Harnack type inequality on Riemannian manifolds of dimensions 5

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HARNACK TYPE INEQUALITY ON RIEMANNIAN MANIFOLDS OF DIMENSION 5.

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ABSTRACT. We give an estimate of type $\sup \times \inf$ on Riemannian manifold of dimension 5 for Yamabe type equation.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we deal with the following Yamabe type equation in dimension $n = 5$:

$$\Delta_g u + h(x)u = n(n-2)u^{N-1},\ u > 0,\ \text{and}\ N = \frac{n+2}{n-2}. \qquad (E)$$

Where h is a continuous function. In the case $\dfrac{4(n-1)h}{n-2} = R_g$ the scalar curvature, (E) is the Yamabe equation. Here, we assume h a bounded function and $h_0 = ||h||_{L^\infty(M)}$. The equation (E) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2-4], [11], [15]. In this case we have a $\sup \times \inf$ inequality. The corresponding equation in two dimensions on open set Ω of $\mathbb{R}^2$, is:

$$\Delta u = V(x)e^u, \qquad (E')$$

The equation (E') was studied by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [20]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c(\inf_\Omega V, ||V||_{L^\infty(\Omega)}, \inf_\Omega u, K, \Omega).$$

And, precisely, in [8], [12], [16], and [20], we have:

$$C \sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, ||V||_{L^\infty(\Omega)}, K, \Omega),$$

and,

$$\sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, ||V||_{C^\alpha(\Omega)}, K, \Omega).$$

where K is a compact subset of Ω, C is a positive constant which depends on $\dfrac{\inf_\Omega V}{\sup_\Omega V}$, and, $\alpha \in (0, 1]$. When $\dfrac{4(n-1)h}{n-2} = R_g$ the scalar curvature, and M compact, the equation (E) is

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Yamabe equation. Yamabe has tried to solve problem but he could not. T.Aubin and R.Schoen have proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary. When M is a compact Riemannian manifold, there exist some compactness result for equation (E) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeormorfic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem. Now, if we suppose M Riemannian manifold (not necessarily compact) Li and Zhang [17] proved that the product $\sup \times \inf$ is bounded. Here we extend the result of [5]. Our proof is an extension of Brezis-Li and Li-Zhang result in dimension 3, see [7] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [10], one of the application of this method. Here, we give an equality of type $\sup \times \inf$ for the equation (E) for any bounded function h. In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3, 5]. There are other estimates of type $\sup + \inf$ on complex Monge-Ampere equation on compact manifolds, see [20-21] . They consider, on compact Kahler manifold (M, g), the following equation:

$$\begin{cases} (\omega_g + \partial\bar{\partial}\varphi)^n = e^{f - t\varphi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\varphi > 0 \text{ on } M \end{cases}$$

And, they prove some estimates of type $\sup_M + m \inf_M \leq C$ or $\sup_M + m \inf_M \geq C$ under the positivity of the first Chern class of M. Here, we have,

Theorem 1.1. *For all compact set K of M, there is a positive constant c, which depends only on, h_0, K, M, g such that:*

$$(\sup_K u)^{1/3} \times \inf_M u \leq c,$$

for all u solution of (E).

This theorem generalizes Li-Zhang and the author 's result, see [17] . Here, we use Li and Zhang and Chen-Lin methods in [11, 17] . Also, we extend the result of [5].

Corollary 1.2. *For all compact set K of M there is a positive constant c, such that:*

$$\sup_K u \leq c = c(g, m, K, M) \text{ if } \inf_M u \geq m > 0,$$

for all u solution of (E).

2. Proof of the theorems

Proof of theorem 1.1: We want to prove that

$$\epsilon^3 (\max_{B(0,\epsilon)} u)^{1/3} \times \min_{B(0,4\epsilon)} u \leq c = c(M, g). \tag{1}$$

We argue by contradiction and we assume that

$$(\max_{B(0,\epsilon_k)} u_k)^{1/3} \times \min_{B(0,4\epsilon_k)} u_k \geq k{\epsilon_k}^{-3}. \tag{2}$$

Step 1: The blow-up analysis The blow-up analysis gives us : For some $\bar{x}_k \in B(0,\epsilon_k)$, $u_k(\bar{x}_k) = \max_{B(0,\epsilon_k)} u_k$, and, from the hypothesis,

$$u_k(\bar{x}_k)^{4/9}\epsilon_k \to +\infty.$$

By a standard selection process, we can find $x_k \in B(\bar{x}_k, \epsilon_k/2)$ and $\sigma_k \in (0, \epsilon_k/4)$ satisfying,

$$\begin{aligned} u_k(x_k)^{4/9}R_k &\to +\infty, && (3)\\ u_k(x_k) &\geq u_k(\bar{x}_k), && (4)\\ \text{and, } u_k(x) &\leq (1+o(1))u_k(x_k), \text{ in } B(x_k,R_k), && (5) \end{aligned}$$

where $o(1) \to 0$ is some universal constant, which tends to 0 with k large enough. It follows from above (2), (4) that

$$(u_k(x_k))^{1/3} \times (\min_{\partial B(x_k,2\epsilon_k)} u_k)R_k^3 \geq (u_k(\bar{x}_k))^{1/3} \times (\min_{B(0,4\epsilon_k)} u_k)\epsilon_k^3 \geq k \to +\infty. \tag{6}$$

We use $\{z^1, \ldots, z^n\}$ to denote some geodesic normal coordinates centered at x_k (we use the exponential map). In the geodesic normal coordinates, $g = g_{ij}(z)dzdz^j$,

$$g_{ij}(z) - \delta_{ij} = O(r^2),\ g := det(g_{ij}(z)) = 1 + O(r^2),\ h(z) = O(1), \tag{7}$$

where $r = |z|$. Thus,

$$\Delta_g u = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j u) = \Delta u + b_i\partial_i u + d_{ij}\partial_{ij}u,$$

where

$$b_j = O(r),\ d_{ij} = O(r^2) \tag{8}$$

We have a new function

$$v_k(y) = M_k^{-1}u_k(M_k^{-2/(n-2)}y) \text{ for } |y| \leq 3\epsilon_k M_k^{2/(n-2)}$$

where $M_k = u_k(0)$. From (5) and (6) we have

$$\left.\begin{array}{lcl} \Delta v_k + \bar{b}_i\partial_i v_k + \bar{d}_{ij}\partial_{ij}v_k - \bar{c}v_k + {v_k}^{N-1} &=& 0 \text{ for } |y| \leq 3\epsilon_k M_k^{2/(n-2)} \\ v_k(0) &=& 1 \\ v_k(y) &\leq& C_1 \text{ for } |y| \leq R_k M_k^{2/(n-2)} \end{array}\right\} \tag{9}$$

