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# Risk estimation for matrix recovery with spectral regularization

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## Abstract

In this paper, we develop an approach to recursively estimate the quadratic risk for matrix recovery problems regularized with spectral functions. Toward this end, in the spirit of the SURE theory, a key step is to compute the (weak) derivative and divergence of a solution with respect to the observations. As such a solution is not available in closed form, but rather through a proximal splitting algorithm, we propose to recursively compute the divergence from the sequence of iterates. A second challenge that we unlocked is the computation of the (weak) derivative of the proximity operator of a spectral function. To show the potential applicability of our approach, we exemplify it on a matrix completion problem to objectively and automatically select the regularization parameter.

## 1. Introduction

Consider the problem of estimating a matrix  $X_0 \in \mathbb{R}^{n_1 \times n_2}$  from  $P$  noisy observations  $y = \mathcal{A}(X_0) + w \in \mathbb{R}^P$ , where  $w \sim \mathcal{N}(0, \sigma^2 \text{Id}_P)$ . The linear bounded operator  $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^P$  entails loss of information such that the problem is ill-posed. This problem arises in various research fields. Because of ill-posedness, side information through a regularizing term is necessary. We thus consider the problem

$$X(y) \in \underset{X \in \mathbb{R}^{n_1 \times n_2}}{\text{Argmin}} \frac{1}{2} \|y - \mathcal{A}(X)\|^2 + \lambda J(X) \quad (1)$$

where the set of minimizers is assumed non-empty,  $\lambda > 0$  is a regularization parameter and  $J : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R} \cup \{\infty\}$  is a proper lower semi-continuous (lsc) convex regularizing function that imposes the desired structure on  $X(y)$ . In this paper, we focus on the case where  $J$  is a convex spectral function, that is a symmetric convex function of the singular values of its argument. Spectral regularization can account for prior knowledge on the spectrum of  $X_0$ , typically low-rank (see e.g. Fazel, 2002).

In practice, the choice of the regularization parameter  $\lambda$  in (1) remains an important problem largely unexplored. Typically, we want to select  $\lambda$  minimizing the quadratic risk  $\mathbb{E}_w \|X(y) - X_0\|^2$ . Since  $X_0$  is unknown and  $X(y)$  is non-unique, one can instead consider an unbiased estimate of the *prediction* risk  $\mathbb{E}_w \|\mathcal{A}(X(y)) - \mathcal{A}(X_0)\|^2$ , where it can be easily shown that  $\mu(y) = \mathcal{A}(X(y))$  is a single-valued mapping. With the proviso that  $\mu(y)$  is weakly differentiable, the SURE (for Stein unbiased risk estimator, Stein, 1981)

$$\text{SURE}(y) = \|y - \mu(y)\|^2 - P\sigma^2 + 2\sigma^2 \text{div } \mu(y) \quad (2)$$

is an unbiased estimate of the prediction risk, where  $\text{div } \mu(y) = \text{Tr}(\partial\mu(y))$ , and  $\partial\mu(y)$  stands for the (weak) Jacobian of  $\mu(y)$ . The SURE depends solely on  $y$ , without prior knowledge of  $X_0$  and then can prove very useful as a basis for automatic ways to choose the regularization parameters  $\lambda$ .

**Contributions.** Our main contribution is to provide the derivative of matrix-valued spectral functions where the matrices have distinct singular values which extends the result of Lewis & Sndov (2001) to non-symmetric matrices. This result is used to recursively compute the derivative of any solution of spectrally regularized inverse problems by solving (1). This is achieved by computing the derivatives of the iterates provided by a proximal splitting algorithm. In particular, this provides an estimate of  $\text{div } \mu(y)$  in (2) which allows to compute  $\text{SURE}(y)$ . A Numerical example on a matrix completion problem is given to support our findings.

