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# Convexity of some spectral functions on Hermitian matrices

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## Abstract

We prove in this note the convexity of the functions  $u \circ \lambda$  and more generally  $u \circ \lambda_B$  on the space of Hermitian matrices, for  $B$  a fixed positive definite hermitian matrix, when  $u : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  is a symmetric convex function which is lower semi-continuous on  $\mathbb{R}^m$ , and finite in at least one point of  $\mathbb{R}^m$ . This is performed by using some optimisation techniques and a generalized Ky Fan inequality. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

## Résumé

**Convexité de certaines fonctions spectrales sur les matrices hermitiennes.** On montre dans cette note la convexité des fonctions  $u \circ \lambda$  et plus généralement  $u \circ \lambda_B$  sur l'espace des matrices hermitiennes, où  $B$  est une matrice hermitienne définie positive fixée, lorsque  $u : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  est une fonction symétrique convexe, semi-continue inférieurement sur  $\mathbb{R}^m$ , et finie en au moins un point de  $\mathbb{R}^m$ , et ce en utilisant des techniques d'optimisation et une inégalité de Ky Fan généralisée. *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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## Version française abrégée

Rappelons que pour  $B \in \mathcal{M}_m(\mathbb{C})$  fixée, et pour tout  $C \in \mathcal{M}_m(\mathbb{C})$ , on dit que  $\lambda \in \mathbb{C}$  est une valeur  $B$ -propre de  $C$  s'il existe  $x$  non nul dans  $\mathbb{C}^m$  tel que  $Cx = \lambda Bx$ ,  $x$  est alors appelé vecteur  $B$ -propre de  $C$ . On note  $\lambda_B(C)$  le vecteur des valeurs  $B$ -propres de  $C$  répétées avec leur multiplicité. Pour  $u$  une fonction symétrique, on montre dans cette note dans le cas des matrices hermitiennes la convexité des fonctions  $u \circ \lambda$  et plus généralement  $u \circ \lambda_B$  lorsque  $u$  satisfait certaines conditions (Théorème 0.1), ce qui fournit en particulier la concavité des fonctions  $F_k = \ln \sigma_k \lambda$  et plus généralement  $\ln \sigma_k \lambda_B$  (Corollaire 0.3) et la convexité des domaines de définition de ces dernières fonctions (Corollaire 0.4). La preuve exposée ici est

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une alternative à celle esquissée dans [3] page 277 ; elle est basée sur la théorie des fonctions spectrales et la conjugaison de Legendre–Fenchel, et généralise la preuve d’un problème sur l’espace des matrices réelles symétriques proposée dans [4] page 300.

L’ensemble  $\mathcal{H}_m(\mathbb{C})$  des matrices carrées complexes d’ordre  $m$  hermitiennes est structuré en **espace euclidien** à l’aide du produit scalaire  $\langle\langle A, B \rangle\rangle = \text{tr}({}^t\overline{AB}) = \text{tr}(AB)$ , dit **produit de Schur**. On note  $\Gamma_0(\mathbb{R}^m)$  l’ensemble des fonctions  $u : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  convexes, semi-continues inférieurement sur  $\mathbb{R}^m$ , et finies en au moins un point. Etant donné  $u \in \Gamma_0(\mathbb{R}^m)$  symétrique et  $B \in \mathcal{H}_m(\mathbb{C})$  **définie positive fixée**, on définit :

$$V_u^B : \mathcal{H}_m(\mathbb{C}) \rightarrow \mathbb{R} \cup \{+\infty\}, \text{ par } C \mapsto V_u^B(C) := u(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))$$

où  $\lambda_{B,1}(C) \geq \lambda_{B,2}(C) \geq \dots \geq \lambda_{B,m}(C)$  désignent les **valeurs  $B$ -propres** de  $C$  chacune répétée selon sa multiplicité. On appellera de telles fonctions  $V_u^B$ , **fonctions de valeurs  $B$ -propres** ou **fonctions  $B$ -spectrales**. Notre premier but sera de déterminer la conjuguée de  $V_u^B$  en fonction de la conjuguée de  $u$ . Rappelons que la conjuguée ou transformée de Legendre–Fenchel de  $u$  est la fonction  $u^* : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  définie par :

$$\forall s \in \mathbb{R}^m, \quad u^*(s) = \sup_{x \in \mathbb{R}^m} \left\{ \langle s, x \rangle - u(x) \right\}$$

$\langle \cdot, \cdot \rangle$  désignant le produit scalaire standard de  $\mathbb{R}^m$ . On prouve ici le résultat suivant :

**Théorème 0.1 (Conjugaison des fonctions de valeurs  $B$ -propres)** *Si  $u \in \Gamma_0(\mathbb{R}^m)$  est symétrique alors :*

- (i) *La conjuguée  $u^*$  ( $\in \Gamma_0(\mathbb{R}^m)$ ) est également symétrique.*
- (ii) *Les fonctions de valeurs  $B$ -propres  $V_u^B$  et  $V_{u^*}^B$  (définies comme ci-dessus) appartiennent à  $\Gamma_0(\mathcal{H}_m(\mathbb{C}))$  avec  $V_{u^*}^{B^{-1}} = (V_u^B)^*$ , donc en particulier la fonction de valeurs  $B$ -propres  $V_u^B$  est convexe semi-continue inférieurement.*

