



HAL
open science

Harnack type inequality on Riemannian manifolds of dimension 5

Samy Skander Bahoura

► **To cite this version:**

Samy Skander Bahoura. Harnack type inequality on Riemannian manifolds of dimension 5. 2013. hal-00695015v3

HAL Id: hal-00695015

<https://hal.science/hal-00695015v3>

Preprint submitted on 30 Oct 2013 (v3), last revised 1 Mar 2023 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HARNACK TYPE INEQUALITY ON RIEMANNIAN MANIFOLDS OF DIMENSION 5.

SAMY SKANDER BAHOURA
JYOTSHANA V. PRAJAPAT

ABSTRACT. We give an estimate of type $\sup \times \inf$ on Riemannian manifold of dimension 5 for the Yamabe equation.

Mathematics Subject Classification: 53C21, 35J60 35B45 35B50

1. INTRODUCTION AND MAIN RESULTS

In this paper, we deal with the following Yamabe equation in dimension $n = 5$:

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = n(n-2)u^{N-1}, \quad u > 0, \quad \text{and } N = \frac{n+2}{n-2}. \quad (1)$$

Here, R_g is the scalar curvature.

The equation (1) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2-4], [11], [15]. In this case we have a $\sup \times \inf$ inequality. The corresponding equation in two dimensions on open set Ω of \mathbb{R}^2 , is:

$$-\Delta u = V(x)e^u, \quad (2)$$

The equation (2) was studied by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c(\inf_{\Omega} V, \|V\|_{L^\infty(\Omega)}, \inf_{\Omega} u, K, \Omega).$$

And, precisely, in [8], [12], [16], and [20], we have:

$$C \sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, \|V\|_{L^\infty(\Omega)}, K, \Omega),$$

and,

$$\sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, \|V\|_{C^\alpha(\Omega)}, K, \Omega).$$

where K is a compact subset of Ω , C is a positive constant which depends on $\frac{\inf_{\Omega} V}{\sup_{\Omega} V}$, and,

$\alpha \in (0, 1]$. When $\frac{4(n-1)h}{n-2} = R_g$ the scalar curvature, and M compact, the equation (1) is Yamabe equation. T. Aubin and R. Schoen have proved the existence of solution in this case, see

Date: October 30, 2013.

for example [1] and [14] for a complete and detailed summary. When M is a compact Riemannian manifold, there exist some compactness result for equation (1) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeomorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem. Now, if we suppose M Riemannian manifold (not necessarily compact) Li and Zhang [17] proved that the product $\sup \times \inf$ is bounded. Here we extend the result of [5]. Our proof is an extension Li-Zhang result in dimension 3, see [3] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [3, 6, 11, 16, 17, 10], some applications of this method, for example an uniqueness result. We refer to [7] for the uniqueness result on the sphere and in dimension 3. Here, we give an equality of type $\sup \times \inf$ for the equation (1) in dimension 5. In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3, 5]. There are other estimates of type $\sup + \inf$ on complex Monge-Ampere equation on compact manifolds, see [20-21]. They consider, on compact Kahler manifold (M, g) , the following equation:

$$\begin{cases} (\omega_g + \partial\bar{\partial}\varphi)^n = e^{f-t\varphi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\varphi > 0 \text{ on } M \end{cases} \quad (3)$$

And, they prove some estimates of type $\sup_M + m \inf_M \leq C$ or $\sup_M + m \inf_M \geq C$ under the positivity of the first Chern class of M . Here, we have,

Theorem 1.1. *For all compact set K of M , there is a positive constant c , which depends only on, K, M, g such that:*

$$(\sup_K u)^{1/3} \times \inf_M u \leq c,$$

for all u solution of (1).

This theorem extend to the dimension 5 the result of Li and Zhang, see [17]. Here, we use the method of Li and Zhang in [17]. Also, we extend a result of [5].

Corollary 1.2. *For all compact set K of M there is a positive constant c , such that:*

$$\sup_K u \leq c = c(g, m, K, M) \text{ if } \inf_M u \geq m > 0,$$

for all u solution of (1).

