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HARNACK TYPE INEQUALITY ON RIEMANNIAN MANIFOLDS OF DIMENSION 5.

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ABSTRACT. We give an estimate of type $\sup \times \inf$ on Riemannian manifold of dimension 5 for the Yamabe equation.

Mathematics Subject Classification: 53C21, 35J60 35B45 35B50

1. INTRODUCTION AND MAIN RESULTS

In this paper, we deal with the following Yamabe equation in dimension $n = 5$:

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = n(n-2)u^{N-1}, \quad u > 0, \quad \text{and } N = \frac{n+2}{n-2}. \quad (1)$$

Here, R_g is the scalar curvature.

The equation (1) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2-4], [11], [15]. In this case we have a $\sup \times \inf$ inequality. The corresponding equation in two dimensions on open set Ω of \mathbb{R}^2 , is:

$$-\Delta u = V(x)e^u, \quad (2)$$

The equation (2) was studied by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c(\inf_{\Omega} V, \|V\|_{L^\infty(\Omega)}, \inf_{\Omega} u, K, \Omega).$$

And, precisely, in [8], [12], [16], and [20], we have:

$$C \sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, \|V\|_{L^\infty(\Omega)}, K, \Omega),$$

and,

$$\sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, \|V\|_{C^\alpha(\Omega)}, K, \Omega).$$

where K is a compact subset of Ω , C is a positive constant which depends on $\frac{\inf_{\Omega} V}{\sup_{\Omega} V}$, and,

$\alpha \in (0, 1]$. When $\frac{4(n-1)h}{n-2} = R_g$ the scalar curvature, and M compact, the equation (1) is Yamabe equation. T. Aubin and R. Schoen have proved the existence of solution in this case, see

for example [1] and [14] for a complete and detailed summary. When M is a compact Riemannian manifold, there exist some compactness result for equation (1) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeomorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem. Now, if we suppose M Riemannian manifold (not necessarily compact) Li and Zhang [17] proved that the product $\sup \times \inf$ is bounded. Here we extend the result of [5]. Our proof is an extension Li-Zhang result in dimension 3, see [3] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [3, 6, 11, 16, 17, 10], some applications of this method, for example an uniqueness result. We refer to [7] for the uniqueness result on the sphere and in dimension 3. Here, we give an equality of type $\sup \times \inf$ for the equation (1) in dimension 5. In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3, 5]. There are other estimates of type $\sup + \inf$ on complex Monge-Ampere equation on compact manifolds, see [20-21]. They consider, on compact Kahler manifold (M, g) , the following equation:

$$\begin{cases} (\omega_g + \partial\bar{\partial}\varphi)^n = e^{f-t\varphi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\varphi > 0 \text{ on } M \end{cases} \quad (3)$$

And, they prove some estimates of type $\sup_M + m \inf_M \leq C$ or $\sup_M + m \inf_M \geq C$ under the positivity of the first Chern class of M . Here, we have,

Theorem 1.1. *For all compact set K of M , there is a positive constant c , which depends only on, K, M, g such that:*

$$(\sup_K u)^{1/3} \times \inf_M u \leq c,$$

for all u solution of (1).

This theorem extend to the dimension 5 the result of Li and Zhang, see [17]. Here, we use the method of Li and Zhang in [17]. Also, we extend a result of [5].

Corollary 1.2. *For all compact set K of M there is a positive constant c , such that:*

$$\sup_K u \leq c = c(g, m, K, M) \text{ if } \inf_M u \geq m > 0,$$

for all u solution of (1).

2. PROOF OF THE THEOREMS

Proof of theorem 1.1: We want to prove that

$$\epsilon^3 (\max_{B(0,\epsilon)} u)^{1/3} \times \min_{B(0,4\epsilon)} u \leq c = c(M, g). \quad (4)$$

We argue by contradiction and we assume that

$$\left(\max_{B(0,\epsilon_k)} u_k\right)^{1/3} \times \min_{B(0,4\epsilon_k)} u_k \geq k\epsilon_k^{-3}. \quad (5)$$