where C_1 is a universal constant and

$$\bar{b}_i(y) = M_k^{-2/(n-2)}b_i(M_k^{-2/(n-2)}y),\ \bar{d}_{ij}(y) = d_{ij}(M_k^{-2/(n-2)}y) \tag{10}$$

and,

$$\bar{c}(y) = M_k^{-4/(n-2)}h(M_k^{-2/(n-2)}y). \tag{11}$$

We can see that for $|y| \leq 3\epsilon_k M_k^{2/(n-2)}$,

$$|\bar{b}_i(y)| \leq CM_k^{-4/(n-2)}|y|,\ |\bar{d}_{ij}(y)| \leq CM_k^{-4/(n-2)}|y|^2,\ |\bar{c}(y)| \leq CM_k^{-4/(n-2)} \tag{12}$$

where C depends on n, M, g.

It follows from (9), (10), (11), (12) and the elliptic estimates, that, along a subsequence, v_k converges in C^2 norm on any compact subset of $\mathbb{R}^2$ to a positive function U satisfying

$$\left.\begin{array}{lll} \Delta U + U^{N-1} & = & 0, \text{ in } \mathbb{R}^n, \text{ with } N = \dfrac{n+2}{n-2} \\ U(0) = 1, & & 0 < U \leq 1. \end{array}\right\} \tag{13}$$

According to Caffarelli-Gidas-Spruck, see [10] we have an explicit form of U

$$U(y) = (1 + |y|^2)^{-(n-2)/2}. \tag{14}$$

Step 2: The Kelvin transform and moving-plane method

For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^*_-$, let,

$$v_k^1(y) := \frac{1}{|y|^{n-2}} v_k \left(e + \frac{y}{|y|^2} \right) \tag{15}$$

denote the Kelvin transformation of v_k with respect to the unit ball, defined on $I\!R^n \cup \{\infty\} \setminus \{0\}$ and $e = (1, 0, ...0)$. If we denote

$$\begin{array}{rcl} \bar{U}_0(y) & = & \dfrac{1}{|y|^{n-2}} U \left(e + \dfrac{y}{|y|^2} \right) \\ & = & \left(\dfrac{1}{(1 + 2y_1 + 2|y|^2)} \right)^{(n-2)/2} \end{array} \tag{16}$$

then

$$v_1^k \to \bar{U}_0 \text{ in } C^2_{loc}(I\!R^n \cup \{\infty\} \setminus \{0\}). \tag{17}$$

Note that $\bar{U}_0$ is still a solution of (13) and $\bar{U}_0(-e/2) = 2^{(n-2)/2}$ is the maximum of $\bar{U}_0(y)$. Hence v_1^k has a non degenerate, local maximum near $-e/2$ for all large k. To arrive at a contradiction to our assumption (2),we use the same method as in [11] of moving-plane, precisely Lemma 2.1 in [11] to show that $\frac{\partial v_k^1}{\partial x_1} < 0$ near the point $e^* = (-1/2, 0, ...0) = -e/2$.

Note, that, if we consider $\tilde{x}_k = \exp_{x_k}(M_k^{-2/(n-2)} e)$, we work in the conformal coordinates in the exponential map around this point, for the blow-up analysis. A computation gives,

$$\partial_i v_k(\frac{y}{|y|^2}) = |y|^{n-2}((n-2)y_i v_k^1(y) + (\delta_{im}|y|^2 - 2y_i y_m)\partial_m v_k^1(y)),$$

and,

$$\partial_{ij} v_k(\frac{y}{|y|^2}) = \alpha_1 v_k^1(y) + \beta_{1m}\partial_m v_k^1(y) + \gamma_{ml}\partial_{ml} v_k^1(y),$$

with,

$$\alpha_1 = (n-2)(\delta_{ij}|y|^n - n y_i y_j |y|^{n-2}),$$

$$\beta_{1m} = |y|^{n-2}((n-2)(\delta_{im}|y|^2 - 2y_i y_m)(y_i - y_j) - 2(\delta_{im} y_j |y|^2 + \delta_{ij} y_m |y|^2 + \delta_{jm} y_i |y|^2 - 2y_i y_j y_m)),$$

and,

$$\gamma_{ml} = |y|^{n-2}(\delta_{im}|y|^2 - 2y_i y_m)(\delta_{jl}|y|^2 - 2y_j y_l).$$

The function v_k^1 satisfies the equation:

$$\Delta v_k^1 + (v_k^1)^{N-1} = E_1(y)$$

where,

$$E_1(y) = -\left(\frac{1}{|y|}\right)^{n+2}\left(\bar{b}_i(\frac{y}{|y|^2})\partial_i v_k(e+\frac{y}{|y|^2}) + \bar{d}_{ij}(\frac{y}{|y|^2})\partial_{ij}v_k(e+\frac{y}{|y|^2}) - \bar{c}(e+\frac{y}{|y|^2})v_k(e+\frac{y}{|y|^2})\right).$$

Thus, v_k^1 is a solution of an equation

$$\Delta v_k^1 + \tilde{d}_{ml}\partial_{ml}v_k^1 + \tilde{b}_m\partial_m v_k^1 + \tilde{c}v_k^1 + (v_k^1)^{N-1} = 0, \tag{18}$$

with,

$$|\tilde{d}_{ml}| \leq C\frac{M_k^{-4/(n-2)}}{|y|^2}, \tag{19}$$

$$|\tilde{b}_m| \leq C\frac{M_k^{-4/(n-2)}}{|y|^3}, \tag{20}$$

$$\text{and, } |\tilde{c}| \leq C\frac{M_k^{-4/(n-2)}}{|y|^4}. \tag{21}$$

Thus, if we denote

$$L_k := \Delta + \tilde{d}_{ml}\partial_{ml} + \tilde{b}_m\partial_m + \tilde{c}, \tag{22}$$

then for any C^2 function g,

$$L_k g \to \Delta g \text{ as } k \to \infty \tag{23}$$

if $\partial_{ml}g$, $\partial_m g$ and $\tilde{c}g$ are all uniformly bounded.

