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QUANTITATIVE AND QUALITATIVE KAC'S CHAOS ON THE BOLTZMANN'S SPHERE

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Abstract. We investigate the construction of chaotic probability measures on the Boltzmann's sphere, which is the state space of the stochastic process of a many-particle system undergoing a dynamics preserving energy and momentum.

Firstly, based on a version of the local Central Limit Theorem (or Berry-Essenn theorem), we construct a sequence of probabilities that is Kac chaotic and we prove a quantitative rate of convergence. Then, we investigate a stronger notion of chaos, namely entropic chaos introduced in [3], and we prove, with quantitative rate, that this same sequence is also entropically chaotic.

Furthermore, we investigate more general class of probability measures on the Boltzmann's sphere. Using the HWI inequality we prove that a Kac chaotic probability with bounded Fisher's information is entropically chaotic and we give a quantitative rate. We also link different notions of chaos, proving that Fisher's information chaos, introduced in [7], is stronger than entropic chaos, which is stronger than Kac's chaos. We give a possible answer to [3, Open Problem 11] in the Boltzmann's sphere's framework.

Finally, applying our previous results to the recent results on propagation of chaos for the Boltzmann equation [12], we prove a quantitative rate for the propagation of entropic chaos for the Boltzmann equation with Maxwellian molecules.

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Keywords: Kac's chaos; entropic chaos; Fisher's information chaos; many-particle jump process; entropy; Fisher's information; mean-field limit; Central Limit Theorem; Berry-Esseen; HWI inequality; Boltzmann equation.

Contents

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1. Introduction

1.1. **Motivation.** In his celebrated paper [8], Kac introduced the notion of propagation of chaos in order to connect a stochastic process of a system of N identical particles undergoing binary collisions to its mean field equation.

Our interest in this paper is to investigate chaotic distributions supported by the phase space of the stochastic process of the N-particle system as we shall explain. We refer to [3] for a detailed introduction on this topic and on Kac's paper [8].

Consider a system of N identical particles of mass $\rho > 0$ such that its evolution is described by a jump process with binary collisions that preserves energy and momentum. Let us denote by i, j the particles undergoing the collision, with pre-collisional velocities $v_i, v_j \in \mathbb{R}^d$ and post-collisional velocities $v_i^*, v_j^* \in \mathbb{R}^d$. We have then the conservation of momentum

$$\rho v_i^* + \rho v_j^* = \rho v_i + \rho v_j,$$

and the conservation of energy

$$\frac{\rho}{2}|v_i^*|^2 + \frac{\rho}{2}|v_j^*|^2 = \frac{\rho}{2}|v_i|^2 + \frac{\rho}{2}|v_j|^2.$$

If the system has initial energy $\mathcal{E} = \frac{1}{2}\sum_{i=1}^N \rho|v_i|^2 \in \mathbb{R}_+$ and initial momentum $M = \rho m = \sum_{i=1}^N \rho v_i \in \mathbb{R}^d$, then both energy and momentum will be unchanged under the dynamics. The phase space of this process is then the manifold $\mathcal{S}^N(\sqrt{\mathcal{E}}, m) \subset \mathbb{R}^{dN}$ defined by

$$\mathcal{S}^N(\sqrt{\mathcal{E}}, m) := \left\{ V = (v_1, \dots, v_N) \in \mathbb{R}^{dN} \,\Big|\, \frac{1}{2}\sum_{i=1}^N \rho|v_i|^2 = \mathcal{E},\ \sum_{i=1}^N \rho v_i = \rho m \right\},$$

which is the intersection of a sphere of radius $\sqrt{2\mathcal{E}/\rho}$ and a hyperplane. This space $\mathcal{S}^N(\sqrt{\mathcal{E}}, m)$ is in fact a sphere in $\mathbb{R}^{dN}$ of dimension $d(N-1)-1$ with radius $\sqrt{2\mathcal{E}/\rho - |m|^2/N}$ and center $(m, \dots, m)/\sqrt{N}$. We remark that we need $|m|^2 \le 2N\mathcal{E}/\rho$ in order to $\mathcal{S}^N(\sqrt{\mathcal{E}}, m)$ be non empty.

Now choosing units such that the mass ρ of each particle is equal to 2, the total value of kinetic energy is dN and, without loss of generality, choosing $m = 0$, the state space of this dynamics is

$$\mathcal{S}_{\mathcal{B}}^N := \mathcal{S}^N(\sqrt{dN}, 0) = \left\{ V = (v_1, \dots, v_N) \in \mathbb{R}^{dN} \,\Big|\, \sum_{i=1}^N |v_i|^2 = dN,\ \sum_{i=1}^N v_i = 0 \right\} \tag{1}$$

and we shall call the manifold $\mathcal{S}_{\mathcal{B}}^N$ the Boltzmann's sphere.

An example of this kind of dynamics is the space homogeneous Boltzmann model that we shall explain. Given a pre-collisional system of velocities $V = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$ and a collision kernel (for more information on the collision kernel we refer to [16, 12])

$$B(z, \cos\theta) = \Gamma(z) b(\cos\theta) \tag{2}$$

the process is:

- for any $i' \neq j'$, pick a random time $T(\Gamma(|v_{i'} - v_{j'}|))$ of collision accordingly to an exponential law of parameter $\Gamma(|v_{i'} - v_{j'}|)$ and choose the minimum time T_1 and the colliding pair (v_i, v_j) such that

$$T_1 = T(\Gamma(|v_i - v_j|)) = \min_{i',j'} T(\Gamma(|v_{i'} - v_{j'}|)),$$

- draw $\sigma \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$ according to the law $b(\cos\theta_{ij})$, with

$$\cos\theta_{ij} = \sigma \cdot \frac{(v_i - v_j)}{|v_i - v_j|},$$

- after collision the new velocities become

$$V_{ij}^* = (v_1, \dots, v_i^*, \dots, v_j^*, \dots, v_N)$$

where the post-collisional velocities v_i^* and v_j^* are given by

$$v_i^* = \frac{v_i + v_j}{2} + \frac{|v_i - v_j|}{2}\sigma, \qquad v_j^* = \frac{v_i + v_j}{2} - \frac{|v_i - v_j|}{2}\sigma. \tag{3}$$

Iterating this construction we built then the associated Markov process $(\mathcal{V}_t)_{t\geq 0}$ on $\mathbb{R}^{dN}$. The equation of the associated law is given by, after a rescaling of time, (see [12])

$$\partial_t G_t^N = L_N G_t^N = \frac{1}{N}\sum_{i<j}\int_{\mathbb{S}^{d-1}} \left[G_t^N(V_{ij}^*) - G_t^N(V)\right] B(|v_i - v_j|, \cos\theta)\, d\sigma \tag{4}$$

with initial data G_0^N and where $V_{ij}^* = (v_1, \dots, v_i^*, \dots, v_j^*, \dots, v_N)$. This equation is known as the *master equation*.

Associated to this process, we have the (limit) spatially homogeneous Boltzmann equation [12, 13, 16]

$$\partial_t f(t,v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B(|v-w|, \cos\theta)\Big(f(w^*)f(v^*) - f(w)f(v)\Big)\, dw\, d\sigma \tag{5}$$

with initial data $f(0,\cdot) = f_0$ and where the post-collisional velocities v^* and w^* are obtained by (3).

The program set by Kac in [8] was to investigate the behavior of solutions of the mean field equation (5) in terms of the behaviour of the solutions of the master equation (4). Moreover, the notion of propagation of chaos introduced by Kac means that if the initial distribution G_0^N is f_0-chaotic (Definition 1 below) then, for all $t > 0$, the solution G_t^N of (4) is f_t-chaotic, where f_t is the solution of (5). For more information on this topic we refer to the recent results of Mischler, Mouhot and Wennberg [12, 13].

This paper is inspired by the works of Carlen, Carvalho, Le Roux, Loss and Villani [3] and also of Hauray and Mischler [7], which investigate chaotic probabilities on the usual sphere in $\mathbb{R}^N$ with radius $\sqrt{N}$ (also called Kac's sphere). This sphere is the phase space of Kac's model, which is a one-dimensional simplification, introduced in [8], of the model presented above, with energy conservation only.

The novelty here is that we investigate chaotic probability sequences in the Boltzmann's sphere $\mathcal{S}_{\mathcal{B}}^N \subset \mathbb{R}^{dN}$ and, furthermore, we prove quantitative rates of chaos convergence.

1.2. **Definitions and main results.** Let E be a Polish space, then we shall denote by $\mathbf{P}(E)$ the space of Borel probability measures on E. Furthermore, through this paper, on the space E^N we will only consider symmetric measures, more precisely, we say that $G^N \in \mathbf{P}(E^N)$ is symmetric if for all $\varphi \in C_b(E^N)$ we have

$$\int_{E^N} \varphi\, dG^N = \int_{E^N} \varphi_\sigma\, dG^N,$$

for any permutation σ of $\{1,\dots,N\}$, and where

$$\varphi_\sigma := \varphi(V_\sigma) = \varphi(v_{\sigma(1)},\dots,v_{\sigma(N)}),$$

for $V=(v_1,\dots,v_N)\in E^N$.

For $G^N\in\mathbf{P}(E^N)$ and a integer $\ell\in[1,N]$ we denote by G^N_ℓ (or $\Pi_\ell(G^N)$) the ℓ-marginal of G^N, defined by

$$\forall\varphi\in C_b(E^\ell),\qquad \int_{E^\ell}\varphi\,dG^N_\ell=\int_{E^N}\varphi\otimes\mathbf{1}^{\otimes(N-\ell)}\,dG^N.$$

We can now give the notion of chaos formalized by Kac in [8], we also refer to [15] for an introduction on this topic with a probabilistic approach and to [11] for a short survey.

Definition 1 (Kac's chaos)**.** *Consider $f\in\mathbf{P}(E)$. We say that $G^N\in\mathbf{P}(E^N)$ is f-chaotic (or f-Kac chaotic), if for each fixed positive integer ℓ, G^N_ℓ converges to $f^{\otimes\ell}$ in the sense of measures in $\mathbf{P}(E^\ell)$ when N goes to infinity, i.e. if for all $\varphi\in C_b(E^\ell)$,*

$$\int_{E^\ell}\varphi\,dG^N_\ell\xrightarrow[N\to\infty]{}\int_{E^\ell}\varphi\,df^{\otimes\ell}. \tag{6}$$

In fact, it is well known that we need condition (6) to hold for only one $\ell\geq 2$ (see for instance [15]).

We also introduce the Monge-Kantorovich-Wasserstein (MKW) distance and for more information about it we refer to [17]. Consider an integer ℓ and $p\in[1,\infty)$, we define then the space

$$\mathbf{P}_p(E^\ell):=\left\{F^\ell\in\mathbf{P}(E^\ell);\ M_p(F^\ell):=\int_{E^\ell}|X|^p\,dF^\ell(X)<\infty\right\}.$$

Then, for $F^\ell,G^\ell\in\mathbf{P}_p(E^\ell)$ we define the MKW distance between F^ℓ and G^ℓ by

$$W_p(F^\ell,G^\ell):=\inf_{\pi\in\Pi(F^\ell,G^\ell)}\left(\int_{E^\ell\times E^\ell}d_{E^\ell}(X,Y)^p\,d\pi(X,Y)\right)^{1/p}, \tag{7}$$

where $\Pi(F^\ell,G^\ell)$ is the set of transfer plan between F^ℓ and G^ℓ, which is the set of probabilty measures on $E^\ell\times E^\ell$ with marginals F^ℓ and G^ℓ respectively, and where we define the distace d_{E^ℓ} as

$$\forall\,X=(x_1,\dots,x_\ell),Y=(y_1,\dots,y_\ell)\in E^\ell,\qquad d_{E^\ell}(X,Y):=\sum_{i=1}^\ell d_E(x_i,y_i).$$

In the paper we will use the Euclidean distance in $E=\mathbb{R}^d$, i.e. $d_E(x_i,y_i)=|x_i-y_i|$ for all $x_i,y_i\in E$. More precisely, we shall use

$$\forall\,f,g\in\mathbf{P}_1(\mathbb{R}^d),\qquad W_1(f,g)=\inf_{\pi\in\Pi(f,g)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|\,d\pi(x,y)$$

and

$$\forall\,f,g\in\mathbf{P}_2(\mathbb{R}^d),\qquad W_2(f,g)=\inf_{\pi\in\Pi(f,g)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\,d\pi(x,y)\right)^{1/2}.$$

Moreover, for $F^N,G^N\in\mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$ we shall use in the definition of $W_p(F^N,G^N)$ the Euclidean distance inherited from $\mathbb{R}^{dN}$, which means that for $X,Y\in\mathcal{S}^N_{\mathcal{B}}$ we shall use $d_{\mathcal{S}^N_{\mathcal{B}}}(X,Y)=|X-Y|$.

Let γ be the Gaussian probability measure on $\mathbb{R}^d$, $\gamma(v) = (2\pi)^{-d/2}\, e^{-|v|^2/2}$, and $\mu \in \mathbf{P}(\mathbb{R}^d)$. We define the relative entropy of μ with respect to γ by

$$H(\mu|\gamma) := \int_{\mathbb{R}^d} \log \frac{d\mu}{d\gamma}\, d\mu, \tag{8}$$

if μ is absolutely continuous with respect to γ, otherwise $H(\mu|\gamma) := +\infty$.

Moreover, for $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ we define the relative entropy with respect to γ^N, the uniform probability measure on $\mathcal{S}_{\mathcal{B}}^N$, by

$$H(G^N|\gamma^N) := \int_{\mathcal{S}_{\mathcal{B}}^N} \left(\log \frac{dG^N}{d\gamma^N} \right) dG^N. \tag{9}$$

We shall now define a stronger notion of chaos, namely the entropic chaos introduced in [3].

Definition 2 (Entropic chaos)**.** *We say that the sequence $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ is entropically f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, if G^N is f-chaotic in Kac's sense (Definition 1) and*

$$\frac{1}{N}\, H(G^N|\gamma^N) \xrightarrow[N\to\infty]{} H(f|\gamma)$$

with $H(f|\gamma) < \infty$.

Finally, with these definitions at hand we can state the main results of the paper.

Theorem 3. *For any $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $1 < p \le \infty$, there exists a sequence of probability measures $F^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ such that*

(i) F^N is f-chaotic. More precisely, for any $\ell \ge 1$ fixed there exists a constant $C = C(\ell) > 0$ such that for $N \ge \ell + 1$ we have

$$W_1(F_\ell^N, f^{\otimes \ell}) \le \frac{C}{\sqrt{N}};$$

(ii) F^N is entropically f-chaotic. More precisely, there exists a constant $C > 0$ such that

$$\left| \frac{1}{N}\, H(F^N|\gamma^N) - H(f|\gamma) \right| \le \frac{C}{\sqrt{N}}.$$

Let us now define the relative Fisher's information of a probability measure $\mu \in \mathbf{P}(\mathbb{R}^d)$ with respect to γ by

$$I(\mu|\gamma) := \int_{\mathbb{R}^d} \left| \nabla \log \frac{d\mu}{d\gamma} \right|^2 d\mu, \tag{10}$$

and, as we did for entropy, we also define for $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ the relative Fisher's information with respect to γ^N by

$$I(G^N|\gamma^N) := \int_{\mathcal{S}_{\mathcal{B}}^N} \left| \nabla_{\mathcal{S}} \log \frac{dG^N}{d\gamma^N} \right|^2 dG^N, \tag{11}$$

where $\nabla_{\mathcal{S}}$ stands for the gradient on the Boltzmann's sphere, i.e. the component of the usual gradient in $\mathbb{R}^{dN}$ that is tangent to the sphere $\mathcal{S}_{\mathcal{B}}^N$.

We define then another stronger notion of chaos, the Fisher's information chaos, in an analogous way of Definition 2.

Definition 4 (Fisher's information chaos)**.** *We say that the sequence $G^N \in \mathbf{P}(\mathcal{S}_B^N)$ is Fisher's information f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, if G^N is f-chaotic in Kac's sense (Definition 1) and*

$$\frac{1}{N} I(G^N|\gamma^N) \xrightarrow[N\to\infty]{} I(f|\gamma)$$

with $I(f|\gamma) < \infty$.

Remark 5. The Fisher's information chaos is introduced in [7] in a weaker way, which is in fact equivalent to Definition 4 thanks to Theorem 6.

Next, we may compare as follows the several notions of chaos:

Theorem 6. *Consider $G^N \in \mathbf{P}(\mathcal{S}_B^N)$, with k-th order moment $M_k(G_1^N)$ bounded, for some $k \geq 6$, and suppose that $G_1^N \rightharpoonup f$ in $\mathbf{P}(\mathbb{R}^d)$.*

Then, each assertion listed below implies the further one:

(i) $N^{-1} I(G^N|\gamma^N) \to I(f|\gamma)$, with $I(f|\gamma) < \infty$.
(ii) $N^{-1} I(G^N|\gamma^N)$ is bounded and G^N is f-chaotic in Kac's sense.
(iii) $N^{-1} H(G^N|\gamma^N) \to H(f|\gamma)$, with $H(f|\gamma) < \infty$.
(iv) G^N is f-chaotic in Kac's sense.

As a consequence, in Definition 2 of the entropic chaos and in Definition 4 of Fisher's information chaos, we only need the convergence of the first marginal, i.e. $G_1^N \rightharpoonup f$, instead of the convergence of all marginals. Hence, this theorem asserts that Fisher's information chaos implies entropic chaos, which in turns implies chaos (or Kac's chaos). Furthermore, we prove a quantitative rate for the implication $(ii) \Rightarrow (iii)$.

Another main result of the paper is the following possible answer to [3, Open Problem 11] in the setting of Boltzmann's sphere.

Theorem 7. *Consider $G^N \in \mathbf{P}(\mathcal{S}_B^N)$ such that G^N is f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, and suppose that*

$$M_k(G_1^N) \leq C, \ k > 2, \qquad \frac{1}{N} I(G^N|\gamma^N) \leq C.$$

Suppose further that $f \in W^{1,\infty}(\mathbb{R}^d)$ and $f(v_1) \geq \exp(-a|v_1|^2)$ for some constant $a > 0$. Then for any fixed ℓ, there exists a constant $C = C(d, \ell, \|f\|_{W^{1,\infty}}, M_k(G_1^N, f)) > 0$ such that for all $N \geq \ell + 1$ we have

$$H(G_\ell^N|f^{\otimes\ell}) \leq C\, W_1(G_\ell^N, f^{\otimes\ell})^{\theta(\ell,d,k)},$$

where $\theta(\ell, d, k)$ is constructive and depends on ℓ, d and k.

We can apply our previous results to the Boltzmann equation theory. Thanks to the work [12] for the Boltzmann equation (4)-(5) we can establish the following theorem.

Theorem 8. *Consider $f_0 \in \mathbf{P}_6(\mathbb{R}^d)$ with $I(f_0|\gamma)$ finite and $G_0^N \in \mathbf{P}(\mathcal{S}_B^N)$ built as in Theorem 3. Consider then, for all $t > 0$, the solution G_t^N of the Master equation* (4) *with initial condition G_0^N, and the solution f_t of the limiting Boltzmann equation* (5) *with initial data f_0 (both with Maxwellian molecules, see* [12, 16]*).*

Then G_t^N is entropically f_t-chaotic. More precisely, for any

$$\epsilon < \frac{48}{(7d+6)^2(5d+24)}$$

there exists a constant $C := C(\epsilon)$ *such that*

$$\sup_{t\geq 0}\left|\frac{1}{N}H(G_t^N|\gamma^N) - H(f_t|\gamma)\right| \leq CN^{-\epsilon}.$$

Theorem 8 improves the results of [12] where Kac's chaos is established with a rate but entropic chaos is proved without any rate.

1.3. **Strategy.** We construct a probability on $\mathcal{S}_{\mathcal{B}}^N$ based on tensorization and conditioning of some probabilty measure on $\mathbb{R}^d$. To this purpose, we use an explicit formula for the marginals of the uniform probablity on $\mathcal{S}_{\mathcal{B}}^N$ and a version of the local Central Limit Theorem (also known as Berry-Esseen), which is the cornerstone of the proof.

In order to study more general probabilities on the Boltzmann's sphere, we use an interpolation-type inequality, relating entropy, Fisher's information and the 2-MKW distance, called HWI inequality from [14, 9, 17], to show that Kac chaotic probabilities with finite Fisher's information are entropically chaotic.