## 2. Recursive risk estimation

**Proximal splitting** Proximal splitting algorithms have become extremely popular to solve non-smooth convex optimization problems that arise often in inverse problems, e.g. (1). These algorithms provide a sequence of iterates  $X^{(\ell)}(y)$  that provably converges to a solution  $X(y)$ . A practical way to compute  $\text{div } \mu(y)$ , hence  $\text{SURE}(y)$ , as initiated by Vonesch et al. (2008), and that we pursue here, consists in differentiating this sequence of iterates. This methodology has been extended to a wide class of proximal splitting schemes in (Deledalle et al., 2012). For the sake of clarity, and without loss of generality, we focus on the case of the forward-backward (FB) splitting algorithm (Combettes & Wajs, 2005).

The FB scheme is a good candidate to solve (1) if  $J$  is simple, meaning that its proximity operator has a closed-form. Recall that the proximity operator of a lsc proper convex function  $G$  on  $\mathbb{R}^{n_1 \times n_2}$  is

$$\text{Prox}_G(X) = \underset{Z \in \mathbb{R}^{n_1 \times n_2}}{\text{argmin}} \frac{1}{2} \|X - Z\|_F^2 + G(Z).$$

The FB algorithm iteration reads

$$X^{(\ell+1)} = \text{Prox}_{\tau\lambda J}(X^{(\ell)} + \tau\mathcal{A}^*(y - \mathcal{A}(X^{(\ell)}))) \quad (3)$$

where  $\mathcal{A}^*$  denotes the adjoint operator of  $\mathcal{A}$ ,  $\tau > 0$  is chosen such that  $\tau\|\mathcal{A}^*\mathcal{A}\| < 2$ , the dependency of the iterate  $X^{(\ell)}$  to  $y$  is dropped to lighten the notation.

**Risk estimation** The divergence term  $\text{div } \mu(y)$  is obtained by differentiating formula (3), which allows, for any vector  $\delta \in \mathbb{R}^P$  to compute iteratively  $\xi^{(\ell)} = \partial X^{(\ell)}(y)[\delta]$  (the derivative of  $y \mapsto X^{(\ell)}(y)$  at  $y$  in the direction  $\delta$ ) as

$$\begin{aligned} \xi^{(\ell+1)} &= \partial \text{Prox}_{\tau\lambda J}(\Xi^{(\ell)})[\zeta^{(\ell)}] \\ \text{where } \Xi^{(\ell)} &= X^{(\ell)} + \tau\mathcal{A}^*(y - \mathcal{A}(X^{(\ell)})) \\ \text{and } \zeta^{(\ell)} &= \xi^{(\ell)} + \tau\mathcal{A}^*(\delta - \mathcal{A}(\xi^{(\ell)})). \end{aligned}$$

Using the Jacobian trace formula of the divergence, it can be easily seen that

$$\text{div } \mu(y) = \mathbb{E}_\delta \langle \partial\mu(y)[\delta], \delta \rangle \approx \frac{1}{k} \sum_{i=1}^k \langle \partial\mu(y)[\delta_i], \delta_i \rangle \quad (4)$$

where  $\delta \sim \mathcal{N}(0, \text{Id}_P)$  and  $\delta_i$  are  $k$  realizations of  $\delta$ . The  $\text{SURE}(y)$  can in turn be iteratively estimated by plugging  $\partial\mu(y)[\delta_i] = \mathcal{A}(\partial X^{(\ell)}(y)[\delta_i])$  in (4).

## 3. Local behavior of spectral functions

This section studies the local behavior of real- and matrix-valued spectral functions. We write the (reduced) singular value decomposition (SVD) of a matrix  $X \in \mathbb{R}^{n_1 \times n_2}$

$$X = V_X \text{diag}(\Lambda_X) U_X^*$$

(which might not be in general unique), where  $\Lambda_X \in \mathbb{R}^n$  is the vector of singular values of  $X$  with  $n = \min(n_1, n_2)$ ,  $\text{diag}(\Lambda_X) \in \mathbb{R}^{n \times n}$  denotes the diagonal matrix with diagonal entries  $\Lambda_X$  and  $V_X \in \mathbb{R}^{n_1 \times n}$  and  $U_X \in \mathbb{R}^{n_2 \times n}$  are the unitary matrices of singular vectors.