For $\lambda < 0$, set $T_\lambda = \{y, y_1 = \lambda\}$, $\Sigma_\lambda = \{y, y_1 > \lambda\}$ and $\Sigma'_\lambda = \Sigma_\lambda - \{y : |y| \leq r_k^{-1/2}\}$. Let

$$v_k^{1,\lambda} = v_k^1(2\lambda - y_1, y_2, \ldots, y_n) \tag{24}$$

and,

$$w_\lambda := v_k^1 - v_k^{1,\lambda}.$$

Then,

$$L_k w_\lambda + \tilde{b}w_\lambda = V_\lambda,$$

where,

$$V_\lambda = (\tilde{d}_{ml} - \tilde{d}_{ml}^\lambda)\partial_{ml}v_k^{1,\lambda} + (\tilde{b}_m - \tilde{b}_m^\lambda)\partial_m v_k^{1,\lambda} + (\tilde{c} - \tilde{c}^\lambda)v_k^{1,\lambda}, \tag{25}$$

and $\tilde{b}(x)$ lies between $v_k^1(x)$ and $v_k^{1,\lambda}(x)$.

From the expression of v_k^1 and $v_k^{1,\lambda}$, we have

$$|\partial_m v_k^{1,\lambda}| \leq C(\frac{1}{|y^\lambda|^{n-1}} + \frac{1}{|y^\lambda|^n}), \tag{26}$$

$$|\partial_{ml} v_k^{1,\lambda}| \leq C(\frac{1}{|y^\lambda|^n} + \frac{1}{|y^\lambda|^{n+1}}). \tag{27}$$

Hence, for $y \in \Sigma'_\lambda$,

$$\begin{aligned} |\tilde{d}_{ml}\partial_{ml}w_\lambda| &\leq C\frac{M_k^{-4/(n-2)}}{|y|^2}(\frac{1}{|y|^n}+\frac{1}{|y|^{n+1}}) \\ |\tilde{b}_m\partial_m w_\lambda| &\leq C\frac{M_k^{-4/(n-2)}}{|y|^3}(\frac{1}{|y|^{n-1}}+\frac{1}{|y|^n}) \\ |\tilde{c}w_\lambda| &\leq C\frac{M_k^{-4/(n-2)}}{|y|^{n-2}}. \end{aligned}$$

Recall that $\frac{1}{2}R_kM_k^{4/9} < r_k^{1/2} \leq \eta_0 R_k M_k^{4/9}$, $R_k \to 0$ and η_0 is a constant chosen later(explain!). Observe that for dimensions $3 \leq n \leq 6$, all of

$$|\tilde{d}_{ml}\partial_{ml}w_\lambda|,\ |\tilde{b}_m\partial_m w_\lambda|,\ |\tilde{c}w_\lambda| \to 0 \text{ uniformly as } k \to \infty, \text{ independent of } \lambda < 0. \tag{28}$$

Furthermore,

$$|(\tilde{d}_{ml}-\tilde{d}^\lambda_{ml})\partial_{ml}v_k^{1,\lambda}| \leq \frac{CM_k^{-4/(n-2)}(y_1-\lambda)}{|y|^4|y^\lambda|^{n-2}},$$

$$|(\tilde{b}_m-\tilde{b}^\lambda_m)\partial_m v_k^{1,\lambda}| \leq \frac{CM_k^{-4/(n-2)}(y_1-\lambda)}{|y|^4|y^\lambda|^{n-2}},$$

and,

$$|(\tilde{c}-\tilde{c}^\lambda)v_k^{1,\lambda}| \leq \frac{CM_k^{-4/(n-2)}(y_1-\lambda)}{|y|^4|y^\lambda|^{n-2}}$$

implies that

$$V_\lambda \leq |V_\lambda| \leq \frac{CM_k^{-4/(n-2)}(y_1-\lambda)}{|y|^4|y^\lambda|^{n-2}}. \tag{29}$$

First of all we have the following lemma as in [11]

Lemma 2.1. *There exists constants $\lambda_0 \leq -2$ and $c_0 = c_0(n,\mu) > 0$, both independent of k such that:*

$$v_k^1(y) - v_k^1(y^{\lambda_0}) \geq c_0(1+|y|)^{-n}(y_1-\lambda)$$

for $y_1 \geq \lambda_0$ and $|y| \geq r_k^{-1/2}$.

Proof To prove our estimate, we consider two sets $|y| \geq \delta > 0$ and $\delta \geq |y| \geq r_k^{-1/2}$. For the first set we use the same technique as in Chen-Lin paper, we use the C^2 convergence of v_k^1 to $\bar{U}_0$ and choose $|\lambda|$ big enough to have our estimate. For two positive constants C_1 and C_2 we write the estimate as follows:

$$v_k^1(y) - v_k^1(y^\lambda) \geq |\lambda|\frac{C_1}{|y|^n}(y_1-\lambda) - \frac{C_2}{|y|^n}(y_1-\lambda).$$

For $|y| > \delta$, $\left|\frac{y}{|y|^2}\right| = \frac{1}{|}y| < 1/\delta$. Using convergence of v_k to U in C^2 norm in $B(0, 1/\delta)$ we have $C' > v_k > C > 0$ for $|y| > \delta$. We write

$$\begin{aligned} & v_k^1(y) - v_k^1(y^\lambda) \\ = & \left(\frac{1}{|y|^{n-2}} - \frac{1}{|y^\lambda|^{n-2}}\right) v_k\left(\frac{y}{|y|^2}\right) + \frac{1}{|y^\lambda|^{n-2}} \frac{v_k\left(\frac{y}{|y|^2}\right) - v_k\left(\frac{y^\lambda}{|y^\lambda|^2}\right)}{\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2}} \left(\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2}\right) \\ =: & \ I_1 + I_2 \end{aligned} \tag{30}$$

Note that

$$\frac{v_k\left(\frac{y}{|y|^2}\right) - v_k\left(\frac{y^\lambda}{|y^\lambda|^2}\right)}{\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2}}$$

appearing in the second term in (30) can be estimated by ∇U in the compact set $|y| \leq \frac{1}{\delta}$ as in Chen-Lin again using the C^2 convergence of v_k. While

$$\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2} = \frac{y - y^\lambda}{|y|^2} + y^\lambda(\frac{1}{|y|^2} - \frac{1}{|y^\lambda|^2}) = 2\frac{y_1 - \lambda}{|y|^2} + \frac{4(y_1 - \lambda)y^\lambda}{|y|^2|y^\lambda|^2}.$$

Since $y_1 > \lambda$ we have $|y^\lambda| > -\lambda$ thus,

$$|\frac{4(y_1 - \lambda)y^\lambda}{|y|^2|y^\lambda|^2}| < \frac{4(y_1 - \lambda)}{-\lambda|y|^2}.$$