2. PROOF OF THE THEOREMS

Proof of theorem 1.1: We want to prove that

$$\epsilon^3 (\max_{B(0,\epsilon)} u)^{1/3} \times \min_{B(0,4\epsilon)} u \leq c = c(M, g). \quad (4)$$

We argue by contradiction and we assume that

$$\left(\max_{B(0, \epsilon_k)} u_k\right)^{1/3} \times \min_{B(0, 4\epsilon_k)} u_k \geq k\epsilon_k^{-3}. \quad (5)$$

Step 1: The blow-up analysis The blow-up analysis gives us : For some $\bar{x}_k \in B(0, \epsilon_k)$, $u_k(\bar{x}_k) = \max_{B(0, \epsilon_k)} u_k$, and, from the hypothesis,

$$u_k(\bar{x}_k)^{4/9} \epsilon_k \rightarrow +\infty.$$

By a standard selection process, we can find $x_k \in B(\bar{x}_k, \epsilon_k/2)$ and $\sigma_k \in (0, \epsilon_k/4)$ satisfying,

$$u_k(x_k)^{4/9} \sigma_k \rightarrow +\infty, \quad (6)$$

$$u_k(x_k) \geq u_k(\bar{x}_k), \quad (7)$$

$$\text{and, } u_k(x) \leq C u_k(x_k), \text{ in } B(x_k, \sigma_k), \quad (8)$$

where C is some universal constant. It follows from above (5), (7) that

$$(u_k(x_k))^{1/3} \times \left(\min_{\partial B(x_k, 2\epsilon_k)} u_k\right) \sigma_k^3 \geq (u_k(\bar{x}_k))^{1/3} \times \left(\min_{B(0, 4\epsilon_k)} u_k\right) \epsilon_k^3 \geq k \rightarrow +\infty. \quad (9)$$

We use $\{z^1, \dots, z^n\}$ to denote some geodesic normal coordinates centered at x_k (we use the exponential map). In the geodesic normal coordinates, $g = g_{ij}(z) dz^i dz^j$,

$$g_{ij}(z) - \delta_{ij} = O(r^2), \quad g := \det(g_{ij}(z)) = 1 + O(r^2), \quad h(z) = O(1), \quad (10)$$

where $r = |z|$. Thus,

$$\Delta_g u = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j u) = \Delta u + b_i \partial_i u + d_{ij} \partial_{ij} u,$$

where

$$b_j = O(r), \quad d_{ij} = O(r^2) \quad (11)$$

We have a new function

$$v_k(y) = M_k^{-1} u_k(M_k^{-2/(n-2)} y) \text{ for } |y| \leq 3\epsilon_k M_k^{2/(n-2)}$$

where $M_k = u_k(0)$. From (6) and (9) we have

$$\left. \begin{aligned} \Delta v_k + \bar{b}_i \partial_i v_k + \bar{d}_{ij} \partial_{ij} v_k - \bar{c} v_k + v_k^{N-1} &= 0 \text{ for } |y| \leq 3\epsilon_k M_k^{2/(n-2)} \\ v_k(0) &= 1 \\ v_k(y) &\leq C_1 \text{ for } |y| \leq \sigma_k M_k^{2/(n-2)} \\ \lim_{k \rightarrow +\infty} \min_{|y|=2\epsilon_k M_k^{4/9}} (v_k(y) |y|^3) &= +\infty. \end{aligned} \right\} \quad (12)$$

where C_1 is a universal constant and

$$\bar{b}_i(y) = M_k^{-2/(n-2)} b_i(M_k^{-2/(n-2)} y), \quad \bar{d}_{ij}(y) = d_{ij}(M_k^{-2/(n-2)} y) \quad (13)$$

and,

$$\bar{c}(y) = M_k^{-4/(n-2)} h(M_k^{-2/(n-2)} y). \quad (14)$$

We can see that for $|y| \leq 3\epsilon_k M_k^{2/(n-2)}$,

$$|\bar{b}_i(y)| \leq C M_k^{-4/(n-2)} |y|, \quad |\bar{d}_{ij}(y)| \leq C M_k^{-4/(n-2)} |y|^2, \quad |\bar{c}(y)| \leq C M_k^{-4/(n-2)} \quad (15)$$

where C depends on n, M, g .

It follows from (12), (13), (14), (15) and the elliptic estimates, that, along a subsequence, v_k converges in C^2 norm on any compact subset of \mathbb{R}^2 to a positive function U satisfying

$$\left. \begin{aligned} \Delta U + U^{N-1} &= 0, \text{ in } \mathbb{R}^n, \text{ with } N = \frac{n+2}{n-2} \\ U(0) &= 1, \quad 0 < U \leq C. \end{aligned} \right\} \quad (16)$$

In the case where $C = 1$, by a result of Caffarelli-Gidas-Spruck, see [10], we have:

$$U(y) = (1 + |y|^2)^{-(n-2)/2}, \quad (17)$$

But, here we do not need this result.