### 3.1. Scalar-valued Spectral Functions

A real-valued spectral function  $J$  can by definition be written as

$$J(X) = \varphi(\Lambda_X) \quad (5)$$

where  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a symmetric function of its argument, meaning  $\varphi(P\Lambda) = \varphi(\Lambda)$  for any permutation matrix  $P \in \mathbb{R}^{n \times n}$  and  $\Lambda$  in the domain of  $\varphi$ . We extend  $\varphi$  to the negative half-line as  $\varphi(\Lambda) = \varphi(|\Lambda|)$ .

We then consider  $J$  a scalar-valued spectral function as defined in (5). From subdifferential calculus on spectral functions (Lewis, 1995), we get the following.

**Proposition 1.** *A spectral function  $J(X) = \varphi(\Lambda_X)$  is convex if and only if  $\varphi$  is convex, and then*

$$\forall \gamma > 0, \quad \text{Prox}_{\gamma J}(X) = V_X \text{diag}(\text{Prox}_{\gamma \varphi}(\Lambda_X))U_X^*.$$

### 3.2. Matrix-valued Spectral Functions

We now turn to matrix-valued spectral functions

$$F(X) = V_X \text{diag}(\Phi(\Lambda_X))U_X^*, \quad (6)$$

where  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is symmetric in its arguments, meaning  $\Phi \circ P = P \circ \Phi$  for any permutation matrix  $P \in \mathbb{R}^{n \times n}$ . We extend  $\Phi$  to negative numbers as  $\Phi(\Lambda) = \text{sign}(\Lambda) \odot \Phi(|\Lambda|)$  and  $\odot$  is the entry-wise matrix multiplication. One can observe that for  $F(X) = \text{Prox}_{\gamma J}(X)$  with  $\Phi = \text{Prox}_{\gamma \varphi}$ , the proximity operator of a convex scalar-valued spectral function is a matrix-valued spectral function. The following theorem provides a closed-form expression of the derivative of  $F$  that generalizes the result of Lewis & Sndov (2001) to any (non-necessarily symmetric) matrix-valued functions. Note that regularity properties and expression of the directional derivative of symmetric matrix-valued spectral functions over non-necessarily symmetric matrices with separable regularizations  $\varphi$  were also derived in (Sun & Sun, 2003).

**Theorem 1.** *Assume that the singular values of  $X$  are all distinct. A matrix-valued spectral function  $F$  is differentiable at  $X$  if and only if  $\Phi$  is differentiable at  $\Lambda_X$ . In this case*

$$\forall \delta \in \mathbb{R}^{n_1 \times n_2}, \quad \partial F(X)[\delta] = V_X (\mathcal{M}(\Lambda_X)[\bar{\delta}] + \Gamma_S(\Lambda_X) \odot \mathcal{P}_S(\bar{\delta}) + \Gamma_A(\Lambda_X) \odot \mathcal{P}_A(\bar{\delta}))U_X^*$$

where  $\bar{\delta} = V_X^* \delta U_X \in \mathbb{R}^{n \times n}$ , the symmetric and anti-symmetric parts are defined as

$$\mathcal{P}_S(Y) = \frac{Y + Y^*}{2} \quad \text{and} \quad \mathcal{P}_A(Y) = \frac{Y - Y^*}{2}$$

and  $\mathcal{M}(\Lambda) : \mathbb{R}^{n \times n} \mapsto \mathbb{R}^{n \times n}$  is

$$\mathcal{M}(\Lambda) = \text{diag} \circ \partial \Phi(\Lambda) \circ \text{diag}.$$