It follows that

$$\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2} \leq \frac{6(y_1 - \lambda)}{|y|^2}$$

and

$$I_2 < \frac{C(y_1 - \lambda)}{|y|^2|y^\lambda|^{n-2}}. \tag{31}$$

To estimate the first term I_1, since $v_k > c > 0$ and $y_1 > \lambda$, we have

$$I_1 = \left(\frac{1}{|y|^{n-2}} - \frac{1}{|y^\lambda|^{n-2}}\right) v_k\left(\frac{y}{|y|^2}\right) > c\frac{|y^\lambda|^{n-2} - |y|^{n-2}}{|y|^{n-2}|y^\lambda|^{n-2}}$$

We now use the binomial formula:

$$|y^\lambda|^{n-2} - |y|^{n-2} = (|y^\lambda| - |y|)(|y|^k + ... + |y^\lambda|^k), k = n - 3.$$

Observe that

$$|y^\lambda| - |y| = \frac{|y^\lambda|^2 - |y|^2)}{|y| + |y^\lambda|} = \frac{-4\lambda(y_1 - \lambda)}{|y| + |y^\lambda|}.$$

Thus,

$$|y^\lambda|^{n-2} - |y|^{n-2} = \frac{-4\lambda(y_1 - \lambda)(|y|^k + ... + |y^\lambda|^k}{|y| + |y^\lambda|}$$

and

$$\left(\frac{1}{|y|^{n-2}}-\frac{1}{|y^\lambda|^{n-2}}\right)v_k\left(\frac{y}{|y|^2}\right)>c\frac{-4\lambda(y_1-\lambda)(|y|^k+...+|y^\lambda|^k}{(|y|+|y^\lambda|)|y|^{n-2}|y^\lambda|^{n-2}}$$

Finally, because $|y^\lambda|>|y|$, we have

$$\frac{|y|^k+...+|y^\lambda|^k}{(|y|+|y^\lambda|)|y|^{n-2}|y^\lambda|^{n-2}}>\frac{|y|^{n-4}(|y|+|y^\lambda|)}{(|y|+|y^\lambda|)|y|^{n-2}|y^\lambda|^{n-2}}=\frac{1}{|y|^2|y^\lambda|^{n-2}}.$$

It follows that

$$\left(\frac{1}{|y|^{n-2}}-\frac{1}{|y^\lambda|^{n-2}}\right)v_k\left(\frac{y}{|y|^2}\right)>\frac{c}{|y|^2|y^\lambda|^{n-2}}.$$

Therefore,

$$I_1+I_2\ >c\frac{-4\lambda(y_1-\lambda)}{|y|^2|y^\lambda|^{n-2}}-C\frac{-4(y_1-\lambda)}{|y|^2|y^\lambda|^{n-2}}$$

with $c,C>0$ and for $-\lambda$ big enough, we have the required inequality. On the annulus $A_k(\delta):=\{y: r_k^{-1/2}\le|y|\le\delta\}$, using the maximum principle for v_k^1 we have

$$\min_{A_k(\delta)}v_k^1=\min_{\partial A_k(\delta)}v_k^1 \tag{32}$$

where the boundary $\partial A_k(\delta)$ is the union of two set $|y|=r_k^{-1/2}$ and $|y|\le\delta$. From (16), $\bar U_0(0)=1$. Hence for given $\varepsilon>0$ small, there exists $\delta_0>0$ such that for all $|y|<\delta_0$,

$$1-\varepsilon=\bar U_0(0)-\varepsilon<\bar U_0(y)<1+\varepsilon=\bar U_0(0)+\varepsilon. \tag{33}$$

Choosing ε sufficiently small, we have

$$\bar U_0(y)>(1-\frac{\varepsilon}{2})\text{ for all }|y|\le\delta_0. \tag{34}$$

While, for $y\in B(0,\delta)$, $y^\lambda\in B(0_\lambda,\delta)$ where $0_\lambda:=(2\lambda,0,\dots,0)$ is the reflection of origin. We have

$$\begin{aligned}\bar U_0(y^\lambda) &= \frac{1}{(1+2(2\lambda-y_1)+2|y^\lambda|^2)^{(n-2)/2}}\\ &\le \frac{1}{(1+2(2\lambda-y_1)+(2\lambda-y_1)^2)^{(n-2)/2}}\\ &\le \frac{1}{(1+2(2\lambda-\delta)+(2\lambda-\delta)^2)^{(n-2)/2}}\\ &= \frac{1}{(1-2(\delta-2\lambda)+(\delta-2\lambda)^2)^{(n-2)/2}}\\ &= \frac{1}{(\delta-2\lambda-1)^{(n-2)}}\end{aligned} \tag{35}$$

for $\lambda\le-2$ and $0<\delta\le\delta_0$. From C^2 convergence of v_k^1 to $\bar U_0$ in $B(0_\lambda,\delta)$, we have

$$v_k^1(y^\lambda)<(1+\frac{\varepsilon}{2})\bar U_0(y^\lambda)\text{ in }B(0,\delta). \tag{36}$$

Note that

$$\begin{aligned}\min_{\{|y|=r_k^{-1/2}\}} v_k^1 &= r_k^{(n-2)/2} \min_{\{|y|=r_k^{1/2}\}} v_k \\ &\geq (1+\epsilon) r_k^{(n-2)/2} \min_{\{|y|=r_k^{1/2}\}} U(e+y) = (1+\epsilon) \min_{\{|y|=r_k^{-1/2}\}} \bar{U}_0(y) \\ &\geq (1+\frac{\epsilon}{2})\bar{U}_0(0) = (1+\frac{\epsilon}{2}). \end{aligned} \tag{37}$$

Using C^2 convergence of v_k^1 to $\bar{U}_0$ on the compact set $|y| = \delta$ we have,

$$\min_{\{|y|=\delta\}} v_k^1 \geq (1-\frac{\epsilon}{10}) \min_{\{|y|=\delta\}} \bar{U}_0. \tag{38}$$

In either case, for $\delta \leq \delta_0$,

$$\begin{aligned}\min_{A_k(\delta)} v_k^1 &\geq (1+\frac{\epsilon}{2})\bar{U}_0(0) = (1+\frac{\epsilon}{2}) \\ &\geq (1+\frac{\epsilon}{2})\bar{U}_0(y^\lambda) \\ &> (1+\frac{\epsilon}{4}) v_k^1(y^\lambda) + \varepsilon/10 \end{aligned} \tag{39}$$

□

Recall that w_λ satisfies

$$L_k w_\lambda + \tilde{b} w_\lambda = V_\lambda \text{ in } \Sigma_\lambda' \tag{40}$$

where

$$V_\lambda \leq \frac{C M_k^{-4/(n-2)}(y_1 - \lambda)}{|y|^4 |y^\lambda|^{n-2}}. \tag{41}$$