Finally, the application of our results to the Boltzmann equation is based on recent results of propagation of chaos from [12] and on the relations of different notions of measuring chaos from the work [7].

1.4. **Previous works.** In [8] it is proved that the N-fold tensorization of a smooth probability on $\mathbb{R}$ conditioned to the Kac's sphere, i.e. the usual sphere $\mathbb{S}^{N-1}(\sqrt{N})$, is Kac chaotic. Then, the work [3] extends this result to a more general class of probabilities on $\mathbb{R}$, introduces the notion of entropic chaos and also proves that the N-fold tensorization conditioned to the Kac's sphere is entropically chaotic. Furthermore, the recent work [7] gives quantitative rates of the results before, introduces the notion of Fisher's information chaos and links these three notions of chaos.

1.5. **Organization of the paper.** In Section 2 we shall study the uniform probability measure on $\mathcal{S}_{\mathcal{B}}^N$. In Section 3 we construct a chaotic distribution on Boltzmann's sphere based on a probability measure on $\mathbb{R}^d$. Furthermore we prove a quantitative chaos convergence rate and we prove point (i) of Theorem 3. Then, in Section 4 we investigate the entropic chaos. First, we study the entropic chaos for the probability distribution built before in Section 3 and we prove point (ii) of Theorem 3. Then, we investigate a more general class of probability measures on $\mathcal{S}_{\mathcal{B}}^N$ and we prove Theorem 6. Finally, in Section 5 we use our previous results to prove a precise version of Theorem 8.

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2. Uniform probability measure

Consider $V = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$, $r \in \mathbb{R}_+$ and $z \in \mathbb{R}^d$. We define the sphere

$$\mathcal{S}^N(r,z) := \left\{ V = (v_1,\dots,v_N) \in \mathbb{R}^{dN} \;\middle|\; \sum_{i=1}^N v_i^2 = r^2,\ \sum_{i=1}^N v_i = z \right\}.$$

We denote by $\gamma^N_{r,z}$ the uniform probability measure on $\mathcal{S}^N(r,z)$. We recall that $\mathcal{S}^N_{\mathcal{B}} := \mathcal{S}^N(\sqrt{dN},0)$ is the Boltzmann sphere and we denote by $\gamma^N := \gamma^N_{\sqrt{dN},0}$ its uniform probability measure. Moreover, we also denote by $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n$ the usual sphere of dimension $n-1$ and radius r, $\mathbb{S}^{n-1} := \mathbb{S}^{n-1}(1)$ and by $\left|\mathbb{S}^{n-1}\right|$ its measure. We can easily compute the measure of $\mathcal{S}^N(r,z)$ by

$$\left|\mathcal{S}^N(r,z)\right| = \left|\mathbb{S}^{d(N-1)-1}\right| \left(r^2 - \frac{|z|^2}{N}\right)_+^{\frac{d(N-1)-1}{2}}, \tag{12}$$

For $V = (v_1,\dots,v_N) \in \mathbb{R}^{dN}$, we shall use through the paper the notation $V_\ell = (v_1,\dots,v_\ell) \in \mathbb{R}^{d\ell}$, $V_{\ell,N} = (v_{\ell+1},\dots,v_N) \in \mathbb{R}^{d(N-\ell)}$ and $\bar{V}_\ell = \sum_{i=1}^\ell v_i \in \mathbb{R}^d$.

We begin with the following result of a change of variables, proved in Appendix A.1.

Lemma 9. *Consider $V \in \mathcal{S}^N(r,z)$. We can make a change of coordinates $(v_1,\dots,v_N) \to (u_1,\dots,u_N)$ in the following way*

$$\begin{aligned} u_N &= \frac{1}{\sqrt{N}}(v_1+\cdots+v_N) \\ u_k &= \frac{1}{\sqrt{k(k+1)}}(v_1+\cdots+v_k - k\,v_{k+1}), \quad 1 \le k \le N-1, \end{aligned} \tag{13}$$

such that the Jacobian is equal to one, $|u_1|^2+\cdots+|u_N|^2 = |v_1|^2+\cdots+|v_N|^2$ and

$$\begin{cases} |v_1|^2+\cdots+|v_N|^2 = r^2 \\ v_{1,\alpha}+\cdots+v_{N,\alpha} = z_\alpha \end{cases} \to \begin{cases} |u_1|^2+\cdots+|u_{N-1}|^2 = r^2 - \dfrac{|z|^2}{N} \\ u_{N,\alpha} = \dfrac{z_\alpha}{\sqrt{N}}, \quad 1 \le \alpha \le d. \end{cases} \tag{14}$$

With these definitions and notations at hand we can study some properties of the uniform probability measure γ^N on $\mathcal{S}^N_{\mathcal{B}}$, which give us the following lemmas:

Lemma 10. *We have the following properties*

(i) for any $\ell \le N-1$ the ℓ-marginal of γ^N is given by $\gamma^N_\ell(dV_\ell) = \gamma^N_\ell(V_\ell)\,dV_\ell$ with

$$\gamma^N_\ell(V_\ell) = \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|} \frac{N^{\frac{d}{2}}}{(N-\ell)^{\frac{d}{2}}} \frac{\left(dN - |V_\ell|^2 - \frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}, \tag{15}$$

where $dV_\ell = dv_1 \dots dv_\ell$ is the Lebesgue measure on $\mathbb{R}^{d\ell}$.

(ii) the moments of γ^N_ℓ are uniformly bounded in N, more precisely, for $k \ge 1$ we have $M_k(\gamma^N_\ell) \le C_{d,k,\ell}$, where $C_{d,k,\ell}$ depends on d,k and ℓ.

Before the proof, we refer to [6] where a Fubini-like theorem on $\mathcal{S}^N(r,z)$ is proved, which yields a generalization of (15) for the ℓ-marginal of $\gamma^N_{r,z}$.

Proof. (i) We can define $\gamma^N_{r,z}$ by

$$\gamma^N_{r,z} := \frac{1}{Z^N_{r,z}} \lim_{h\to 0} \frac{1}{h}\left(\mathbf{1}_{B^N_z(r+h)} - \mathbf{1}_{B^N_z(r)}\right), \quad B^N_z(r) := \{V \in \mathbb{R}^{dN}; |V| \le r,\ \sum_{i=1}^N v_i = z\},$$

where $Z^N_{r,z}$ is the normalization constant so that the integral of $\gamma^N_{r,z}$ is one.

Consider $\varphi \in C(\mathbb{R}^{d\ell})$, for $\ell \leq N-1$, then

$$\begin{aligned}
&\left\langle \mathbf{1}_{B^N_z(r)}, \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle \\
&\quad = \int_{\mathbb{R}^{dN}} \mathbf{1}_{|V_\ell|^2+|V_{\ell,N}|^2 \leq r^2}\, \mathbf{1}_{\bar{V}_\ell + v_{\ell+1}+\cdots+v_N = z}\, \varphi(V_\ell)\, dV_\ell\, dV_{\ell,N} \\
&\quad = \int_{\mathbb{R}^{d\ell}} \varphi(V_\ell) \left(\int_{\mathbb{R}^{d(N-\ell)}} \mathbf{1}_{|V_{\ell,N}|^2 \leq r^2 - |V_\ell|^2} \mathbf{1}_{v_{\ell+1}+\cdots+v_N = z-\bar{V}_\ell}\, dV_{\ell,N} \right) dV_\ell \\
&\quad = \int_{\mathbb{R}^{d\ell}} \varphi(V_\ell) \left| \mathbb{B}^{d(N-\ell-1)} \right| \left(r^2 - |V_\ell|^2 - \frac{|z-\bar{V}_\ell|^2}{N-\ell} \right)_+^{\frac{d(N-\ell-1)}{2}} dV_\ell,
\end{aligned}$$

where $|\mathbb{B}^{d(N-\ell-1)}|$ is the measure of the unit ball in dimension $d(N-\ell-1)$. We deduce then that the ℓ-marginal of $\gamma^N_{r,z}$, denoted by $\Pi_\ell(\gamma^N_{r,z})$, is given by

$$\begin{aligned}
\Pi_\ell(\gamma^N_{r,z}) &= \frac{1}{Z^N_{r,z}} \frac{d}{dr} \left[\left| \mathbb{B}^{d(N-\ell-1)} \right| \left(r^2 - |V_\ell|^2 - \frac{|z-\bar{V}_\ell|^2}{N-\ell} \right)_+^{\frac{d(N-\ell-1)}{2}} \right] \\
&= \frac{\left| \mathbb{B}^{d(N-\ell-1)} \right|}{Z^N_{r,z}} d(N-\ell-1)\, r \left(r^2 - |V_\ell|^2 - \frac{|z-\bar{V}_\ell|^2}{N-\ell} \right)_+^{\frac{d(N-\ell-1)-2}{2}} \\
&= \frac{\left| \mathbb{S}^{d(N-\ell-1)-1} \right|}{Z^N_{r,z}} r \left(r^2 - |V_\ell|^2 - \frac{|z-\bar{V}_\ell|^2}{N-\ell} \right)_+^{\frac{d(N-\ell-1)-2}{2}}
\end{aligned}$$

and in the particular case $r^2 = dN$, $z = 0$

$$\Pi_\ell(\gamma^N) = \gamma^N_\ell = \frac{\left| \mathbb{S}^{d(N-\ell-1)-1} \right|}{Z^N_{\sqrt{dN},0}} (dN)^{1/2} \left(dN - |V_\ell|^2 - \frac{|\bar{V}_\ell|^2}{N-\ell} \right)_+^{\frac{d(N-\ell-1)-2}{2}}. \tag{16}$$

Now we shall compute $Z^N := Z^N_{\sqrt{dN},0}$, with

$$Z^N = \left| \mathbb{S}^{d(N-\ell-1)-1} \right| (dN)^{1/2} \int_{\mathbb{R}^{d\ell}} \left(dN - |V_\ell|^2 - \frac{|\bar{V}_\ell|^2}{N-\ell} \right)_+^{\frac{d(N-\ell-1)-2}{2}} dV_\ell. \tag{17}$$

We start by the integral

$$A = \int_{\mathbb{R}^{d\ell}} \left(dN - |V_\ell|^2 - \frac{|\bar{V}_\ell|^2}{N-\ell} \right)_+^{\frac{d(N-\ell-1)-2}{2}} dV_\ell,$$

with the changement of variable (13)-(14) (replacing N by ℓ), with the notation $U = U_{\ell-1} = (u_1, \ldots, u_{\ell-1})$ and $x = u_\ell$ to simplify, we obtain

$$A = \int_{\mathbb{R}^{d\ell}} \left(dN - |U|^2 - \frac{N}{N-\ell}|x|^2 \right)_+^{\frac{d(N-\ell-1)-2}{2}} dU dx.$$

Changing U to spherical coordinates in dimension $d(\ell-1)$, we have

$$
\begin{aligned}
A &= \int_{\mathbb{R}^d}\int_0^\infty |\mathbb{S}^{d(\ell-1)-1}| \left(dN-\rho^2-\frac{N}{N-\ell}|x|^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} \rho^{d(\ell-1)-1}\,d\rho\,dx \\
&= |\mathbb{S}^{d(\ell-1)-1}| \int_0^\infty \left(\int_{\mathbb{R}^d} \left(dN-\rho^2-\frac{N}{N-\ell}|x|^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} dx\right) \rho^{d(\ell-1)-1}\,d\rho.
\end{aligned} \tag{18}
$$

Looking first to the integral over $\mathbb{R}^d$ we obtain, changing x to spherical coordinates in dimension d,

$$
\begin{aligned}
B &= \int_{\mathbb{R}^d} \left(dN-\rho^2-\frac{N}{N-\ell}|x|^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} dx \\
&= |\mathbb{S}^{d-1}| \int_0^\infty \left(dN-\rho^2-\frac{N}{N-\ell}y^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} y^{d-1}dy,
\end{aligned}
$$

and after some computations we get

$$
\begin{aligned}
B &= \frac{|\mathbb{S}^{d-1}|}{2}\left(\frac{N-\ell}{N}\right)^{d/2} (dN-\rho^2)_+^{\frac{d(N-\ell)-2}{2}} \int_0^1 (1-y)^{\frac{d(N-\ell-1)-2}{2}} y^{\frac{d-2}{2}}\,dy \\
&= \frac{|\mathbb{S}^{d-1}|}{2}\left(\frac{N-\ell}{N}\right)^{d/2} (dN-\rho^2)_+^{\frac{d(N-\ell)-2}{2}} \frac{\Gamma\left(\frac{d(N-\ell-1)-2}{2}+1\right)\Gamma\left(\frac{d-2}{2}+1\right)}{\Gamma\left(\frac{d(N-\ell-1)-2}{2}+\frac{d-2}{2}+2\right)}.
\end{aligned}
$$

Plugging this expression in (18) we get

$$
\begin{aligned}
A &= |\mathbb{S}^{d(\ell-1)-1}| \frac{|\mathbb{S}^{d-1}|}{2}\left(\frac{N-\ell}{N}\right)^{d/2} \frac{\Gamma\left(\frac{d(N-\ell-1)-2}{2}+1\right)\Gamma\left(\frac{d-2}{2}+1\right)}{\Gamma\left(\frac{d(N-\ell-1)-2}{2}+\frac{d-2}{2}+2\right)} \\
&\quad \times \int_0^\infty (dN-\rho^2)_+^{\frac{d(N-\ell)-2}{2}} \rho^{d(\ell-1)-1}\,d\rho,
\end{aligned}
$$

and we can compute the last integral

$$
\begin{aligned}
C &:= \int_0^\infty (dN-\rho^2)_+^{\frac{d(N-\ell)-2}{2}} \rho^{d(\ell-1)-1}\,d\rho \\
&= \frac{1}{2}(dN)^{\frac{d(N-1)-2}{2}} \frac{\Gamma\left(\frac{d(N-\ell)-2}{2}+1\right)\Gamma\left(\frac{d(\ell-1)-2}{2}+1\right)}{\Gamma\left(\frac{d(N-\ell)-2}{2}+\frac{d(\ell-1)-2}{2}+2\right)}.
\end{aligned}
$$

Finally, plugging this in (17), we obtain

$$
\begin{aligned}
Z^N &= \left|\mathbb{S}^{d(N-\ell-1)-1}\right| \left|\mathbb{S}^{d(\ell-1)-1}\right| \frac{\left|\mathbb{S}^{d-1}\right|}{2}\left(\frac{N-\ell}{N}\right)^{d/2} \frac{1}{2}(dN)^{\frac{d(N-1)-1}{2}} \\
&\quad \times \frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d(N-\ell)}{2}\right)} \frac{\Gamma\left(\frac{d(N-\ell)}{2}\right)\Gamma\left(\frac{d(\ell-1)}{2}\right)}{\Gamma\left(\frac{d(N-1)}{2}\right)}
\end{aligned}
$$

and using the fact that

$$|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \tag{19}$$

we have

$$Z^N = \left|\mathbb{S}^{d(N-1)-1}\right| \, (dN)^{\frac{d(N-1)-1}{2}} \, \left(\frac{N-\ell}{N}\right)^{d/2}, \tag{20}$$

then we conclude by plugging (20) in (16).

(ii) Let $k \geq 1$ be a even integer. We have then to compute $M_k(\gamma_\ell^N)$

$$\begin{aligned}\int_{\mathbb{R}^{d\ell}} |V_\ell|^k \, \gamma_\ell^N(V_\ell) \, dV_\ell = \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|} \, \frac{\left(\frac{N}{N-\ell}\right)^{\frac{d}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}} \\ \times \int_{\mathbb{R}^{d\ell}} |V_\ell|^k \, \left(dN - |V_\ell|^2 - \frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}} \, dV_\ell.\end{aligned} \tag{21}$$

As in the proof of (i), we use the change of coordinates (13)-(14), then to simplify we denote $U = U_{\ell-1} = (u_1, \ldots, u_{\ell-1})$ and $x = u_\ell$. Hence we can compute the integral

$$\begin{aligned}A_k &= \int_{\mathbb{R}^{d\ell}} |V_\ell|^k \, \left(dN - |V_\ell|^2 - \frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}} \, dV_\ell \\ &= \int_{\mathbb{R}^{d\ell}} \left(|U|^2 + |x|^2\right)^{\frac{k}{2}} \, \left(dN - |U|^2 - \frac{N}{N-\ell}|x|^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} \, dU \, dx.\end{aligned}$$

With another change of coordinates, U to spherical coordinates in dimension $d(\ell-1)$, x also to spherical coordinates in dimension d we have

$$\begin{aligned}A_k &= \left|\mathbb{S}^{d(\ell-1)-1}\right| \left|\mathbb{S}^{d-1}\right| \int_0^\infty \int_0^\infty \left(\rho^2 + y^2\right)^{\frac{k}{2}} \, \left(dN - \rho^2 - \frac{N}{N-\ell} y^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} \, \rho^{d(\ell-1)-1} y^{d-1} d\rho \, dy \\ &\leq C \left|\mathbb{S}^{d(\ell-1)-1}\right| \left|\mathbb{S}^{d-1}\right| \int_0^\infty \rho^k \left\{ \int_0^\infty \left(dN - \rho^2 - \frac{N}{N-\ell} y^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} \, y^{d-1} \, dy \right\} \rho^{d(\ell-1)-1} \, d\rho \\ &+ C \left|\mathbb{S}^{d(\ell-1)-1}\right| \left|\mathbb{S}^{d-1}\right| \int_0^\infty \left\{ \int_0^\infty y^k \left(dN - \rho^2 - \frac{N}{N-\ell} y^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} \, y^{d-1} \, dy \right\} \rho^{d(\ell-1)-1} \, d\rho \\ &=: I_1 + I_2.\end{aligned}$$

For the first term we have (already computed in (i))

$$\begin{aligned} I_1 &= \frac{1}{2}\left|\mathbb{S}^{d(\ell-1)-1}\right|\left|\mathbb{S}^{d-1}\right|\left(\frac{N-\ell}{N}\right)^{\frac{d}{2}}\frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d(N-\ell)}{2}\right)} \\ &\quad\times\int_0^\infty (dN-\rho^2)^{\frac{d(N-\ell)-2}{2}}\rho^{d(\ell-1)-1+k}\,d\rho \\ &= \frac{1}{2}\left|\mathbb{S}^{d(\ell-1)-1}\right|\left|\mathbb{S}^{d-1}\right|\left(\frac{N-\ell}{N}\right)^{\frac{d}{2}}\frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d(N-\ell)}{2}\right)} \\ &\quad\times\frac{1}{2}(dN)^{\frac{d(N-1)-2+k}{2}}\frac{\Gamma\left(\frac{d(N-\ell)}{2}\right)\Gamma\left(\frac{d(\ell-1)+k}{2}\right)}{\Gamma\left(\frac{d(N-1)+k}{2}\right)}. \end{aligned}$$

In the same way, we can compute the second term to get

$$\begin{aligned} I_2 &= \frac{1}{2}\left|\mathbb{S}^{d(\ell-1)-1}\right|\left|\mathbb{S}^{d-1}\right|\left(\frac{N-\ell}{N}\right)^{\frac{d+k}{2}}\frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d+k}{2}\right)}{\Gamma\left(\frac{d(N-\ell)+k}{2}\right)} \\ &\quad\times\int_0^\infty (dN-\rho^2)^{\frac{d(N-\ell)-2+k}{2}}\rho^{d(\ell-1)-1}\,d\rho \\ &= \frac{1}{2}\left|\mathbb{S}^{d(\ell-1)-1}\right|\left|\mathbb{S}^{d-1}\right|\left(\frac{N-\ell}{N}\right)^{\frac{d+k}{2}}\frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d+k}{2}\right)}{\Gamma\left(\frac{d(N-\ell)+k}{2}\right)} \\ &\quad\times\frac{1}{2}(dN)^{\frac{d(N-1)-2+k}{2}}\frac{\Gamma\left(\frac{d(N-\ell)+k}{2}\right)\Gamma\left(\frac{d(\ell-1)}{2}\right)}{\Gamma\left(\frac{d(N-1)+k}{2}\right)}. \end{aligned}$$

Plugging this two estimates in (21) we obtain after some simplifications

$$\begin{aligned} M_k(\gamma_\ell^N) &\le \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|}\frac{\left(\frac{N}{N-\ell}\right)^{\frac{d}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}\,(I_1+I_2) \\ &\le (dN)^{\frac{k}{2}}\frac{\Gamma\left(\frac{d(N-1)}{2}\right)}{\Gamma\left(\frac{d(N-1)+k}{2}\right)}\frac{\Gamma\left(\frac{d(\ell-1)+k}{2}\right)}{\Gamma\left(\frac{d(\ell-1)}{2}\right)} + (dN)^{\frac{k}{2}}\frac{\Gamma\left(\frac{d(N-1)}{2}\right)}{\Gamma\left(\frac{d(N-1)+k}{2}\right)}\frac{\Gamma\left(\frac{d+k}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \end{aligned}$$

Using the fact that for k even we have

$$\begin{aligned} \Gamma\left(\frac{n}{2}+\frac{k}{2}\right) &= \frac{(n+k-2)}{2}\,\frac{(n+k-4)}{2}\cdots\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right) \\ &= \frac{1}{2^{\frac{k}{2}}}\underbrace{(n+k-2)(n+k-4)\cdots n}_{k/2\text{ terms}}\,\Gamma\left(\frac{n}{2}\right), \end{aligned}$$

we conclude that

$$
\begin{aligned}
M_k(\gamma_\ell^N) &\leq \frac{(dN)^{\frac{k}{2}}}{[d(N-1)+k-2][d(N-1)+k-4]\cdots[d(N-1)]} \\
&\quad \times \Big([d(\ell-1)+k-2][d(\ell-1)+k-4]\cdots[d(\ell-1)] \\
&\qquad + (d+k-2)(d+k-4)\cdots d\Big) \\
&\leq \frac{(dN)^{\frac{k}{2}}}{[d(N-1)]^{\frac{k}{2}}}\Big([d(\ell-1)+k-2][d(\ell-1)+k-4]\cdots[d(\ell-1)] \\
&\qquad\qquad + (d+k-2)(d+k-4)\cdots d\Big) \qquad (22) \\
&\leq 2^{\frac{k}{2}}\Big([d(\ell-1)+k-2][d(\ell-1)+k-4]\cdots[d(\ell-1)] \\
&\qquad\qquad + (d+k-2)(d+k-4)\cdots d\Big) \\
&\leq C_{d,k,\ell},
\end{aligned}
$$

where $C_{d,k,\ell}$ depends only on d, k and ℓ.

