2,\alpha}$ estimates, there exists a subsequence which converges in $C_{loc}^{2,\alpha}$ and pointwise to a non-decreasing function u . Moreover, u satisfies

$$\begin{cases} \mathcal{M}[u] + f(u) = 0 & \text{in } \mathbb{R}, \\ u(0) = \rho. \end{cases}$$

□

We are now in position to construct a solution of (3.1).

Proof of Theorem 3.1:

From Proposition 3.1 the solution u is bounded and non-decreasing. Thus

$$l^- := \lim_{x \rightarrow -\infty} u(x) \quad \text{and} \quad l^+ := \lim_{x \rightarrow +\infty} u(x)$$

exist and the only things left to prove are

$$(3.17) \quad l^- = 0$$

and

$$(3.18) \quad l^+ = 1.$$

By construction, using Lemma 2.1, we see that $l^+ \in \{\rho, 1\}$ and $l^- \leq \rho$.

Now, for convenience, we break down our proof into three Steps.

Step 1.

Recall that u is the limit of the sequence $u_\epsilon^{r_n}$. Using Lemma 3.2, there exists a positive constant τ such that $\forall n \geq n_0$,

$$(3.19) \quad \epsilon(u_\epsilon^{r_n})'(0) + \int_0^1 \int_{-r_n}^{+\infty} J(-z) z u_\epsilon^{r_n}(tz) dz dt - M_1 + (1 - \rho) c_\epsilon^{r_n} \geq \tau.$$

Step 2.

Next, we establish some *a priori* estimates on the speed c . Namely, we have change claim to lemma

Lemma 3.3. *If $l^+ \leq \rho$ then $u \equiv \rho$ and $c - M_1 > 0$.*

Proof:

If $l^+ \leq \rho$, using Lemma 2.1 and the maximum principle, we see that $u \equiv \rho$. So it only remains to prove that $c - M_1 > 0$.

For convenience, let us define $(u_n, c_n) := (u_\epsilon^{r_n}, c_\epsilon^{r_n})$. Since $u \equiv \rho$, then u_n converges to ρ pointwise and in $C_{loc}^{2,\alpha}$. Therefore $u_n'(0) \rightarrow 0$ as $n \rightarrow +\infty$.

Observe that $|J(-z) z u_n(tz)| \leq J(-z) |z| \in L^1(\mathbb{R} \times [0, 1])$. Then using the Lebesgue's Dominated Convergence Theorem, to pass to the limit in (3.19) yields

$$\begin{aligned} c(1 - \rho) - M_1 + \int_0^1 \int_{-\infty}^{+\infty} J(-z) z \rho dz dt &\geq \tau, \\ c(1 - \rho) - (1 - \rho) M_1 &\geq \tau, \\ (1 - \rho)(c - M_1) &\geq \tau. \end{aligned}$$

Hence $c - M_1 > 0$.

□

Step 3.

We are now in a position to prove that $l^+ = 1$ and $l^- = 0$.

Let us start with $l^+ = 1$. We argue by contradiction. Assume $l^+ = \rho$. Then from Lemma 3.3 $u \equiv \rho$ and one can choose $\delta > 0$ such that $\bar{c} - M_1 > 0$ for any $\bar{c} \in (c - \delta, c + \delta)$.

Now let us compute $\mathcal{S}[e^{\lambda x}]$ in $(-r_n, +\infty)$ with $\mathcal{S} := \mathcal{S}(\epsilon, J, r_n, c_n)$.

$$(3.20) \quad \mathcal{S}[e^{\lambda x}] = e^{\lambda x} [\epsilon \lambda^2 - c_n \lambda + \int_{-r_n-x}^{+\infty} J(-z) e^{\lambda z} dz - 1],$$

$$(3.21) \quad \mathcal{S}[e^{\lambda x}] = e^{\lambda x} h_n(x, \lambda).$$

Since $c_n \rightarrow c$, then $c_n \in (c - \delta, c + \delta)$ for n large enough. We will show that for n large enough $e^{\lambda x}$ is a supersolution of (3.11) for some positive λ .

Namely we have change claim to lemma

Lemma 3.4. *There exists $\lambda_1 > 0$ such that $h_n(\lambda_1, x) \leq 0$ uniformly in x for n large enough.*

Proof:

Since J has a compact support, using Lebesgue's Theorems, we easily see that

$$h_n(0, x) = \int_{-r_n-x}^{+\infty} J(-z) dz - 1 \leq 0 \quad \text{for } x \in [-r_n, +\infty)$$

and

$$\frac{\partial h_n}{\partial \lambda}(0, x) = \int_{-r_n-x}^{+\infty} J(-z) z dz - c_n \quad \text{for } x \in [-r_n, +\infty).$$

Moreover, for r_n large enough, we have

$$\begin{cases} \int_{-r_n-x}^{+\infty} J(-z) z dz \leq \int_{-r_n}^{+\infty} J(-z) z dz = M_1 & \text{for } x \in [-r_n, 0], \\ \int_{-r_n-x}^{+\infty} J(-z) z dz = \int_{-r_n}^{+\infty} J(-z) z dz = M_1 & \text{for } x > 0. \end{cases}$$

Hence for r_n large enough

$$(3.22) \quad \frac{\partial h_n}{\partial \lambda}(0, x) = \int_{-r_n-x}^{+\infty} J(-z) z dz - c_n \leq M_1 - c_n \quad \text{for } x \in [-r_n, +\infty).$$

Since $c_n - M_1 > 0$ for n sufficiently large, using (3.22), we deduce that there exists a $\lambda_1 > 0$ independent of x and n such that

$$\forall n \geq n_0, \quad h_n(\lambda_1, x) \leq 0 \quad \text{in } [-r_n, +\infty).$$

□

Define now $w := e^{\lambda_1 x} - u_n$. Observe that $w > 0$ in \mathbb{R}^+ . Using Lemma 3.4, we see that for $n \geq n_0$, w also satisfies the following

$$\begin{cases} \mathcal{S}[w] \leq 0 \text{ for } x \in [-r_n, 0], \\ w(0) = 1 - \rho > 0, \\ w(-r_n) = e^{-\lambda_1 r_n} > 0, \end{cases}$$

where $\mathcal{S} := \mathcal{S}(\epsilon, J, r_n, c_n)$.

Now, using the maximum principle (see Appendix), we find that $w \geq 0$ in $[-r_n, +\infty)$. Hence

$$(3.23) \quad \forall n \geq n_0, \quad u_n \leq e^{\lambda_1 x}.$$

By passing to the limit $n \rightarrow +\infty$ in (3.23) it follows that

$$u \leq e^{\lambda_1 x}.$$

This contradicts $u \equiv \rho$. Thus $l^+ = 1$.

We now show that $l^- = 0$.

By construction and using Lemma 2.1, it follows that $l^- \in \{0, \rho\}$ (resp. $l^- \in [0, \rho]$) when f is bistable (resp. ignition).

Again we argue by contradiction. Assume that $l^- > 0$. By construction and from the monotonicity of u we see that $l^- = \rho$ and $u \equiv \rho$ in \mathbb{R}^- (resp. $l^- \in (0, \rho]$ and $u \leq \rho$ in \mathbb{R}^-) when f is bistable (resp. ignition). Note that in both case, the following holds:

$$(3.24) \quad f(u) \equiv 0 \quad \text{in} \quad \mathbb{R}^-,$$

$$(3.25) \quad f(u) > 0 \quad \text{in} \quad \mathbb{R}^+.$$

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Since u satisfies (3.16), using (iv) of Lemma 2.1, we achieve

$$(c - M_1)[l^+ - l^-] = \int_{\mathbb{R}} f(u) = \int_{\mathbb{R}^+} f(u) \geq 0.$$

Using that $l^+ = 1$, $l^- \leq \rho$ and u is non-trivial in \mathbb{R}^+ , we deduce that $\int_{\mathbb{R}^+} f(u) > 0$ and therefore

$$c - M_1 > 0.$$

We can now argue as above to conclude that $u \leq e^{\lambda_2 x}$ for some positive λ_2 . Therefore $0 < l^- = 0$, which is the desired contradiction. Hence $l^- = 0$, which ends the construction of the solution of the problem (3.1) and the proof of Theorem 3.1.

□

4. Construction of solutions of (3.1) when $\epsilon > 0$

In this section we prove Theorem 1.5. Namely, we construct solutions of (3.1) for $\epsilon > 0$ and for J satisfying (H2). For convenience let us recall (H2), some notation and Theorem 1.5.

$$(H2) \quad J \in C^0(\mathbb{R}), \quad J \geq 0, \quad \int_{\mathbb{R}} J = 1 \quad \text{and} \quad \int_{\mathbb{R}} J(z)|z|dz < +\infty,$$

$$M_1 := \int_{\mathbb{R}} J(-z)zdz \quad \Sigma := \{z \in \mathbb{R} | J(-z) > 0\}.$$

Theorem 4.1.

Let $\epsilon > 0$ and assume that J satisfies (H2). Then there exists a smooth solution (u_ϵ, c_ϵ) of (3.1). Moreover c_ϵ is unique and u_ϵ is strictly increasing and unique up to translation.

The monotonicity of u_ϵ and the uniqueness of (u_ϵ, c_ϵ) are a consequence of the nonlinear comparison principle proved in [14, 15]. So we refer the interested reader to these references for more details of the proofs.