Now, we need a precision in the previous estimates, we take a conformal change of metric such that, the Ricci tensor vanish,

$$R_{jp} = 0. \quad (18)$$

We have by the expressions for g and g_{ij} , as in the paper of Li-Zhang,

$$b_j = O(r^2), \quad R = O(r), \quad d_{ij} = -\frac{1}{3}R_{ipqj}z^p z^q + O(r^3). \quad (19)$$

Thus,

$$|\bar{c}| \leq C|y|M_k^{-2}, \quad |\bar{b}_i| \leq C|y|^2M_k^{-2}, \quad (20)$$

and,

$$\bar{d}_{ij} = -\frac{1}{3}M_k^{-4/3}R_{ipqj}y^p y^q + O(1)M_k^{-2}|y|^3. \quad (21)$$

As, in the paper of Li-Zhang, we have:

$$v_k(y) \geq C|y|^{-3}, \quad 1 \leq |y| \leq 2\epsilon_k M_k^{2/3}. \quad (22)$$

with $C > 0$.

For $x \in \mathbb{R}^2$ and $\lambda > 0$, let,

$$v_k^{\lambda,x}(y) := \frac{\lambda}{|y-x|} v_k \left(x + \frac{\lambda^2(y-x)}{|y-x|^2} \right), \quad (23)$$

denote the Kelvin transformation of v_k with respect to the ball centered at x and of radius λ .

We want to compare for fixed x , v_k and $v_k^{\lambda,x}$. For simplicity we assume $x = 0$. We have:

$$v_k^\lambda(y) := \frac{\lambda}{|y|} v_k(y^\lambda), \text{ with } y^\lambda = \frac{\lambda^2 y}{|y|^2}.$$

For $\lambda > 0$, we set,

$$\Sigma_\lambda = B(0, \epsilon_k M_k^2) - \bar{B}(0, \lambda).$$

The boundary condition, (12), become:

$$\lim_{k \rightarrow +\infty} \min_{|y|=\epsilon_k M_k^{4/9}} (v_k(y)|y|^3) = \lim_{k \rightarrow +\infty} \min_{|y|=2\epsilon_k M_k^{4/9}} (v_k(y)|y|^3) = +\infty. \quad (24)$$

As in the paper of Li-Zhang, we have:

$$\Delta w_\lambda + \bar{b}_i \partial_i w_\lambda + \bar{d}_{ij} \partial_{ij} w_\lambda - \bar{c} w_\lambda + \frac{(n+2)}{(n-2)} \xi^{4/(n-2)} w_\lambda = E_\lambda \text{ in } \Sigma_\lambda. \quad (25)$$

where ξ stay between v_k and v_k^λ . Here,

$$E_\lambda = -\bar{b}_i \partial_i v_k^\lambda - \bar{d}_{ij} \partial_{ij} v_k^\lambda + \bar{c} v_k^\lambda - E_1,$$

with,

$$E_1(y) = - \left(\frac{\lambda}{|y|} \right)^{n+2} (\bar{b}_i(y^\lambda) \partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda) \partial_{ij} v_k(y^\lambda) - \bar{c}(y^\lambda) v_k(y^\lambda)). \quad (26)$$

Lemma 2.1. *We have,*

$$|E_\lambda| \leq C_1 \lambda |y|^{-1} M_k^{-2} + C_2 \lambda^3 |y|^{-3} M_k^{-4/3}. \quad (27)$$

Proof: as in the paper of Li-Zhang, we have a nonlinear term E_λ with the following property,

$$|E_\lambda| \leq C_1 \lambda^3 M_k^{-2} |y|^{-2} + C_2 \lambda^5 M_k^{-4/3} |y|^{-4} \leq C_1 \lambda |y|^{-1} M_k^{-2} + C_2 \lambda^3 |y|^{-3} M_k^{-4/3}.$$