We proved then a uniform bound in N for k even. If k is odd we use $|v|^k \leq |v|^{k-1} + |v|^{k+1}$ with the last estimate to conclude.

□

Now, using this explicit formula for γ_ℓ^N computed above, we prove that γ^N is γ-chaotic, where γ is the Gaussian probability measure in $\mathbb{R}^d$, i.e. $\gamma(v) = (2\pi)^{-d/2}\, e^{-|v|^2/2}$, for $v \in \mathbb{R}^d$. The proof presented here is an adaptation of [5], where it is proved that the uniform probability measure on the sphere $\mathbb{S}^{n-1}(\sqrt{n}) \subset \mathbb{R}^n$ is γ_1-chaotic, with $\gamma_1(x) = (2\pi)^{-1/2}\, e^{-x^2/2}$ the one-dimensional Gaussian measure.

Lemma 11. *The sequence of probability measure $\gamma^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ is γ-chaotic, more precisely, for any ℓ such that $d\ell \leq d(N-2)-3$ we have*

$$
\|\gamma_\ell^N - \gamma^{\otimes \ell}\|_{L^1} \leq 2\,\frac{d(\ell+2)+2}{dN - d(\ell+2) - 2}.
$$

Proof. Let ℓ be an even integer. Then we have

$$
\begin{aligned}
\frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|} &= \frac{1}{\pi^{\frac{d\ell}{2}}}\,\frac{\Gamma\left(\frac{d(N-1)}{2}\right)}{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)} \\
&= \frac{(dN)^{\frac{d\ell}{2}}}{(2\pi)^{\frac{d\ell}{2}}}\left(1-\frac{d+2}{dN}\right)\left(1-\frac{d+4}{dN}\right)\cdots\left(1-\frac{d(\ell+1)}{dN}\right).
\end{aligned}
$$

By the explicit formula of γ_ℓ^N in Lemma 10 we obtain

$$
\gamma_\ell^N = \frac{\left(\frac{N}{N-\ell}\right)^{\frac{d}{2}}}{(2\pi)^{\frac{d\ell}{2}}}\left(1-\frac{d+2}{dN}\right)\cdots\left(1-\frac{d(\ell+1)}{dN}\right)\left(1-\frac{|V_\ell|^2}{dN}-\frac{|\bar{V}_\ell|^2}{dN(N-\ell)}\right)_+^{\frac{d(N-\ell-1)-2}{2}}.
$$

Since γ_ℓ^N and $\gamma^{\otimes\ell}$ are probability densities, the L^1 norm of their difference can be computed in the following way

$$\|\gamma_\ell^N - \gamma^{\otimes\ell}\|_{L^1} = 2\int_{\mathbb{R}^{d\ell}} \left(\frac{\gamma_\ell^N}{\gamma^{\otimes\ell}} - 1\right)_+ \gamma^{\otimes\ell}\, dV_\ell, \tag{23}$$

and we shall denote

$$\frac{\gamma_\ell^N}{\gamma^{\otimes\ell}} = \left(\frac{N}{N-\ell}\right)^{\frac{d}{2}} h(V_\ell)\, A$$

with

$$h(V_\ell) := e^{\frac{|V_\ell|^2}{2}} \left(1 - \frac{|V_\ell|^2}{dN} - \frac{|\bar V_\ell|^2}{dN(N-\ell)}\right)_+^{\frac{d(N-\ell-1)-2}{2}}$$

and

$$A := \left(1 - \frac{d+2}{dN}\right)\cdots\left(1 - \frac{d(\ell+1)}{dN}\right).$$

We obtain that

$$\begin{aligned}\log h(V_\ell) &= \frac{|V_\ell|^2}{2} + \frac{d(N-\ell-1)-2}{2}\log\left(1 - \frac{|V_\ell|^2}{dN} - \frac{|\bar V_\ell|^2}{dN(N-\ell)}\right)\\ &\le \frac{|V_\ell|^2}{2} + \frac{d(N-\ell-1)-2}{2}\log\left(1 - \frac{|V_\ell|^2}{dN}\right),\end{aligned}$$

and since the function $\alpha(z) = z/2 + [(d(N-\ell-1)-2)/2]\log(1-z/dN)$ has a maximum for $z = d(\ell+1)+2$, we deduce

$$\log h(V_\ell) \le \frac{d(\ell+1)+2}{2} + \frac{d(N-\ell-1)-2}{2}\log\left(1 - \frac{d(\ell+1)+2}{dN}\right), \tag{24}$$

for $d\ell \le d(N-1)-3$.

On the other hand, for the quantity A, we have

$$\begin{aligned}\log\left[\left(1 - \frac{d(\ell+1)+2}{dN}\right)A\right] &= \sum_{j=1}^{(d(\ell+1)+2)/2} \log\left(1 - \frac{2j}{dN}\right)\\ &\le \int_0^{(d(\ell+1)+2)/2} \log\left(1 - \frac{2x}{dN}\right) dx\\ &= -\frac{d(N-\ell-1)-2}{2}\log\left(1 - \frac{d(\ell+1)+2}{dN}\right) - \frac{d(\ell+1)+2}{2},\end{aligned} \tag{25}$$

again for $d\ell \le d(N-1)-3$.

Combining (24) and (25) we obtain

$$\log\left[h(V_\ell)\left(1 - \frac{d(\ell+1)+2}{dN}\right)A\right] \le 0$$

and then

$$\left(1 - \frac{d(\ell+1)+2}{dN}\right)\frac{\gamma_\ell^N}{\gamma^{\otimes\ell}} \le \frac{(N-\ell)^{\frac{d}{2}}}{N^{\frac{d}{2}}},$$

which implies

$$\frac{\gamma_\ell^N}{\gamma^{\otimes\ell}} - 1 \leq \frac{d(\ell+1)+2}{dN - d(\ell+1) - 2}.$$

Plugging this expression in (23) we deduce

$$\|\gamma_\ell^N - \gamma^{\otimes\ell}\|_{L^1} \leq \frac{2d(\ell+1)+4}{dN - d(\ell+1) - 2},$$

which is valid if ℓ is even.

Finally, if ℓ is odd, then $\ell+1$ is even and we shall write

$$\|\gamma_\ell^N - \gamma^{\otimes\ell}\|_{L^1} \leq \|\gamma_{\ell+1}^N - \gamma^{\otimes\ell+1}\|_{L^1} \leq 2\,\frac{d(\ell+2)+2}{dN - d(\ell+2) - 2}$$

for $d\ell \leq d(N-2) - 3$, which concludes the proof.

□

3. Chaotic sequences in Kac's sense

In this section, inpired by the work [3], we shall construct a chaotic sequence of probability measure on the Boltzmann's sphere based on the tensorization of some suitable probabilty f on $\mathbb{R}^d$ and conditioning to $\mathcal{S}_{\mathcal{B}}^N$. We shall give a quantitative rate of the chaos convergence, proving a precise version of point (i) in Theorem 3.

First of all, we define

$$Z_N(f;r,z) = \int_{\mathcal{S}^N(r,z)} f^{\otimes N}\, d\gamma_{r,z}^N, \quad \text{and} \quad Z'_N(f;r,z) = \int_{\mathcal{S}^N(r,z)} \frac{f^{\otimes N}}{\gamma^{\otimes N}}\, d\gamma_{r,z}^N, \tag{26}$$

for $r \in \mathbb{R}_+$ and $z \in \mathbb{R}^d$, and we shall investigate their asymptotic behaviour. We remark that, since $\gamma^{\otimes N}$ is constant on $\mathcal{S}^N(r,z)$, we have

$$Z'_N(f;r,z) = \frac{Z_N(f;r,z)}{\gamma^{\otimes N}}$$

and we shall study in the sequel only the behaviour of $Z'_N(f;r,z)$.

Define the space $\mathbf{P}_k(\mathbb{R}^d) := \{f \in \mathbf{P}(\mathbb{R}^d);\ M_k(f) := \int |v|^k f\, dv < \infty\}$, for some $k \geq 1$. Let us consider $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, for some $p > 1$, a probability measure that verifies

$$\begin{aligned} &\int_{\mathbb{R}^d} v f(v)\, dv = 0\,, && \int_{\mathbb{R}^d} v \otimes v\, f(v)\, dv = \mathcal{E} I_d, \\ &\int_{\mathbb{R}^d} |v|^2\, f(v)\, dv = d\mathcal{E} = E\,, && \int_{\mathbb{R}^d} (|v|^2 - E)^2 f(v)\, dv = \Sigma^2, \end{aligned} \tag{27}$$

where I_d is the d-dimensional identity matrix.

3.1. Preliminary results.

Before study the asymptotic behaviour of Z'_N, we shall state some preliminary results that will be useful in the sequel.

Consider $(\mathcal{V}_j)_{j\in\mathbb{N}^*}$ a sequence of random variables i.i.d. in $\mathbb{R}^d$ with same law f, then the law of the couple $(\mathcal{V}_1, \mathcal{V}_1^2)$ is

$$h(v,u) = f(v)\, \delta_{u=|v|^2} \in \mathbf{P}(\mathbb{R}^d \times \mathbb{R}_+). \tag{28}$$

Moreover, we have the following lemma.

Lemma 12. *The random variable* $S_N := \sum_{j=1}^N (\mathcal{V}_j, |\mathcal{V}_j|^2)$ *has law* $s^N(z,u)\,dzdu$ *with*

$$s^N(z,u) := \frac{\left|\mathcal{S}^N(\sqrt{u},z)\right|}{2\left(u-\frac{|z|}{N^2}\right)^{1/2} N^{d/2}}\, Z_N(f;\sqrt{u},z),$$

where $z \in \mathbb{R}^d$ *and* $u \in \mathbb{R}_+$.

Proof. Let $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}_+)$, with the change of coordinates (13)-(14) $v \to u$, we have

$$\begin{aligned}\mathbb{E}\left[\varphi\left(\sum_{j=1}^N \mathcal{V}_j, \sum_{j=1}^N |\mathcal{V}_j|^2\right)\right] &= \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{j=1}^N v_j, \sum_{j=1}^N |v_j|^2\right) f^{\otimes N}\, dV \\ &= \int_{\mathbb{R}^{dN}} \varphi\left(\sqrt{N}u_N, \sum_{j=1}^N |u_j|^2\right) f^{\otimes N}\, dU.\end{aligned}$$

Denoting $r^2 = \sum_{j=1}^{N-1} |u_j|^2$ and splitting the integral, the last equation is equal to

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi(\sqrt{N}u_N, r^2+|u_N|^2)\left\{\left|\mathbb{S}^{d(N-1)-1}(r)\right| \int_{\mathbb{S}^{d(N-1)-1}(r)} f^{\otimes N}\, d\sigma_r^{d(N-1)-1}\right\} du_N\, dr$$

where σ_R^{n-1} is the uniform probability measure on $\mathbb{S}^{n-1}(R)$. Making the change of coordinates $w = r^2 + |u_N|^2$ and $z = \sqrt{N}u_N$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^d} \varphi(z,w)\left\{\frac{\left|\mathbb{S}^{d(N-1)-1}\left(\sqrt{w-\frac{|z|^2}{N}}\right)\right|}{2\left(w-\frac{|z|}{N^2}\right)^{1/2} N^{d/2}} \int_{\mathbb{S}^{d(N-1)-1}\left(\sqrt{w-\frac{|z|^2}{N}}\right)} f^{\otimes N}\, d\sigma^{d(N-1)-1}_{\sqrt{w-|z|^2/N}}\right\} dz\, dw$$

$$= \int_0^\infty \int_{\mathbb{R}^d} \varphi(z,w)\left\{\frac{\left|\mathcal{S}^N(\sqrt{w},z)\right|}{2\left(w-\frac{|z|}{N^2}\right)^{1/2} N^{d/2}}\, Z_N(f;\sqrt{w},z)\right\} dz\, dw,$$

from which we conclude. □

Since S_N is the summation of independent random variables, its law's density is also given by

$$s^N(z,u) = h^{*N}(z,u), \tag{29}$$

and we deduce from the lemma above

$$Z_N(f;\sqrt{u},z) = \frac{2\left(u-\frac{|z|}{N^2}\right)^{1/2} N^{d/2}\, h^{(*N)}(z,u)}{|\mathcal{S}^N(\sqrt{u},z)|}\,. \tag{30}$$

Lemma 13. *If* $f \in \mathbf{P}_{2k}(\mathbb{R}^d)$ *then* $h \in \mathbf{P}_k(\mathbb{R}^{d+1})$.

Proof. Let $y = (v,u) \in \mathbb{R}^{d+1}$ with $v \in \mathbb{R}^d$ and $u \in \mathbb{R}$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} |y|^k\, h(y)\, dy &= \int_{\mathbb{R}^{d+1}} \left(|v|^2+|u|^2\right)^{k/2} f(v)\delta_{u=|v|^2}\, dv\, du \\ &\le C_k\left(\int_{\mathbb{R}^d} |v|^k\, f(v)\, dv + \int_{\mathbb{R}^d} |v|^{2k}\, f(v)\, dz\right),\end{aligned}$$

from which we conclude. □

Lemma 14. *Suppose $f \in L^p(\mathbb{R}^d)$ for some $p > 1$. Then $h^{*2} \in L^q(\mathbb{R}^{d+1})$ if*

(i) for $d = 1$: $1 < q < p$ and $q < \frac{2p}{p+1}$
(ii) for $d = 2$: $q \le p$
(iii) for $d \ge 3$: if $f \in L_s(\mathbb{R}^d)$ $(s > 0)$, for $q < p$ and

$$q = \frac{(d-2)(p-1) + sp}{(d-2)(p-1) + s} > 1.$$

Proof. We compute first $h^{*2}(v, u)$ with $v, v' \in \mathbb{R}^d$ and $u, u' \in \mathbb{R}$.

$$\begin{aligned} h^{*2}(v,u) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}} h(v - v', u - u')\, h(v', u')\, du'dv' \\ &= \int_{\mathbb{R}^d} f(v - v')\, f(v') \left\{ \int_{\mathbb{R}} \delta_{u-u'=|v-v'|^2} \delta_{u'=|v'|^2} du' \right\} dv' \\ &= \int_{\mathbb{R}^d} f(v - v')\, f(v')\, \delta_{u=|v-v'|^2-|v'|^2}\, dv'. \end{aligned}$$

Moreover, we have

$$\delta_{u=|v-v'|^2-|v'|^2} = \delta_{u=2\left|\frac{v}{2}-v'\right|^2+\frac{|v|^2}{2}}.$$

Then we can compute the L^q norm of h^{*2},

(31)
$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}} \left|h^{*2}(v,u)\right|^q dv\, du \\ &\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}} \left| \int_{\mathbb{R}^d} f(v - v')\, f(v')\, \delta_{u=2\left|\frac{v}{2}-v'\right|^2+\frac{|v|^2}{2}}\, dv' \right|^q dv\, du \\ &\quad \le \int_{\mathbb{R}^d}\int_{\mathbb{R}} \left| \left(\int_{\mathbb{R}^d} \delta_{\left|\frac{v}{2}-v'\right|^2=\frac{u}{2}-\frac{|v|^2}{4}}\, dv' \right)^{(q-1)/q} \left(\int_{\mathbb{R}^d} f(v - v')^q\, f(v')^q\, \delta_{\left|\frac{v}{2}-v'\right|^2=\frac{u}{2}-\frac{|v|^2}{4}}\, dv' \right)^{1/q} \right|^q dv\, du. \end{aligned}$$

where we used Holder's inequality.

We look to the integral over δ, using $w = \frac{v}{2} - v'$

$$\int_{\mathbb{R}^d} \delta_{|w|^2=\frac{u}{2}-\frac{|v|^2}{4}}\, dw = |\mathbb{S}^{d-1}| \int_{\mathbb{R}} \delta_{r^2=\frac{u}{2}-\frac{|v|^2}{4}}\, r^{d-1}\, dr$$

where we changed to polar coordinates and then, with $z = r^2$

(32)
$$\begin{aligned} \int_{\mathbb{R}^d} \delta_{|w|^2=\frac{u}{2}-\frac{|v|^2}{4}}\, dw &= \frac{|\mathbb{S}^{d-1}|}{2} \int_{\mathbb{R}} \delta_{z=\frac{u}{2}-\frac{|v|^2}{4}}\, z^{(d-2)/2}\, dz \\ &= \frac{|\mathbb{S}^{d-1}|}{2} \left(\frac{u}{2} - \frac{|v|^2}{4} \right)^{(d-2)/2}. \end{aligned}$$

Therefore we obtain, plugging (32) in (31) and using Fubbini,

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}} \left|h^{*2}(v,u)\right|^q dv\, du \\ &\quad \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v - v')^q\, f(v')^q \left\{ \int_{\mathbb{R}} \left[\frac{|\mathbb{S}^{d-1}|}{2} \left(\frac{u}{2} - \frac{|v|^2}{4} \right)^{(d-2)/2} \right]^{q-1} \delta_{u=2\left|\frac{v}{2}-v'\right|^2+\frac{|v|^2}{2}}\, du \right\} dv\, dv' \\ &\quad = \frac{|\mathbb{S}^{d-1}|^{q-1}}{2^{q-1}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left| \frac{v}{2} - v' \right|^{(d-2)(q-1)} f(v - v')^q\, f(v')^q\, dv\, dv' \quad =: A \end{aligned}$$

Now we have the cases $d=1$, $d=2$ and $d\geq 3$:

(i) for $d=1$: Splitting the expression, we have

$$\begin{aligned} A &\leq \int_{\left|\frac{v}{2}-v'\right|\leq 1} \frac{f(v-v')^q\, f(v')^q}{\left|\frac{v}{2}-v'\right|^{q-1}}\, dv\, dv' + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v-v')^q\, f(v')^q dv\, dv' \\ &=: T_1 + T_2. \end{aligned}$$

For the last estimate we have $T_2 \leq \|f\|_{L^q}^{2q} \leq \|f\|_{L^p}^{2q}$ (because $q<p$ and f is a probability measure), and for the first term we use Holder's inequality

$$T_1 \leq \left(\int_{\left|\frac{v}{2}-v'\right|\leq 1} \frac{1}{\left|\frac{v}{2}-v'\right|^{(q-1)p/(p-q)}}\, dv\, dv'\right)^{(p-q)/p} \left(\int_{\left|\frac{v}{2}-v'\right|\leq 1} f(v-v')^p\, f(v')^p\, dv\, dv'\right)^{q/p}.$$

Then, the first integral converges if $(q-1)p/(p-q)<1$, which give us $T_1 \leq C\|f\|_{L^p}^{2q}$ if

$$q < \frac{2p}{p+1}.$$

(ii) for $d=2$: In this case we have

$$\begin{aligned} A &\leq \frac{|\mathbb{S}^1|^{q-1}}{2^{q-1}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v-v')^q\, f(v')^q dv\, dv' \\ &= \frac{|\mathbb{S}^1|^{q-1}}{2^{q-1}} \|f\|_{L^q}^{2q} \leq \frac{|\mathbb{S}^1|^{q-1}}{2^{q-1}} \|f\|_{L^p}^{2q}. \end{aligned}$$

(iii) for $d\geq 3$: We have, using $w=v-v'$ and $u=v'$

$$\begin{aligned} A &= \frac{|\mathbb{S}^{d-1}|^{q-1}}{2^{q-1}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\frac{v}{2}-v'\right|^{(d-2)(q-1)} f(v-v')^q\, f(v')^q dv\, dv' \\ &= \frac{|\mathbb{S}^{d-1}|^{q-1}}{2^{q-1}} \frac{1}{2^{(d-2)(q-1)}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |w-u|^{(d-2)(q-1)}\, f(w)^q\, f(u)^q dw\, du \\ &\leq \frac{|\mathbb{S}^{d-1}|^{q-1}}{2^{(d-1)(q-1)}} \left\{ 2C \left(\int_{\mathbb{R}^d} |w|^{(d-2)(q-1)}\, f(w)^q\, dw\right) \left(\int_{\mathbb{R}^d} f(u)^q\, du\right) \right\} \\ &\leq C\|f\|_{L^q}^q \|f\|_{L^q_m}^q \end{aligned}$$

where we have used $|w-u|^{(d-2)(q-1)} \leq C\left(|w|^{(d-2)(q-1)} + |u|^{(d-2)(q-1)}\right)$ and $m=(d-2)(q-1)$.