The construction of a solution of (3.1) in this case uses a standard approximation procedure of J by kernels J_n with compact support. First, let j_0 be a positive symmetric function defined by

$$(4.1) \quad j_0(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{for } x \in (-1, 1), \\ 0 & \text{elsewhere.} \end{cases}$$

Let $(\chi_n)_{n \in \mathbb{N}}$ be the following sequence of “cut-off” functions:

- $\chi_n \in C_0^\infty(\mathbb{R})$,
- $0 \leq \chi_n \leq 1$,
- $\chi_n(s) \equiv 1$ for $|s| \leq n$ and $\chi_n(s) \equiv 0$ for $|s| \geq 2n$.

Define the following approximation kernel

$$J_n := \frac{1}{m_n} \left(\frac{j_0}{n} + J(z)\chi_n(z) \right),$$

where $m_n := \frac{1}{n} \int_{\mathbb{R}} j_0(z) dz + \int_{\mathbb{R}} J\chi_n(z) dz$. Observe that since $\int_{\mathbb{R}} j_0 > 0$, J_n is well-defined, $J_n(z) \rightarrow J(z)$ pointwise and for n big enough say $n \geq n_0$, we have $J_n \leq 2(j_0 + J)$.

By construction, we easily see that there exists $a \in \mathbb{R}$ such that for all n , $J_n(a) > 0$ and $J_n(-a) > 0$. Therefore J_n satisfies the assumptions of Theorem 3.1. Furthermore, using Theorem 3.1, there exists for each $n \in \mathbb{N}$ a smooth increasing normalized solution (u_n, c_n) of the problem (4.2) below:

$$(4.2) \quad \begin{cases} \mathcal{M}(\epsilon, J_n, c) + f(u) = 0 & \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(0) = \rho, \\ u(+\infty) = 1. \end{cases}$$

Before proving Theorem 4.1, let us prove some *a priori* estimates on u_n and c_n .

Proposition 4.1.

Let (u_n, c_n) be a normalised solution of (4.2). Then

- (i) There exists $K > 0$ such that $\forall n, |c_n| \|u'_n\|_{L^\infty(\mathbb{R})} < K$,
- (ii) There exists $C > 0$ such that $\forall n, |c_n| < C$.

Proof:

Let (u_n, c_n) be the normalized solution of (4.2).

Observe that (i) is a straight forward application of Lemma 2.1 and Remark 2.3.

To obtain (ii) we argue as follows. From Lemma 2.2 and Remark 2.3 we have

$$(4.3) \quad \mu c_n^2 - |c_n| \kappa_n \leq \epsilon K,$$

where $\mu := \min\{\rho, 1 - \rho\}$, $K := 2 + \sup_{[0,1]} |f|$ and $\kappa_n := \int_{\mathbb{R}} J_n(z) |z| dz$.

Observe that $m_n \rightarrow 1$ and let n_0 be such that $m_n \geq \frac{1}{2}$ for $n \geq n_0$. Therefore, for $n \geq n_0$,

$$(4.4) \quad \kappa_n = \frac{1}{m_n} \left(\frac{\int_{\mathbb{R}} j_0(z) |z| dz}{n} + \int_{\mathbb{R}} J\chi_n(z) |z| dz \right) \leq 2\kappa,$$

where $\kappa := \int_{\mathbb{R}} J(z) |z| dz$.

Combining (4.4) with (4.3), we obtain

$$(4.5) \quad \mu c_n^2 - 2|c_n| \kappa \leq \epsilon K.$$

Hence (ii) follows from (4.5). □

Remark 4.1. Observe that the constant K, μ are independant of the approximation. Therefore, Proposition 4.1 holds as soon as we have a uniform control of κ_n . In particular, Proposition 4.1 holds if J is a probability measure with a finite first moment.

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1:

Let (u_n, c_n) denote the sequence of normalized solutions of (4.2). By Proposition 4.1, $(c_n)_{n \in \mathbb{N}}$ is bounded and $(u_n)_{n \in \mathbb{N}}$ is uniformly bounded. We may now argue as in the proof of Theorem 3.3. Choose a subsequence of c_n , which converges to some number \tilde{c} . Since $(u_n)_{n \in \mathbb{N}}$ is a uniformly bounded sequence of increasing functions, using Helly's Theorem and local $C^{2,\alpha}$ estimates, there exists a subsequence which converges in $C_{loc}^{2,\alpha}$ and pointwise to a non-decreasing function \tilde{u} . Moreover, \tilde{u} satisfies

$$(4.6) \quad \begin{cases} \mathcal{M}[\tilde{u}] + f(\tilde{u}) = 0 & \text{in } \mathbb{R}, \\ \tilde{u}(0) = \rho. \end{cases}$$

To complete the proof, it remains to show that \tilde{u} satisfies the right boundary conditions. Our argument follows the one developed in subsection 3.3. However, we stress the fact that in this case J does not necessarily have compact support. Therefore, for general J , we cannot expect that \tilde{u} has an exponential behavior near $-\infty$.

As in subsection 3.3, since \tilde{u} is bounded and non-decreasing, we can define

$$l^- := \lim_{x \rightarrow -\infty} \tilde{u}(x) \quad \text{and} \quad l^+ := \lim_{x \rightarrow +\infty} \tilde{u}(x).$$

Since $\epsilon > 0$ and u_n converges in $C_{loc}^{2,\alpha}$, using Proposition 4.1 and Lemma 2.1, we deduce that $\|u'_n\|_{L^\infty(\mathbb{R})} \leq C_1$ with C_1 independent of n . Thus, arguing as in Lemma 3.2, one can easily construct $\tau > 0$ such that

$$(4.7) \quad \forall n \quad \int_{\mathbb{R}^+} f(u_n) \geq \tau,$$

$$(4.8) \quad \forall n \in \mathbb{N}, \quad \epsilon u'_n(0) + (1 - \rho)c_n + \int_0^1 \int_{\Sigma_n} J_n(-z) z u_n(tz) dt dz - \int_{\Sigma_n} J_n(-z) z dz \geq \tau,$$

where $\Sigma_n = \{z \in \mathbb{R} | J_n(-z) > 0\}$.

We now prove some useful estimates:

Lemma 4.1. *Let (u_n, c_n) be the above sequence of solutions. Then*

$$(4.9) \quad \forall n \in \mathbb{N}, \quad \epsilon u'_n(0) - \rho c_n + \int_0^1 \int_{\Sigma_n} J_n(-z) z u_n(tz) dt dz \geq 0.$$

Proof:

Integrating equation (4.2) over \mathbb{R}^- yields

$$\epsilon u'_n(0) - c_n \rho + \int_0^1 \int_{-\infty}^0 \int_{-\infty}^{+\infty} J_n(-z) z u'_n(x + tz) dt dz dx = - \int_{-\infty}^0 f(u_n) dx.$$

Hence

$$\epsilon u'_n(0) - c_n \rho + \int_0^1 \int_{\Sigma_n} J_n(-z) z u_n(tz) dt dz \geq 0,$$

since $f(u_n) \leq 0$ in \mathbb{R}^- .

□

We are now in a position to show that $l^+ = 1$. Assume by contradiction that $l^+ < 1$. Then $l^+ = \rho$ and from Lemma 2.1 $\tilde{u} \equiv \rho$ and $\tilde{u}' \equiv 0$ in $\Sigma \cup \mathbb{R}^+$. Thus $u'_n(0) \rightarrow 0$.

Recall that $u_n \rightarrow \tilde{u}$ pointwise and for n big enough $|J_n(z)zu_n(tz)| \leq 2(j_0(z) + J(z))|z|$. Therefore passing to the limit in (4.8) and (4.9) using Lebesgue's Dominated Convergence Theorem, we get

$$(1 - \rho)\tilde{c} + \int_0^1 \int_{\Sigma} J(-z)z\tilde{u}(tz) dt dz - M_1 \geq \tau,$$

$$-\rho\tilde{c} + \int_0^1 \int_{\Sigma} J(-z)z\tilde{u}(tz) dt dz \geq 0.$$

Using now that $\tilde{u} \equiv \rho$ in Σ , we get the following contradiction:

$$(1 - \rho)(\tilde{c} - M_1) \geq \tau > 0,$$

$$\rho(\tilde{c} - M_1) \leq 0.$$

Hence $l^+ = 1$.

□

For simplicity, we treat the case of bistable and ignition nonlinearities separately. Let us first assume that f is bistable. From the above computation, we can see that $l^- = 0$. Indeed, from Lemma 2.1 if $l^- \neq 0$, then $\tilde{u} \equiv \rho$ and $\tilde{u}' \equiv 0$ in $\Sigma \cup \mathbb{R}^-$, and the previous argument holds.

The case of ignition nonlinearity needs further investigation. In that case, we argue as follow. We claim that

Claim 4.1. *Let (\tilde{u}, \tilde{c}) be a solution of (4.6) and f be of ignition type. Then the following holds:*

$$(4.10) \quad (M_1 - \tilde{c})[1 - l^-] < 0$$

and

$$(4.11) \quad \epsilon \tilde{u}'(0) + \int_0^1 \int_{-\infty}^{+\infty} J(-z)z\tilde{u}(tz) dz dt - \rho\tilde{c} = 0.$$

From this claim we get easily $l^- = 0$. Indeed, since \tilde{u} satisfies equation (4.6), is smooth and non-trivial, after integration of (4.6) over \mathbb{R}^- , a quick computation using the Fundamental Theorem of Calculus and Fubini's Theorem yields

$$(4.12) \quad \epsilon \tilde{u}'(0) - (\rho - l^-)\tilde{c} + \int_0^1 \int_{-\infty}^0 \int_{-\infty}^{+\infty} J(-z)z\tilde{u}'(x + tz) dt dz dx = 0,$$

$$(4.13) \quad \epsilon \tilde{u}'(0) - (\rho - l^-)\tilde{c} + \int_0^1 \int_{-\infty}^{+\infty} J(-z)z \left(\int_{-\infty}^0 \tilde{u}'(x + tz) dx \right) dt dz = 0,$$

$$(4.14) \quad \epsilon \tilde{u}'(0) - (\rho - l^-)\tilde{c} + \int_0^1 \int_{-\infty}^{+\infty} J(-z)z[\tilde{u}(tz) - l^-] dt dz = 0$$

Using (4.11) in (4.14) leads to

$$l^-(\tilde{c} - M_1) = 0.$$

Thus $l^- = 0$ since $(\tilde{c} - M_1) \neq 0$ by (4.10).