Now, consider the "auxiliary function"

$$h_\lambda = A r_k^{(2-n)/2} G^\lambda(y, 0) - \int_{\Sigma_\lambda} G^\lambda(y, \eta) \tilde{Q}_\lambda(\eta) d\eta, \tag{42}$$

with

$$\tilde{Q}_\lambda := \frac{C_1 M_k^{-4/(n-2)}}{(|y| + r_k^{-1/2})^4 (|y| - \lambda)^{(n-2)}} = C_1 M_k^{-4/(n-2)} Q_\lambda \tag{43}$$

where we define

$$Q_\lambda = \frac{1}{(|y| + r_k^{-1/2})^4 (|y| - \lambda)^{(n-2)}}$$

for simplicity of notations. Here we choose constant $C_1 > 0$ (big enough) later. Note that

$$\Delta h_\lambda = \tilde{Q}_\lambda = \frac{C_1 M_k^{-4/(n-2)}}{(|y| + r_k^{-1/2})^4 (|y| - \lambda)^{(n-2)}}.$$

It can be verified that

$$V_\lambda \leq \frac{C M_k^{-4/(n-2)}(y_1 - \lambda)}{|y|^4 |y^\lambda|^{n-2}} \leq \tilde{Q}_\lambda. \tag{44}$$

The function h_k^λ satisfies the following properties:

Lemma 2.2. *The functions $\tilde{b}$ and h_λ satisfy the following properties*
(i)For all $\lambda_0 \leq \lambda \leq -\frac{1}{4}$,

$$0 \leq \tilde{b}(y) \leq \frac{C}{|y|^4} \text{ in } \Sigma'_\lambda \tag{45}$$

(ii)The auxiliary function $h_k^\lambda(x) = 0$ on $x_1 = \lambda$, $\lambda_0 \leq \lambda \leq -\frac{1}{4}$ and $h_k^\lambda(x) = O(|x|^{-\tau})$ for a constant $\tau > 0$;
(iii) $h_k^\lambda(x) \in C^1(\bar{\Sigma}'_\lambda)$ and

$$\begin{aligned} 0 < h_\lambda(y) &\leq A r_k^{-(n-2)/2} G^\lambda(y,0) + C_1' M_k^{-2/(n-2)} \frac{(y_1-\lambda)}{|y|^n} o(1) &(46)\\ L_k h_\lambda &\geq C_1 \tilde{Q}_\lambda. &(47)\end{aligned}$$

(iv)$h_k^{\lambda_0}(x) \leq w_{\lambda_0}$and that both h_k^λ and $\nabla_x h_k^\lambda$ are continuous with respect to both the variables x and λ in Σ'_λ. Moreover,

$$L_k w_\lambda + \tilde{b} w_\lambda \leq C_1 \tilde{Q}_\lambda \leq L_k h_\lambda \tag{48}$$

and hence,

$$L_k(w_\lambda - h_\lambda) + \tilde{b}(w_\lambda - h_\lambda) \leq -\tilde{b} h_\lambda \leq 0. \tag{49}$$

Most of the estimates mentioned from (i)-(iv) above are similar, but much simpler than those of [11], and we refer the reader to that paper for details.

However, observe that we cannot apply the technique of [11] directly, as our operator is L_k. Here the crucial step is to have correct estimates for the perturbation terms in $L_k h_k^\lambda$, which we obtain in the following lemmas.

3. Estimates for $L_k h_k^\lambda$

Lemma 3.1. *(**Estimate of $G^\lambda(y,0)$**) The function $G^\lambda(y,0)$ satisfies*

$$|\tilde{d}_{ml}\partial_{ml} G^\lambda(y,0) + \tilde{b}_m \partial_m G^\lambda(y,0) + \tilde{c} G^\lambda(y,0)| \leq r_k^{(n-2)/2} \frac{C M_k^{-4/(n-2)}}{|y|^4 |y^\lambda|^{n-2}}, \tag{50}$$

and hence

$$L_k G^\lambda(y,0) \leq C_2 r_k^{(n-2)/2} Q_\lambda. \tag{51}$$

Proof From the fact that,

$$G^\lambda(y,0) = c_n(|y|^{2-n} - |y^\lambda|^{2-n}),$$

we have, around 0 and $+\infty$,

$$|\tilde{c} G^\lambda(y,0)| \leq \frac{C M_k^{-4/(n-2)}}{|y|^4 |y|^{n-2}} + \frac{C M_k^{-4/(n-2)}}{|y|^4 |y^\lambda|^{n-2}},$$

$$|\tilde{b}_m \partial_m G^\lambda(y,0)| \leq \frac{C M_k^{-4/(n-2)}}{|y|^3 |y|^{n-1}} + \frac{C M_k^{-4/(n-2)}}{|y|^3 |y^\lambda|^{n-1}},$$

and,

$$|\tilde{d}_{ml}\partial_{ml}G^{\lambda}(y,0)| \leq \frac{CM_k^{-4/(n-2)}}{|y|^2|y|^n} + \frac{CM_k^{-4/(n-2)}}{|y|^2|y^{\lambda}|^n},$$

For the previous expression, we remark that, around 0

$$|y|^3|y|^{n-1} = |y|^2|y|^n = |y|^4|y|^{n-2} \geq |y|^4 r_k^{-(n-2)/2},$$

Thus,

$$|\tilde{d}_{ml}\partial_{ml}G^{\lambda}(y,0) + \tilde{b}_m\partial_m G^{\lambda}(y,0) + \tilde{c}G^{\lambda}(y,0)| \leq r_k^{(n-2)/2}\frac{CM_k^{-4/(n-2)}}{|y|^4|y^{\lambda}|^{n-2}},$$

and hence

$$L_k G^{\lambda}(y,0) \leq C_2 r_k^{(n-2)/2}\tilde{Q}_{\lambda}$$

□

Now, we look at the second term in h_{λ} by setting,

$$u = -\int_{\Sigma'_{\lambda}} G^{\lambda}(y,\eta)Q_{\lambda}(\eta)d\eta$$

We have that u is a solution of a Dirichlet problem on Σ'_{λ}.