Finally, we have $\|f\|_{L^q}^q \leq \|f\|_{L^p}^q$ and with the hypothesis $f\in L^p\cap L_s$, we have $\|f\|_{L^q_m}<\infty$ for $m=s(p-q)/(p-1)$ and $q<p$ (see Lemma 31 in Appendix A.2), more precisely for

$$q = \frac{(d-2)(p-1)+sp}{(d-2)(p-1)+s} > 1.$$

□

3.2. **Asymptotic of Z'_N.** In this section we shall study the behaviour of Z'_N when N goes to infinity. First of all, let us state a version of the Central Limit Theorem, also known as Berry-Esseen type theorem, which is the main ingredient of the proof of the asymptotic of Z'_N in Theorem 17. The proof of the CLT presented here is a slightly adaptation of [7, Theorem 4.6] (see also [3, Theorem 27]).

Theorem 15 (Central Limit Theorem)**.** *Let $g \in \mathbf{P}_3(\mathbb{R}^D)$ such that, for some integer $k \geq 1$, we have $g^{*k} \in L^p(\mathbb{R}^D)$ for some $p > 1$. Moreover, assume that*

$$\int_{\mathbb{R}^D} x\, g(x)\, dx = 0, \qquad \int_{\mathbb{R}^D} (x \otimes x)\, g(x)\, dx = I_D, \qquad \int_{\mathbb{R}^D} |x|^3 g(x)\, dx \leq C_3. \tag{33}$$

Then there exists a constant $C = C(D, p, \|g^{*k}\|_{L^p}) > 0$ and $N(k, p)$ such that for all $N > N(k, p)$ we have

$$\|g_N - \gamma\|_{L^\infty} = \sup_{x \in \mathbb{R}^D} |g_N(x) - \gamma(x)| \leq \frac{C}{\sqrt{N}},$$

where $g_N(x) = N^{D/2} g^{*N}(\sqrt{N}x)$ is the normalized N-convolution power of g.

In the sequel we will need the following lemma, and we refer again to [3, Proposition 26] and [7, Lemma 4.8] for its proof.

Lemma 16. *(i) Consider $g \in \mathbf{P}_3(\mathbb{R}^D)$ satisfying (33). Then, there exists $\delta \in (0, 1)$ such that*

$$\forall\, \xi \in B(0, \delta) \qquad |\widehat{g}(\xi)| \leq e^{-|\xi|^2/4}.$$

(ii) Consider $g \in \mathbf{P}(\mathbb{R}^D) \cap L^p(\mathbb{R}^D)$ for $1 < p \leq \infty$. For any $\delta > 0$ there exists $\kappa(\delta) = \kappa(M_3(g), \|g\|_{L^p}, \delta) \in (0, 1)$ such that

$$\sup_{|\xi| \geq \delta} |\widehat{g}(\xi)| \leq \kappa(\delta).$$

Proof of Theorem 15. We remark that

$$\widehat{g}_N(\xi) = \widehat{g}\left(\frac{\xi}{\sqrt{N}}\right)^N, \qquad \widehat{\gamma}_N(\xi) = \widehat{\gamma}\left(\frac{\xi}{\sqrt{N}}\right)^N.$$

We have $g^{*k} \in L^1 \cap L^p$, for $p \in (1, \infty]$, and then by the Hausdorff-Young inequality we deduce that $\widehat{(g^{*k})} = (\widehat{g})^k$ lies in $L^{p'} \cap L^\infty$ with $p' \in (1, \infty]$. Furthermore, $\widehat{g}_N(\xi) \in L^1$ for any $N \geq kp'$. Hence we shall use the inverse Fourier transform to write

$$\begin{aligned} |g_N(x) - \gamma(x)| &= (2\pi)^D \left| \int_{\mathbb{R}^D} e^{i\xi \cdot x} \left(\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)\right) d\xi \right| \\ &\leq (2\pi)^D \int_{\mathbb{R}^D} |\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)|\, d\xi. \end{aligned} \tag{34}$$

Spliting the last integral in low and high frequencies, we obtain

$$\begin{aligned} \int_{\mathbb{R}^D} |\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)|\, d\xi &\leq \int_{|\xi| \geq \sqrt{N}\delta} |\widehat{g}_N(\xi)|\, d\xi + \int_{|\xi| \geq \sqrt{N}\delta} |\widehat{\gamma}(\xi)|\, d\xi \\ &\quad + \int_{|\xi| < \sqrt{N}\delta} |\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)|\, d\xi \\ &=: T_1 + T_2 + T_3, \end{aligned}$$

for some $\delta \in (0, 1)$.

For the first term, we write

$$\begin{aligned} T_1 &\leq \int_{|\xi|\geq\sqrt{N}\delta} \left|\widehat{g}\left(\frac{\xi}{\sqrt{N}}\right)\right|^N d\xi = N^{D/2}\int_{|\eta|\geq\delta} |\widehat{g}(\eta)|\, d\eta \\ &\leq N^{D/2}\left(\sup_{\eta\geq\delta}|\widehat{g}(\eta)^k|\right)^{N/k-p'}\int_{|\eta|\geq\delta}|\widehat{g}(\eta)^k|^{p'}\, d\eta \\ &\leq N^{D/2}\kappa(\delta)^{N/k-p'}C_{D,p}\|g^{*k}\|_{L^p}^{p'} \end{aligned}$$

where $\delta\in(0,1)$ is given by Lemma 16-(i) and $\kappa(\delta)$ is given by Lemma 16-(ii) applied to g^{*k} (because we have supposed only $g^{*k}\in L^p$). We get the same estimate for the second term, then we obtain that there exists a constant $C=C(D,p,\|g^{*k}\|_{L^p})$ such that

$$T_1+T_2\leq\frac{C}{\sqrt{N}}.$$

Finally, for the third term we have

$$T_3=\int_{|\xi|<\sqrt{N}\delta}\frac{|\widehat{g}_N(\xi)-\widehat{\gamma}(\xi)|}{|\xi|^3}|\xi|^3\, d\xi$$

and we can estimate

$$\begin{aligned} \frac{|\widehat{g}_N(\xi)-\widehat{\gamma}(\xi)|}{|\xi|^3} &= \frac{1}{N^{3/2}}\,\frac{|\widehat{g}(\xi/\sqrt{N})^N-\widehat{\gamma}(\xi/\sqrt{N})^N|}{|\xi/\sqrt{N}|^3} \\ &= \frac{1}{N^{3/2}}\,\frac{|\widehat{g}(\xi/\sqrt{N})^N-\widehat{\gamma}(\xi/\sqrt{N})^N|}{|\xi/\sqrt{N}|^3}\times\left|\sum_{k=0}^{N-1}\widehat{g}(\xi/\sqrt{N})^k\widehat{\gamma}(\xi/\sqrt{N})^(N-k-1)\right|. \end{aligned}$$

Moreover, point (i) in Lemma 16 implies

$$\left|\sum_{k=0}^{N-1}\widehat{g}(\xi/\sqrt{N})^k\widehat{\gamma}(\xi/\sqrt{N})^{(N-k-1)}\right|\leq\sum_{k=0}^{N-1}e^{-\frac{k|\xi|^2}{4N}}e^{-\frac{(N-k-1)|\xi|^2}{4N}}\leq Ne^{-\frac{|\xi|^2}{8}}.$$

Hence, we obtain

$$\begin{aligned} T_3 &\leq \frac{1}{N^{3/2}}\left(\sup_\eta\frac{|\widehat{g}(\eta)-\widehat{\gamma}(\eta)|}{|\eta|^3}\right)\int_{\mathbb{R}^D}Ne^{-\frac{|\xi|^2}{8}}|\xi|^3\, d\xi \\ &\leq \frac{1}{\sqrt{N}}\left(M_3(g)+M_3(\gamma)\right)C_D, \end{aligned}$$

and we finish the proof gathering the estimates of T_1, T_2 and T_3 togheter with (34). □

With these results we are able to state the following theorem about the asymptotic behaviour of Z'_N.

Theorem 17. *Consider $f\in\mathbf{P}_6(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ satisfying (27). Then we have*

$$\begin{aligned} Z'_N(f;r,z) = \frac{\sqrt{2d}}{\Sigma\mathcal{E}^{d/2}}\,\frac{(dN)^{\frac{d(N-1)-2}{2}}}{\left(r^2-\frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}}}\,\frac{e^{-\frac{dN}{2}}}{e^{-\frac{r^2}{2}}} \\ \times\left[\exp\left(-\frac{|z|^2}{2\mathcal{E}N}-\frac{(r^2-NE)^2}{2\Sigma^2N}\right)+O\left(1/\sqrt{N}\right)\right] \end{aligned}$$

and in the particular case $r^2 = dN$ and $z = 0$, we have

$$Z'_N(f;\sqrt{dN},0) = \frac{\sqrt{2d}}{\Sigma\mathcal{E}^{d/2}}\left[\exp\left(-\frac{N(d-E)^2}{2\Sigma^2}\right) + O\left(1/\sqrt{N}\right)\right].$$

Proof. Let us introduce

$$g(v,u) = \Sigma\,\mathcal{E}^{d/2}\,h(\mathcal{E}^{1/2}v, E+\Sigma u) \in \mathbf{P}(\mathbb{R}^{d+1}),$$

with $v \in \mathbb{R}^d$ and $u \in \mathbb{R}$. Since h lies in $\mathbf{P}_3(\mathbb{R}^{d+1})$ by Lemma 13 and $h^{*2} \in L^q(\mathbb{R}^{d+1})$ for some $q \in (1,p)$ thanks to Lemma 14, we have $g \in \mathbf{P}_3(\mathbb{R}^{d+1})$ and $g^{*2} \in L^q(\mathbb{R}^{d+1})$.

Moreover g verifies (by construction)

$$\int_{\mathbb{R}^{d+1}} y\,g(y)\,dy = 0\,, \quad \int_{\mathbb{R}^{d+1}} (y\otimes y)\,g(y)\,dy = I_{d+1},$$

where I_{d+1} is the identity matrix in dimension $d+1$.

We can now apply Theorem 15 to g, which implies that there exists $C > 0$ and N_0 such that for all $N > N_0$,

$$\sup_{(v,u)\in\mathbb{R}^d\times\mathbb{R}} |g_N(v,u) - \gamma(v,u)| \le \frac{C}{\sqrt{N}},$$

where $g_N(v,u) = N^{(d+1)/2} g^{*N}(\sqrt{N}v, \sqrt{N}u)$ is the normalized N-convolution power of g, with

$$g^{*N}(\sqrt{N}v,\sqrt{N}u) = \Sigma\,\mathcal{E}^{d/2}\,h^{*N}(\mathcal{E}^{1/2}\sqrt{N}v, NE+\Sigma\sqrt{N}u),$$

and

$$\gamma(v,u) = \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}\frac{e^{-u^2/2}}{(2\pi)^{1/2}}$$

is the Gaussian measure in dimension $d+1$ (recall that we have $v \in \mathbb{R}^d$ and $u \in \mathbb{R}$). It follows that

$$\sup_{(v,u)\in\mathbb{R}^d\times\mathbb{R}} \left| h^{*N}(v,u) - \frac{\Sigma^{-1}\mathcal{E}^{-d/2}}{N^{(d+1)/2}}\,\gamma\left(\mathcal{E}^{-1/2}N^{-1/2}v, \frac{u-NE}{\Sigma\sqrt{N}}\right)\right| \le \frac{C}{\sqrt{N}}\,\frac{\Sigma^{-1}\mathcal{E}^{-d/2}}{N^{(d+1)/2}}. \tag{35}$$

Gathering (35) and (30) we obtain

$$Z_N(f;r,z)$$
$$= \frac{2\,N^{d/2}\left(r^2 - \frac{|z|}{N^2}\right)^{1/2}}{|\mathcal{S}^N(r,z)|}\,\frac{\Sigma^{-1}\mathcal{E}^{-d/2}}{N^{(d+1)/2}}\,\frac{1}{(2\pi)^{(d+1)/2}}\left[\exp\left(-\frac{|z|^2}{2\mathcal{E}N} - \frac{(r^2-NE)^2}{2\Sigma^2 N}\right) + O\left(1/\sqrt{N}\right)\right].$$

Using (12) we have

$$Z_N(f;r,z) = \frac{2\,N^{d/2}\left(r^2 - \frac{|z|}{N^2}\right)^{1/2}}{\left|\mathbb{S}^{d(N-1)-1}\right|}\left(r^2 - \frac{|z|^2}{N}\right)_+^{-\frac{d(N-1)-1}{2}}\frac{\Sigma^{-1}\mathcal{E}^{-d/2}}{N^{(d+1)/2}}\,\frac{1}{(2\pi)^{(d+1)/2}}$$
$$\times\left[\exp\left(-\frac{|z|^2}{2\mathcal{E}N} - \frac{(r^2-NE)^2}{2\Sigma^2 N}\right) + O\left(1/\sqrt{N}\right)\right].$$

Thanks to the formula

$$|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and to Stirling's formula,

$$\Gamma\left(an+b\right)=\sqrt{2\pi}\left(an\right)^{\frac{an+b-1}{2}}e^{-an}\left(1+O(1/n)\right),$$

we have

$$\Gamma\left(\frac{d(N-1)}{2}\right)=\sqrt{2\pi}\left(dN\right)^{\frac{d(N-1)-1}{2}}2^{-\frac{d(N-1)-1}{2}}e^{-\frac{dN}{2}}\left(1+O(1/N)\right)$$

and then

$$Z_N(f;r,z)=\frac{\sqrt{2d}}{\Sigma\mathcal{E}^{d/2}}\left(\frac{e^{-\frac{r^2}{2}}}{(2\pi)^{\frac{dN}{2}}}\right)\frac{(dN)^{\frac{d(N-1)-2}{2}}}{\left(r^2-\frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}}}\frac{e^{-\frac{dN}{2}}}{e^{-\frac{r^2}{2}}}$$
$$\times\left[\exp\left(-\frac{|z|^2}{2\mathcal{E}N}-\frac{(r^2-NE)^2}{2\Sigma^2N}\right)+O\left(1/\sqrt{N}\right)\right],$$

which implies for the case $r^2=dN$ and $z=0$

$$Z'_N(f;\sqrt{dN},0)=\frac{\sqrt{2d}}{\Sigma\mathcal{E}^{d/2}}\left[\exp\left(-\frac{N(d-E)^2}{2\Sigma^2}\right)+O\left(1/\sqrt{N}\right)\right].$$

□

3.3. **Conditioned tensor product.** Consider now

$$F^N=[f^{\otimes N}]_{\mathcal{S}_B^N}=\frac{f^{\otimes N}}{Z_N(f;\sqrt{dN},0)}\,\gamma^N$$

the restriction of the N-fold tensor of f to the Boltzmann's sphere $\mathcal{S}_B^N$. Suppose also that $E=\int|v|^2f=d$, i.e. f has the same second order moment that γ.

We have then the following theorem, which is a precise version of point (i) in Theorem 3.

Theorem 18. *Consider $f\in\mathbf{P}_6(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$, with $p>1$. Then, the sequence of probability measure $F^N\in\mathbf{P}(\mathcal{S}_B^N)$ defined by $F^N=[f^{\otimes N}]_{\mathcal{S}_B^N}$ is f-chaotic.*

More precisely, for any fixed ℓ there exists a constant $C:=C(\ell)>0$ such that for $N\geq\ell+1$ we have

$$W_1(F_\ell^N,f^{\otimes\ell})\leq\|F_\ell^N-f^{\otimes\ell}\|_{L_1^1}\leq\frac{C}{\sqrt{N}}.$$

Proof. With the notation $V=(v_1,\ldots,v_N)\in\mathbb{R}^{dN}$, $V_\ell=(v_i)_{1\leq i\leq\ell}$, $V_{\ell,N}=(v_i)_{\ell+1\leq i\leq N}$ and $\bar{V}_\ell=\sum_{i=1}^\ell v_i$, we have from the definition of F^N

$$F^N(dV)=\frac{f^{\otimes N}(V)\gamma^N(dV)}{Z_N(f;\sqrt{dN},0)}$$
$$=\frac{f^{\otimes\ell}}{\gamma^{\otimes\ell}}(V_\ell)\,\frac{1}{Z'_N(f;\sqrt{dN},0)}\,\frac{f^{\otimes N-\ell}}{\gamma^{\otimes N-\ell}}(V_{\ell,N})\,\gamma^N(dV).$$

We recall that $\gamma^N=\gamma^N_{\sqrt{dN},0}$ and we have

$$\gamma^N_{\sqrt{dN},0}(dV)=\gamma^N_\ell(dV_\ell)\,\gamma^{N-\ell}_{\sqrt{dN-|V_\ell|^2},z}(dV_{\ell,N})$$

where $z = -\sum_{i=1}^{\ell} v_i = -\bar{V}_\ell$. We fix $\ell \geq 1$ and $N \geq \ell + 1$, then we have

$$\begin{aligned} F_\ell^N(V_\ell) &= \int_{\mathbb{R}^{d(N-\ell)}} F^N(V)\, dV_{\ell,N} \\ &= \frac{f^{\otimes \ell}}{\gamma^{\otimes \ell}}(V_\ell)\, \frac{\gamma_\ell^N(V_\ell)}{Z'_N(f;\sqrt{dN},0)} \int_{\mathcal{S}^{N-\ell}\left(\sqrt{dN-|V_\ell|^2},z\right)} \frac{f^{\otimes N-\ell}}{\gamma^{\otimes N-\ell}}(V_{\ell,N})\, \gamma^{N-\ell}_{\sqrt{dN-|V_\ell|^2},z}(dV_{\ell,N}) \\ &= \frac{f^{\otimes \ell}}{\gamma^{\otimes \ell}}(V_\ell)\, \frac{Z'_{N-\ell}\left(f;\sqrt{dN-|V_\ell|^2},-\bar{V}_\ell\right)}{Z'_N(f;\sqrt{dN},0)}\, \gamma_\ell^N(V_\ell). \end{aligned}$$

Let us first compute the ratio betwenn $Z'_{N-\ell}$ and Z'_N, by Theorem 17 we have

$$\begin{aligned} \frac{Z'_{N-\ell}\left(f;\sqrt{dN-|V_\ell|^2},-\bar{V}_\ell\right)}{Z'_N(f;\sqrt{dN},0)} &= \frac{(d(N-\ell))^{\frac{d(N-\ell-1)-2}{2}}}{\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)^{\frac{d(N-\ell-1)-2}{2}}}\, \frac{e^{-\frac{d(N-\ell)}{2}}}{e^{-\frac{(dN-|V_\ell|^2)}{2}}} \\ &\quad \times \left[\exp\left(-\frac{|\bar{V}_\ell|^2}{2\mathcal{E}(N-\ell)} - \frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right) + O\left(N^{-1/2}\right)\right]. \end{aligned}$$