We turn now our attention to the proof of Claim.

Proof of Claim 4.1

The first equality is a straight forward consequence of Lemma 2.1 and of the definition of f . Indeed, since \tilde{u} satisfies (4.6), from Lemma 2.1

$$\left(\int_{\mathbb{R}} J(-z)z \, dz - \tilde{c} \right) [l^+ - l^-] = - \int_{\mathbb{R}} f(\tilde{u}).$$

Using the definition of f (f is of ignition type) we get $\int_{\mathbb{R}} f(u) \, dx = \int_{\mathbb{R}^+} f(\tilde{u}) \, dx$. Since \tilde{u} is smooth, non-constant, $\tilde{u} \rightarrow 1$ as $x \rightarrow +\infty$ and $\int_{\mathbb{R}^+} f(\tilde{u}) > 0$, we end up with

$$(M_1 - \tilde{c})[1 - l^-] < 0.$$

Let us now prove the second equality. From the assumption on f we have, for all $n \in \mathbb{N}$, $f(u_n) \equiv 0$ in \mathbb{R}^- . Thus, u_n satisfies the linear equation below

$$(4.15) \quad \epsilon u_n'' + J_n \star u_n - u_n - c_n \tilde{u}_n' = 0 \quad \text{in } \mathbb{R}^-.$$

Integrating over \mathbb{R}^- (4.15) yields

$$\epsilon u_n'(0) + \int_0^1 \int_{-\infty}^0 \int_{-\infty}^{+\infty} J_n(-z)z u_n'(x + tz) \, dz \, dt \, dx - \rho c_n = 0.$$

Using Fubini's Theorem we find that

$$(4.16) \quad \epsilon u_n'(0) + \int_0^1 \int_{-\infty}^{+\infty} J_n(-z)z u_n(tz) \, dz \, dt - \rho c_n = 0.$$

Since for n big enough $J_n(-z)|z|u_n(tz) \leq 2(j_0(-z) + J(-z))|z|$ and $(j_0 + J(z))z \in L^1(\mathbb{R})$, we can apply the Lebesgue's Dominated Convergence Theorem and pass to the limit in (4.16) to obtain the desired equality

$$\epsilon \tilde{u}'(0) + \int_0^1 \int_{-\infty}^{+\infty} J(-z)z \tilde{u}(tz) \, dz \, dt - \rho \tilde{c} = 0.$$

□

5. Singular limits and construction of a solution of (3.1) for $\epsilon = 0$

In this section we prove Theorem 1.2. More precisely, we focus our attention on the solution of (3.1) when $\epsilon = 0$, i.e., the solution of:

$$(5.1) \quad \begin{cases} J \star u - u - cu' + f(u) = 0 & \text{in } \mathbb{R}, \\ u(x) \rightarrow 0 & x \rightarrow -\infty, \\ u(x) \rightarrow 1 & x \rightarrow +\infty. \end{cases}$$

In this case two different situations may occur, either $c \neq 0$ and the solution is smooth or $c = 0$ and the solution may be discontinuous. However, we still have existence of a travelling front solution with unique speed. Let us recall Theorem 1.2:

Theorem 5.1.

Assume that J satisfies (H2). Then there exists a solution (u, c) of (5.1). Moreover the speed c is unique and u is non-decreasing.

Since the proof of Theorem 5.1 is rather long, it is split up into three parts. In the first part, Section 5.1, we construct a non-decreasing front solution of (5.1). Then in the second part, Section 5.3, when J does not satisfy (H1), some asymptotic and monotone properties of the solutions are established. Finally, the uniqueness of the speed is obtained in the last part, Section 5.4.

Again the proof of existence essentially uses the same type of argumentation as in the above sections. Let $(\epsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive number which converges to 0 and (u_n, c_n) be the unique normalized solution given by Theorem 4.1, e.g. (u_n, c_n) satisfies the following equation

$$(5.2) \quad \begin{cases} \epsilon_n u_n'' + J \star u_n - u_n - c_n u_n' + f(u_n) = 0 & \text{in } \mathbb{R}, \\ u_n(-\infty) = 0, \\ u_n(+\infty) = 1. \end{cases}$$

and $u_n(0) = \sigma$ for some $\sigma \in (0, 1)$. From Lemma 2.2 and Remark 2.3, one has

$$\mu c_n^2 - |c_n| \kappa \leq \epsilon_n K.$$

Therefore c_n is bounded. Since for each n , u_n is a uniformly bounded increasing function, using Helly's Theorem, there exists a subsequence which converges pointwise to some (u, c) . So at least, in the sense of distribution, (u, c) satisfies:

$$\begin{aligned} J \star u - u - cu' + f(u) &= 0, \\ u(0) &= \sigma = u_n(0). \end{aligned}$$

Remark 5.1. Note that the normalization of the sequence is not a priori prescribed. To obtain the right boundary conditions, one must choose the appropriate normalization, which depends on f , J and c .

Before proving the existence of a front, let us establish some useful estimates which occur when f is of ignition type.

Lemma 5.1. Let f be of ignition type, assume that $M_1 := \int_{\mathbb{R}} J(-z)z \, dz \geq 0$, then there exists a positive constant c_0 and an integer n_0 such that $c_n \geq c_0$ for all $n \geq n_0$.

Proof:

Observe that the proof is trivial when $M_1 > 0$. Indeed, using Lemma 2.1, we have

$$c_n = M_1 + \int_{\mathbb{R}} f(u_n) \, dx.$$

Therefore $c_n > \frac{M_1}{2}$ for all $n \in \mathbb{N}$.

The case $M_1 = 0$ is more delicate. We argue by contradiction. Assume that there exists a sequence of speeds c_n that converges to 0.

Let (u_n, c_n) be the sequence of solutions normalized by $u_n(0) = \rho + \delta$ that converges pointwise to u . Since u is monotone it satisfies

$$J \star u - u = -f(u) \quad \text{almost everywhere.}$$

Using Fatou's Lemma, we easily see that

$$0 \leq \int_{\mathbb{R}} f(u) \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}} f(u_n) = \lim_{n \rightarrow +\infty} c_n = 0.$$

Therefore u is discontinuous at zero and satisfies $u \equiv 1$ in \mathbb{R}^+ .

Now note that since $M_1 = 0$, J satisfies the assumption (H1) and therefore \mathcal{M} satisfies the strong maximum principle. Let $x_0 \in \mathbb{R}^+$ be a point where u achieves a global maximum, at this point we have

$$0 \geq J \star u(x_0) - 1 = -f(1) = 0.$$

Using the strong maximum principle and the monotonicity of u , we get $u \equiv 1$ in \mathbb{R} , which contradicts the normalization $u(0) = \rho + \delta$. \square

5.1. Existence of a travelling wave. We are now in a position to get the existence of a front. From the above calculation there exists (u, c) , satisfying :

$$(5.3) \quad J \star u - u - cu' + f(u) = 0,$$

$$(5.4) \quad u(0) = \sigma,$$

in the sense of distribution. For simplicity we consider two cases, $c \neq 0$ and $c = 0$.

5.1.1. *The case $c \neq 0$:* If $c \neq 0$, then the solution u normalized by $u(0) = \rho$ is smooth and (u, c) is a strong solution of (5.3). Indeed, from (v) of Lemma 2.1, we have

$$\begin{aligned} \|u'_n\|_{L^\infty(\mathbb{R})} &\leq \frac{K}{c_n}, \\ \|u''_n\|_{L^\infty(\mathbb{R})} &\leq \left(\frac{K}{c_n}\right)^2. \end{aligned}$$

Therefore for some $n_0 \in \mathbb{N}$,

$$\|u_n\|_{C^2(\mathbb{R})} \leq C \text{ for any } n \geq n_0$$

and for some $\tau > 0$

$$(5.5) \quad \forall n \geq n_0, \quad \int_0^{+\infty} f(u_n) \geq \tau.$$

Thus, we can extract a subsequence of $(u_n)_{n \in \mathbb{N}}$ that converges pointwise and in $C^1_{loc}(\mathbb{R})$ to some smooth function u . Furthermore, (u, c) satisfies (5.3) with $u(0) = \rho$. Since u is a smooth solution, Lemma 2.1 holds and we have

- (i) $u'(x) \rightarrow 0$ as $x \rightarrow \pm\infty$ and $f(l^\pm) = 0$.
- (ii) If $l^+ \leq \rho$ then $u \equiv \rho$ in \mathbb{R}^+ and $u' \equiv 0$ in \mathbb{R}^+ .
If $l^- \geq \rho$ then $u \equiv \rho$ in \mathbb{R}^- and $u' \equiv 0$ in \mathbb{R}^- .
- (iii) $\int_{\mathbb{R}} f(u) < \infty$, $\int_{\mathbb{R}} (u')^2 < \infty$.

Moreover, we also have the following:

Lemma 5.2.