The matrices  $\Gamma_S(\Lambda)$  and  $\Gamma_A(\Lambda)$  are defined, for all  $1 \leq i \leq n$  and  $1 \leq j \leq n$ , as

$$\Gamma_S(\Lambda)_{i,j} = \begin{cases} 0 & \text{if } i = j \\ \frac{\Phi(\Lambda)_i - \Phi(\Lambda)_j}{\Lambda_i - \Lambda_j} & \text{if } \Lambda_i \neq \Lambda_j \\ \partial \Phi(\Lambda)_{i,i} - \partial \Phi(\Lambda)_{i,j} & \text{otherwise,} \end{cases}$$

$$\Gamma_A(\Lambda)_{i,j} = \begin{cases} 0 & \text{if } i = j \\ \frac{\Phi(\Lambda)_i + \Phi(\Lambda)_j}{\Lambda_i + \Lambda_j} & \text{if } \Lambda_i > 0 \quad \text{or} \quad \Lambda_j > 0 \\ \partial \Phi(\Lambda)_{i,i} - \partial \Phi(\Lambda)_{i,j} & \text{otherwise.} \end{cases}$$

A proof is given in Appendix A.

Note that for symmetric matrices  $X$  and  $\delta$ , we recover the result of Lewis & Sndov (2001) when  $X$  has distinct singular values. For matrices with repeated singular values, we conjecture that our theorem and the expression of the derivative remain valid. For the symmetric case with repeated eigenvalues, this was formally proved in Lewis & Sndov (2001).

## 4. Numerical applications

### 4.1. Nuclear norm regularization

We here consider the problem of recovering a low-rank matrix  $X_0 \in \mathbb{R}^{n_1 \times n_2}$ . To this end,  $J$  is taken as the nuclear norm (a.k.a., trace or Schatten 1-norm) which is in some sense the tightest convex relaxation to the NP-hard rank minimization problem (Candès & Recht, 2009). The nuclear norm is defined by

$$J(X) = \|X\|_* \triangleq \|\Lambda_X\|_1. \quad (7)$$

Taking  $J(\cdot)$  as  $\|\cdot\|_*$  and  $\varphi$  as  $\|\cdot\|_1$  in Proposition 1 gives:

**Corollary 1.** *The proximal operator of  $\gamma\|\cdot\|_*$  is*

$$\forall \gamma > 0, \quad \text{Prox}_{\gamma\|\cdot\|_*}(X) = V_X \text{diag}(T_\gamma(\Lambda_X))U_X^*, \quad (8)$$

where  $T_\gamma = \text{Prox}_{\gamma\|\cdot\|_1}$  is the component-wise soft-thresholding, defined for  $i = 1, \dots, n$  as

$$T_\gamma(t)_i = \max(0, 1 - \gamma/\|t_i\|)t_i.$$

We now turn to the derivative of  $F = \text{Prox}_{\gamma\|\cdot\|_*}$ . A straightforward attempt is to take  $\Phi = \text{Prox}_{\gamma\|\cdot\|_1} = T_\gamma$  and apply Theorem 1 with

$$\partial\Phi(X)[\delta]_i = \partial T_\gamma(t)[\delta]_i = \begin{cases} 0 & \text{if } \|t_i\| \leq \gamma \\ \delta_i & \text{otherwise.} \end{cases} \quad (9)$$

However, strictly speaking, Theorem 1 does not apply since a proximity mapping is 1-Lipschitz in general, hence not necessarily differentiable everywhere. Thus, its derivative may be set-valued, as is the case for soft-thresholding at  $\pm\gamma$ .

A direct consequence of Corollary 1 is that  $J$  is a simple function allowing for the use of the FB algorithm. Moreover, the expression of the derivative (9) provides an estimation of the SURE as explained in Section 2.