Using the later expression with Lemma 10 one obtains

$$\begin{aligned} F_\ell^N(V_\ell) &= \frac{f^{\otimes \ell}}{\gamma^{\otimes \ell}}(V_\ell)\, \frac{(d(N-\ell))^{\frac{d(N-\ell-1)-2}{2}}}{\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)^{\frac{d(N-\ell-1)-2}{2}}}\, \frac{e^{\frac{d\ell}{2}}}{e^{\frac{|V_\ell|^2}{2}}} \\ &\quad \times \left[\exp\left(-\frac{|\bar{V}_\ell|^2}{2\mathcal{E}(N-\ell)} - \frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right) + O\left(N^{-1/2}\right)\right] \\ &\quad \times \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|}\, \frac{\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}\left(\frac{N-\ell}{N}\right)^{\frac{d}{2}}} \\ &= f^{\otimes \ell}\left[\exp\left(-\frac{|\bar{V}_\ell|^2}{2\mathcal{E}(N-\ell)} - \frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right) + O\left(N^{-1/2}\right)\right] \mathbf{1}_{dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}>0} \\ &\quad \times \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|}\, \frac{(d(N-\ell))^{\frac{d(N-\ell-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}\, \left(\frac{N}{N-\ell}\right)^{d/2} (2\pi e)^{\frac{d\ell}{2}}. \end{aligned}$$

Since

$$\left(\frac{N}{N-\ell}\right)^{d/2} = O(1),$$

we have

$$F_\ell^N(V_\ell) = f^{\otimes \ell}(V_\ell)\, \theta_1^N(V_\ell)\, \theta_2^N(V_\ell) \tag{36}$$

with

$$
\begin{aligned}
\theta_1^N &= \left[\exp\left(-\frac{|\bar V_\ell|^2}{2\mathcal{E}(N-\ell)} - \frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right) + O\left(N^{-1/2}\right)\right]\mathbf{1}_{dN-|V_\ell|^2-\frac{|\bar V_\ell|^2}{N-\ell}>0}, \\
\theta_2^N &= \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|}\,\frac{(d(N-\ell))^{\frac{d(N-\ell-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}\,(2\pi e)^{\frac{d\ell}{2}}.
\end{aligned}
\tag{37}
$$

Thanks to Stirling's formula again, we obtain

$$
\frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|} = \left(\frac{dN}{2\pi}\right)^{\frac{dl}{2}}\left(1+O(N^{-1})\right), \qquad \theta_2^N = 1+O(N^{-1}).
$$

Moreover we can easily see by (37) that $\|\theta_1^N\|_{L^\infty} \le C$ uniformly in N, and

$$
\begin{aligned}
|\theta_1^N(V_\ell)-1| &= |\theta_1^N(V_\ell)-1|\mathbf{1}_{|V_\ell|\le R} + |\theta_1^N(V_\ell)-1|\mathbf{1}_{|V_\ell|\ge R} \\
&\le \left|\left(\frac{|\bar V_\ell|^2}{2\mathcal{E}(N-\ell)} + \frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right) + O\left(1/\sqrt{N}\right)\right|\mathbf{1}_{|V_\ell|\le R} + C\frac{|V_\ell|^b}{R^b}\mathbf{1}_{|V_\ell|\ge R} \\
&\le C\left(\frac{R^2}{N}+\frac{R^4}{N}+O\left(1/\sqrt{N}\right)\right)\mathbf{1}_{|V_\ell|\le R} + C\frac{|V_\ell|^b}{R^b}\mathbf{1}_{|V_\ell|\ge R},
\end{aligned}
\tag{38}
$$

for some $R>0$ and $b\ge 0$.

Finally, choosing $R=N^{1/8}$ and $b=4$ one has

$$
\begin{aligned}
\|F_\ell^N - f^{\otimes\ell}\|_{L_1^1} &= \|(\theta_1^N\theta_2^N-1)f^{\otimes\ell}\|_{L_1^1} \\
&\le (\theta_2^N-1)\|\theta_1^N\, f^{\otimes\ell}\|_{L_1^1} + \|(\theta_1^N-1)f^{\otimes\ell}\|_{L_1^1} \\
&\le \frac{C}{N}\|f^{\otimes\ell}\|_{L_1^1} + \frac{C}{\sqrt{N}}\|f^{\otimes\ell}\|_{L_1^1} + \frac{C}{\sqrt{N}}\|f^{\otimes\ell}\|_{L_5^1} \\
&\le \frac{C\ell}{N}\|f\|_{L_1^1} + \frac{C\ell}{\sqrt{N}}\|f\|_{L_1^1} + \frac{C\ell}{\sqrt{N}}\|f\|_{L_5^1}.
\end{aligned}
$$

□

4. Entropic and Fisher's information chaos

We recall that in the Subsection 1.2 we defined the relative entropy and relative Fisher's information of a probability measure. Moreover, we defined strongers notions of chaos, namely the entropic chaos in Definition 2 and the Fisher's information chaos in Definition 4. We prove in this section precise versions of point (ii) in Theorem 3, Theorem 6 and Theorem 7.

4.1. **Relations between the different notions of chaos.**

First of all, we start with the following lemma and we refer to [3, 7, 10] and the references therein for a proof.

Lemma 19. *For all probabilities $\mu, \nu \in \mathbf{P}(Z)$ on a locally compact metric space, we have*

$$H(\mu|\nu) = \sup_{\varphi \in C_b(Z)} \left\{ \int_Z \varphi \, d\mu - \log\left(\int_Z e^{\varphi} \, d\nu \right) \right\}$$
$$= \sup_{\varphi \in C_b(Z),\, \int_Z e^{\varphi}\, d\nu = 1} \int_Z \varphi \, d\mu.$$

The following theorem is an adaptation of [3, Theorem 17], where the same result is proved for probability measures on the usual sphere in $\mathbb{R}^N$.

Theorem 20. *Consider $g \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, for some $p \in (1, \infty]$, where g satisfies*

$$\int |v|^2 \, g = d.$$

Consider G^N a probability measure on $\mathcal{S}_{\mathcal{B}}^N$ such that for some positive integer ℓ, we have $G_\ell^N \rightharpoonup \pi_\ell$ in $\mathbf{P}(\mathbb{R}^{d\ell})$ when N goes to infinity.

Then, we have

$$\frac{1}{\ell} H(\pi_\ell | g^{\otimes \ell}) \leq \liminf_{N \to \infty} \frac{1}{N} H\left(G^N | \left[g^{\otimes N} \right]_{\mathcal{S}_{\mathcal{B}}^N} \right).$$

Proof. Let fix a function $\varphi := \varphi(v_1, \ldots, v_\ell) \in C_b(\mathbb{R}^{d\ell})$ such that

$$\int_{\mathbb{R}^{d\ell}} e^{\varphi} \, g^{\otimes \ell} = 1, \qquad H(\pi_\ell | g^{\otimes \ell}) \leq \int_{\mathbb{R}^{d\ell}} \varphi \, d\pi_\ell + \varepsilon \tag{39}$$

for some $\varepsilon > 0$, which is possible thanks to Lemma 19. We introduce the function

$$\Phi(v_1, ..., v_N) := \varphi(v_1, \ldots, v_\ell) + \cdots + \varphi(v_{(m-1)\ell+1}, \ldots, v_{m\ell}),$$

where m is the integer part of N/ℓ, i.e. $N = m\ell + r$ with $0 \leq r \leq \ell - 1$. Thanks again to Lemma 19 we have

$$\frac{1}{N} H\left(G^N | \left[g^{\otimes N} \right]_{\mathcal{S}_{\mathcal{B}}^N} \right) \geq \frac{1}{N} \int_{\mathcal{S}_{\mathcal{B}}^N} \Phi \, G^N(dV) - \frac{1}{N} \log\left(\int_{\mathcal{S}_{\mathcal{B}}^N} e^{\Phi} \, d \left[g^{\otimes N} \right]_{\mathcal{S}_{\mathcal{B}}^N} \right).$$

For the first term of the right-hand side, using the symmetry of G^N and the convergence of its ℓ-marginal, we have

$$\frac{1}{N} \int_{\mathcal{S}_{\mathcal{B}}^N} \Phi \, G^N(dV) = \frac{m}{N} \int_{\mathbb{R}^{d\ell}} \varphi \, dG_\ell^N \xrightarrow[N \to \infty]{} \frac{1}{\ell} \int_{\mathbb{R}^{d\ell}} \varphi \, d\pi_\ell.$$

We note that the second term of the right-hand side can be written in the following way

$$\int_{\mathcal{S}_{\mathcal{B}}^N} e^{\Phi} \, d \left[g^{\otimes N} \right]_{\mathcal{S}_{\mathcal{B}}^N} = \frac{1}{Z'_N(g; \sqrt{dN}, 0)} \int_{\mathcal{S}_{\mathcal{B}}^N} e^{\Phi} \left(\frac{g}{\gamma} \right)^{\otimes N} d\gamma^N$$

since

$$\left[g^{\otimes N} \right]_{\mathcal{S}_{\mathcal{B}}^N} = \frac{g^{\otimes N}}{Z_N(g; \sqrt{dN}, 0)} \, \gamma^N.$$

Applying Theorem 17 and thanks to $\int |v|^2 \, g = d$ we get

$$Z'_N(g; \sqrt{dN}, 0) = \frac{\sqrt{2d}}{\Sigma(g)} \left(1 + O(1/\sqrt{N}) \right),$$

where $\Sigma(g)$ is given by (27) applied to g, and then

$$\lim_{N\to\infty}\left(\frac{1}{N}\,\log Z'_N(g;\sqrt{dN},0)\right)=0. \tag{40}$$

For the other term, denoting $u=(v_1,\dots,v_{m\ell})$, $w=(v_{m\ell+1},\dots,v_N)$ and $\bar w=v_{m\ell+1}+\cdots+v_N$, we write

$$\begin{aligned}&\int_{\mathcal{S}^N(\sqrt{dN},0)} e^{\Phi}\left(\frac{g}{\gamma}\right)^{\otimes N}\,d\gamma^N\\&=\int_{\mathbb{R}^{dr}}\frac{|\mathbb{S}^{d(N-r-1)-1}|}{|\mathbb{S}^{d(N-1)}|}\,\frac{\left(dN-|w|^2-\frac{|\bar w|^2}{N-r}\right)^{\frac{d(N-r-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}\,\left(\frac{N}{N-r}\right)^{\frac{d}{2}}\left(\frac{g}{\gamma}\right)^{\otimes r}\\&\quad\times\left\{\int_{\mathcal{S}^{\ell m}(\sqrt{dN-|w|^2},-\bar w)}\left(\frac{e^{\varphi}\,g^{\otimes\ell}}{\gamma^{\otimes\ell}}\right)^{\otimes m}d\gamma^N_{\sqrt{dN-|w|^2},-\bar w}\right\}dw\end{aligned}$$

where the integral in dw have to be taken over the region

$$\{w\in\mathbb{R}^{dr}\,|\,dN-|w|^2-|\bar w|^2/(\ell m)>0\}.$$

We recognize that the last integral is equal to $Z'_m\left(e^{\varphi}\,g^{\otimes\ell};\sqrt{dN-|w|^2},-\bar w\right)$ (where Z'_m is a multi-dimensional version of Z'_N, obtained replacing N by $m\ell$) and by Theorem 17 we have

$$\begin{aligned}&Z'_m\left(e^{\varphi}\,g^{\otimes\ell};\sqrt{dN-|w|^2},-\bar w\right)\\&\quad=O(1)\times\frac{(d\ell m)^{\frac{d(\ell m-1)-2}{2}}}{\left(dN-|w|^2-\frac{|\bar w|^2}{\ell m}\right)^{\frac{d(\ell m-1)-2}{2}}}\,\frac{e^{-\frac{d\ell m}{2}}}{e^{-\frac{(dN-|w|^2)}{2}}}\end{aligned}$$

and using (19), we get

$$\begin{aligned}\int_{\mathcal{S}^N(\sqrt{dN},0)} e^{\Phi}\left(\frac{g}{\gamma}\right)^{\otimes N}d\gamma^N&=C\int_{\mathbb{R}^{dr}}e^{-\frac{|w|^2}{2}}\left(\frac{g}{\gamma}\right)^{\otimes r}dw\\&=O(1)\times(2\pi)^{dr/2}\int_{\mathbb{R}^{dr}}g^{\otimes r}dw=O(1).\end{aligned}$$

With these estimate at hand, we can deduce

$$\liminf_{N\to\infty}\left(-\frac{1}{N}\,\log\int_{\mathcal{S}^N(\sqrt{dN},0)}e^{\Phi}\left(\frac{g}{\gamma}\right)^{\otimes N}d\gamma^N\right)\geq 0$$

and together with (40) we obtain

$$\liminf_{N\to\infty}\frac{1}{N}\,H(G^N|\left[g^{\otimes N}\right]_{\mathcal{S}^N_{\mathcal{B}}})\geq\frac{1}{\ell}\int_{\mathbb{R}^{d\ell}}\varphi\,d\pi_\ell\geq\frac{1}{\ell}\,H(\pi_\ell|g^{\otimes\ell})-\varepsilon.$$

Since ε is arbitrary, we can conclude letting $\varepsilon\to 0$.

□

Our aim now is to give an analogous result of Theorem 20 for the Fisher's information. However the strategy here is different, it is not based on the asymptotic behaviour of Z'_N like before, but on a geometric approach following [7], where this analogous result

is proved in the Kac's sphere setting. To this purpose, firstly we shall present some results to conclude with the Theorem 22.

Consider $W = (w_1, \dots, w_N) \in \mathbb{R}^{dN}$ and $V = (v_1, \dots, v_N) \in \mathcal{S}_{\mathcal{B}}^N$, where we recall that $v_i = (v_{i,\alpha})_{1 \le \alpha \le d}$, $w_i = (w_{i,\alpha})_{1 \le \alpha \le d} \in \mathbb{R}^d$ for all $1 \le i \le N$.

Let P_h be the projection on the hyperplane $\{X \in \mathbb{R}^{dN} \; ; \; \sum_{i=1}^N x_i = 0\}$, then it can be computed in the following way

$$P_h W = W - \sum_{\alpha=1}^{d} \left(W \cdot \frac{e_\alpha^N}{|e_\alpha^N|} \right) \frac{e_\alpha^N}{|e_\alpha^N|},$$

where $e_\alpha^N = (e_\alpha, \dots, e_\alpha) \in \mathbb{R}^{dN}$ with $e_\alpha = (\delta_{\alpha\beta})_{1 \le \beta \le d} \in \mathbb{R}^d$. Since $|e_\alpha^N| = \sqrt{N}$ we obtain

$$P_h W = W - \frac{1}{N} \sum_{\alpha=1}^{d} (W \cdot e_\alpha^N)\, e_\alpha^N. \tag{41}$$

Moreover, the projection P_s on the sphere $\{X \in \mathbb{R}^{dN} \; ; \; \sum_{i=1}^N |x_i|^2 = dN\}$ is given by

$$P_s W = \sqrt{dN}\, \frac{W}{|W|}. \tag{42}$$

Hence the projection $P_{\mathcal{S}}$ on the Boltzmann's sphere $\mathcal{S}_{\mathcal{B}}^N$ can be computed as the composition of the others, i.e. $P_{\mathcal{S}} = P_s \circ P_h$, more precisely

$$\begin{aligned} P_{\mathcal{S}} W &= (P_s \circ P_h) W \\ &= \sqrt{dN}\, \frac{P_h W}{|P_h W|} \\ &= \sqrt{dN}\, \frac{W - \frac{1}{N} \sum_{\alpha=1}^{d} (W \cdot e_\alpha^N)\, e_\alpha^N}{\left| W - \frac{1}{N} \sum_{\alpha=1}^{d} (W \cdot e_\alpha^N)\, e_\alpha^N \right|}, \end{aligned} \tag{43}$$

or in coordinates, for $1 \le j \le N$ and $1 \le \beta \le d$,

$$(P_{\mathcal{S}} W)_{j,\beta} = \frac{\sqrt{dN}}{\left| W - \frac{1}{N} \sum_{\alpha=1}^{d} (W \cdot e_\alpha^N)\, e_\alpha^N \right|} \left(w_{j,\beta} - \frac{1}{N} \sum_{k=1}^{N} w_{k,\beta} \right). \tag{44}$$

Consider $V \in \mathcal{S}_{\mathcal{B}}^N$ and a smooth function F defined on $\mathcal{S}_{\mathcal{B}}^N$. Then the gradient ∇_h on $\{X \in \mathbb{R}^{dN} \; ; \; \sum_{i=1}^N x_i = 0\}$ is (recall that ∇ stands for the usual gradient on $\mathbb{R}^{dN}$)

$$\nabla_h F(V) = \nabla F(V) - \frac{1}{N} \sum_{i=1}^{N} \sum_{\alpha=1}^{d} \partial_{v_{i,\alpha}} F(V)\, e_\alpha^N.$$

Moreover, the gradient ∇_s on the sphere $\{X \in \mathbb{R}^{dN} \; ; \; \sum_{i=1}^N |x_i|^2 = dN\}$ is given by

$$\nabla_s F(V) = \nabla F(V) - \left(\frac{V}{|V|} \cdot \nabla F(V) \right) \frac{V}{|V|}.$$

Combining them we can compute the gradient on $\mathcal{S}_{\mathcal{B}}^N$, which is given by

$$\begin{aligned}\nabla_{\mathcal{S}} F(V) &= \nabla_h F(V) - \left(\frac{V}{|V|}\cdot\nabla_h F(V)\right)\frac{V}{|V|}\\ &= \nabla F(V) - \frac{1}{N}\sum_{i=1}^N\sum_{\alpha=1}^d \partial_{v_{i,\alpha}} F(V)\, e_\alpha^N\\ &\quad - \left[V\cdot\nabla F(V) - \frac{1}{N}\sum_{i=1}^N\sum_{\alpha=1}^d \partial_{v_{i,\alpha}} F(V)\,(e_\alpha^N\cdot V)\right]\frac{V}{|V|^2}\\ &= \nabla F(V) - \frac{1}{N}\sum_{i=1}^N\sum_{\alpha=1}^d \partial_{v_{i,\alpha}} F(V)\, e_\alpha^N - [V\cdot\nabla F(V)]\,\frac{V}{|V|^2},\end{aligned} \tag{45}$$

since $e_\alpha^N\cdot V = \sum_{i=1}^N v_{i,\alpha} = 0$ because $V\in\mathcal{S}_{\mathcal{B}}^N$.