Let (u, c) be a solution of (5.3) normalized by $u(0) = \rho$. Then

$$(5.6) \quad -\rho c + \int_0^1 \int_{-\infty}^{+\infty} J(-z)zu(tz) dt dz \geq 0,$$

$$(5.7) \quad (1 - \rho)c + \int_0^1 \int_{-\infty}^{+\infty} J(-z)zu(tz) dt dz - M_1 \geq \tau.$$

Remark 5.2. When f is of ignition type, then $-\rho c + \int_0^1 \int_{-\infty}^{+\infty} J(-z)zu(tz) dt dz = 0$.

Proof:

By integrating equation (5.2) over \mathbb{R}^+ and \mathbb{R}^- respectively and using (5.5), we obtain the two inequalities

$$\begin{aligned}\epsilon_n u'_n(0) + (1 - \rho)c_n + \int_0^1 \int_{-\infty}^{+\infty} J(-z) z u_n(tz) dz dt - M_1 &= \int_{\mathbb{R}^+} f(u_n) \geq \tau, \\ \epsilon_n u'_n(0) - \rho c_n + \int_0^1 \int_{-\infty}^{+\infty} J(-z) z u_n(tz) dt dz &= - \int_{\mathbb{R}^-} f(u_n) \geq 0.\end{aligned}$$

Since $\|u'_n\|_{L^\infty(\mathbb{R})} < K$ and $J(-z)|z||u_n(tz)| \leq J(-z)|z|$, letting $n \rightarrow \infty$ in the above equations yields

$$\begin{aligned}(1 - \rho)c + \int_0^1 \int_{-\infty}^{+\infty} J(-z) z u(tz) dz dt - M_1 &\geq \tau, \\ -\rho c + \int_0^1 \int_{-\infty}^{+\infty} J(-z) z u(tz) dt dz &\geq 0.\end{aligned}$$

□

Now, to obtain $l^+ = 1$ and $l^- = 0$, we argue as in Section 4. If $l^+ < 1$ then $u \equiv \rho$ in Σ . Using Lemma 5.2 we obtain the following contradiction

$$\begin{aligned}(1 - \rho)(c - M_1) &\geq \tau, \\ -\rho(c - M_1) &\geq 0.\end{aligned}$$

Thus $l^+ = 1$. For a bistable nonlinearity a similar argument holds, since if $l^- > 0$, then $u \equiv \rho$ in Σ . For ignition type nonlinearities f the same argument as in Section 4 holds, except that we use Lemma 5.2 and Remark 5.2 instead of Claim 4.1.

5.1.2. *The case $c = 0$:* In this case u may be discontinuous. However, u is monotone and satisfies

$$(5.8) \quad J \star u - u + f(u) = 0 \quad \text{almost everywhere.}$$

Since u is uniformly bounded and monotone, we can still define l^\pm and show that $f(l^\pm) = 0$. The main difficulty is then proving that u is non-trivial and that $l^+ = 1$ and $l^- = 0$.

Let us first prove a useful lemma.

Lemma 5.3.

Let u be the solution of (5.8). Then the following holds:

$$\int_{\mathbb{R}} f(u) dx = -(l^+ - l^-)M_1.$$

Proof:

Integrating (5.8) over $(-R, R)$ for some positive R gives

$$(5.9) \quad \int_{-R}^R (J \star u - u) dx = - \int_{-R}^R f(u) dx.$$

Using the Fundamental Theorem of Calculus and Fubini's Theorem, we get

$$(5.10) \quad \int_{-R}^R (J \star u_n - u_n) dx = \int_{-R}^R \int_{\mathbb{R}} \int_0^1 J(-z) z u_n'(x + tz) dt dz dx$$

$$(5.11) \quad = \int_{\mathbb{R}} \int_0^1 J(-z) z [u_n(R + tz) - u_n(-R + tz)] dt dz.$$

Since u_n converges pointwise to u and u_n is uniformly bounded, by using Lebesgue's Dominated Convergence Theorem we obtain

$$(5.12) \quad \int_{-R}^R (J \star u - u) dx = \int_{\mathbb{R}} \int_0^1 J(-z) z [u(R + tz) - u(-R + tz)] dt dz.$$

Thus we have

$$(5.13) \quad \int_{\mathbb{R}} \int_0^1 J(-z) z [u(R + tz) - u(-R + tz)] dt dz = - \int_{-R}^R f(u) dx.$$

From the above equation, the boundedness of u , and $|M_1| < +\infty$, we deduce that $f(u)$ has a finite integral. Moreover, by letting $R \rightarrow +\infty$ in the above equation yields:

$$(5.14) \quad (l^+ - l^-)M_1 = - \int_{\mathbb{R}} f(u) dx.$$

□

Since the construction of a solution (u, c) with the right boundary conditions is different for bistable and ignition type nonlinearities f , we treat these cases separately.

5.1.3. *The bistable case:* Let us first assume that $M_1 := \int_{\mathbb{R}} J(-z) z dz \geq 0$. Let (u_n) be the sequence of solutions normalized by $u_n(0) = \rho + \delta$ for some small positive δ . With the above normalization, the solution u of (5.8) satisfies

$$\begin{aligned} J \star u - u + f(u) &= 0, \\ u(0) &= \rho + \delta. \end{aligned}$$

Since $f(l^+) = 0$ and $u(0) > \rho$, then u is non-trivial and $l^+ = 1$. Moreover we have

$$\int_{\mathbb{R}^+} f(u) > 0.$$

So the main difficulty is to show that $l^- = 0$. Using the fact that f is bistable, the only possible values for l^- are 0 and ρ . Now argue by contradiction: Assume that $l^- = \rho$. Then $f(u) \geq 0$ in \mathbb{R} . By using Lemma 5.3, we have our desired contradiction

$$0 < \int_{\mathbb{R}} f(u) = -(1 - \rho)M_1 \leq 0.$$

Thus $l^- = 0$.

When $M_1 < 0$, a similar argument will also work. In this case we use the normalization $u_n(0) = \rho - \delta$ instead of $u_n(0) = \rho + \delta$.

5.1.4. The ignition case: Using Lemma 5.1, the only case to consider is when $M_1 < 0$, since otherwise $c \neq 0$.

According to the previous computation, for every $\sigma \in (0, \rho]$, we get a solution u_σ of

$$\begin{aligned} J \star u_\sigma - u_\sigma + f(u_\sigma) &= 0, \\ u_\sigma(0) &= \sigma. \end{aligned}$$

We claim that for all these solutions u_σ , $l_\sigma^+ = 1$.

Claim 5.1. $\forall \sigma \in (0, 1), l_\sigma^+ = 1$.

Proof:

When $\sigma > \rho$ then $l_\sigma^+ = 1$ follows easily from $f(l_\sigma^+) = 0$ and the monotone behavior of u_σ . When $\sigma \leq \rho$, we argue by contradiction. Assume that $l_\sigma^+ < 1$. Since $f(l_\sigma^+) = 0$, then $l^+ \leq \rho$ and $f(u_\sigma) \equiv 0$ in \mathbb{R} . By using Lemma 5.3, it follows that

$$(l_\sigma^+ - l_\sigma^-)M_1 = 0.$$

This implies that $l_\sigma^+ = l_\sigma^-$. Therefore $u_\sigma \equiv \sigma$ in \mathbb{R} by monotonicity.

From $u_\sigma \equiv \sigma$, we get that there exists a constant K and a sequences of points $(x_n)_{n \in \mathbb{N}}$, such that for all n , $x_n \in (-1, 0)$ and $u'_n(x_n) < K$. Let us now integrate equation (5.2) over $(-\infty, x_n]$. Since $\sigma \leq \rho$ and $x_n \in (-1, 0)$, it follows that

$$\epsilon_n u'_n(x_n) - c_n u(x_n) + \int_0^1 \int_{\mathbb{R}} J(-z) z u_n(x_n + tz) dz dt = 0.$$

Since $(x_n)_{n \in \mathbb{N}}$ is bounded, using the monotonicity of u_n , we see that $u_n(x_n + z) \rightarrow \sigma$ for any $z \in \mathbb{R}$. Letting $n \rightarrow +\infty$ in the above equation and using Lebesgue's Dominated Convergence Theorem, we end up with the following contradiction

$$0 > \sigma \int_{\mathbb{R}} J(-z) z dz = 0.$$

□

We are now in a position to construct a solution with the right boundary conditions. Let us denote $u_\sigma^\tau(x) := u_\sigma(x + \tau)$ and define the following set for $\sigma < \rho$:

$$A_\sigma^\tau := \{\tau \geq 0 \mid u_\sigma^\tau(0) < \rho\}.$$

Observe that from Claim 5.1, for a fixed σ ,

$$\tau^-(\sigma) := \sup A_\sigma^\sigma$$

is well defined. We easily check that the function $u_{\sigma}^{\tau^-(\sigma)}$ satisfies

$$(5.15) \quad J \star u_{\sigma}^{\tau^-} - u_{\sigma}^{\tau^-} + f(u_{\sigma}^{\tau^-}) = 0,$$

$$(5.16) \quad u_{\sigma}^{\tau^-(\sigma)} > \rho \quad \text{in} \quad \mathbb{R}^{+,*},$$

$$(5.17) \quad u_{\sigma}^{\tau^-(\sigma)}(0) \leq \rho.$$

Let $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence of real numbers converging to 0. By using Helly's Theorem, we can extract a subsequence of $(u_{\sigma_n}^{\tau^-(\sigma_n)})_{n \in \mathbb{N}}$, that converges pointwise to some function u . The

function u satisfies

$$(5.18) \quad J \star u - u + f(u) = 0,$$

$$(5.19) \quad u \geq \rho \quad \text{in} \quad \mathbb{R}^{+,*},$$

$$(5.20) \quad u(0) \leq \rho.$$

By Claim 5.1, we see that $u \rightarrow 1$ as $x \rightarrow +\infty$ and u is non-trivial since $u(0) \leq \rho$. Now it remains to show that $l^- = 0$.