### 4.2. Application to matrix completion

We now exemplify the proposed SURE computation approach on a matrix completion problem encountered in recommendation systems such as the popular Netflix problem. We therefore consider the forward model  $y = \mathcal{A}(X_0) + w \in \mathbb{R}^P$ ,  $w \sim \mathcal{N}(0, \sigma^2 \text{Id}_P)$ , where  $X_0$  is a dense but low-rank (or approximately so) matrix and  $\mathcal{A}$  is binary masking operator.

We have taken  $(n_1, n_2) = (1000, 100)$  and  $P = 25000$  observed entries (i.e., 25%). The underlying dense matrix  $X_0$  has been chosen to be approximately low-rank with a rapidly decaying spectrum  $\Lambda_{X_0} = \{k^{-1}\}_{k=1}^n$ . The standard deviation  $\sigma$  has been set such that the resulting minimum least-square estimate has a relative error  $\|X_{LS} - X_0\|_F / \|X_0\|_F = 0.9$ . Figure 1 depicts the prediction risk and its SURE estimate as a function of  $\lambda$ . For each value of  $\lambda$  in the tested range,  $\text{SURE}(y)$  in (2) has been computed for a single realization of  $y$  with  $k = 4$  realizations  $\delta_i$  in (4)<sup>1</sup>. At the optimal  $\lambda$  value,  $X(y)$  has a rank of 55 with a relative error of 0.46 (i.e., a gain of about a factor 2 w.r.t. the least-square estimator).

## 5. Conclusion

The core theoretical contribution of this paper is the derivative of (rectangular) matrix-valued spectral functions. This was a key step to compute the derivative of the proximal operator associated to the nuclear norm, and finally to use the SURE to recursively estimate the quadratic prediction risk of matrix recovery problems involving the nuclear norm regularization. The SURE was also used to automatically select the optimal regularization parameter.

<sup>1</sup>Without impacting the optimal choice of  $\lambda$ , the two curves have been vertically shifted for visualization.

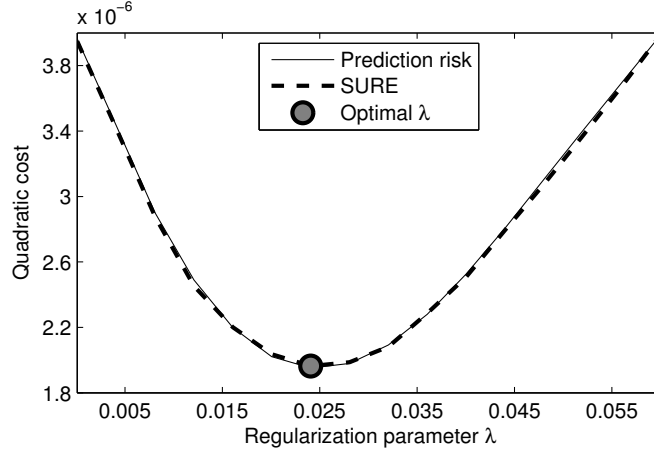


Figure 1. Predicted risk and its SURE estimate<sup>1</sup>.

## A. Summary of the proof of Theorem 1

The following lemma derives the expression of the derivative of the SVD mapping  $X \mapsto (V_X, \Lambda_X, U_X)$ . Note that this mapping is not well defined because even if the  $\Lambda_X$  are distinct, one can apply arbitrary sign changes and permutations to the set of singular vectors. The lemma should thus be interpreted in the sense that one can locally write a Taylor expansion using the given differential for any particular choice of SVD. We point out that a proof of this lemma using wedge products can be found in (Edelman, 2005), but for the sake of completeness, we provide our own proof here.