Lemma 3.2. *(* **Estimate for u:***) For the function u, we have*

$$\Delta u = C_1 Q_{\lambda} \quad (52)$$

$$|\tilde{d}_{ml}\partial_{ml}u + \tilde{b}_m\partial_m u + \tilde{c}u| = o(1)Q_{\lambda}. \quad (53)$$

Proof We want to prove that,

$$|\tilde{d}_{ml}\partial_{ml}u + \tilde{b}_m\partial_m u + \tilde{c}u| = o(1)Q_{\lambda}$$

Because of the expression of G^{λ}, we consider u as a difference of two convolution product on Σ_{λ}. Thus, to differentiate u is equivalent to differentiating Q_{λ} inside the integral plus the boundary term. Our aim is to estimate the auxiliary function and its derivatives of order less than two near infinity. Write,

$$-\tilde{u} = \int_{\Sigma_{\lambda}} G^{\lambda}(y,\eta)Q_{\lambda}(\eta)d\eta$$

and,

$$u - \tilde{u} = -\int_{B(0,r_k^{-1/2})} G^{\lambda}(y,\eta)Q_{\lambda}(\eta)d\eta$$

We denote Σ_λ^s the reflection of Σ_λ, with respect to the hyperplane $T_\lambda = \{y_1 = \lambda\}$, i.e., $\Sigma_\lambda^s := \{y \in I\!R^n : y_1 < \lambda\}$. Thus,

$$\begin{aligned} -\tilde{u} &= \int_{\Sigma_\lambda} c_n(|y-\eta|^{2-n} - |y-\eta^\lambda|^{2-n}) Q_\lambda(\eta) d\eta \\ &= \int_{\Sigma_\lambda} |y-\eta|^{2-n} Q_\lambda(\eta) d\eta - \int_{\Sigma_\lambda^s} |y-\eta|^{2-n} Q_\lambda(\eta^\lambda) d\eta, \\ &= \int_{\mathbb{R}^n} |y-\eta|^{2-n} Q_\lambda(\eta) d\eta - \int_{\Sigma_\lambda^s} |y-\eta|^{2-n} (Q_\lambda(\eta) + Q_\lambda(\eta^\lambda)) d\eta, \qquad (54) \\ &=: f + v \qquad (55) \end{aligned}$$

where

$$f \in C^\infty(\mathbb{R}^n),\ \Delta f = Q_\lambda \text{ in } I\!R^n \qquad (56)$$

$$\text{and } \Delta v = 0 \text{ in } \Sigma_\lambda. \qquad (57)$$

1) Behavior of $\tilde{u}$ and $u - \tilde{u}$ near infinity.

Let φ a cutoff function in the unit ball, i.e.,

$$0 \le \varphi \le 1,\ \varphi \equiv 1 \text{ on } B(0,1/2) \text{ and, } \varphi \equiv 0 \text{ in } \mathbb{R}^n \setminus B(0,1)$$

We write,

$$f = f_1 + f_2,$$

where,

$$\begin{aligned} f_1 &= \int_{\mathbb{R}^n} |y-\eta|^{2-n} Q_\lambda(\eta) \varphi \, d\eta \text{ and} \\ f_2 &= \int_{\mathbb{R}^n} |y-\eta|^{2-n} Q_\lambda(\eta)(1-\varphi)\, d\eta. \end{aligned}$$

Then,

$$\Delta f_1 = Q_\lambda \varphi \text{ and } \Delta f_2 = Q_\lambda(1-\varphi),$$

We use the Fourier transform to prove that f_2 is in the Schwartz space and thus,

$$|f_2| \le C|y|^{2-n},\ |\partial f_2| \le C|y|^{1-n},\ |\partial^2 f_2| \le C|y|^{-n}$$

For f_1 we use the fact that in the present case, $|y|$ is big enough, and we differentiate inside the integral, ($|\eta| < |y|/2 \Rightarrow |y-\eta| > |y|/2$) to conclude

$$|f_1| \le C|y|^{2-n},\ |\partial f_1| \le C|y|^{1-n},\ |\partial^2 f_1| \le C|y|^{-n}$$

Thus, for $|y|$ large,

$$|f| \le C|y|^{2-n},\ |\partial f| \le C|y|^{1-n},\ |\partial^2 f| \le C|y|^{-n}. \qquad (58)$$

For v, we have $y \in \Sigma_\lambda$ and the integral is taken over the reflected set Σ_λ^s. We set,

$$R_\lambda(\eta) = Q_\lambda(\eta) + Q_\lambda(\eta^\lambda),$$

so that

$$\begin{aligned} v(y) &= \int_{\Sigma^s_\lambda} |y-\eta|^{2-n} R_\lambda(\eta) d\eta, \\ &= \int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} |y-\eta|^{2-n} R_\lambda(\eta) d\eta - \int_{\Sigma^s_\lambda \cap \{|\eta| \leq |y|/2\}} |y-\eta|^{2-n} R_\lambda(\eta). \end{aligned} \tag{59}$$

Second integral: If $|\eta| \leq |y|/2$ then $|y-\eta| \geq |y|/2$ and thus,

$$\left| \int_{\Sigma^s_\lambda \cap \{|\eta| \leq |y|/2\}} |y-\eta|^{2-n} R_\lambda(\eta) d\eta \right| \leq \begin{cases} C r_k |y|^{2-n} \text{ if } n = 4 \\ C|y|^{2-n} \text{ if } n = 5, 6. \end{cases} \tag{60}$$

First integral: First, we have,

$$|R_\lambda(\eta)| \leq C(1+|\eta|)^{-2-n},$$

If $|y-\eta| \leq |y|/2$, then,

$$|\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} |y-\eta|^{2-n} R_\lambda(\eta) d\eta| \leq |R_\lambda(|y|/2)| \int_{\{|\eta - y| \leq |y|/2\}} |y-\eta|^{2-n} d\eta \leq C(1+|y|)^{-n},$$

If $|y|/2 \leq |y-\eta| \leq 3|y|$,

$$|\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} |y-\eta|^{2-n} R_\lambda(\eta) d\eta| \leq |R_\lambda(|y|/2)| \int_{\{|y|/2 \leq |\eta - y| \leq 3|y|\}} |y-\eta|^{2-n} d\eta \leq C(1+|y|)^{-n},$$

If $3|y| \leq |y-\eta|$, then $|\eta| = |y - \eta - y| \geq 2|y|$, and thus, $|\eta - y| = |\eta|(|\theta_\eta - y/|\eta||) \geq |\eta|/2$. With $|\theta_\eta| = 1$ the angular part of η. Thus

$$\begin{aligned} |\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} |y-\eta|^{2-n} R_\lambda(\eta) d\eta| &\leq C|y|^{1-n} \int_{\{|\eta| \geq |y|/2\}} |\eta|^{-1-n} d\eta \\ &= C|y|^{1-n} \int_{\{r \geq |y|/2\}} r^{-2} dr = C|y|^{-n}. \end{aligned}$$