Let Φ be a smooth vector field on $\mathbb{R}^{dN}$, which written in composants is $\Phi(V) = (\Phi_1(V),\dots,\Phi_N(V))$ with $\Phi_i(V) = (\Phi_{i,1}(V),\dots,\Phi_{i,d}(V))$ for $1\le i\le N$. We denote by $\operatorname{div}_{\mathcal{S}}$ the divergence on $\mathcal{S}_{\mathcal{B}}^N$, then it can be computed in the following way

$$\operatorname{div}_{\mathcal{S}}\Phi(V) = \sum_{j=1}^N\sum_{\beta=1}^d \nabla_{\mathcal{S}}\Phi_{j,\beta}(V)\cdot e_{j,\beta},$$

where $e_{j,\beta} = (\delta_{jk}\delta_{\beta\gamma})_{(1\le k\le N)(1\le\gamma\le d)}\in\mathbb{R}^{dN}$. Using (45) and after some simplifications we obtain

$$\operatorname{div}_{\mathcal{S}}\Phi(V) = \operatorname{div}\Phi(V) - \frac{1}{N}\sum_{j=1}^N\sum_{\beta=1}^d\sum_{i=1}^N \partial_{v_{i,\beta}}\Phi_{j,\beta}(V) - \sum_{j=1}^N\sum_{\beta=1}^d V\cdot\nabla\Phi_{j,\beta}(V)\,\frac{v_{j,\beta}}{|V|^2}. \tag{46}$$

Lemma 21. *Consider a function F and a vector field Φ, smooth enough, defined on $\mathcal{S}_{\mathcal{B}}^N$. Then we have the following integration by parts formula on $\mathcal{S}_{\mathcal{B}}^N$*

$$\int_{\mathcal{S}_{\mathcal{B}}^N}\left\{\nabla_{\mathcal{S}}F(V)\cdot\Phi(V) + F(V)\operatorname{div}_{\mathcal{S}}\Phi(V) - \frac{d(N-1)-1}{dN}F(V)\Phi(V)\cdot V\right\}d\gamma^N(V) = 0.$$

Proof. The proof presented here is an adaptation of [7, Lemma 4.16]. Let χ be a smooth function with compact support on $\mathbb{R}_+$ and define for $V\in\mathbb{R}^{dN}$

$$\phi(V) := \chi(|P_hV|)\,(F\circ P_{\mathcal{S}})(V)\,(\Phi\circ P_{\mathcal{S}})(V).$$

We can compute $\operatorname{div}\phi(V)$ and after some simplifications using the formulæ for the projections (41) and (43), the gradient (45) and the divergence (46) on $\mathcal{S}_{\mathcal{B}}^N$ we get

$$\begin{aligned}\operatorname{div}\phi(V) &= \frac{\chi'(|P_hV|)}{\sqrt{dN}}\,F(P_{\mathcal{S}}V)\,P_{\mathcal{S}}V\cdot\Phi(P_{\mathcal{S}}V)\\ &\quad + \chi(|P_hV|)\,\nabla_{\mathcal{S}}F(P_{\mathcal{S}}V)\cdot\Phi(P_{\mathcal{S}}V)\,\frac{\sqrt{dN}}{|P_hV|}\\ &\quad + \chi(|P_hV|)\,F(P_{\mathcal{S}}V)\,\operatorname{div}_{\mathcal{S}}\Phi(P_{\mathcal{S}}V)\,\frac{\sqrt{dN}}{|P_hV|}.\end{aligned} \tag{47}$$

Integrating (47) we get

$$\int_{\mathbb{R}^{dN}} F(P_{\mathcal{S}}V)\, P_{\mathcal{S}}V \cdot \Phi(P_{\mathcal{S}}V)\, \frac{\chi'(|P_hV|)}{\sqrt{dN}}\, dV$$
$$+ \int_{\mathbb{R}^{dN}} \left[\nabla_{\mathcal{S}} F(P_{\mathcal{S}}V) \cdot \Phi(P_{\mathcal{S}}V) + F(P_{\mathcal{S}}V)\ \mathrm{div}_{\mathcal{S}}\, \Phi(P_{\mathcal{S}}V)\right] \chi(|P_hV|)\, \frac{\sqrt{dN}}{|P_hV|}\, dV = 0.$$

Using the change of coordinates $V = (v_1, \dots, v_N) \to U = (u_1, \dots, u_N)$ given by Lemma 9 and then the variables $w = \sum_{i=1}^N |u_i|^2$ and $z = \sqrt{N} u_N$, we obtain that the last expression is equal to

$$\int_0^\infty \int_{\mathbb{R}^d} \left\{ \frac{|\mathbb{S}^{d(N-1)-1}|}{2\, N^{d/2}} \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}} \int_{\mathcal{S}^N(w,z)} F(V)\, V \cdot \Phi(V)\, d\gamma^N_{w,z} \right\} \frac{\chi'\left(\sqrt{w - \frac{|z|^2}{N}}\right)}{\sqrt{dN}}\, dz\, dw$$
$$+ \int_0^\infty \int_{\mathbb{R}^d} \left\{ \frac{|\mathbb{S}^{d(N-1)-1}|}{2\, N^{d/2}} \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}} \right.$$
$$\left. \int_{\mathcal{S}^N(w,z)} \left[\nabla_{\mathcal{S}} F(P_{\mathcal{S}}V) \cdot \Phi(P_{\mathcal{S}}V) + F(P_{\mathcal{S}}V)\ \mathrm{div}_{\mathcal{S}}\, \Phi(P_{\mathcal{S}}V)\right] d\gamma^N_{w,z} \right\} \chi\left(\sqrt{w - \frac{|z|^2}{N}}\right) \frac{\sqrt{dN}}{\sqrt{w - \frac{|z|^2}{N}}}\, dz\, dw,$$

and then we get

$$\int_0^\infty \int_{\mathbb{R}^d} \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}} \frac{\chi'\left(\sqrt{w - \frac{|z|^2}{N}}\right)}{dN}\, dz\, dw \left(\int_{\mathcal{S}^N_{\mathcal{B}}} F(V)\, V \cdot \Phi(V)\, d\gamma^N\right)$$
$$+ \int_0^\infty \int_{\mathbb{R}^d} \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-3}{2}} \chi\left(\sqrt{w - \frac{|z|^2}{N}}\right)\, dz\, dw \left(\int_{\mathcal{S}^N_{\mathcal{B}}} \left[\nabla_{\mathcal{S}} F(V) \cdot \Phi(V) + F(V)\ \mathrm{div}_{\mathcal{S}}\, \Phi(V)\right] d\gamma^N\right)$$
$$= 0.$$

Since we have

$$\int \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}} \chi'\left(\sqrt{w - \frac{|z|^2}{N}}\right) =$$
$$- [d(N-1)-1] \int \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-3}{2}} \chi\left(\sqrt{w - \frac{|z|^2}{N}}\right),$$

we obtain the result

$$\int_{\mathcal{S}^N_{\mathcal{B}}} \left\{ \nabla_{\mathcal{S}} F(V) \cdot \Phi(V) + F(V)\, \mathrm{div}_{\mathcal{S}}\, \Phi(V) - \frac{d(N-1)-1}{dN}\, F(V) \Phi(V) \cdot V \right\} d\gamma^N(V) = 0.$$

□

With these results at hand we are able to state the following theorem, which is the Fisher's information version of Theorem 20 and the proof is an adaptation of [7, Theorem 4.15].

Theorem 22. *Consider G^N a probability measure on $\mathcal{S}_{\mathcal{B}}^N$ such that for some positive integer ℓ, we have $G_\ell^N \rightharpoonup \pi_\ell$ in $\mathbf{P}(\mathbb{R}^{d\ell})$ when N goes to infinity.*

Then, we have

$$\frac{1}{\ell}\, I(\pi_\ell|\gamma^{\otimes\ell}) \le \liminf_{N\to\infty} \frac{1}{N}\, I(G^N|\gamma^N).$$

Proof. Let us denote $G^N =: g^N\gamma^N$. Using [7] we have the following representation formula

$$\begin{aligned} I(G^N|\gamma^N) &= \int_{\mathcal{S}_{\mathcal{B}}^N} |\nabla_{\mathcal{S}} \log g^N|^2 g^N \, d\gamma^N \\ &= \sup_{\Phi\in C_b^1(\mathbb{R}^{dN};\mathbb{R}^{dN})} \int_{\mathcal{S}_{\mathcal{B}}^N} \left(\nabla_{\mathcal{S}} \log g^N \cdot \Phi - \frac{|\Phi|^2}{4} \right) g^N \, d\gamma^N \end{aligned}$$

and we obtain by Lemma 21

(48)

$$I(G^N|\gamma^N) = \sup_{\Phi\in C_b^1(\mathbb{R}^{dN};\mathbb{R}^{dN})} \int_{\mathcal{S}_{\mathcal{B}}^N} \left(\frac{d(N-1)-1}{dN}\, \Phi(V)\cdot V - \operatorname{div}_{\mathcal{S}} \Phi(V) - \frac{|\Phi(V)|^2}{4} \right) g^N \, d\gamma^N.$$

Furthermore for π_ℓ we have, also from [7],

$$I(\pi_\ell|\gamma^{\otimes\ell}) = \sup_{\varphi\in C_b^1(\mathbb{R}^{d\ell};\mathbb{R}^{d\ell})} \int_{\mathbb{R}^{d\ell}} \left(\varphi\cdot V_\ell - \operatorname{div}\varphi - \frac{|\varphi|^2}{4} \right) \pi_\ell.$$

Let us fix $\varepsilon > 0$ and choose φ such that

$$\frac{1}{\ell}\, I(\pi_\ell|\gamma^{\otimes\ell}) - \varepsilon \le \frac{1}{\ell} \int_{\mathbb{R}^{d\ell}} \left(\varphi\cdot V_\ell - \operatorname{div}\varphi - \frac{|\varphi|^2}{4} \right) \pi_\ell$$

Denote $N = q\ell + r$, $0 \le r < \ell$, and define $V_N = (V_{\ell,1}, \dots, V_{\ell,q}, V_r)$. Choosing $\Phi(V_N) := (\varphi(V_{\ell,1}), \dots, \varphi(V_{\ell,q}), 0) \in C_b^1(\mathbb{R}^{dN};\mathbb{R}^{dN})$ we obtain from (48) and the symmetry of G^N

$$\begin{aligned} \frac{1}{N}\, I(G^N|\gamma^N) &\ge \frac{1}{N} \int_{\mathcal{S}_{\mathcal{B}}^N} \left(\frac{d(N-1)-1}{dN}\, \Phi(V_N)\cdot V_N - \operatorname{div}_{\mathcal{S}} \Phi(V_N) - \frac{|\Phi(V_N)|^2}{4} \right) G^N(dV_N) \\ &\ge \frac{q}{N} \int_{\mathbb{R}^{d\ell}} \left(\frac{d(N-1)-1}{dN}\, \varphi(V_\ell)\cdot V_\ell - \operatorname{div}\varphi(V_\ell) - \frac{|\varphi(V_\ell)|^2}{4} \right) G_\ell^N(dV_\ell) + \frac{R(N)}{N}, \end{aligned}$$

with

$$R(N) = \int_{\mathbb{R}^{d\ell}} \sum_{k=1}^{\ell} \sum_{i=1}^{\ell} \sum_{\beta=1}^{d} \left(\frac{1}{N}\, \partial_{v_{i,\beta}} \varphi_{k,\beta} + \frac{1}{dN} (\partial_{v_{i,\beta}} \varphi_{k,\beta}) v_{i,\beta} v_{k,\beta} \right) G_\ell^N(dV_\ell).$$

The last expression is bounded if $\nabla\varphi$ decreases rapidly enough at infinity. Hence, passing to the limit we obtain

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N}\, I(G^N|\gamma^N) &\ge \frac{1}{\ell} \int_{\mathbb{R}^{d\ell}} \left(\varphi\cdot V_\ell - \operatorname{div}\varphi - \frac{|\varphi|^2}{4} \right) \pi_\ell \\ &\ge \frac{1}{\ell} I(\pi_\ell|\gamma^{\otimes\ell}) - \varepsilon, \end{aligned}$$

and we conclude letting $\varepsilon \to 0$. □

We can prove now precise versions of implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ of Theorem 6 as follows.

Theorem 23. *Consider $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ such that $G_1^N \rightharpoonup f$ in $\mathbf{P}(\mathbb{R}^d)$. We have the following properties:*

(i) If $H(f|\gamma) < \infty$ and

$$\frac{1}{N} H(G^N|\gamma^N) \xrightarrow[N\to\infty]{} H(f|\gamma),$$

then G^N is f-chaotic (in Kac's sense).

(ii) If $I(f|\gamma) < \infty$ and

$$\frac{1}{N} I(G^N|\gamma^N) \xrightarrow[N\to\infty]{} I(f|\gamma),$$

then G^N is f-chaotic (in Kac's sense).

Proof. Since $G_1^N \rightharpoonup f$ in $\mathbf{P}(\mathbb{R}^d)$, by [15] there exists a subsequence $G_\ell^{N'}$ and $\pi_\ell \in \mathbf{P}(\mathbb{R}^{d\ell})$ such that $G_\ell^{N'} \rightharpoonup \pi_\ell$ in $\mathbf{P}(\mathbb{R}^{d\ell})$, when N' goes to infinity, for all $\ell \geq 1$ (and in particular $\pi_1 = f$).

(i). By Theorem 20 we have

$$\frac{1}{\ell} H(\pi_\ell|\gamma^{\otimes\ell}) \leq \liminf_{N\to\infty} \frac{1}{N} H(G^N|\gamma^N) = H(f|\gamma).$$

Since we also have the reverse inequality by superadditivity of the entropy functional, we obtain

$$\begin{aligned} H(\pi_\ell|\gamma^{\otimes\ell}) - \ell H(f|\gamma) &= \int \pi_\ell \log \frac{\pi_\ell}{\gamma^{\otimes\ell}} - \ell \int f \log \frac{f}{\gamma} \\ &= \int \pi_\ell \log \frac{\pi_\ell}{\gamma^{\otimes\ell}} - \int \pi_\ell \log \frac{f^{\otimes\ell}}{\gamma^{\otimes\ell}} \\ &= \int f^{\otimes\ell} \left(\frac{\pi_\ell}{f^{\otimes\ell}} \log \frac{\pi_\ell}{f^{\otimes\ell}} - \frac{\pi_\ell}{f^{\otimes\ell}} + 1 \right) \\ &= 0, \end{aligned}$$

which implies $\pi_\ell = f^{\otimes\ell}$ a.e. on $\{f^{\otimes\ell} > 0\}$, since the function $z \mapsto z \log z - z + 1$ is equal to 0 in $z = 1$. Thanks to $\pi_\ell, f^{\otimes\ell} \in \mathbf{P}(\mathbb{R}^{d\ell})$, we obtain

$$\int_{\{f^{\otimes\ell}>0\}} \pi_\ell = \int_{\{f^{\otimes\ell}>0\}} f^{\otimes\ell} = 1.$$

It follows that $\pi_\ell = f^{\otimes\ell}$ a.e on $\mathbb{R}^{d\ell}$, so the whole sequence G_ℓ^N converges to $f^{\otimes\ell}$ and thus G^N is f-chaotic.

(ii). The proof of point (ii) being similar, thanks to Theorem 22 and the superadditivity of the Fisher's information [2], we skip it.

□

4.2. **Entropic chaos for the conditioned tensor product.** We shall study now the entropic chaoticity of the probability measure $F^N = [f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}$ with quantitative rate in the following theorem, which is a precise version of point (ii) of Theorem 3.

Theorem 24. *Let $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $p > 1$. Then, the sequence of probabilities $F^N := [f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N} \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ is entropically f-chaotic. More precisely, there exists $C > 0$ such that we have*

$$\left| \frac{1}{N} H(F^N|\gamma^N) - H(f|\gamma) \right| \leq \frac{C}{\sqrt{N}}.$$

Proof. We write

$$\begin{aligned} \frac{1}{N} H(F^N|\gamma^N) &= \frac{1}{N} \int_{\mathcal{S}_{\mathcal{B}}^N} \left(\log \frac{dF^N}{d\gamma^N} \right) dF^N \\ &= \frac{1}{N} \int_{\mathcal{S}_{\mathcal{B}}^N} \left(\log \frac{f^{\otimes N}}{Z'_N(f; \sqrt{dN}, 0)\, \gamma^{\otimes N}} \right) dF^N \\ &= \int_{\mathbb{R}^d} \left(\log \frac{f}{\gamma} \right) dF_1^N - \frac{1}{N} \log Z'_N(f; \sqrt{dN}, 0). \end{aligned}$$

Thanks to the assumptions on f, we can use Theorem 17 to obtain

$$\frac{1}{N} H(F^N|\gamma^N) = \int_{\mathbb{R}^d} \left(\log \frac{f}{\gamma} \right) dF_1^N + O(1/N).$$

Using (36)-(37) we have $F_1^N(v) = \theta_1^N(v)\, \theta_2^N(v)\, f(v)$ or more precisely

$$F_1^N(v) = f(v) \left(e^{-\frac{|v|^2}{2N} - \frac{|v|^4}{2N}} + O\left(1/\sqrt{N}\right) \right) (1 + O(1/N)) =: \theta^N(v)\, f(v),$$

and then

$$\frac{1}{N} H(F^N|\gamma^N) - H(f|\gamma) = \int_{\mathbb{R}^d} (\theta^N - 1)\, f \left(\log \frac{f}{\gamma} \right) + O(1/N). \tag{49}$$

We estime now the first term of the right-hand side, denoted by T,

$$\begin{aligned} |T| &\leq \int_{\mathbb{R}^d} |\theta^N - 1|\, f\, |\log \gamma|\, dv + \int_{\mathbb{R}^d} |\theta^N - 1|\, f\, |\log f|\, dv \\ &\leq \int_{\mathbb{R}^d} |\theta^N - 1|\, f\, C(1 + |v|^2)\, dv + \int_{\mathbb{R}^d} |\theta^N - 1|\, f |\log f|\, dv \\ &=: T_1 + T_2. \end{aligned}$$

We recall that (already computed in equation (38))

$$|\theta^N - 1| \leq C \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}} \right) \mathbf{1}_{|V_\ell| \leq R} + C \frac{|V_\ell|^k}{R^k} \mathbf{1}_{|V_\ell| \geq R}$$

for some $k \geq 0$ and $R > 0$. Then, for the first term we have

$$\begin{aligned}|T_1| &\leq \int_{B_R} |\theta^N - 1|\, f\, (1+|v|^2) + \int_{B_R^C} |\theta^N - 1|\, f\, (1+|v|^2) \\ &\leq \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}}\right) \|f\|_{L^1_2} + \frac{1}{R^k}\Big(M_k(f) + M_{k+2}(f)\Big) \\ &\leq \frac{C_f}{\sqrt{N}}\end{aligned}$$

where we have chosen $R = N^{1/8}$ and $k = 4$.

For the last term T_2, define $A > 1$ and $B_R = \{v \in \mathbb{R}^d;\ |v| \leq R\}$, then we have

$$\begin{aligned}|T_2| \leq &\int_{B_R} |\theta^N - 1|\, f\, |\log f| + \int_{B_R^C} |\theta^N - 1|\, f\, |\log f|\, \mathbf{1}_{f \geq A} \\ &+ \int_{B_R^C} |\theta^N - 1|\, f\, |\log f|\, \mathbf{1}_{1 \leq f \leq A} + \int_{B_R^C} |\theta^N - 1|\, f\, |\log f|\, \mathbf{1}_{e^{-|v|^2} \leq f \leq 1} \\ &+ \int_{B_R^C} |\theta^N - 1|\, f\, |\log f|\, \mathbf{1}_{0 \leq f \leq e^{-|v|^2}}.\end{aligned}$$

Now we compute each one of this five terms. First, we deduce that

$$|T_{2,1}| \leq \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}}\right) \int_{B_R} f\, |\log f| = \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}}\right) C_f.$$

For the second term, we use that $f\, |\log f| \leq f^{(1+p)/2} \leq f^p / A^{(p-1)/2}$ over $\{f \geq A, |V_\ell| \geq R\}$, and then

$$|T_{2,2}| \leq \frac{\|f\|^p_{L^p}}{A^{(p-1)/2}}.$$

Using $f\, |\log f| \leq f\, |\log A|$ over $\{1 \leq f \leq A, |V_\ell| \geq R\}$ for the third one, we obtain

$$|T_{2,3}| \leq \frac{\log A}{R^k}\, M_k(f).$$

Thanks to $f\, |\log f| \leq f|v|^2 \leq f|v|^{m+2}/R^m$ over $\{e^{-|v|^2} \leq f \leq 1, |V_\ell| \geq R\}$, we get

$$|T_{2,4}| \leq \frac{1}{R^m}\, M_{m+2}(f).$$

Finally, by $f\, |\log f| \leq 4\sqrt{f} \leq 4e^{-|v|^2/2}$ over $\{0 \leq f \leq e^{-|v|^2}, |V_\ell| \geq R\}$

$$|T_{2,4}| \leq C\, e^{-R}.$$

Putting togheter all this terms, we have

$$\begin{aligned}|T_2| &\leq \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}}\right) C_f + \frac{\|f\|^p_{L^p}}{A^{(p-1)/2}} + \frac{\log A}{R^k}\, M_k(f) + \frac{M_{m+2}(f)}{R^m} + C\, e^{-R} \\ &\leq \frac{C_f}{\sqrt{N}}\end{aligned}$$

choosing $A^{(p-1)/2} = R^k$, $R = N^{1/8}$, $k = 6$ and $m = 4$.

We have then $|T| \leq C\, N^{-1/2}$ and we conclude plugging it in (49).

□

4.3. **On a more general class of chaotic probabilities.** In the subsection before we have constructed a particular probability measure on $\mathcal{S}^N_{\mathcal{B}}$ that is entropically chaotic. Hence, a natural question is whether it is true for a more general class of probabilities on the Boltzmann's sphere. Theorem 28, which is a precise version of $(ii) \Rightarrow (iii)$ in Theorem 6, gives an answer with a quantitative rate.