Claim 5.2. $l^- = 0$.

Proof:

Observe that since $\sigma \rightarrow 0$, it follows that $l_\sigma^- \rightarrow 0$. By integrating the equations (5.15) and (5.18) over \mathbb{R}^- , we get

$$(5.21) \quad \int_{\mathbb{R}} \int_0^1 J(-z) z u(tz) dt dz - M_1 l^- = 0,$$

$$(5.22) \quad \int_{\mathbb{R}} \int_0^1 J(-z) z u_\sigma^{\tau^-(\sigma)}(tz) dt dz - M_1 l_\sigma^- = 0.$$

Now letting $\sigma \rightarrow 0$ in the second equation and using Lebesgue's Dominated Convergence Theorem, we end up with

$$\int_{\mathbb{R}} \int_0^1 J(-z) z u(tz) dt dz = 0.$$

Therefore $M_1 l^- = 0$ and $l^- = 0$.

□

Remark 5.3. Observe that the equalities or inequalities satisfied by a solution (u, c) can be obtained as well when J is assumed to be a Borel probability measure μ with a finite first moment. Therefore the argument to obtain the right boundary condition for u stands as well for this case.

5.2. Some useful asymptotic behavior. In this section, provided an extra assumption on the sign of the speed, we establish some a priori bounds on any solution u of (5.1) when J does not satisfy the strong maximum condition. Namely, we establish

Proposition 5.1.

(i) When $\text{supp}(J) \subset \mathbb{R}^+$:

Let (u, c) be a solution of (5.1) with $c < 0$. Then for some positive δ ,

$$\frac{\rho}{2} e^{-\delta|x|} \leq u < 1.$$

(ii) When $\text{supp}(J) \subset \mathbb{R}^-$:

Let (u, c) be a solution of (5.1) with $c > 0$. Then for some positives γ, μ ,

$$0 < u \leq 1 - \frac{1-\rho}{2} e^{-\mu|x|}.$$

Before proving Proposition, let us recall the following results that we prove in the appendix.

Theorem 5.2. Pre-Maximum-Principle

Let $u \in C^1(\mathbb{R})$ be such that

$$J \star u - u - cu' \geq 0.$$

Assume that u achieves a maximum at a point x_0 . Then the following holds

- When $\text{supp}(J) \subset \mathbb{R}^+$, $u \equiv u(x_0)$ in a neighbourhood of $-\infty$.
- When $\text{supp}(J) \subset \mathbb{R}^-$, $u \equiv u(x_0)$ in a neighbourhood of $+\infty$.

For a general measure μ , the above pre-maximum principle is replaced by the following Theorem where the notion of the support is understood as follow:

$$\text{supp}(\mu) := \{x \in \mathbb{R} \mid \forall N_x \text{ open neighbourhood of } x \implies \mu(N_x) > 0\}.$$

Theorem 5.3. Let $u \in C^1(\mathbb{R})$ be such that

$$J \star u - u - cu' \geq 0.$$

Assume that u achieves a maximum at a point x_0 . Then the following holds

- When $\text{supp}(\mu) \subset \mathbb{R}^+$, $u \equiv u(x_0)$ in a unbounded set of $-\infty$.
- When $\text{supp}(\mu) \subset \mathbb{R}^-$, $u \equiv u(x_0)$ in a unbounded set of $+\infty$.

Let us now prove Proposition 5.1

Proof of Proposition 5.1:

Let us start with (i). Since $c < 0$, u is smooth. By a standard argument using the Pre-Maximum Principle, we have $u < 1$. Since u is smooth and satisfies $u(+\infty) = 1$, without loss of generality, we can assume that $u(x) \geq \frac{\rho}{2}$ in \mathbb{R}^+ . Define now the operator L by:

$$L[v] := J \star v - v - cv' - kv.$$

Choose $k > 0$ such that $-k < \frac{f(u)}{u} < k$ and $-k < \frac{f(u)}{u-1} < k$. Such a k exists because f is Lipschitz continuous in $[0,1]$ and $f(0) = f(1) = 0$. Let us define

$$\gamma_1(x) := \begin{cases} \frac{f(u)}{u} & \text{when } u(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that u and any translation of u satisfy

$$L[u] = -(\gamma_1(x) + k)u \leq 0 \quad \text{in } \mathbb{R}.$$

Now let us compute $L[g_\delta]$ in $\mathbb{R}^{*, -}$ with $g_\delta(x) := e^{-\delta|x|}$. This gives

$$(5.23) \quad L[g_\delta] = e^{-\delta|x|} \int_{\mathbb{R}} J(x-y) e^{-\delta(|y|-|x|)} dy - e^{\delta x} - c\delta e^{\delta x} - k e^{\delta x}$$

$$(5.24) \quad = e^{-\delta|x|} \left(\int_{\mathbb{R}} J(-z) e^{-\delta(|x+z|-|x|)} dz - 1 - c\delta - k \right)$$

$$(5.25) \quad \geq g_\delta \left(\int_{\mathbb{R}} J(-z) e^{-\delta|z|} dz - 1 - c\delta - k \right).$$

Since $c < 0$, let δ_1 be such that

$$\int_{\mathbb{R}} J(-z) e^{-\delta_1|z|} dz - 1 - c\delta_1 - k \geq 0.$$

Therefore $Lg_{\delta_1} \geq 0$ in $\mathbb{R}^{*, -}$.

By construction we have $\frac{\rho}{2}g_{\delta_1} \leq \frac{\rho}{2} < u(x)$ in \mathbb{R}^+ . Using the comparison principle (Theorem A.2) with u and $\frac{\rho}{2}g_{\delta_1}$ in $\mathbb{R}^{*,*}$, it follows that $\frac{\rho}{2}g_{\delta_1} \leq u$ in \mathbb{R} , which proves (i).

The proof of (ii) is in the same spirit. Again by a standard argument using the pre-maximum principle we have $u > 0$. Since u is smooth and satisfies $u(-\infty) = 0$, without loss of generality, we can this time assume that $u(x) \leq \frac{\rho}{2}$ in \mathbb{R}^- .

Let us now define

$$\gamma_2(x) := \begin{cases} \frac{-f(u)}{1-u} & \text{when } u(x) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $1 - u$ satisfies

$$\begin{aligned} L[1 - u] &= f(u) - k(1 - u) \\ &= -(\gamma_2(x) + k)(1 - u) \leq 0 \quad \text{in } \mathbb{R}. \end{aligned}$$

Let us now compute $L[g_\delta]$ in $\mathbb{R}^{*,+}$. This gives

$$(5.26) \quad L[g_\delta] = e^{-\delta|x|} \int_{\mathbb{R}} J(x - y) e^{-\delta(|y| - |x|)} dy - e^{-\delta x} + c\delta e^{-\delta x} - k e^{-\delta x}$$

$$(5.27) \quad = e^{-\delta|x|} \left(\int_{\mathbb{R}} J(-z) e^{-\delta(|x+z| - |x|)} dz - 1 + c\delta - k \right)$$

$$(5.28) \quad \geq g_\delta \left(\int_{\mathbb{R}} J(-z) e^{-\delta|z|} dz - 1 + c\delta - k \right).$$

Since $c > 0$, let δ_2 be such that

$$\int_{\mathbb{R}} J(-z) e^{-\delta_2|z|} dz - 1 + c\delta_2 - k \geq 0.$$

Therefore $L[g_{\delta_2}] \geq 0$ in $\mathbb{R}^{*,+}$.

By construction, u and g_{δ_2} satisfies $\frac{1-\rho}{2}g_{\delta_2} \leq \frac{1-\rho}{2} < 1 - u$ in \mathbb{R}^- . Using the comparison principle (Theorem A.2) with $1 - u$ and $\frac{1-\rho}{2}g_{\delta_2}$ in $\mathbb{R}^{*,*}$, it follows that $\frac{1-\rho}{2}g_{\delta_2} \leq 1 - u$ in \mathbb{R} , which proves (ii). □

5.3. Some monotony properties of solutions of (5.1). In this section we establish some monotonicity properties of solutions of equation (5.1) and give some *a priori* bounds on the speed c when the strong maximum principle does not hold. Let us start with the following

Theorem 5.4. *Let (u, c) be a continuous solution of (5.1). Then the following holds:*

- When $\text{supp}(J) \subset \mathbb{R}^+$, u is monotone in a neighbourhood of $-\infty$.
- When $\text{supp}(J) \subset \mathbb{R}^-$, u is monotone in a neighbourhood of $+\infty$.

The proof of this result uses improved techniques and ideas developed in [14, 15]. In particular, the proof relies on the following nonlinear comparison principle that was established in [14]:

Theorem 5.5. Nonlinear Comparison Principle

Assume that f is as in Theorem 1.2 and let $0 < u$ and $v < 1$ be two continuous functions such that

$$(5.29) \quad J \star u - u - \alpha u' + f(u) \leq 0 \quad \text{in } \mathbb{R},$$

$$(5.30) \quad J \star v - v - \alpha v' + f(v) \geq 0 \quad \text{in } \mathbb{R},$$

$$(5.31) \quad \lim_{x \rightarrow -\infty} u(x) \geq 0, \quad \lim_{x \rightarrow -\infty} v(x) \leq 0,$$

$$(5.32) \quad \lim_{x \rightarrow +\infty} u(x) \geq 1, \quad \lim_{x \rightarrow +\infty} v(x) \leq 1.$$

Then there exists a positive real number τ such that $u_\tau \geq v$.

and the following technical Lemma :

Lemma 5.4.