Next, we need an auxiliary function which correct the nonlinear term. Here we take the following auxiliary function:

$$h_\lambda = -C_1 \lambda M_k^{-2} (|y| - \lambda) - C_2 \lambda^2 M_k^{-4/3} \left(\left(1 - \left(\frac{\lambda}{|y|} \right)^3 \right) - \left(1 - \left(\frac{\lambda}{|y|} \right) \right) \right), \quad (28)$$

we have,

$$h_\lambda \leq 0, \quad (29)$$

$$\Delta h_\lambda = -C_1 \lambda |y|^{-1} M_k^{-2} - C_2 \lambda^3 |y|^{-3} M_k^{-4/3}, \quad (30)$$

and, thus,

$$\Delta h_\lambda + |E_\lambda| \leq 0.$$

As in the paper of Li-Zhang, we can prove the following lemma:

Lemma 2.2. *We have,*

$$w_\lambda + h_\lambda > 0, \text{ in } \Sigma_\lambda \quad \forall 0 < \lambda \leq \lambda_1. \quad (31)$$

Before to prove the lemma, note that, here, we consider the fact that,

$$\lambda \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}. \quad (32)$$

And, as in the paper of Li-Zhang, we need the estimate (22):

$$v_k(y) \geq C|y|^{-3}, \quad 1 \leq |y| \leq 2\epsilon_k M_k^{2/3}.$$

with $C > 0$.

Proof :

Step 1: There exists $\lambda_0 > 0$ independent of k such the assertion of the lemma holds for all $0 < \lambda < \lambda_0$.

To see this, we write:

$$w_\lambda = v_k(y) - v_k^\lambda(y) = |y|^{-3/2}(|y|^{3/2}v_k(y) - |y^\lambda|^{3/2}v_k(y^\lambda)).$$

Let, in polar coordinates,

$$f(r, \theta) = r^{3/2}v_k(r, \theta).$$

By the properties of v_k , there exist $r > 0$ and $C > 0$ independent of k such that:

$$\partial_r f(r, \theta) > Cr^{1/2}, \text{ for } 0 < r < r_0.$$

Thus,

$$\begin{aligned} w_\lambda(y) &= |y|^{-3/2}(f(|y|, y/|y|) - f(|y^\lambda|, y/|y|)) = |y|^{-3/2} \int_{|y^\lambda|}^{|y|} \partial_r f(r, y/|y|) dr > \\ &> C'|y|^{-3/2}(|y|^{3/2} - |y^\lambda|^{3/2}) > C''(|y| - \lambda) \text{ for } 0 < \lambda < |y| < r_0, \end{aligned}$$

with, $C', C'' > 0$.

It follows that,

$$w_\lambda + h_\lambda \geq (C'' - o(1))(|y| - \lambda), \text{ for } 0 < \lambda < |y| < r_0, \quad (33)$$

Now, for

$$r_0 \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}.$$

we have by the definition of h_λ , and, as in the paper of Li-Zhang, we need the estimate (22):

$$v_k(y) \geq C|y|^{-3}, \quad 1 \leq |y| \leq 2\epsilon_k M_k^{2/3}.$$

to have,

$$|h_\lambda| < \frac{1}{2}v_k(y).$$

Thus, as in the paper of Li-Zhang,

$$w_\lambda + h_\lambda > 0, \text{ for } 0 < r_0 < |y| < 2\epsilon_k M_k^{4/9}. \quad (34)$$

Step 2: Set,

$$\bar{\lambda}^k = \sup\{0 < \lambda \leq \lambda_1, w_\mu + h_\mu \geq 0, \text{ in } \Sigma_\mu \forall 0 < \mu \leq \lambda\}, \quad (35)$$

We claim that, $\bar{\lambda}^k = \lambda_1$.

In order to apply the maximum principle and the Hopf lemma, we need to prove that:

$$(\Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c})(w_\lambda + h_\lambda) \leq 0 \text{ in } \Sigma_\lambda \quad (36)$$

In other words, we need to prove that:

$$\Delta h_\lambda + \bar{b}_i \partial_i h_\lambda + \bar{d}_{ij} \partial_{ij} h_\lambda - \bar{c} h_\lambda + E_\lambda \leq 0 \text{ in } \Sigma_\lambda. \quad (37)$$