Step 1: The blow-up analysis The blow-up analysis gives us : For some $\bar{x}_k \in B(0, \epsilon_k)$, $u_k(\bar{x}_k) = \max_{B(0,\epsilon_k)} u_k$, and, from the hypothesis,

$$u_k(\bar{x}_k)^{4/9} \epsilon_k \rightarrow +\infty.$$

By a standard selection process, we can find $x_k \in B(\bar{x}_k, \epsilon_k/2)$ and $\sigma_k \in (0, \epsilon_k/4)$ satisfying,

$$u_k(x_k)^{4/9} \sigma_k \rightarrow +\infty, \quad (6)$$

$$u_k(x_k) \geq u_k(\bar{x}_k), \quad (7)$$

$$\text{and, } u_k(x) \leq C u_k(x_k), \text{ in } B(x_k, \sigma_k), \quad (8)$$

where C is some universal constant. It follows from above (5), (7) that

$$(u_k(x_k))^{1/3} \times \left(\min_{\partial B(x_k, 2\epsilon_k)} u_k\right) \sigma_k^3 \geq (u_k(\bar{x}_k))^{1/3} \times \left(\min_{B(0,4\epsilon_k)} u_k\right) \epsilon_k^3 \geq k \rightarrow +\infty. \quad (9)$$

We use $\{z^1, \dots, z^n\}$ to denote some geodesic normal coordinates centered at x_k (we use the exponential map). In the geodesic normal coordinates, $g = g_{ij}(z) dz^i dz^j$,

$$g_{ij}(z) - \delta_{ij} = O(r^2), \quad g := \det(g_{ij}(z)) = 1 + O(r^2), \quad h(z) = O(1), \quad (10)$$

where $r = |z|$. Thus,

$$\Delta_g u = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j u) = \Delta u + b_i \partial_i u + d_{ij} \partial_{ij} u,$$

where

$$b_j = O(r), \quad d_{ij} = O(r^2) \quad (11)$$

We have a new function

$$v_k(y) = M_k^{-1} u_k(M_k^{-2/(n-2)} y) \text{ for } |y| \leq 3\epsilon_k M_k^{2/(n-2)}$$

where $M_k = u_k(0)$. From (6) and (9) we have

$$\left. \begin{aligned} \Delta v_k + \bar{b}_i \partial_i v_k + \bar{d}_{ij} \partial_{ij} v_k - \bar{c} v_k + v_k^{N-1} &= 0 \text{ for } |y| \leq 3\epsilon_k M_k^{2/(n-2)} \\ v_k(0) &= 1 \\ v_k(y) &\leq C_1 \text{ for } |y| \leq \sigma_k M_k^{2/(n-2)} \\ \lim_{k \rightarrow +\infty} \min_{|y|=2\epsilon_k M_k^{4/9}} (v_k(y) |y|^3) &= +\infty. \end{aligned} \right\} \quad (12)$$

where C_1 is a universal constant and

$$\bar{b}_i(y) = M_k^{-2/(n-2)} b_i(M_k^{-2/(n-2)} y), \quad \bar{d}_{ij}(y) = d_{ij}(M_k^{-2/(n-2)} y) \quad (13)$$

and,

$$\bar{c}(y) = M_k^{-4/(n-2)} h(M_k^{-2/(n-2)} y). \quad (14)$$

We can see that for $|y| \leq 3\epsilon_k M_k^{2/(n-2)}$,

$$|\bar{b}_i(y)| \leq C M_k^{-4/(n-2)} |y|, \quad |\bar{d}_{ij}(y)| \leq C M_k^{-4/(n-2)} |y|^2, \quad |\bar{c}(y)| \leq C M_k^{-4/(n-2)} \quad (15)$$

where C depends on n, M, g .

It follows from (12), (13), (14), (15) and the elliptic estimates, that, along a subsequence, v_k converges in C^2 norm on any compact subset of \mathbb{R}^2 to a positive function U satisfying

$$\left. \begin{aligned} \Delta U + U^{N-1} &= 0, \text{ in } \mathbb{R}^n, \text{ with } N = \frac{n+2}{n-2} \\ U(0) &= 1, \quad 0 < U \leq C. \end{aligned} \right\} \quad (16)$$

In the case where $C = 1$, by a result of Caffarelli-Gidas-Spruck, see [10], we have:

$$U(y) = (1 + |y|^2)^{-(n-2)/2}, \quad (17)$$

But, here we do not need this result.