Let u and v be as in the above Theorem 5.5 and let $0 < \delta \leq \frac{\epsilon}{2}$ be such that

$$(5.33) \quad f(p) \quad \text{non increasing for } p < \delta \quad \text{and} \quad 1 - p < \delta.$$

Then there exists $M > 0$ so that

$$(5.34) \quad 1 - u(x) < \frac{\delta}{2} \quad \forall x > M$$

$$(5.35) \quad \text{and} \quad v(x) < \frac{\delta}{2} \quad \forall x < -M.$$

If furthermore there exists a positive constant b such that u and v satisfy:

$$(5.36) \quad u(x + b) > v(x) \quad \forall x \in [-M - 1, M + 1]$$

$$(5.37) \quad \text{and} \quad u(x + b) + \frac{\delta}{2} > v(x) \quad \forall x \in \mathbb{R}$$

Then we have $u(x + b) \geq v(x) \quad \forall x \in \mathbb{R}$.

Remark 5.4. The proof of the Theorem 5.5 and the Lemma 5.4 uses only the comparison property of the operator $J \star u - u$ and an explicit construction. Therefore Theorem 5.5 and the Lemma 5.4 hold also when J is replaced by any Borel probability measure μ .

Before proving Theorem 5.4, let us establish the two useful Lemmas

Lemma 5.5. Let u be a bounded continuous solution of (5.1) and assume that there exists $\tau > 0$ such that $u_\tau \geq u$. Then we have the following alternative

- Either u is non decreasing in a neighbourhood of $-\infty$ when $\text{supp}(J) \subset \mathbb{R}^+$, respectively in a neighbourhood of $+\infty$ when $\text{supp}(J) \subset \mathbb{R}^-$.
- Or $u_\tau > u$.

Proof:

By assumption we have $u_\tau \geq u$. Since u is continuous, either $u_\tau > u$ in \mathbb{R} or there exists $x_0 \in \mathbb{R}$ such that $u_\tau(x_0) = u(x_0)$. In the later case, at this point the following $w := u_\tau - u$ achieves a global minimum and satisfies

$$0 \leq J \star w(x_0) - w(x_0) = 0.$$

Therefore $u_\tau \equiv u$ in $x_0 - \text{supp}(J)$.

Assume for the moment that $\text{supp}(J) \subset \mathbb{R}^+$.

Using the above computation with any point of $x_0 - \text{supp}(J)$ and iterating this process yields

$u_\tau \equiv u$ in some neighbourhood $(-\infty, R)$. By rewriting $u_\tau \equiv u$ as $u \equiv u_{-\tau}$, we easily show that $u(R) = u(R - k\tau)$ for any $k \in \mathbb{N}$. Thus $u(R) = 0$ and $u \equiv 0$ in some neighbourhood $(-\infty, R)$, which proves the monotonicity of u in some neighbourhood $(-\infty, R)$ of $-\infty$.

In the other case, $\text{supp}(J) \subset \mathbb{R}^-$, we end up with $u_\tau \equiv u$ in some neighbourhood $(R_1, +\infty)$ instead of $(-\infty, R)$. Then $u(R_1) = u(R_1 + k\tau^*)$ for any $k \in \mathbb{N}$. Thus $u(R_1) = 1$ and $u \equiv 1$ in some neighbourhood $(R_1, +\infty)$, which proves the monotonicity of u in some neighbourhood $(R_1, +\infty)$.

□

We are now in a position to prove Theorem 5.4.

Proof of Theorem 5.4:

The proof being similar in both cases, we only expose the case $\text{supp}(J) \subset \mathbb{R}^+$. Let (u, c) be a continuous bounded solution of (5.1). Our proof follows three steps.

Step 1 – According to Theorem 5.5, since u is a sub and supersolution of problem (5.1) with speed c , there exists $\tau > 0$ such that $u_\tau \geq u$. Moreover using Lemma 5.4, there exists $M > 0$ such that

$$(5.38) \quad 1 - u(x) < \frac{\delta}{2} \quad \forall x > M$$

$$(5.39) \quad \text{and} \quad u(x) < \frac{\delta}{2} \quad \forall x < -M.$$

Using Lemma 5.5 we also have that either u is monotone a neighbourhood of $-\infty$ or $u_\tau > u$.

Step 2 – Now, let us show that for all $\tau' \geq \tau$ we have the following alternative:

- Either u is monotone a neighbourhood of $-\infty$
- Or $u_{\tau'} > u$.

From the previous step, we are to reduce to analyse the case $u_\tau > u$. By the previous step since u satisfies (5.38) and (5.39) we have for all $\epsilon > 0$,

$$u_{\tau+\epsilon} + \frac{\delta}{2} \geq u \quad \text{in} \quad \mathbb{R} \setminus [-M, M].$$

Since u is continuous on $[-M, M]$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, we have

$$u_{\tau+\epsilon} \geq u \quad \text{in} \quad [-M, M].$$

Therefore for all $\epsilon \in [0, \epsilon_0]$, $u_{\tau+\epsilon}$ and u satisfies the assumptions of Lemma 5.4 and it follows that $u_{\tau+\epsilon} \geq u$ for all $\epsilon \in [0, \epsilon_0]$. Using now Lemma 5.5 it follows that for all $\epsilon \in [0, \epsilon_0]$ either u is monotone a neighbourhood of $-\infty$ or $u_{\tau+\epsilon} > u$. Observe that the only case to be dealt with is when for all $\epsilon \in [0, \epsilon_0]$ we have $u_{\tau+\epsilon} > u$. Indeed, otherwise u is monotone a neighbourhood of $-\infty$ and there is nothing left to prove. So let us assume that $u_{\tau+\epsilon} > u$ for all $\epsilon \in [0, \epsilon_0]$, then $u_{\tau+\epsilon_0} > u$ and by repeating the above argument with $u_{\tau+\epsilon_0}$ instead of u_τ the desired alternative follows by induction.

Step 3 – From the two previous step, we end up with the following alternative

- (i) Either u is monotone a neighbourhood of $-\infty$
- (ii) Or for all τ such that $u_\tau \geq u$, we have $u_{\tau'} > u$ for all $\tau' \geq \tau$.

As in the above step, the only case to analyse is (ii). Let us define the following quantity

$$\tau^* := \inf\{\tau > 0 \mid \forall \tau' \geq \tau \ u_{\tau'} \geq \tau\}.$$

This quantity is well defined from the above steps. Observe that if $\tau^* = 0$, we are done, since it means that u is monotone in \mathbb{R} . So let us assume that $\tau^* > 0$. From the definition of τ^* , we see that there exists a point x_0 such that $u_{\tau^*}(x_0) = u(x_0)$. Indeed, if not then we have $u_{\tau^*} > u$ and using the continuity of u it follows that for some $\epsilon > 0$,

$$u_{\tau^*-\epsilon} \geq u \quad \text{in} \quad [-M, M]$$

and

$$u_{\tau^*-\epsilon} + \frac{\delta}{2} \geq u \quad \text{in} \quad \mathbb{R} \setminus [-M, M].$$

Using now Lemma 5.5 we have that $u_{\tau^*-\epsilon} \geq u$ which using Step 2 leads to a contradiction of the definition of τ^* . Therefore, there exists a point x_0 such that $w := u_{\tau^*} - u$ achieves a global minimum and satisfies

$$0 = J \star w(x_0) - w(x_0).$$

Arguing now as in the proof of Lemma 5.5, it follows that $u \equiv 0$ in some neighbourhood $(-\infty, R')$, which proves the monotonicity of u in some neighbourhood $(-\infty, R)$ of $-\infty$.

□

Remark 5.5. A consequence of the above proofs is that we have the following characterization of a continuous solution (u, c) of (5.1). More precisely we have

- Either $u \equiv 0$ in a neighbourhood of $-\infty$ when $\text{supp}(J) \subset \mathbb{R}^+$, respectively $u \equiv 1$ in a neighbourhood of $+\infty$ when $\text{supp}(J) \subset \mathbb{R}^-$.
- Or u is monotone.

Remark 5.6. Observe that for a general probability measure μ Theorem 5.4 holds as well provided we have $\text{supp}(\mu) \cap \mathbb{R}^+ \neq \emptyset$ and $\text{supp}(\mu) \cap \mathbb{R}^- \neq \emptyset$. Indeed, for a smooth solution (u, c) , we have in this case $0 < u < 1$. Therefore in the above proof the cases $u \equiv 0$ or $u \equiv 1$ in some unbounded set are always ruled out and we have $u_\tau > u$ for any positive real τ .

Now let us show some *a priori* estimates on the speed c .

Lemma 5.6. Let (u, c) be a solution of (5.1). Then

- $c \leq 0$ when $\text{supp}(J) \subset \mathbb{R}^+$.
- $c \geq 0$ when $\text{supp}(J) \subset \mathbb{R}^-$.

Remark 5.7. Observe that to obtain the sign of the speed, no regularity on u is required. Therefore, the Lemma holds for solution which are only L^∞ .

Proof:

First observe that in both cases, it is sufficient to prove the above statement for speed $c \neq 0$. Let us first start with the first case. In this situation u is smooth and an easy computation shows that

$$(5.40) \quad cu'(x) = \int_{-\infty}^x J(x-y)[u(y) - u(x)] dy + f(u).$$

By Theorem 5.4 u is non-decreasing in a neighbourhood $(-\infty, R)$ of $-\infty$. Therefore

$$\int_{-\infty}^x J(x-y)[u(y) - u(x)] dy \leq 0 \quad \text{in} \quad (-\infty, R).$$

Since u is smooth and satisfies the boundary conditions $u(+\infty) = 1$, there exists a point $x_0 \in (-\infty, R)$ such that $u(x_0) \leq \rho$ and $u'(x_0) > 0$. Using (5.40) at x_0 , it follows that $c \leq 0$ since $f(u(x_0)) \leq 0$.