First note that, $h_\lambda < 0$. Here, we consider the fact that,

$$\lambda \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}.$$

We have,

$$|\bar{c}| \leq C|y|M_k^{-2},$$

Thus,

$$|y||\bar{c}h_\lambda| \leq C_1 M_k^{-4} \lambda |y|^2 (|y| - \lambda) + C_2 M_k^{-10/3} \lambda |y|^2 \leq o(1) M_k^{-2} \lambda,$$

which we can write as,

$$|\bar{c}h_\lambda| \leq C_1 M_k^{-2} \lambda |y|^{-1}. \quad (38)$$

We have,

$$|\bar{b}_i| \leq C|y|^2 M_k^{-2},$$

Thus,

$$|\bar{b}_i \partial_i h_\lambda| \leq C_1 M_k^{-4} \lambda |y|^2 + C_2 M_k^{-10/3} (\lambda^5 |y|^{-2} + \lambda^3),$$

$$|y| C_1 M_k^{-4} \lambda |y|^2 = o(1) M_k^{-2} \lambda,$$

which we can write as,

$$C_1 M_k^{-4} \lambda |y|^2 = o(1) M_k^{-2} \lambda |y|^{-1}. \quad (39)$$

and,

$$|y|^3 C_2 M_k^{-10/3} \lambda^5 |y|^{-2} = C_2 M_k^{-10/3} \lambda^3 |y| = o(1) M_k^{-4/3} \lambda^3,$$

which we can write as,

$$C_2 M_k^{-10/3} \lambda^5 |y|^{-2} = o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \quad (40)$$

and,

$$|y|^3 C_2 M_k^{-10/3} \lambda^3 = o(1) M_k^{-2} \lambda^3,$$

which we can write as,

$$C_2 M_k^{-10/3} \lambda^3 = o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \quad (41)$$

Thus,

$$|\bar{b}_i \partial_i h_\lambda| \leq o(1) M_k^{-2} \lambda |y|^{-1} + o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \quad (42)$$

We have,

$$|\bar{d}_{ij}| \leq |y|^2 M_k^{-4/3},$$

Thus,

$$|\bar{d}_{ij} \partial_{ij} h_\lambda| \leq \lambda |y| M_k^{-10/3} + C_2 M_k^{-8/3} (\lambda^5 |y|^{-3} + \lambda^3 |y|^{-1}),$$

Thus,

$$|\bar{d}_{ij}\partial_{ij}h_\lambda| \leq \lambda|y|^{-1}M_k^{-10/3} + o(1)M_k^{-8/3}\lambda^5|y|^{-3} + o(1)M_k^{-8/3}\lambda^3|y|^{-1},$$

Finally,

$$|\bar{d}_{ij}\partial_{ij}h_\lambda| \leq o(1)\lambda|y|^{-1}M_k^{-2} + o(1)M_k^{-4/3}\lambda^3|y|^{-3} + o(1)M_k^{-2}\lambda|y|^{-1}. \quad (43)$$

Finally,

$$|\bar{d}_{ij}\partial_{ij}h_\lambda + \bar{b}_i\partial_i h_\lambda + \bar{c}h_\lambda| \leq o(1)\lambda|y|^{-1}M_k^{-2} + o(1)\lambda^3|y|^{-3}M_k^{-4/3}. \quad (44)$$

Finally, we have,

$$\Delta h_\lambda + \bar{b}_i\partial_i h_\lambda + \bar{d}_{ij}\partial_{ij}h_\lambda - \bar{c}h_\lambda + E_\lambda \leq 0 \text{ in } \Sigma_\lambda,$$

And, thus (36),

$$(\Delta + \bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c})(w_\lambda + h_\lambda) \leq 0 \text{ in } \Sigma_\lambda.$$

Also, we have from the boundary condition and the definition of v_k^λ and h_λ , we have:

$$|h_\lambda(y)| + v_k^\lambda(y) \leq \frac{C(\lambda_1)}{|y|^3}, \quad \forall |y| = \epsilon_k M_k^{4/9}, \quad (45)$$

and, thus,

$$w_{\bar{\lambda}^k}(y) + h_{\bar{\lambda}^k}(y) > 0 \quad \forall |y| = \epsilon_k M_k^{4/9}, \quad (46)$$

We can use the maximum principle and the Hopf lemma to have:

$$w_{\bar{\lambda}^k} + h_{\bar{\lambda}^k} > 0, \text{ in } \Sigma_\lambda, \quad (47)$$

and,

$$\frac{\partial}{\partial \nu}(w_{\bar{\lambda}^k} + h_{\bar{\lambda}^k}) > 0, \text{ in } \Sigma_\lambda. \quad (48)$$

From the previous estimates we conclude that $\bar{\lambda}^k = \lambda_1$ and the lemma is proved.