Now, we need a precision in the previous estimates, we take a conformal change of metric such that, the Ricci tensor vanish,

$$R_{jp} = 0. \quad (18)$$

We have by the expressions for g and g_{ij} , as in the paper of Li-Zhang,

$$b_j = O(r^2), \quad R = O(r), \quad d_{ij} = -\frac{1}{3}R_{ipqj}z^p z^q + O(r^3). \quad (19)$$

Thus,

$$|\bar{c}| \leq C|y|M_k^{-2}, \quad |\bar{b}_i| \leq C|y|^2 M_k^{-2}, \quad (20)$$

and,

$$\bar{d}_{ij} = -\frac{1}{3}M_k^{-4/3}R_{ipqj}y^p y^q + O(1)M_k^{-2}|y|^3. \quad (21)$$

As, in the paper of Li-Zhang, we have:

$$v_k(y) \geq C|y|^{-3}, \quad 1 \leq |y| \leq 2\epsilon_k M_k^{2/3}. \quad (22)$$

with $C > 0$.

For $x \in \mathbb{R}^2$ and $\lambda > 0$, let,

$$v_k^{\lambda,x}(y) := \frac{\lambda}{|y-x|} v_k \left(x + \frac{\lambda^2(y-x)}{|y-x|^2} \right), \quad (23)$$

denote the Kelvin transformation of v_k with respect to the ball centered at x and of radius λ .

We want to compare for fixed x , v_k and $v_k^{\lambda,x}$. For simplicity we assume $x = 0$. We have:

$$v_k^\lambda(y) := \frac{\lambda}{|y|} v_k(y^\lambda), \quad \text{with } y^\lambda = \frac{\lambda^2 y}{|y|^2}.$$

For $\lambda > 0$, we set,

$$\Sigma_\lambda = B(0, \epsilon_k M_k^2) - \bar{B}(0, \lambda).$$

The boundary condition, (12), become:

$$\lim_{k \rightarrow +\infty} \min_{|y|=\epsilon_k M_k^{4/9}} (v_k(y)|y|^3) = \lim_{k \rightarrow +\infty} \min_{|y|=2\epsilon_k M_k^{4/9}} (v_k(y)|y|^3) = +\infty. \quad (24)$$

As in the paper of Li-Zhang, we have:

$$\Delta w_\lambda + \bar{b}_i \partial_i w_\lambda + \bar{d}_{ij} \partial_{ij} w_\lambda - \bar{c} w_\lambda + \frac{(n+2)}{(n-2)} \xi^{4/(n-2)} w_\lambda = E_\lambda \quad \text{in } \Sigma_\lambda. \quad (25)$$

where ξ stay between v_k and v_k^λ . Here,

$$E_\lambda = -\bar{b}_i \partial_i v_k^\lambda - \bar{d}_{ij} \partial_{ij} v_k^\lambda + \bar{c} v_k^\lambda - E_1,$$

with,

$$E_1(y) = - \left(\frac{\lambda}{|y|} \right)^{n+2} (\bar{b}_i(y^\lambda) \partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda) \partial_{ij} v_k(y^\lambda) - \bar{c}(y^\lambda) v_k(y^\lambda)). \quad (26)$$

Lemma 2.1. *We have,*

$$|E_\lambda| \leq C_1 \lambda |y|^{-1} M_k^{-2} + C_2 \lambda^3 |y|^{-3} M_k^{-4/3}. \quad (27)$$

Proof: as in the paper of Li-Zhang, we have a nonlinear term E_λ with the following property,

$$|E_\lambda| \leq C_1 \lambda^3 M_k^{-2} |y|^{-2} + C_2 \lambda^4 M_k^{-4/3} |y|^{-4} \leq C_1 \lambda |y|^{-1} M_k^{-2} + C_2 \lambda^3 |y|^{-3} M_k^{-4/3}.$$

Next, we need an auxiliary function which correct the nonlinear term. Here we take the following auxiliary function:

$$h_\lambda = -C_1 \lambda M_k^{-2} (|y| - \lambda) - C_2 \lambda^2 M_k^{-4/3} \left(\left(1 - \left(\frac{\lambda}{|y|} \right)^3 \right) - \left(1 - \left(\frac{\lambda}{|y|} \right) \right) \right), \quad (28)$$

we have,

$$h_\lambda \leq 0, \quad (29)$$

$$\Delta h_\lambda = -C_1 \lambda |y|^{-1} M_k^{-2} - C_2 \lambda^3 |y|^{-3} M_k^{-4/3}, \quad (30)$$

and, thus,

$$\Delta h_\lambda + |E_\lambda| \leq 0.$$

As in the paper of Li-Zhang, we can prove the following lemma:

Lemma 2.2. *We have,*

$$w_\lambda + h_\lambda > 0, \text{ in } \Sigma_\lambda \quad \forall 0 < \lambda \leq \lambda_1. \quad (31)$$

Before to prove the lemma, note that, here, we consider the fact that,

$$\lambda \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}. \quad (32)$$

And, as in the paper of Li-Zhang, we need the estimate (22):

$$v_k(y) \geq C |y|^{-3}, \quad 1 \leq |y| \leq 2\epsilon_k M_k^{2/3}.$$

with $C > 0$.