First of all, let us present some results concerning different forms of measuring chaos that will be useful in te sequel.

Lemma 25. *Consider $f, g \in \mathbf{P}(\mathbb{R}^d)$ and $F^N, G^N \in \mathbf{P}(\mathbb{R}^{dN})$. Let us define $M_k(F,G) := M_k(F) + M_k(G)$.*

For any $k > 2$ we have

$$W_2(f,g) \leq 2^{\frac{3}{2}}\, M_k(f,g)^{\frac{1}{2(k-1)}}\, W_1(f,g)^{\frac{k-2}{2(k-1)}} \tag{50}$$

and

$$\frac{W_2(F^N,G^N)}{\sqrt{N}} \leq 2^{\frac{3}{2}} \left(\frac{M_k(F^N,G^N)}{N}\right)^{\frac{1}{2(k-1)}} \left(\frac{W_1(F^N,G^N)}{N}\right)^{\frac{k-2}{2(k-1)}}. \tag{51}$$

The proof of Lemma 25 come from [12, Lemma 4.1] for (50) and (51) is a simple generalization of (50) to the case of N variables.

We denote by $\overline{W}_1$ the MKW distance (7) defined with a bounded distance in $\mathbb{R}^d$, more precisely, for all $f, g \in \mathbf{P}_1(\mathbb{R}^d)$,

$$\overline{W}_1(f,g) = \inf_{\pi \in \Pi(f,g)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \min\{|x-y|, 1\}\, \pi(dx,dy).$$

Consider $G^N \in \mathbf{P}(\mathbb{R}^{dN})$ and $f \in \mathbf{P}(\mathbb{R}^d)$. We define then $\widehat{G}^N$, $\delta_f \in \mathbf{P}(\mathbf{P}(\mathbb{R}^d))$ by, for all $\Phi \in C_b(\mathbf{P}(\mathbb{R}^d))$,

$$\begin{aligned} &\int_{\mathbf{P}(\mathbb{R}^d)} \Phi(\rho)\, \widehat{G}^N(d\rho) = \int_{\mathbb{R}^{dN}} \Phi(\mu^N_V)\, G^N(dV), \qquad \mu^N_V = \frac{1}{N}\sum_{i=1}^N \delta_{v_i} \in \mathbf{P}(\mathbb{R}^d), \\ &\int_{\mathbf{P}(\mathbb{R}^d)} \Phi(\rho)\, \delta_f(d\rho) = \Phi(f). \end{aligned} \tag{52}$$

Furthermore, $\mathcal{W}$ stands for the Wasserstein distance on $\mathbf{P}(\mathbf{P}(\mathbb{R}^d))$. More precisely, for some distance D on $\mathbf{P}(\mathbb{R}^d)$ we define

$$\forall \mu, \nu \in \mathbf{P}(\mathbf{P}(\mathbb{R}^d)), \quad \mathcal{W}_D(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbf{P}(\mathbb{R}^d)\times\mathbf{P}(\mathbb{R}^d)} D(f,g)\, d\pi(f,g).$$

In the particular case of $\widehat{G}^N$ and δ_f we have $\Pi(\widehat{G}^N, \delta_f) = \{\widehat{G}^N \otimes \delta_f\}$ and then

$$\mathcal{W}_D\Big(\widehat{G}^N, \delta_f\Big) = \int_{\mathbb{R}^{dN}} D(\mu^N_V, f)\, G^N(dV). \tag{53}$$

We have the following result from [7].

Lemma 26. *Consider $f, g \in \mathbf{P}(\mathbb{R}^d)$ and $F^N, G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$. Let us define $M_k(F,G) := M_k(F) + M_k(G)$.*

For any $k > 2$ we have

$$W_2(f,g) \leq 2^{\frac{3}{2}}\, M_k(f,g)^{\frac{1}{k}}\, \overline{W}_1(f,g)^{\frac{1}{2}-\frac{1}{k}} \tag{54}$$

and

$$\frac{W_2(F^N,G^N)}{\sqrt{N}} \leq 2^{\frac{3}{2}} \left(\frac{M_k(F^N,G^N)}{N}\right)^{\frac{1}{k}} \left(\frac{\overline{W}_1(F^N,G^N)}{N}\right)^{\frac{1}{2}-\frac{1}{k}}. \tag{55}$$

For any $0 < \alpha_1 < 1/(d+1)$ *and* $k > d(\alpha_1^{-1} - d - 1)^{-1}$ *there exists a constant* $C := C(d,\alpha_1,k) > 0$ *such that*

$$\mathcal{W}_{\overline{W}_1}(\widehat{G}^N,\delta_f) \leq C M_k(G_1^N,f)^{1/k} \left(\overline{W}_1(G_2^N,f^{\otimes 2}) + \frac{1}{N}\right)^{\alpha_1}. \tag{56}$$

For any $0 < \alpha_2 < 1/d'$ *and* $k > d'(\alpha_2^{-1} - d')^{-1}$, *with* $d' := \max(d,2)$, *there exists a constant* $C := C(d,\alpha_2,k) > 0$ *such that*

$$\left|\overline{W}_1(G^N,f^{\otimes N}) - \mathcal{W}_{\overline{W}_1}(\widehat{G}^N,\delta_f)\right| \leq C\,\frac{M_k(f)^{1/k}}{N^{\alpha_2}}. \tag{57}$$

The equations (54) and (55) come from [7, Lemmas 2.1 and 2.2], and (56)-(57) are proved in [7, Theorem 1.2].

As a consequence of Lemma 26 we have the following result.

Lemma 27. *Consider* $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ *and* $f \in \mathbf{P}(\mathbb{R}^d)$ *such that* $M_k(G_1^N)$ *and* $M_k(f)$ *are finite. Let us denote* $\mathcal{M}_k := M_k(G_1^N) + M_k(f)$.

Then for any $0 < \alpha_1 < 1/(d+1)$ *and* $\alpha_1 < k(dk+d+k)^{-1}$, $0 < \alpha_2 < 1/d'$ *and* $\alpha_2 < k(d'k+d')^{-1}$, *with* $d' := \max(d,2)$, *there exists a constant* $C := C(d,k,\alpha_1,\alpha_2)$ *such that*

$$\frac{W_2(G^N,f^{\otimes N})}{\sqrt{N}} \leq C\mathcal{M}_k^{\frac{1}{k}} \left(\overline{W}_1(G_2^N,f^{\otimes 2})^{\alpha_1} + N^{-\alpha_1} + N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}}$$

Proof. First of all, we remark that $N^{-1}M_k(G^N)$ is equivalent to $M_k(G_1^N)$ since G^N is symmetric. Then, using Lemma 26 we have

$$\begin{aligned}
\frac{W_2(G^N,f^{\otimes N})}{\sqrt{N}} &\leq 2^{\frac{2}{3}}\mathcal{M}_k^{\frac{1}{k}} \left(\frac{\overline{W}_1(G^N,f^{\otimes N})}{N}\right)^{\frac{1}{2}-\frac{1}{k}} \\
&\leq 2^{\frac{2}{3}}\mathcal{M}_k^{\frac{1}{k}} \left(C\frac{M_k(f)^{\frac{1}{k}}}{N^{\alpha_2}} + \mathcal{W}_{\overline{W}_1}\left(\widehat{G}^N,\delta_f\right)\right)^{\frac{1}{2}-\frac{1}{k}} \\
&\leq 2^{\frac{2}{3}}C\mathcal{M}_k^{\frac{1}{k}} \left(N^{-\alpha_2} + \left(\overline{W}_1(G_2^N,f^{\otimes 2}) + N^{-1}\right)^{\alpha_1}\right)^{\frac{1}{2}-\frac{1}{k}}
\end{aligned}$$

where we have used successively (55), (57) and (56), with α_1 and α_2 defined as above. □

We can now state a precise version of $(ii) \Rightarrow (iii)$ in Theorem 6.

Theorem 28. *Consider* $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$. *Moreover we suppose that* G^N *is* f*-chaotic, for some* $f \in \mathbf{P}(\mathbb{R}^d)$, *and also that*

$$M_k(G_1^N) \leq C_1,\ k \geq 6, \qquad \frac{1}{N}\,H(G^N|\gamma^N) \leq C_2, \qquad \frac{1}{N}\,I(G^N|\gamma^N) \leq C_3.$$

Then G^N is entropically f-chaotic. More precisely, there exists $C = C(C_1, C_2, C_3) > 0$ and for any $\beta < (k-2)[4(dk+d+k)]^{-1}$ a constant $C' := C'(\beta)$ such that

$$\left| \frac{1}{N} H(G^N|\gamma^N) - H(f|\gamma) \right| \leq C \left(\frac{W_2(G^N, f^{\otimes N})}{\sqrt{N}} + C' N^{-\beta} \right).$$

Proof. First of all, thanks to Theorem 20 (with $g = \gamma$ and $\ell = 1$) we have

$$H(f|\gamma) \leq \liminf_{N\to\infty} \frac{1}{N} H(G^N|\gamma^N) \leq C_2.$$

Thanks to [1, Example 2] (see also [4] for related inequalities) we obtain

$$I(G_1^N|\gamma_1^N) \leq C \, \frac{I(G^N|\gamma^N)}{N}$$

and passing to the limit inferior we deduce

$$I(f|\gamma) \leq C,$$

which implies that $I(f) < \infty$. Indeed, $I(f|\gamma) = I(f) + M_2(f) - 2d$, from which we conclude.

Furthermore, since $I(f) \leq C$, f lies in $L^p(\mathbb{R}^d)$ for some $p > 1$ by Sobolev embeddings. Moreover $M_k(f) < \infty$ for some $k \geq 6$ since $M_k(G_1^N)$ is bounded and $G_1^N \rightharpoonup f$ weakly in $\mathbf{P}(\mathbb{R}^d)$. We have then all the conditions on f to construct $F^N = [f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}$ satisfying Theorems 18 and 24.

Let us denote

$$F^N = \frac{f^{\otimes N}}{Z_N(f; \sqrt{dN}, 0)} \, \gamma^N =: f^N \gamma^N$$

and we compute the relative Fisher's information with respect to γ^N

$$\frac{1}{N} I(F^N|\gamma^N) = \frac{1}{N} \int_{\mathcal{S}_{\mathcal{B}}^N} \frac{\left|\nabla_{\mathcal{S}} f^N\right|^2}{f^N} \, d\gamma^N$$

where we recall that $\nabla_{\mathcal{S}}$ is the tangent component to the sphere $\mathcal{S}_{\mathcal{B}}^N$ of the usual gradient ∇ in $\mathbb{R}^{dN}$. Since $|\nabla_{\mathcal{S}} f^N|^2 \leq |\nabla f^N|^2$, let us compute the usual gradient of f^N

$$\begin{aligned} \frac{\left|\nabla f^N\right|^2}{f^N} &= \sum_{i=1}^N \frac{\left|\nabla_{\mathbb{R}^d} f^N\right|^2}{f^N} \\ &= \frac{1}{Z_N(f; \sqrt{dN}, 0)} \sum_{i=1}^N \frac{|\nabla_i f_i|^2}{f_i} f_1 \cdots f_{i-1} \, f_{i+1} \cdots f_N \end{aligned}$$

where $f_i = f(v_i)$.

We can return to the Fisher's information to obtain

$$\begin{aligned}\frac{1}{N}\, I(F^N|\gamma^N) &\le \frac{1}{N}\int_{\mathcal{S}_{\mathcal{B}}^N} \frac{\left|\nabla f^N\right|^2}{f^N}\, d\gamma^N \\ &= \frac{1}{N}\int_{\mathcal{S}_{\mathcal{B}}^N} \frac{1}{Z_N(f;\sqrt{dN},0)} \sum_{i=1}^N \frac{|\nabla_i f_i|^2}{f_i} f_1\cdots f_{i-1}\, f_{i+1}\cdots f_N\, d\gamma^N \\ &= \int_{\mathbb{R}^d} \frac{|\nabla_{v_1} f_1|^2}{f_1}\, \frac{Z_{N-1}(f;\sqrt{dN-|v_1|^2},-v_1)}{Z_N(f;\sqrt{dN},0)}\, d\gamma_1^N.\end{aligned}$$

In the proof of Theorem 18 we computed the quantity

$$\frac{Z'_{N-1}(f;\sqrt{dN-|v_1|^2},-v_1)}{Z'_N(f;\sqrt{dN},0)}\,\gamma_1^N(v_1) = \theta_1^N(v_1)\gamma(v_1)$$

with $|\theta_1^N(v_1)| \le C'$. Now, we use the fact that

$$\frac{Z_{N-1}(f;\sqrt{dN-|v_1|^2},-v_1)}{Z_N(f;\sqrt{dN},0)} = \frac{1}{\gamma(v_1)}\,\frac{Z'_{N-1}(f;\sqrt{dN-|v_1|^2},-v_1)}{Z'_N(f;\sqrt{dN},0)}$$

to obtain

$$\frac{1}{N}\, I(F^N|\gamma^N) \le \int_{\mathbb{R}^d} \frac{|\nabla_{v_1} f_1|^2}{f_1}\,\theta_1^N(v_1)\, dv_1 \le C. \tag{58}$$

Since $\mathcal{S}_{\mathcal{B}}^N$ has positive Ricci curvature (because it has positive curvature), by [17, Theorem 30.22] and [9] the following HWI inequalities hold

$$\begin{aligned} H(F^N|\gamma^N) - H(G^N|\gamma^N) &\le \frac{\pi}{2}\sqrt{I(F^N|\gamma^N)}\, W_2(F^N,G^N), \\ H(G^N|\gamma^N) - H(F^N|\gamma^N) &\le \frac{\pi}{2}\sqrt{I(G^N|\gamma^N)}\, W_2(F^N,G^N).\end{aligned} \tag{59}$$

Remark 29. In the original HWI inequality, the 2-MKW distance is defined with the geodesic distance on $\mathcal{S}_{\mathcal{B}}^N$, however here we use on $\mathcal{S}_{\mathcal{B}}^N$ the Euclidean distance inherited from $\mathbb{R}^{dN}$. Fortunately, these distance are equivalent, hence the HWI inequality holds in our case adding a factor $\pi/2$ on the right-hand side of the inequality.

Multiplying both sides by $1/N$ we obtain

$$\begin{aligned} \frac{1}{N}\, H(F^N|\gamma^N) - \frac{1}{N}\, H(G^N|\gamma^N) &\le \frac{\pi}{2}\sqrt{\frac{I(F^N|\gamma^N)}{N}}\,\frac{W_2(F^N,G^N)}{\sqrt{N}}, \\ \frac{1}{N}\, H(G^N|\gamma^N) - \frac{1}{N}\, H(F^N|\gamma^N) &\le \frac{\pi}{2}\sqrt{\frac{I(G^N|\gamma^N)}{N}}\,\frac{W_2(F^N,G^N)}{\sqrt{N}}.\end{aligned}$$

Since $N^{-1}I(F^N|\gamma^N)$ and $N^{-1}I(G^N|\gamma^N)$ are bounded, we deduce

$$\left|\frac{1}{N}\, H(F^N|\gamma^N) - H(G^N|\gamma^N)\right| \le C\,\frac{W_2(F^N,G^N)}{\sqrt{N}}. \tag{60}$$

Finally, we write

$$\left|\frac{1}{N}H(G^N|\gamma^N) - H(f|\gamma)\right| \le \left|\frac{1}{N}H(G^N|\gamma^N) - \frac{1}{N}H(F^N|\gamma^N)\right| + \left|\frac{1}{N}H(F^N|\gamma^N) - H(f|\gamma)\right|$$

and thanks to the later estimate (60) with the triangle inequality for the first term of the right-hand side and Theorem 24 for the second one, we obtain

$$\left|\frac{1}{N}H(G^N|\gamma^N) - H(f|\gamma)\right| \le C\left(\frac{W_2(G^N, f^{\otimes N})}{\sqrt{N}} + \frac{W_2(F^N, f^{\otimes N})}{\sqrt{N}} + \frac{1}{\sqrt{N}}\right). \tag{61}$$

Now we have to estimate the second term of the right-hand side. Hence, thanks to Lemma 27 we have

$$\frac{W_2(F^N, f^{\otimes N})}{\sqrt{N}} \le C'\mathcal{M}_k^{\frac{1}{k}}\left(\overline{W}_1(F_2^N, f^{\otimes 2})^{\alpha_1} + N^{-\alpha_1} + N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}},$$

and from Theorem 18 we have $\overline{W}_1(F_2^N, f^{\otimes 2}) \le W_1(F_2^N, f^{\otimes 2}) \le CN^{-1/2}$, which yields

$$\begin{aligned}\frac{W_2(F^N, f^{\otimes N})}{\sqrt{N}} &\le C'\mathcal{M}_k^{\frac{1}{k}}\left(N^{-\alpha_1/2} + N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}} \\ &\le C'N^{-\frac{\alpha_1}{2}(\frac{1}{2}-\frac{1}{k})},\end{aligned}$$

with $\alpha_1 < k(dk + d + k)^{-1}$. We conclude putting this last estimate in (61). □

We give a possible answer to [3, Open problem 11] in the Boltzmann's sphere framework, which is a precise version of Theorem 7.

Theorem 30. *Consider $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ such that G^N is f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, and suppose that*

$$M_k(G_1^N) \le C,\ k > 2, \qquad \frac{1}{N}I(G^N|\gamma^N) \le C.$$

Suppose further that

$$f \in W^{1,\infty}(\mathbb{R}^d) \qquad \text{and} \qquad f(v_1) \ge \exp(-a|v_1|^2) \tag{62}$$

for some constant $a > 0$.

Then for any fixed ℓ, there exists a constant $C = C(d, \ell, \|f\|_{W^{1,\infty}}, M_k(G_1^N, f)) > 0$ such that for all $N \ge \ell + 1$ we have

$$H(G_\ell^N|f^{\otimes\ell}) \le C\,W_1(G_\ell^N, f^{\otimes\ell})^{\theta(\ell,d,k)},$$

where $\theta(\ell, d, k) = k[d(\ell + k + 2) + 2k + 4]^{-1}$. As a consequence, $H(G_\ell^N|f^{\otimes\ell}) \to 0$ when $N \to \infty$.

As remarked in [7], the assumptions (62) of Theorem 30 are natural since they are propagated in time for the Boltzmann model presented in Section 1. However, the conditioned tensor product assumption can be assumed at initial time for the Boltzmann model but it is not propagated along time.

Proof. We write

$$\begin{aligned} H(G_\ell^N|f^{\otimes\ell}) &= \Big[H(G_\ell^N|\gamma^{\otimes\ell}) - H(f^{\otimes\ell}|\gamma^{\otimes\ell})\Big] + \int (G_\ell^N - f^{\otimes\ell}) \log \gamma^{\otimes\ell} \\ &\quad + \int (f^{\otimes\ell} - G_\ell^N) \log f^{\otimes\ell} \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

Let us split the proof in several steps.