Thus, in this case too, we have

$$|\partial_i v| \leq \begin{cases} C r_k |y|^{2-n} \text{ if } n = 4 \\ C|y|^{2-n} \text{ if } n = 5, 6. \end{cases} \tag{61}$$

Estimate of the first derivatives of v: We have,

$$\partial_i v = \int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} (y_i - \eta_i)|y-\eta|^{-n} R_\lambda(\eta) d\eta - \int_{\Sigma^s_\lambda \cap \{|\eta| \leq |y|/2\}} (y_i - \eta_i)|y-\eta|^{-n} R_\lambda(\eta)$$

Second integral If $|\eta| \leq |y|/2$ then $|y-\eta| \geq |y|/2$ and thus,

$$\left| \int_{\Sigma^s_\lambda \cap \{|\eta| \leq |y|/2\}} (y_i - \eta_i)|y-\eta|^{-n} R_\lambda(\eta) d\eta \right| \leq \begin{cases} C r_k |y|^{1-n} \text{ if } n = 4, \\ C|y|^{1-n} \text{ if } n = 5, 6. \end{cases} \tag{62}$$

First integral First, we have,

$$|R_\lambda(\eta)| \leq C(1+|\eta|)^{-2-n},$$

If $|y-\eta| \leq |y|/2$, then,

$$|\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} (y_i - \eta_i)|y-\eta|^{-n} R_\lambda(\eta) d\eta| \leq |R_\lambda(|y|/2)| \int_{\{|\eta - y| \leq |y|/2\}} |y-\eta|^{1-n} d\eta \leq C(1+|y|)^{-1-n},$$

If $|y|/2 \leq |y-\eta| \leq 3|y|$,

$$|\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} (y_i - \eta_i)|y-\eta|^{-n} R_\lambda(\eta) d\eta| \leq |R_\lambda(|y|/2)| \int_{\{|y|/2 \leq |\eta - y| \leq 3|y|\}} |y-\eta|^{1-n} d\eta \leq C(1+|y|)^{-1-n},$$

If $3|y| \leq |y-\eta|$, then $|\eta| = |y - \eta - y| \geq 2|y|$, and thus, $|\eta - y| = |\eta|(|\theta_\eta - y/|\eta|| \geq |\eta|/2$. With $|\theta_\eta| = 1$ the angular part of η. Thus

$$|\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} (y_i - \eta_i)|y-\eta|^{-n} R_\lambda(\eta) d\eta| \leq C|y|^{-n} \int_{\{|\eta| \geq |y|/2\}} |\eta|^{-1-n} d\eta$$

$$= C|y|^{-n} \int_{\{r \geq |y|/2\}} r^{-2} dr = C|y|^{-1-n}.$$

Thus,

$$|\partial_i v| \leq \begin{cases} Cr_k |y|^{1-n} \text{ if } n = 4 \\ C|y|^{1-n} \text{ if } n = 5, 6. \end{cases}$$

Estimate of the second derivatives: We write,

$$\partial_{ij} v = \int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} \partial_j((y_i - \eta_i)|y-\eta|^{-n}) R_\lambda(\eta) d\eta - \int_{\Sigma^s_\lambda \cap \{|\eta| \leq |y|/2\}} \partial_j((y_i - \eta_i)|y-\eta|^{-n}) R_\lambda(\eta)$$

Second integral we have, If $|\eta| \leq |y|/2$ then $|y-\eta| \geq |y|/2$ and $|\partial_j((y_i - \eta_i)|y-\eta|^{-n}| \leq |y-\eta|^{-n}$, thus,

$$|\int_{\Sigma^s_\lambda \cap \{|\eta| \leq |y|/2\}} \partial_j((y_i - \eta_i)|y-\eta|^{-n}) R_\lambda(\eta) d\eta| \leq \begin{cases} Cr_k |y|^{-n} \text{ if } n = 4 \\ C|y|^{-n} \text{ if } n = 5, 6. \end{cases}$$

First integral

We use an integration by part,

$$\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} \partial_j \frac{(y_i - \eta_i)}{|y-\eta|^{-n}} R_\lambda(\eta) d\eta = -\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} \frac{(y_i - \eta_i)}{|y-\eta|^{-n}} \partial_j R_\lambda(\eta) d\eta +$$

$$+ \int_{\partial(\Sigma^s_\lambda \cap \{|\sigma| \geq |y|/2\})} \frac{(y_i - \sigma_i)}{|y-\sigma|^{-n}} R_\lambda(\sigma) \nu_j(\sigma) d\sigma$$

From, the computation for the first derivatives, we have,

$$|\int_{\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}} \frac{(y_i - \eta_i)}{|y-\eta|^{-n}} \partial_j R_\lambda(\eta) d\eta| \leq C|y|^{-n},$$

The boundary term has the following decomposition, $\partial(\Sigma^s_\lambda \cap \{|\eta| \geq |y|/2\}) = (\partial\Sigma^s_\lambda) \cap \{|\eta| \geq |y|/2\})) \cup (\Sigma^s_\lambda \cap \{|\eta| = |y|/2\})$. For the first boundary, $\nu_j(\sigma) = 0$ for $j \neq 1$, and thus,

$$\int_{\partial(\Sigma^s_\lambda \cap \{|\sigma| \geq |y|/2\})} \frac{(y_i - \sigma_i)}{|y-\sigma|^{-n}} R_\sigma(\sigma) \nu_j(\sigma) d\sigma = \int_{\Sigma^s_\lambda \cap \{|\sigma| = |y|/2\}} \frac{(y_i - \sigma_i)}{|y-\sigma|^{-n}} R_\sigma(\sigma) \nu_j(\sigma) d\sigma$$

Clearly, we have,

$$\int_{\Sigma_\lambda^s \cap \{|\sigma|=|y|/2\}} \frac{|y_i-\sigma_i|}{|y-\sigma|^{-n}} |R_\sigma(\sigma)||\nu_j(\sigma)| d\sigma \leq C|y|^{-n},$$

Thus, for $j \neq 1$, we have:

$$|\partial_{ij} v| \leq \begin{cases} Cr_k|y|^{-n} \text{ if } n=4 \\ C|y|^{-n} \text{ if } n=5,6. \end{cases}$$

But, $\Delta v = 0$, thus,

$$|\partial_{11} v| = |\sum_{i=2}^{n} \partial_{ii} v| \leq \begin{cases} Cr_k|y|^{-n} \text{ if } n=4 \\ C|y|^{-n} \text{ if } n=5,6. \end{cases}$$