Proof :

Step 1: There exists $\lambda_0 > 0$ independent of k such the assertion of the lemma holds for all $0 < \lambda < \lambda_0$.

To see this, we write:

$$w_\lambda = v_k(y) - v_k^\lambda(y) = |y|^{-3/2}(|y|^{3/2}v_k(y) - |y^\lambda|^{3/2}v_k(y^\lambda)).$$

Let, in polar coordinates,

$$f(r, \theta) = r^{3/2}v_k(r, \theta).$$

By the properties of v_k , there exist $r_0 > 0$ and $C > 0$ independent of k such that:

$$\partial_r f(r, \theta) > Cr^{1/2}, \text{ for } 0 < r < r_0.$$

Thus,

$$\begin{aligned} w_\lambda(y) &= |y|^{-3/2}(f(|y|, y/|y|) - f(|y^\lambda|, y/|y|)) = |y|^{-3/2} \int_{|y^\lambda|}^{|y|} \partial_r f(r, y/|y|) dr > \\ &> C'|y|^{-3/2}(|y|^{3/2} - |y^\lambda|^{3/2}) > C''(|y| - \lambda) \text{ for } 0 < \lambda < |y| < r_0, \end{aligned}$$

with, $C', C'' > 0$.

It follows that,

$$w_\lambda + h_\lambda \geq (C'' - o(1))(|y| - \lambda), \text{ for } 0 < \lambda < |y| < r_0, \quad (33)$$

Now, for

$$r_0 \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}.$$

we have by the definition of h_λ , and, as in the paper of Li-Zhang, we need the estimate (22):

$$v_k(y) \geq C|y|^{-3}, \quad 1 \leq |y| \leq 2\epsilon_k M_k^{2/3}.$$

to have,

$$|h_\lambda| < \frac{1}{2}v_k(y).$$

Thus, as in the paper of Li-Zhang,

$$w_\lambda + h_\lambda > 0, \text{ for } 0 < r_0 < |y| < 2\epsilon_k M_k^{4/9}. \quad (34)$$

Step 2: Set,

$$\bar{\lambda}^k = \sup\{0 < \lambda \leq \lambda_1, w_\mu + h_\mu \geq 0, \text{ in } \Sigma_\mu \ \forall 0 < \mu \leq \lambda\}, \quad (35)$$

We claim that, $\bar{\lambda}^k = \lambda_1$.

In order to apply the maximum principle and the Hopf lemma, we need to prove that:

$$(\Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c})(w_\lambda + h_\lambda) \leq 0 \text{ in } \Sigma_\lambda \quad (36)$$

In other words, we need to prove that:

$$\Delta h_\lambda + \bar{b}_i \partial_i h_\lambda + \bar{d}_{ij} \partial_{ij} h_\lambda - \bar{c} h_\lambda + E_\lambda \leq 0 \text{ in } \Sigma_\lambda. \quad (37)$$

First note that, $h_\lambda < 0$. Here, we consider the fact that,

$$\lambda \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}.$$

We have,

$$|\bar{c}| \leq C|y|M_k^{-2},$$

Thus,

$$|y||\bar{c}h_\lambda| \leq C_1 M_k^{-4} \lambda |y|^2 (|y| - \lambda) + C_2 M_k^{-10/3} \lambda |y|^2 \leq o(1) M_k^{-2} \lambda,$$

which we can write as,

$$|\bar{c}h_\lambda| \leq C_1 M_k^{-2} \lambda |y|^{-1}. \quad (38)$$

We have,

$$|\bar{b}_i| \leq C|y|^2 M_k^{-2},$$

Thus,

$$|\bar{b}_i \partial_i h_\lambda| \leq C_1 M_k^{-4} \lambda |y|^2 + C_2 M_k^{-10/3} (\lambda^5 |y|^{-2} + \lambda^3),$$