**Lemma 1.** *We consider  $X_0$  with distinct singular values. For any matrix  $X$  in a neighborhood of  $X_0$ , we can define without ambiguity the SVD mapping  $X \mapsto (V_X, \Lambda_X, U_X)$  by sorting the values in  $\Lambda_X$  and imposing sign constraints on  $U_X$ . The singular value mapping  $X \mapsto (V_X, \Lambda_X, U_X)$  is  $C^1$  and for a given matrix  $\delta$ , its directional derivative is*

$$\begin{aligned}\partial\Lambda_X[\delta] &= \text{diag}(V_X^* \delta U_X), \\ \partial V_X[\delta] &= V_X \iota_V, \\ \partial U_X[\delta] &= U_X \iota_U\end{aligned}$$

where  $\iota_V \in \mathbb{R}^{n \times n}$  and  $\iota_U \in \mathbb{R}^{n \times n}$  are defined, for all  $1 \leq i \leq n$  and  $1 \leq j \leq n$ , as

$$(\iota_V)_{i,j} = \frac{(\Lambda_X)_j \bar{\delta}_{i,j} + (\Lambda_X)_i \bar{\delta}_{j,i}}{(\Lambda_X)_j^2 - (\Lambda_X)_i^2} \quad \text{and} \quad (\iota_U)_{i,j} = \frac{(\Lambda_X)_i \bar{\delta}_{i,j} + (\Lambda_X)_j \bar{\delta}_{j,i}}{(\Lambda_X)_j^2 - (\Lambda_X)_i^2}. \quad (10)$$

and where  $\bar{\delta} = V_X^* \delta U_X \in \mathbb{R}^{n \times n}$ .

*Proof.* Let  $S_n$  be the sub-space of Hermitian matrix in  $\mathbb{R}^{n \times n}$ . Let  $\psi : \mathbb{R}^{n_1 \times n_2} \times \mathcal{Y} \rightarrow \mathbb{R}^{n_1 \times n_2} \times S_n \times S_n$ , where  $\mathcal{Y} = \mathbb{R}^{n_1 \times n} \times \mathbb{R}^{n_2 \times n} \times \mathbb{R}^n$ , be defined for  $Y = (V, U, \Lambda) \in \mathcal{Y}$  as

$$\psi(X, Y) = ( X - V \text{diag}(\Lambda) U^*, \quad V^* V - \text{Id}, \quad U^* U - \text{Id} ).$$

We have for any vector  $\zeta_X \in \mathbb{R}^{n_1 \times n_2}$

$$\partial_1 \psi(X, Y)[\zeta_X] = ( \zeta_X, \quad 0, \quad 0 ) \quad (11)$$

and for any vector  $\zeta_Y = (\zeta_U, \zeta_V, \zeta_\Lambda) \in \mathcal{Y}$

$$\partial_2 \psi(X, Y)[\zeta_Y] = ( -\zeta_V \text{diag}(\Lambda) U^* - V \text{diag}(\Lambda) \zeta_U^* - V \text{diag}(\zeta_\Lambda) U^*, \quad V^* \zeta_V + \zeta_V^* V, \quad U^* \zeta_U + \zeta_U^* U ).$$

Let  $X_0$  have distinct singular values and any of its SVD  $Y_0 = (U_0, V_0, \Lambda_0)$ . We have  $\psi(X_0, Y_0) = 0$ . Moreover, denoting  $\iota_V = V_0^* \zeta_V \in \mathbb{R}^{n \times n}$  and  $\iota_U = U_0^* \zeta_U \in \mathbb{R}^{n \times n}$ , for any  $z = (z_1, z_2, z_3) \in \mathbb{R}^{n_1 \times n_2} \times S_n \times S_n$ , solving  $\partial_2 \psi(X_0, Y_0)[\zeta_Y] = z$  is equivalent to solving

$$\iota_V \text{diag}(\Lambda_0) + \text{diag}(\Lambda_0) \iota_U^* + \text{diag}(\zeta_\Lambda) = -V_0^* z_1 U_0 = -\bar{z}_1 \quad (12)$$