Step 1. For the first term we use the HWI inequality on $\mathbb{R}^{d\ell}$ [14],

$$T_1 = H(G_\ell^N|\gamma^{\otimes\ell}) - H(f^{\otimes\ell}|\gamma^{\otimes\ell}) \leq \sqrt{I(G_\ell^N|\gamma^{\otimes\ell})}\, W_2(G_\ell^N, f^{\otimes\ell}).$$

The Fisher's information $I(G_\ell^N|\gamma^{\otimes\ell})$ is bounded thanks to $N^{-1}I(G^N|\gamma^N) \leq C$. Moreover, we have thanks to Lemma 25 applied for $G_\ell^N, f^{\otimes\ell} \in \mathbf{P}(\mathbb{R}^{d\ell})$

$$W_2(G_\ell^N, f^{\otimes\ell}) \leq C\, M_k(G_\ell^N, f^{\otimes\ell})^{\frac{1}{2(k-1)}}\, W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k-2}{2(k-1)}},$$

where $M_k(G_\ell^N, f^{\otimes\ell}) := M_k(G_\ell^N) + M_k(f^{\otimes\ell})$. We conclude then

$$T_1 \leq C\, M_k(G_\ell^N, f^{\otimes\ell})^{\frac{1}{2(k-1)}}\, W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k-2}{2(k-1)}}. \tag{63}$$

Step 2. Let us denote by B_R the ball centered at origin with radius $R > 0$ on $\mathbb{R}^{d\ell}$, by B_R^c its complementary and let $v = (v_1, \dots, v_\ell) \in \mathbb{R}^{d\ell}$. Since $\log \gamma^{\otimes\ell} = -(d/2)\log 2\pi - |v|^2/2$, we can write

$$T_2 = \frac{1}{2}\int_{B_R} (f^{\otimes\ell} - G_\ell^N)|v|^2 + \frac{1}{2}\int_{B_R^c} (f^{\otimes\ell} - G_\ell^N)|v|^2.$$

The function $\phi(v) = |v|^2$ lies in $\mathrm{Lip}(B_R)$ with $\|\nabla\phi\|_{L^\infty(B_R)} = 2R$. We obtain

$$\begin{aligned} \int_{B_R} (f^{\otimes\ell} - G_\ell^N)|v|^2 &\leq 2R \sup_{\|\phi\|_{\mathrm{Lip}(B_R)}\leq 1} \left\{\int \phi (f^{\otimes\ell} - G_\ell^N)\right\} \\ &\leq 2R \sup_{\|\phi\|_{\mathrm{Lip}(\mathbb{R}^{d\ell})}\leq 1} \left\{\int \phi (f^{\otimes\ell} - G_\ell^N)\right\} \\ &= 2R\, W_1(G_\ell^N, f^{\otimes\ell}), \end{aligned} \tag{64}$$

where the last equality comes from the duality form for the W_1 distance (see for instance [17]). Next we write

$$\int_{B_R^c} (f^{\otimes\ell} - G_\ell^N)|v|^2 \leq \frac{1}{R^{k-2}}\int_{B_R^c} (f^{\otimes\ell} + G_\ell^N)|v|^k = \frac{M_k(G_\ell^N, f^{\otimes\ell})}{R^{k-2}}. \tag{65}$$

Choosing R such that (64) is equal to (65) we get

$$T_2 \leq 2^{\frac{k-2}{k-1}}\, M_k(G_\ell^N, f^{\otimes\ell})^{\frac{1}{k-1}}\, W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k-2}{k-1}}. \tag{66}$$

Step 3. Finally, let us investigate the third term T_3. We write

$$T_3 = \int_{B_R} (f^{\otimes \ell} - G_\ell^N) \log f^{\otimes \ell} + \int_{B_R^c} (f^{\otimes \ell} - G_\ell^N) \log f^{\otimes \ell}. \tag{67}$$

For the first integral in (67) we have, since $f \in W^{1,\infty}$ and $f^{\otimes \ell}(v) \geq e^{-a|v|^2}$,

$$\int_{B_R} (f^{\otimes \ell} - G_\ell^N) \log f^{\otimes \ell} \leq \Big(\ell \log \|f\|_{L^\infty(B_R)} + aR^2 \Big) \|f^{\otimes \ell} - G_\ell^N\|_{L^1(B_R)}.$$

Let $g = f^{\otimes \ell} - G_\ell^N$, $\rho \in C^\infty(\mathbb{R}^{d\ell})$ be a mollifier and $\rho_\varepsilon(v) = \varepsilon^{-d} \rho(\varepsilon^{-1} v)$, then we have

$$\|g\|_{L^1(B_R)} \leq \|g * \rho_\varepsilon\|_{L^1(B_R)} + \|g * \rho_\varepsilon - g\|_{L^1(B_R)}.$$

The first term

$$\begin{aligned} \|g * \rho_\varepsilon\|_{L^1(B_R)} &= \int_{B_R} \left\{ \int |\rho_\varepsilon(w - v)| \, |f^{\otimes \ell}(v) - G_\ell^N(v)| \, dv \right\} dw \\ &\leq \|\nabla \rho_\varepsilon\|_{L^\infty(B_R)} \, W_1(G_\ell^N, f^{\otimes \ell}) \int_{B_R} dw \\ &\leq \frac{C}{\varepsilon^{d+1}} \, R^{d\ell} \, W_1(G_\ell^N, f^{\otimes \ell}). \end{aligned}$$

The second term

$$\|g * \rho_\varepsilon - g\|_{L^1(B_R)} \leq \varepsilon \|\nabla g\|_{L^1},$$

and $\|\nabla g\|_{L^1}$ is finite since $N^{-1} I(G^N | \gamma^N)$ is finite and then also is $I(f|\gamma)$ (see proof of Theorem 28). We have then

$$\begin{aligned} \|g\|_{L^1(B_R)} &\leq \frac{C}{\varepsilon^{d+1}} \, R^{d\ell} \, W_1(G_\ell^N, f^{\otimes \ell}) + C\varepsilon \\ &\leq C \, R^{\frac{d\ell}{d+2}} \, W_1(G_\ell^N, f^{\otimes \ell})^{\frac{1}{d+2}}, \end{aligned}$$

where we have optimized ε. For the second integral in (67) we have

$$\int_{B_R^c} (f^{\otimes \ell} - G_\ell^N) \log f^{\otimes \ell} \leq \ell \log \|f\|_{L^\infty} \, \frac{M_k(G_\ell^N, f^{\otimes \ell})}{R^k}.$$

We conclude then, optimizing in R,

$$\begin{aligned} T_3 &\leq C \Big(\ell \log \|f\|_{L^\infty(B_R)} + aR^2 \Big) \, R^{\frac{d\ell}{d+2}} \, W_1(G_\ell^N, f^{\otimes \ell})^{\frac{1}{d+2}} + \ell \log \|f\|_{L^\infty} \, \frac{M_k(G_\ell^N, f^{\otimes \ell})}{R^k} \\ &\leq C \, W_1(G_\ell^N, f^{\otimes \ell})^{\frac{k}{d(\ell+k+2)+2k+4}}. \end{aligned} \tag{68}$$

Finally, gathering (63), (66) and (68), we obtain

$$\begin{aligned} H(G_\ell^N | f^{\otimes \ell}) &\leq C \left(W_1(G_\ell^N, f^{\otimes \ell})^{\frac{k-2}{2(k-1)}} + W_1(G_\ell^N, f^{\otimes \ell})^{\frac{k-2}{k-1}} + W_1(G_\ell^N, f^{\otimes \ell})^{\frac{k}{d(\ell+k+2)+2k+4}} \right) \\ &\leq C \, W_1(G_\ell^N, f^{\otimes \ell})^{\frac{k}{d(\ell+k+2)+2k+4}}, \end{aligned}$$

where $C = C(d, \ell, \|f\|_{W^{1,\infty}}, M_k(G_1^N, f))$.

□

5. Application to the Boltzmann equation

We can apply our results to the spatially homogeneous Boltzmann equation (equations (5) and (4) in Section 1) with Maxwellian molecules (see [16, 12]).

We prove now Theorem 8.

Proof of Theorem 8. We split the proof in several steps.

Step 1. Let G_0^N be built as in Theorem 18, i.e. $G_0^N = [f_0^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}$, which is possible since $f_0 \in \mathbf{P}_6(\mathbb{R}^d)$ and $I(f_0|\gamma)$ is finite. We know from [12, Lemma 7.13] that for all $t \geq 0$ the normalized Fisher's information $N^{-1}I(G_t^N|\gamma^N)$ is bounded since $N^{-1}I(G_t^N|\gamma^N) \leq N^{-1}I(G_0^N|\gamma^N)$ and the later one is bounded by construction (see equation (58)). Since the Boltzmann equation propagates moments and the Fisher's information's bound, we have, for all $t > 0$, $M_6(f_t)$ and $I(f_t|\gamma)$ bounded.

We can then apply Theorem 28 to G_t^N (taking $G^N = G_t^N$ and $f = f_t$ in the notation of that theorem) and we obtain that for any $\beta < (k-2)[4(dk+d+k)]^{-1}$ there exists $C' = C'(\beta)$ such that

$$\left| \frac{1}{N} H(G_t^N|\gamma^N) - H(f_t|\gamma) \right| \leq CC' \left(\frac{W_2(G_t^N, f_t^{\otimes N})}{\sqrt{N}} + N^{-\beta} \right). \tag{69}$$

We have then to estimate the first term of the right-hand side and we shall use the result of propagation of chaos proved in [12].

Step 2. Thanks to the result of propagation of chaos in [12, Theorems 5.1 and 5.3] we have, for $s > 2 + d/4$,

$$\sup_{t \geq 0} \left\| \Pi_2(G_t^N) - f_t^{\otimes 2} \right\|_{H^{-s}} \leq C\, \mathcal{W}_{W_2}\left(\widehat{G}_0^N, \delta_{f_0} \right) \tag{70}$$

where we recall that $\widehat{G}_0^N$, $\delta_{f_0} \in \mathbf{P}(\mathbf{P}(\mathbb{R}^d))$ are defined in (52) and $\mathcal{W}_{W_2}(\widehat{G}_0^N, \delta_{f_0})$ in (53), more precisely

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N, \delta_{f_0} \right) = \int_{\mathbb{R}^{dN}} W_2(\mu_V^N, f_0)\, G_0^N(dV).$$

We recall that we want to estimate the first term of the right-hand side of (69) and we shall explain how we can obtain it from (70). On the one hand, for the right-hand side of (70) we shall obtain a estimate of the type

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N, \delta_{f_0} \right) \leq C \left[W_1(\Pi_2(G_0^N), f_0^{\otimes 2}) + N^{-\theta_2} \right]^{\theta_1}$$

since we can estimate $W_1(\Pi_2(G_0^N), f_0^{\otimes 2})$ from Theorem 18. On the other hand, for the left-hand side of (70), we shall deduce an estimate like

$$\frac{1}{\sqrt{N}} W_2(G_t^N, f_t^{\otimes N}) \leq C \left\| \Pi_2(G_t^N) - f_t^{\otimes 2} \right\|_{H^{-s}}^{\theta_3}$$

to be able to conclude.

Step 3. First of all, we deduce from (54) in Lemma 26,

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N,\delta_{f_0}\right)\leq 2^{\frac{2}{3}}\mathcal{M}_k^{\frac{1}{k}}\mathcal{W}_{\overline{W}_1}\left(\widehat{G}_0^N,\delta_{f_0}\right)^{\frac{1}{2}-\frac{1}{k}}.$$

Then, thanks to (56) in Lemma 26 we obtain

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N,\delta_{f_0}\right)\leq 2^{\frac{2}{3}}\mathcal{M}_k^{\frac{1}{k}}\left(C_{\alpha_1}\mathcal{M}_k^{\frac{1}{k}}\left(\overline{W}_1(\Pi_2(G_0^N),f_0^{\otimes 2})+N^{-1}\right)^{\alpha_1}\right)^{\frac{1}{2}-\frac{1}{k}},$$

and using Theorem 18, which tell us $\overline{W}_1(\Pi_2(G_0^N),f_0^{\otimes 2})\leq CN^{-1/2}$, we deduce

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N,\delta_{f_0}\right)\leq C_{\alpha_1}N^{-\frac{\alpha_1}{2}(\frac{1}{2}-\frac{1}{k})}, \tag{71}$$

where we recall that $\alpha_1<k(dk+d+k)^{-1}$.

Step 3. Thanks to [7, Lemma 2.1] applied to $\Pi_2(G_t^N)$ and $f_t^{\otimes 2}\in\mathbf{P}(\mathbb{R}^{2d})$, for any $s>d/2$ (with $d\geq 2$) there exists $C:=C(d,s)$ such that

$$\overline{W}_1(\Pi_2(G_t^N),f_t^{\otimes 2})\leq CM_k(\Pi_2(G_t^N),f_t^{\otimes 2})^{\frac{2d}{2d+2ks}}\left\|\Pi_2(G_t^N)-f_t^{\otimes 2}\right\|_{H^{-s}}^{\frac{2k}{2d+2ks}}.$$

Furthermore, from Lemma 27 we obtain that there exists a constant $C:=C(d,k,\alpha_1,\alpha_2)$ such that

$$\frac{W_2(G_t^N,f_t^{\otimes N})}{\sqrt{N}}\leq C\mathcal{M}_k^{\frac{1}{k}}\left(\overline{W}_1(\Pi_2(G_t^N),f_t^{\otimes 2})^{\alpha_1}+N^{-\alpha_1}+N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}}.$$

Finally, gathering these two estimates with (70) and (71) we obtain that there exists $C:=C(d,s,\alpha_1,\alpha_2,M_k(f_0),M_k(\Pi_1(G_0^N)))$ such that

$$\begin{aligned}\frac{W_2(G_t^N,f_t^{\otimes N})}{\sqrt{N}}&\leq C\left(N^{-\alpha_1^2(\frac{k}{d+ks})(\frac{1}{2}-\frac{1}{k})}+N^{-\alpha_1}+N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}}\\&\leq CN^{-\epsilon},\end{aligned} \tag{72}$$

where

$$\begin{aligned}\epsilon&=\alpha_1^2\left(\frac{k}{d+ks}\right)\left(\frac{1}{2}-\frac{1}{k}\right)^2\\&<\left(\frac{k-2}{2(dk+d+k)}\right)^2\frac{k}{d+ks}\\&<\left(\frac{k-2}{2(dk+d+k)}\right)^2\frac{4k}{dk+4d+8k}\end{aligned}$$

using $\alpha_1<k(dk+d+k)^{-1}$ and $s>2+d/4$ from (70). We conclude taking $k=6$ and gathering (72) with (69).

□

Appendix A. Auxiliary results

We prove here some auxiliary results used in Section 2 and Section 3.

A.1. **Change of variables.** We present the proof of Lemma 9 in Section 2.

Proof of Lemma 9. Thanks to (13) we have

$$|u_N|^2 = \frac{1}{N}\left(\sum_{i=1}^{N}|v_i|^2 + 2\sum_{i=1}^{N-1}\sum_{j>i}^{N} v_i\cdot v_j\right)$$

and, for $1 \le k \le N-1$,

$$|u_k|^2 = \frac{1}{k(k+1)}\left(\sum_{i=1}^{k}|v_i|^2 + 2\sum_{i=1}^{k-1}\sum_{j>i}^{k} v_i\cdot v_j + k^2|v_{k+1}|^2 - 2k\sum_{i=1}^{k} v_i\cdot v_{k+1}\right).$$

We deduce from these estimates that $|u_1|^2+\cdots+|u_N|^2 =: I_1 + I_2$ with

$$\begin{aligned} I_1 &= \sum_{k=1}^{N-1}\left(\frac{1}{k(k+1)}\sum_{i=1}^{k}|v_k|^2 + \frac{k}{k+1}|v_{k+1}|^2\right) + \frac{1}{N}\sum_{i=1}^{N}|v_i|^2 \\ &=: \sum_{k=1}^{N-1} A_k + A_N \end{aligned}$$

and

$$\begin{aligned} I_2 &= 2\left[\sum_{k=1}^{N-1}\left(\frac{1}{k(k+1)}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} v_i\cdot v_j - \frac{1}{k+1}\sum_{i=1}^{k} v_i\cdot v_{k+1}\right) - \frac{1}{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} v_i\cdot v_j\right] \\ &=: 2\left[\sum_{k=1}^{N-1} B_k + B_N\right]. \end{aligned}$$

First of all, looking to I_1 we easily see that $|v_N|^2$ appears only in A_{N-1} and A_N, so its coefficient is $(N-1)/N + 1/N = 1$. For m such that $2 \le m \le N-1$, $|v_m|^2$ appears in $A_{m-1}, A_m, \dots, A_{N-1}$ and A_N, hence its coefficient is given by

$$\frac{m-1}{m} + \sum_{j=m}^{N-1}\frac{1}{j(j+1)} + \frac{1}{N} = 1.$$

The coefficient of $|v_1|^2$ is the same of $|v_2|^2$ since there is no A_0. We conclude then

$$I_1 = |v_1|^2 + \cdots + |v_N|^2.$$

We can compute I_2 in the same way. For $1 \le m \le N-1$, $v_m \cdot v_N$ appears only in B_{N-1} and B_N, so its coefficient is $-1/N + 1/N = 0$. Moreover, for $1 \le m < p \le N-1$, $v_m \cdot v_p$ appears in $B_{p-1}, B_p, \dots, B_{N-1}$ and B_N, hence its coefficient is given by

$$-\frac{1}{p} + \sum_{j=p}^{N-1}\frac{1}{p(p+1)} + \frac{1}{N} = 0.$$

Finally, we conclude that $|u_1|^2+\cdots+|u_N|^2 = |v_1|^2+\cdots+|v_N|^2 = r^2$ and $u_N = z/\sqrt{N}$ follows easily from (13).

The last point to prove is that the Jacobien is equal to one. To simplify we consider $d = 1$, the general case being the same. Consider the matrix M_N that represents the linear application in (13), i.e. $M_N u = v$, where $u = (u_1, \dots, u_N) \in \mathbb{R}^N$ and $v = (v_1, \dots, v_N) \in \mathbb{R}^N$.

We claim that $\det(M_N) = 1$. Indeed we have

$$M_N = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \frac{1}{\sqrt{(N-1)N}} & \cdots & \cdots & \frac{1}{\sqrt{(N-1)N}} & -\frac{(N-1)}{\sqrt{(N-1)N}} \\ \frac{1}{\sqrt{N}} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{N}} \end{pmatrix}$$

and it can be written in the form $M_N = D_N A_N$ with a diagonal matrix D_N,

$$M_N = \begin{pmatrix} \frac{1}{\sqrt{2}} & & & & \\ & \frac{1}{\sqrt{6}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{(N-1)N}} & \\ & & & & \frac{1}{\sqrt{N}} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & -2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & -(N-1) \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

Let us prove the claim by recurrence. For $N = 2$ is clear that $\det(D_2) = 1/2$ and $\det(A_2) = 2$, which implies $\det(M_2) = 1$. Then, supposing that $\det(M_{N-1}) = 1$ we have

$$\det(M_{N-1}) = \left(\prod_{k=1}^{N-2} \frac{1}{\sqrt{k(k+1)}} \times \frac{1}{\sqrt{(N-1)}} \right) \det(A_{N-1}) = 1 \tag{73}$$

since $\det(D_{N-1})$ is easily computed. Moreover, we have the following relation $\det(A_N) = N \det(A_{N-1})$. Hence we deduce that

$$\begin{aligned} \det(M_N) &= \left(\prod_{k=1}^{N-1} \frac{1}{\sqrt{k(k+1)}} \times \frac{1}{\sqrt{N}} \right) \det(A_N) \\ &= \left(\prod_{k=1}^{N-2} \frac{1}{\sqrt{k(k+1)}} \times \frac{1}{\sqrt{(N-1)N}} \times \frac{1}{\sqrt{N}} \right) N \det(A_{N-1}) \\ &= 1 \end{aligned}$$

thanks to (73), which concludes the proof of the claim.

□

A.2. **Regularity lemma.**

Lemma 31. *Let $f \in \mathbf{P}(\mathbb{R}^d)$. Suppose $f \in L^p \cap L_s(\mathbb{R}^d)$ for $p > 1$ and $s > 0$. Then $f \in L^q_m(\mathbb{R}^d)$ with $q < p$ and $m = s(p-q)(p-1)$.*

Proof. Let us compute the L^q_m norm of f,

$$\begin{aligned} \|f\|^q_{L^q_m} &= \int (1 + |v|^2)^{m/2} f(v)^q \, dv \\ &\leq C \left(\int f(v)^q \, dv + \int |v|^m f(v)^q \, dv \right). \end{aligned}$$

For the first term we have $\|f\|_{L^q}^q \leq \|f\|_{L^p}^q$ and for the second one we obtain

$$\int |v|^m f(v)^q \, dv \leq \left(\int |v|^{mr/(r-1)} f(v)^{(q-\alpha)r/(r-1)} \right)^{(r-1)/r} \left(\int f(v)^{\alpha r} \right)^{1/r}$$

by Holder's inequality for some $r > 1$ and $0 < \alpha < q$. Now choosing $r = p/\alpha$ and choosing α such that $(q-\alpha)r/(r-1) = 1$, i.e. $\alpha = p(q-1)/(p-1)$ we obtain

$$\int |v|^m f(v)^q \, dv \leq \left(\int |v|^{m(p-1)/(p-q)} f(v) \right)^{(p-q)/(p-1)} \left(\int f(v)^p \right)^{(q-1)/(p-1)}.$$

Finally, choosing $m = s(p-q)/(p-1)$ we conclude with

$$\|f\|_{L^q_m}^q \leq C \left(\|f\|_{L^p}^q + \|f\|_{L_s}^{(p-q)/(p-1)} \|f\|_{L^p}^{p(q-1)/(p-1)} \right).$$

□

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