$$|y| C_1 M_k^{-4} \lambda |y|^2 = o(1) M_k^{-2} \lambda,$$

which we can write as,

$$C_1 M_k^{-4} \lambda |y|^2 = o(1) M_k^{-2} \lambda |y|^{-1}. \quad (39)$$

and,

$$|y|^3 C_2 M_k^{-10/3} \lambda^5 |y|^{-2} = C_2 M_k^{-10/3} \lambda^3 |y| = o(1) M_k^{-4/3} \lambda^3,$$

which we can write as,

$$C_2 M_k^{-10/3} \lambda^5 |y|^{-2} = o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \quad (40)$$

and,

$$|y|^3 C_2 M_k^{-10/3} \lambda^3 = o(1) M_k^{-2} \lambda^3,$$

which we can write as,

$$C_2 M_k^{-10/3} \lambda^3 = o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \quad (41)$$

Thus,

$$|\bar{b}_i \partial_i h_\lambda| \leq o(1) M_k^{-2} \lambda |y|^{-1} + o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \quad (42)$$

We have,

$$|\bar{d}_{ij}| \leq |y|^2 M_k^{-4/3},$$

Thus,

$$|\bar{d}_{ij}\partial_{ij}h_\lambda| \leq \lambda|y|M_k^{-10/3} + C_2M_k^{-8/3}(\lambda^5|y|^{-3} + \lambda^3|y|^{-1}),$$

Thus,

$$|\bar{d}_{ij}\partial_{ij}h_\lambda| \leq \lambda|y|^{-1}M_k^{-10/3} + o(1)M_k^{-8/3}\lambda^5|y|^{-3} + o(1)M_k^{-8/3}\lambda^3|y|^{-1},$$

Finally,

$$|\bar{d}_{ij}\partial_{ij}h_\lambda| \leq o(1)\lambda|y|^{-1}M_k^{-2} + o(1)M_k^{-4/3}\lambda^3|y|^{-3} + o(1)M_k^{-2}\lambda|y|^{-1}. \quad (43)$$

Finally,

$$|\bar{d}_{ij}\partial_{ij}h_\lambda + \bar{b}_i\partial_i h_\lambda + \bar{c}h_\lambda| \leq o(1)\lambda|y|^{-1}M_k^{-2} + o(1)\lambda^3|y|^{-3}M_k^{-4/3}. \quad (44)$$

Finally, we have,

$$\Delta h_\lambda + \bar{b}_i\partial_i h_\lambda + \bar{d}_{ij}\partial_{ij}h_\lambda - \bar{c}h_\lambda + E_\lambda \leq 0 \text{ in } \Sigma_\lambda,$$

And, thus (36),

$$(\Delta + \bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c})(w_\lambda + h_\lambda) \leq 0 \text{ in } \Sigma_\lambda.$$

Also, we have from the boundary condition and the definition of v_k^λ and h_λ , we have:

$$|h_\lambda(y)| + v_k^\lambda(y) \leq \frac{C(\lambda_1)}{|y|^3}, \quad \forall |y| = \epsilon_k M_k^{4/9}, \quad (45)$$

and, thus,

$$w_{\bar{\lambda}^k}(y) + h_{\bar{\lambda}^k}(y) > 0 \quad \forall |y| = \epsilon_k M_k^{4/9}, \quad (46)$$

We can use the maximum principle and the Hopf lemma to have:

$$w_{\bar{\lambda}^k} + h_{\bar{\lambda}^k} > 0, \text{ in } \Sigma_\lambda, \quad (47)$$

and,

$$\frac{\partial}{\partial \nu}(w_{\bar{\lambda}^k} + h_{\bar{\lambda}^k}) > 0, \text{ in } \Sigma_\lambda. \quad (48)$$

From the previous estimates we conclude that $\bar{\lambda}^k = \lambda_1$ and the lemma is proved.

Given any $\lambda > 0$, since the sequence v_k converges to U and $h_{\bar{\lambda}^k}$ converges to 0 on any compact subset of \mathbb{R}^2 , we have:

$$U(y) \geq U^\lambda(y), \quad \forall |y| \geq \lambda, \quad \forall 0 < \lambda < \lambda_1. \quad (49)$$

Since $\lambda_1 > 0$ is arbitrary, and since we can apply the same argument to compare v_k and $v_k^{\lambda, x}$, we have:

$$U(y) \geq U^{\lambda, x}(y), \quad \forall |y - x| \geq \lambda > 0. \quad (50)$$

Thus implies that U is a constant which is a contradiction.

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