Given any $\lambda > 0$, since the sequence v_k converges to U and $h_{\bar{\lambda}_k}$ converges to 0 on any compact subset of \mathbb{R}^2 , we have:

$$U(y) \geq U^\lambda(y), \quad \forall |y| \geq \lambda, \quad \forall 0 < \lambda < \lambda_1. \quad (49)$$

Since $\lambda_1 > 0$ is arbitrary, and since we can apply the same argument to compare v_k and $v_k^{\lambda, x}$, we have:

$$U(y) \geq U^{\lambda, x}(y), \quad \forall |y - x| \geq \lambda > 0. \quad (50)$$

Thus implies that U is a constant which is a contradiction.

REFERENCES

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag 1998
- [2] S.S Bahoura. Majorations du type $\sup u \times \inf u \leq c$ pour l'équation de la courbure scalaire sur un ouvert de \mathbb{R}^n , $n \geq 3$. J. Math. Pures. Appl.(9) 83 2004 no, 9, 1109-1150.
- [3] S.S. Bahoura. Harnack inequalities for Yamabe type equations. Bull. Sci. Math. 133 (2009), no. 8, 875-892
- [4] S.S. Bahoura. Lower bounds for sup+inf and $\sup \times \inf$ and an extension of Chen-Lin result in dimension 3. Acta Math. Sci. Ser. B Engl. Ed. 28 (2008), no. 4, 749-758
- [5] S.S. Bahoura. Estimations uniformes pour l'equation de Yamabe en dimensions 5 et 6. J. Funct. Anal. 242 (2007), no. 2, 550-562.
- [6] S.S. Bahoura. $\sup \times \inf$ inequality on manifold of dimension 3, to appear in MATHEMATICA AETERNA
- [7] H. Brezis, YY. Li. Some nonlinear elliptic equations have only constant solutions. J. Partial Differential Equations 19 (2006), no. 3, 208217.
- [8] H. Brezis, YY. Li , I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [9] H.Brezis and F.Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V e^u$ in two dimensions, Commun Partial Differential Equations 16 (1991), 1223-1253.
- [10] L. Caffarelli, B. Gidas, J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 37 (1984) 369-402.
- [11] C-C.Chen, C-S. Lin. Estimates of the conformal scalar curvature equation via the method of moving planes. Comm. Pure Appl. Math. L(1997) 0971-1017.
- [12] C-C.Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \mathbb{R}^2 . Commun. Anal. Geom. 6, No.1, 1-19 (1998).
- [13] B. Gidas, W-Y. Ni, L. Nirenberg. Symmetry and related properties via the maximum principle. Comm. Math. Phys. 68 (1979), no. 3, 209-243.
- [14] J.M. Lee, T.H. Parker. The Yamabe problem. Bull.Amer.Math.Soc (N.S) 17 (1987), no.1, 37 -91.
- [15] YY. Li. Prescribing scalar curvature on \mathbb{S}_n and related Problems. C.R. Acad. Sci. Paris 317 (1993) 159-164. Part I: J. Differ. Equations 120 (1995) 319-410. Part II: Existence and compactness. Comm. Pure Appl.Math.49 (1996) 541-597.
- [16] YY. Li. Harnack Type Inequality: the Method of Moving Planes. Commun. Math. Phys. 200,421-444 (1999).
- [17] YY. Li, L. Zhang. A Harnack type inequality for the Yamabe equation in low dimensions. Calc. Var. Partial Differential Equations 20 (2004), no. 2, 133-151.
- [18] YY.Li, M. Zhu. Yamabe Type Equations On Three Dimensional Riemannian Manifolds. Commun.Contem.Mathematics, vol 1. No.1 (1999) 1-50.

- [19] I. Shafirir. A sup+inf inequality for the equation $-\Delta u = Ve^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.
- [20] Y-T. Siu. The existence of Kahler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group. Ann. of Math. (2) 127 (1988), no. 3, 585627
- [21] G. Tian. A Harnack type inequality for certain complex Monge-Ampre equations. J. Differential Geom. 29 (1989), no. 3, 481488.

DEPARTEMENT DE MATHEMATIQUES, UNIVERSITE PIERRE ET MARIE CURIE, 2 PLACE JUSSIEU, 75005, PARIS, FRANCE.

E-mail address: samybahoura@yahoo.fr