Finally, we have:

$$|\partial_i u| \leq \begin{cases} Cr_k|y|^{1-n} \text{ if } n=4 \\ C|y|^{1-n} \text{ if } n=5,6. \end{cases}$$

and,

$$|\partial_{ij} u| \leq \begin{cases} Cr_k|y|^{-n} \text{ if } n=4, \\ C|y|^{-n} \text{ if } n=5,6. \end{cases}$$

Estimate for $u-\tilde{u}$:

Around infinity, we use the fact that, y is big enough ($|\eta| < |y|/2 \Rightarrow |y-\eta| > |y|/2$) and we differentiate inside the integral, to have:

$$|u-\tilde{u}| \leq C|y|^{2-n}, \ |\partial(u-\tilde{u})| \leq C|y|^{1-n}, \ |\partial^2(u-\tilde{u})| \leq C|y|^{-n}. \tag{63}$$

(2) Behavior of u and $u-\tilde{u}$ near 0. The function f_2 is smooth and solution of an elliptic equation with $Q_\lambda(1-\varphi) \in C^\infty(\mathbb{R}^n)$, thus, by the elliptic estimates, we have,

$$|f_2|_{C^2(B(0,1))} \leq C,$$

We write the function f_1 as,

$$f_1(y) = \int_{B(0,1)} |y-\eta|^{2-n} Q_\lambda(\eta)\varphi d\eta$$

note that, $|Q_\lambda(\eta)| \leq Cr_k^2$ and thus,

$$|f_1(y)| \leq Cr_k^2$$

Moreover, we can write:

$$\partial_i f_1 = \int_{B(0,1)} (y_i-\eta_i)|y-\eta|^{-n} Q_\lambda(\eta)\varphi d\eta$$

Thus,

$$|\partial_i f_1| \leq \int_{B(0,1)} |y-\eta|^{1-n} Q_\lambda(\eta)\varphi d\eta \leq Cr_k^2,$$

Also, we can write, (see, Gilbarg-Trudinger),

$$\partial_{ij} f_1(y) = \int_{B(0,1)} \partial_j((y_i-\eta_i)|y-\eta|^{-n})(Q_\lambda(\eta)\varphi(\eta)d\eta + Q_\lambda(y)\varphi(y) \int_{\partial B(0,1)} (y_i-\sigma_i)|y-\sigma|^{-n} d\sigma,$$

Thus,

$$|\partial_{ij} f_1(y)| \leq C \int_{B(0,1)} |y-\eta|^{-n} |Q_\lambda(\eta)\varphi(\eta) - Q_\lambda(y)\varphi(y)| d\eta + Cr_k^2$$

We write,

$$\int_{B(0,1)} |y-\eta|^{-n} |Q_\lambda(\eta)\varphi(\eta) - Q_\lambda(y)\varphi(y)| = \int_{B(0,1)\cap\{|\eta|\geq|y|/2\}} |y-\eta|^{-n} |Q_\lambda(\eta)\varphi(\eta) - Q_\lambda(y)\varphi(y)| d\eta +$$

$$+ \int_{B(0,1)\cap\{|\eta|\leq|y|/2\}} |y-\eta|^{-n} |Q_\lambda(\eta)\varphi(\eta) - Q_\lambda(y)\varphi(y)| d\eta$$

Second integral We have $|\eta| \leq |y|/2$, thus $|y-\eta| \geq |y|/2$, and thus,

$$\int_{B(0,1)\cap\{|\eta|\leq|y|/2\}} |y-\eta|^{-n} |Q_\lambda(\eta)\varphi(\eta) - Q_\lambda(y)\varphi(y)| \leq \frac{Cr_k^2}{|y|} \int_{B(0,1)\cap\{|\eta|\leq|y|/2\}} |y-\eta|^{1-n} d\eta \leq \frac{Cr_k^2}{|y|}$$

First integral We write,

$$Q_\lambda(\eta)\varphi(\eta) - Q_\lambda(y)\varphi(y) = (\eta - y)\nabla Q_\lambda(\xi), \quad \text{with, } \xi \text{ between } \eta \text{ and } y,$$

We remark that,

$$|\nabla Q_\lambda(\xi)| \leq C(|\xi| + r_k^{-1/2})^{-5},$$

If $|y| \leq |\xi| \leq |\eta|$, then,

$$|\nabla Q_\lambda(\xi)| \leq C(|y| + r_k^{-1/2})^{-5} \leq Cr_k^2/|y|$$

If $|y|/2 \leq |\eta| \leq |\xi| \leq |y|$,

$$|\nabla Q_\lambda(\xi)| \leq C(|\eta| + r_k^{-1/2})^{-5} \leq Cr_k^2/|y|$$

Finally, we have:

$$|\partial_{ij} f_1(y)| \leq Cr_k^2/|y|$$

Now, we estimate v near 0, as for f we decompose v in two functions v_1 and v_2, and we see that $y \in \Sigma_\lambda$ small enough is far from the symmetral Σ_λ^s of Σ_λ. And we differentiate inside the integral to have:

$$|\partial_{ij} v_1(y)| \leq Cr_k^2 \text{ and } |\partial_{ij} v_2(y)| \leq C.$$

Now, for $u - \tilde{u}$, we use the fact that $|y| \geq \sigma r_k^{-1/2}$ with $\sigma > 1$ and the elliptic interior estimates to have (we differentiate inside the integral)

$$|u-\tilde{u}|_{C^0(B(0,1))} \leq Cr_k, \ |u-\tilde{u}|_{C^1(B(0,1))} \leq Cr_k^{3/2}, \ |u-\tilde{u}|_{C^2(B(0,1))} \leq Cr_k^2$$

It follows that

$$|\tilde{d}_{ml}\partial_{ml} u + \tilde{b}_m \partial_m u + \tilde{c}u| = o(1)Q_\lambda \tag{64}$$

and that

$$L_k u = (C_1 + o(1))Q_\lambda. \tag{65}$$

□

As in Chen-Lin paper [11], we have the following lemma (which we state without proof)

Lemma 3.3. . *For Q_λ, we have in Σ'_λ for $\lambda \leq -1/4$ and for large k:*

$$r_k^{(n-2)/2} \int_{\Sigma'_\lambda} G^\lambda(y,\eta)\tilde{Q}_\lambda(\eta)d\eta = o(1)G^\lambda(y,0).$$

If , we choose$A > 0$ small enough in the definition of h_λ, we have,

$$h_\lambda > 0,$$

$$L_k h_\lambda \geq C_1 \tilde{Q}_\lambda.$$

We can now use Lemma of [11] to obtain a contradiction, and this completes the proof of Theorem

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