The proof in the second case is rather similar. Indeed, in this case we have

$$(5.41) \quad cu'(x) = \int_x^{+\infty} J(x-y)[u(y) - u(x)] dy + f(u),$$

and by Theorem 5.4 u is non-decreasing in a neighbourhood $(R', +\infty)$ of $+\infty$. Therefore

$$\int_{-\infty}^x J(x-y)[u(y) - u(x)] dy \geq 0 \quad \text{in } (R', +\infty).$$

Since u is smooth and satisfies the boundary conditions $u(-\infty) = 0$, there exists a point $x_1 \in (R', +\infty)$ such that $u(x_1) \geq \rho$ and $u'(x_1) > 0$. Using (5.41) at x_1 , it follows that $c \geq 0$ since $f(u(x_1)) \geq 0$.

□

Finally, let us show the following result:

Theorem 5.6.

Let (u, c) be a bounded solution of (5.1) and assume that $c \neq 0$. Then u is monotone non decreasing.

Proof:

First observe that since $c \neq 0$, u is a smooth function. For convenience, let us first assume that J satisfies (H1). In this situation, the operator $\mathcal{M}[u] = J \star u - u$ satisfies a strong maximum principle and the monotonicity of u immediately follows from the arguments developed in [14, 15]. So we omit the proof here. Let us now treat the two other cases, $\text{supp}(J) \subset \mathbb{R}^+$ and $\text{supp}(J) \subset \mathbb{R}^-$. Observe that in both cases, from Proposition 5.1 we have $0 < u < 1$. The monotonicity of u then follows from Remark 5.5.

□

5.4. Uniqueness of the speed of the travelling wave . In this section we deal with the uniqueness of the speed c of the travelling wave solutions of (5.1). More precisely, we show the following

Theorem 5.7. Assume that J satisfies (H2). If there exists a non-decreasing travelling wave solution (u, c) of (5.1), then c is unique. The uniqueness is understood as follow: Let (v, c') be another solution of (5.1), then $c = c'$.

The proof uses improved techniques and ideas developed in [14, 15] and especially the nonlinear comparison principle established in [14], Theorem 5.5. For convenience we split the proof in three parts, corresponding respectively to the following three possible cases :

- (1) J satisfies (H1).
- (2) $\text{supp}(J) \subset \mathbb{R}^+$.
- (3) $\text{supp}(J) \subset \mathbb{R}^-$.

Let us first introduce the following notation :

$$L_c[u] := J \star u - u - cu'.$$

In the first case, the operator L_c satisfies a strong maximum principle and the uniqueness of the speed immediately follows from the argument developed in [14, 15]. So we omit the proof here. Since the proof in cases 2 and 3 are similar, we will only present the proof in the second case.

Proof of Theorem 5.7:

We argue by contradiction. Let (u, c) and (v, c') be two solutions of (5.1). Without loss of generality we can assume that (u, c) is a non-decreasing solution. From Lemma 5.6, Remark 5.7 and Proposition 5.1, we have $u < 1$, $c \leq 0$, $c' \leq 0$, and $v < 1$. Assume that for a moment that $c < 0$, then u is smooth and we have

$$(5.42) \quad L_{c'}[u] + f(u) = (c - c')u'.$$

Therefore u will be either a subsolution or supersolution of equation (5.42) according to the sign of $(c - c')$. First assume by contradiction that $c - c' < 0$. Then u is a supersolution of (5.42) and according to Theorem 5.5 the following quantity is well defined:

$$(5.43) \quad \tau^* := \inf\{\tau \in \mathbb{R} \mid u_\tau \geq v\}.$$

Using the boundary conditions, we easily check that $\tau^* > -\tau_0$ for some positive τ_0 . Arguing as in the proof of Theorem 5.4 there exists a point $x_0 \in \mathbb{R}$, such that $u_{\tau^*}(x_0) = v(x_0)$. Therefore the function $w := u_{\tau^*} - v \geq 0$ achieves a non positive minimum at x_0 and satisfies

$$0 \leq L_{c'}[w](x_0) = J \star w(x_0) - w(x_0) - c'w(x_0) = (c - c')u'(x_0) \leq 0.$$

Thus

$$J \star w(x_0) - w(x_0) = 0$$

and $u_{\tau^*} \equiv v$ in $x_0 - \text{supp}(J)$. Arguing as in Theorem 5.5, we have $u_{\tau^*} \equiv v$ in $(-\infty, R)$ for some $R \in \mathbb{R}$. Since $1 > u > 0$, we can choose x_1 in $(-\infty, R)$ such that $u'_{\tau^*}(x_1) > 0$. Using the above argument with x_1 instead of x_0 it yields $u_{\tau^*} \equiv v$ in $x_1 - \text{supp}(J)$. Since $x_1 \in (-\infty, R)$, by construction we also have $v'(x_1) = u'_{\tau^*}(x_1)$. Using now that (v, c') satisfies (5.1), at the point x_1 we have

$$\begin{aligned} & J \star v(x_1) - v(x_1) - c'v'(x_1) + f(v(x_1)) = 0, \\ \Leftrightarrow & \int_{\mathbb{R}} J(x_1 - y)v(y) dy - u_{\tau^*}(x_1) - c'u'_{\tau^*}(x_1) + f(u_{\tau^*}(x_1)) = 0, \\ \Leftrightarrow & \int_{x_1 - \text{supp}(J)} J(x_1 - y)v(y) dy - u_{\tau^*}(x_1) - c'u'_{\tau^*}(x_1) + f(u_{\tau^*}(x_1)) = 0, \\ \Leftrightarrow & \int_{x_1 - \text{supp}(J)} J(x_1 - y)u_{\tau^*}(y) dy - u_{\tau^*}(x_1) - c'u'_{\tau^*}(x_1) + f(u_{\tau^*}(x_1)) = 0, \\ \Leftrightarrow & (c - c')u'_{\tau^*}(x_1) = 0, \end{aligned}$$

which is a contradiction since $u'_{\tau^*}(x_1) > 0$ and $c \neq c'$. Therefore, $c - c' \geq 0$. Let us now assume that $c - c' > 0$, then u is a subsolution of (5.42). As above according to Theorem 5.5 the following quantity is well defined :

$$(5.44) \quad \tau^* := \inf\{\tau \in \mathbb{R} \mid v \geq u_{-\tau}\}.$$

Now following the a simillar argument as above yields the desired contradiction. Thus $c - c' \leq 0$ and we achieve $c' \leq c \leq c'$, which means $c = c'$. To finish the proof, it remains to treat the case $c = 0$. In this situation, since $c' \leq 0$, the only case to treat is when $c' < 0$. In this situation, by Theorem 5.6 v is monotone increasing and we can interchange the role of

u and v in the above analysis to obtain the desired contraction. Hence, in all situation we have $c = c'$.

□

Remark 5.8. Note that the argument to obtain the uniqueness of the speed only relies on the nonlinear comparison principle and the maximum principle. Observe that a strong maximum principle is satisfied for a Borel probability measure μ satisfying the conditions

- $\text{supp}(\mu) \cap \mathbb{R}^+ \neq \emptyset$ and $\text{supp}(\mu) \cap \mathbb{R}^- \neq \emptyset$.
- There exists two real numbers $a, b \in \text{supp}(\mu)$ such that $\frac{a}{b} \in \mathbb{R} \setminus \mathbb{Q}$.

Therefore, following the above argument leads to the uniqueness of the speed of the front in this context. Moreover, since the strong maximum is satisfied on \mathbb{R} , we also have the uniqueness up to translation of the profile u , i.e if (v, c) is another solution then $u \equiv v_\tau$ for some real τ .

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APPENDIX A. Appendix

We present in this section a collection of basic results ranging from maximum principles to existence of solutions for a nonlinear problem. For convenience we present only some of the proofs and give references otherwise. The first two subsections recall several maximum principles and comparison principles that we use throughout this paper. The last subsection deals with the linear and nonlinear theory.

A.1. Maximum principles in bounded domains. We start this section with maximum principles for operators \mathcal{L} defined by (3.3):

Theorem A.1. *Strong Maximum Principle*

Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be such that

$$\mathcal{L}[u] \geq 0 \text{ in } \Omega \quad (\text{resp. } \mathcal{L}[u] \leq 0 \text{ in } \Omega).$$

Then u cannot achieve a non-negative maximum (resp. non-positive minimum) in Ω without being constant.

Similar theorems holds for the operators \mathcal{S} and \mathcal{M} defined by (3.10) and (1.5). Next we present a comparison result.

Theorem A.2. *Comparison Principle*

Assume that J satisfies (H1), $b, \gamma \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ and $\gamma \geq 0$.

Let u and v be two smooth functions ($C^{1,\alpha}(\mathbb{R})$) and ω be a connected subset of \mathbb{R} . Assume that u and v satisfy the following conditions :

- $\mathcal{M}[v] - b(x)v' - \gamma(x)v \geq 0$ in $\omega \subset \mathbb{R}$,
- $\mathcal{M}[u] - b(x)u' - \gamma(x)u \leq 0$ in $\omega \subset \mathbb{R}$,
- $u \geq_{\neq} v$ in $\mathbb{R} - \omega$,
- if ω is an unbounded domain, assume also that $\lim_{\pm\infty} u - v \geq 0$.

Then $u \geq v$ in \mathbb{R} .

A similar theorem holds for the operator \mathcal{S} defined by (3.10). See [15] for a proof of Theorems A.1 and A.2.