where  $\iota_V + \iota_V^* = z_2$  and  $\iota_U + \iota_U^* = z_3$ . Considering  $z_2 = 0$  and  $z_3 = 0$  shows that  $\iota_V$  and  $\iota_U$  are antisymmetric. In particular they are zero along the diagonal. Thus applying the operator  $\text{diag}$  to both sides of (12) shows  $\zeta_\Lambda = -\text{diag}(V_0^* z_1 U_0)$ . Now considering the entries  $(i, j)$  and  $(j, i)$  of the linear system (12) shows that for any  $1 \leq i \leq n$  and  $1 \leq j \leq n$

$$\begin{pmatrix} (\Lambda_0)_j & -(\Lambda_0)_i \\ -(\Lambda_0)_i & (\Lambda_0)_j \end{pmatrix} \begin{pmatrix} (\iota_V)_{i,j} \\ (\iota_U)_{i,j} \end{pmatrix} = \begin{pmatrix} -(\bar{z}_1)_{i,j} \\ -(\bar{z}_1)_{j,i} \end{pmatrix}. \quad (13)$$

Since for  $i \neq j$ ,  $(\Lambda_0)_i \neq (\Lambda_0)_j$ , these  $2 \times 2$  symmetric linear systems can be solved. Then  $\partial_2 \psi(X_0, Y_0)$  is invertible on  $\mathbb{R}^{n_1 \times n_2} \times 0_n \times 0_n$  and for  $z = (z_1, 0, 0)$ , its inverse is

$$(\partial_2 \psi(X_0, Y_0))^{-1}[z] = (V_0 \iota_V, U_0 \iota_U, -\text{diag}(V_0^* z_1 U_0)) \quad (14)$$

where  $\iota_V$  and  $\iota_U$  are given by the solutions of the above series of  $2 \times 2$  symmetric linear systems.

Since  $\text{Im}(\partial_1 \psi(X, Y(X))) \subset \mathbb{R}^{n_1 \times n_2} \times 0_n \times 0_n$ , we can apply the implicit function theorem (Rockafellar & Wets, 2005). Hence, for any  $X \in \mathbb{R}^{n_1 \times n_2}$  in the neighborhood of  $X_0$ , there exists a function  $Y(X) = (U_X, V_X, \Lambda_X)$  such that  $\psi(X, Y(X)) = 0$ , i.e.  $X$  admits an SVD. Moreover, this function is  $C^1$  in the neighborhood of  $X_0$  and its differential is

$$\partial Y(X) = -\partial_2 \psi(X, Y(X))^{-1} \circ \partial_1 \psi(X, Y(X)).$$

Injecting (11) and (14) gives the desired formula by solving (13) in closed form. Since  $X_0$  is any matrix with distinct singular values, we can conclude.  $\square$

We now turn to the proof of the theorem.

*Proof.* Since the singular values of  $X$  are all distinct, by composition of differentiable functions, we can derive the relationship (6) that defines  $F$  which gives

$$V_X^* \partial F(X)[\delta] U_X = \iota_V \text{diag}(\Phi(\Lambda_X)) + \text{diag}(\Phi(\Lambda_X)) \iota_U^* + \mathcal{M}(\Lambda_X)[\bar{\delta}]$$

where we have used the notation introduced in Lemma 1. Using the expression (10) for  $\iota_U$  and  $\iota_V$  shows that the matrix  $W = \iota_V \text{diag}(\Phi(\Lambda_X)) + \text{diag}(\Phi(\Lambda_X)) \iota_U^*$  is computed as

$$W_{i,j} = \frac{1}{\Lambda_j^2 - \Lambda_i^2} (\varphi_j(\Lambda_j \bar{\delta}_{i,j} + \Lambda_i \bar{\delta}_{j,i}) - \varphi_i(\Lambda_i \bar{\delta}_{i,j} + \Lambda_j \bar{\delta}_{j,i}))$$

where  $\varphi = \Phi(\Lambda)$ . Rearranging this expression using the symmetric and anti-symmetric parts shows the desired formula.  $\square$

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