Let us now prove a useful maximum principle which holds for L^∞ sub and supersolutions of the following problem

$$(A.1) \quad J \star u - u - cu' - \lambda u = 0 \quad \text{almost everywhere in } \mathbb{R}$$

$$(A.2) \quad u(-\infty) = 0$$

$$(A.3) \quad u(+\infty) = 0$$

Theorem A.3. *L^∞ Maximum Principle*

Assume that J satisfies (H1) and λ is nonnegative. Let $u \in L^\infty$ be such that

$$J \star u - u - cu' - \lambda u \leq 0 \quad \text{almost everywhere in } \mathbb{R}$$

$$u(-\infty) \geq 0$$

$$u(+\infty) \geq 0$$

Then $u \geq 0$ in \mathbb{R} .

Proof:

Let ϵ_n be a sequence of real numbers which tends to 0 and let ρ_{ϵ_n} be a sequence of C^∞ mollifier such that for all $x \in \mathbb{R}$, $\rho_{\epsilon_n} \star u(x) \rightarrow u(x)$. For a fixed ϵ let us define $u_\epsilon := \rho_\epsilon \star u$. Since u satisfies

$$J \star u - u - cu' - \lambda u \leq 0 \quad \text{almost everywhere in } \mathbb{R},$$

it follows that

$$J \star u_\epsilon - u_\epsilon - cu'_\epsilon - \lambda u_\epsilon \leq 0 \quad \text{in } \mathbb{R}.$$

Using now the classical maximum principle, we have that $u_\epsilon \geq 0$ in \mathbb{R} . Letting ϵ goes to 0 along the sequence ϵ_n , it follows that for all $x \in \mathbb{R}$

$$u(x) = \lim_{n \rightarrow \infty} u_{\epsilon_n}(x) \geq 0.$$

Hence, $u \geq 0$ in \mathbb{R} . □

A.2. Linear Theory. Next we provide an elementary lemma to construct solutions to constant-coefficient Dirichlet problems of the form

$$\begin{cases} \mathcal{L}[u] - \lambda v = f & \text{in } \Omega = (-r, R), \\ u(R) = 0, \\ u(-r) = 0. \end{cases}$$

Lemma A.1. Let $f \in C^0(\Omega) \cap L^2(\Omega)$ and recall that $\mathcal{L} - \lambda Id$ is defined by

$$\mathcal{L}[v] - \lambda v = \epsilon v'' + \int_{-r}^R J(x-y)v(y)dy - v - cv' - \lambda v,$$

where $\epsilon > 0$, $c, \lambda \in \mathbb{R}$, $\lambda > 0$. Assume furthermore that $(|c| + \int_{\mathbb{R}} J(z)|z|dz) < \sqrt{\lambda\epsilon}$. Then there exists a unique solution $v \in C^0(\mathbb{R}) \cap L^2(\Omega)$ of

$$(A.4) \quad \begin{cases} \mathcal{L}[v] - \lambda v = f & \text{in } \Omega, \\ v(-r) = 0, \\ v(R) = 0. \end{cases}$$

Moreover, $v \in C^2(\Omega)$.

Proof

Uniqueness follows from the maximum principle. Let $X = H_0^1(\Omega)$ and define the following bilinear form $\mathcal{A}(u, v)$ for $u, v \in X$:

$$\mathcal{A}(u, v) = \epsilon \int_{\Omega} u' v' - \int_{\Omega} \int_{\Omega} J(x-y)[u(y) - u(x)]v(x) dy dx + c \int_{\Omega} u' v + (\lambda + (1 - h_1^+ - h_1^-)) \int_{\Omega} uv.$$

We will show that \mathcal{A} is coercive and continuous in X . Existence will then be given by the Lax-Milgram Lemma.

We start with the continuity of \mathcal{A} . Observe that by a density argument, it is sufficient to prove the continuity for smooth functions with compact support. Let ϕ and ψ be two smooth functions with compact support in Ω . Let \mathcal{B} the following bilinear form :

$$\mathcal{B}(\phi, \psi) = \int_{\Omega} \int_{\Omega} J(x-y)[\phi(y) - \phi(x)]\psi(x) dy dx.$$

From basic computations and the Cauchy-Schwartz inequality we have

$$\begin{aligned} \mathcal{A}(\phi, \psi) &\leq \epsilon \|\phi'\|_{L^2(\Omega)} \|\psi'\|_{L^2(\Omega)} + |c| \|\phi'\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} + (\lambda + 3) \|\phi\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} + \mathcal{B}(\phi, \psi) \\ &\leq C \|\phi\|_{H_0^1(\Omega)} \|\psi\|_{H_0^1(\Omega)} + |\mathcal{B}(\phi, \psi)|. \end{aligned}$$

Therefore, to obtain the continuity of \mathcal{A} , it remains to show that \mathcal{B} is a continuous bilinear form for the H_0^1 norm.

From the Fundamental Theorem of Calculus, Fubini's Theorem and the Cauchy-Schwartz inequality we have:

$$\begin{aligned} |\mathcal{B}(\phi, \psi)| &\leq \int_{\Omega} \int_{\Omega} J(x-y) |\phi(y) - \phi(x)| |\psi(x)| dy dx \\ &\leq \int_{\Omega} \int_{\Omega} J(z) |z| \left(\int_0^1 |\phi'(x + tz)| |\psi(x)| dt \right) dz dx \\ &\leq \int_{\Omega} \int_0^1 J(z) |z| \left(\int_{\Omega} |\phi'(x + tz)| |\psi(x)| dx \right) dz dt \\ &\leq \int_{\Omega} \int_0^1 J(z) |z| dz dt \|\phi'\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \\ &\leq \left(\int_{\mathbb{R}} J(z) |z| dz \right) \|\phi'\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}. \end{aligned}$$

Thus

$$(A.5) \quad \mathcal{B}(\phi, \psi) \leq \left(\int_{\mathbb{R}} J(z) |z| dz \right) \|\psi\|_{L^2(\Omega)} \|\phi'\|_{L^2(\Omega)} \leq C \|\phi\|_{H_0^1(\Omega)} \|\psi\|_{H_0^1(\Omega)},$$

which shows the continuity of \mathcal{B} and therefore the continuity of \mathcal{A} .

To complete the proof it remains to prove the coercivity of \mathcal{A} . Again, by a density argument, it is sufficient to prove the coercivity for smooth functions with compact support. Let $u \in C_0^\infty(\Omega)$. Since $(1 - h_1^+ - h_1^-) \geq 0$, using Hölder's inequality we observe that

$$(A.6) \quad \mathcal{A}(u, u) \geq \epsilon \int_{\Omega} (u')^2 - |c| \|u'\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \lambda \int_{\Omega} u^2 - \mathcal{B}(u, u).$$

Now using (A.5) with u instead of ϕ and ψ yields

$$\mathcal{B}(u, u) \leq \left(\int_{\mathbb{R}} J(z)|z| dz \right) \|u\|_{L^2(\Omega)} \|u'\|_{L^2(\Omega)}.$$

Therefore we have

$$(A.7) \quad \mathcal{A}(u, u) \geq \epsilon \|u'\|_{L^2(\Omega)}^2 - \left(\int_{\mathbb{R}} J(z)|z| dz + |c| \right) \|u'\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \lambda \|u\|_{L^2(\Omega)}^2.$$

Since $(|c| + \int_{\mathbb{R}} J(z)|z| dz) \leq \sqrt{\lambda\epsilon}$, we have

$$(A.8) \quad \frac{\epsilon}{2} \|u'\|_{L^2(\Omega)}^2 - (|c| + \int_{\mathbb{R}} J(z)|z| dz) \|u'\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \frac{\lambda}{2} \|u\|_{L^2(\Omega)}^2 \geq 0.$$

Combine (A.7) and (A.8) to find that

$$(A.9) \quad \mathcal{A}(u, u) \geq \frac{\epsilon}{2} \|u'\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|u\|_{L^2(\Omega)}^2 \geq c_0 \|u\|_{H_0^1(\Omega)},$$

with $c_0 = \frac{\min\{\epsilon, \lambda\}}{2}$, which shows the coercivity of \mathcal{A} . □

Remark A.1.

A consequence of Lemma A.1 is the unique solvability of the following problem :

$$(A.10) \quad \begin{cases} \mathcal{L}[u] - \lambda u = f & \text{in } \Omega, \\ u(-r) = \alpha, \\ u(R) = \beta. \end{cases}$$

for λ large enough. Indeed, let $g \in C_0^\infty(\mathbb{R})$ with $g(-r) = \alpha$, $g(r) = \beta$, and let v be the unique solution of (A.4) with right handside $f - \mathcal{L}[g] - \lambda g \in C^0(\Omega) \cap L^2(\Omega)$. Then $u = g + v$ is the unique solution of (A.10).

Since the maximum principle holds for equation (A.10), using standard sub and supersolution scheme, we can easily construct a solution for the following semilinear problem:

$$(A.11) \quad \begin{cases} \mathcal{L}[u] = f(u) & \text{in } \Omega, \\ u(-r) = \alpha, \\ u(R) = \beta. \end{cases}$$

where f is a lipschitz continuous function. Namely, we have the following

Theorem A.4.

Let \bar{u} and \underline{u} be respectively a supersolution and a subsolution of (A.11) and assume that $\underline{u} \leq \bar{u}$. Then there exists a solution u of (A.11) such that $\underline{u} \leq u \leq \bar{u}$.

The proof is rather standard and we leave it to the reader.

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¹ MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTRASSE 22, D-04103 LEIPZIG, DEUTSCHLAND

Current address: ² INRA Avignon -BioSP, Domaine St Paul, Agroparc, 84914 Avignon cedex 9, France

E-mail address: coville@avignon.inra.fr