On établit pour la preuve le lemme suivant :

**Lemme 0.2 (Inégalité de Ky Fan généralisée)** *Soit  $B$  une matrice hermitienne définie positive fixée d’ordre  $m$ . Alors l’inégalité suivante est satisfaite :*

$$\forall C, S \in \mathcal{H}_m(\mathbb{C}), \quad \text{tr}(CS) = \sum_{i=1}^m \lambda_i(CS) \leq \sum_{i=1}^m \lambda_{B,i}(C) \lambda_{B^{-1},i}(S)$$

où  $\lambda_{B,1}(C) \geq \dots \geq \lambda_{B,m}(C)$  désigne le vecteur des valeurs  $B$ -propres de  $C$  répétées avec leur multiplicité et  $\lambda_{B^{-1},1}(S) \geq \dots \geq \lambda_{B^{-1},m}(S)$  le vecteur des valeurs  $B^{-1}$ -propres de  $S$ . En outre, le cas d’égalité est vérifié si et seulement si il existe  $U \in U_m(\mathbb{C})$  tel que  $C = B^{\frac{1}{2}} {}^t\overline{U} \left( \text{diag}(\lambda_B(C)) \right) U B^{\frac{1}{2}}$  et  $S = B^{-\frac{1}{2}} {}^t\overline{U} \left( \text{diag}(\lambda_{B^{-1}}(S)) \right) U B^{-\frac{1}{2}}$  avec  $\lambda_B(C) := (\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))$  et  $\lambda_{B^{-1}}(S) := (\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S))$ .

Du théorème découlent deux corollaires :

**Corollaire 0.3** *Pour tout entier  $k \in \{1, \dots, m\}$ , la fonction :*

$$F_k^B : \mathcal{H}_m(\mathbb{C}) \rightarrow \mathbb{R} \cup \{+\infty\}, \quad C \mapsto F_k^B(C) = \begin{cases} -\ln \sigma_k \left( \lambda_B(C) \right) & \text{si } C \in \lambda_B^{-1}(\Gamma_k) \\ +\infty & \text{sinon} \end{cases}$$

où  $\lambda_B^{-1}(\Gamma_k) := \{C \in \mathcal{H}_m(\mathbb{C}) / \lambda_B(C) \in \Gamma_k\}$ , avec  $\Gamma_k := \{\lambda \in \mathbb{R}^m / \forall 1 \leq j \leq k, \sigma_j(\lambda) > 0\}$  est convexe. Ici  $\sigma_j$  désigne la  $j$ -ième fonction symétrique élémentaire.

**Corollaire 0.4** *Si  $\Gamma$  est un convexe fermé (non vide) symétrique de  $\mathbb{R}^m$ , alors l’ensemble  $\lambda_B^{-1}(\Gamma) := \{C \in \mathcal{H}_m(\mathbb{C}) / \lambda_B(C) \in \Gamma\}$  est un convexe fermé de  $\mathcal{H}_m(\mathbb{C})$ . En particulier,  $\lambda_B^{-1}(\overline{\Gamma}_k)$  est un convexe fermé de*

$\mathcal{H}_m(\mathbb{C})$  et donc l'ensemble des fonctions  $k$ -admissibles  $\mathcal{A}_k := \{\varphi \in C^2(M, \mathbb{R}) / \lambda_\omega(\omega + i\partial\bar{\partial}\varphi) \in \Gamma_k\}$  (au sens de [5,6] où  $(M, J, g, \omega)$  est une variété kählérienne) est convexe.

## 1 Introduction and statement of results

Let  $\mathcal{H}_m(\mathbb{C})$  be the space of complex Hermitian matrices of order  $m$ . We recall that for any two matrices  $B$  and  $C$  of  $\mathcal{H}_m(\mathbb{C})$ ,  $\lambda \in \mathbb{C}$  is called a  $B$ -eigenvalue of  $C$  if there exists  $x \neq 0$  in  $\mathbb{C}^m$  such that  $Cx = \lambda Bx$ ,  $x$  is then called a  $B$ -eigenvector of  $C$ . Moreover, given a fixed positive definite matrix  $B \in \mathcal{H}_m(\mathbb{C})$ , the following result holds :

**Proposition 1.1** *Let  $C \in \mathcal{H}_m(\mathbb{C})$ , then :*

- (i) *The spectrum of  $B^{-1}C$  (i.e. the  $B$ -spectrum of  $C$ ) is entirely real.*
- (ii) *The greatest eigenvalue of  $B^{-1}C$  (i.e. the greatest  $B$ -eigenvalue of  $C$ ) equals  $\sup_{u \neq 0} \frac{\langle Cu, u \rangle}{\langle Bu, u \rangle}$ , where  $\langle \cdot, \cdot \rangle$  denotes the standard Hermitian product of  $\mathbb{C}^m$ .*
- (iii)  *$B^{-1}C$  is diagonalizable.*

Since the spectrum of  $B^{-1}C$  is the spectrum of the Hermitian matrix  $B^{-\frac{1}{2}}CB^{-\frac{1}{2}}$ , the proof is an easy adaptation of the standard one for symmetric matrices.

In this note, we will prove for  $u$  a symmetric function and in the case of Hermitian matrices the convexity of the functions  $u \circ \lambda$  and more generally  $u \circ \lambda_B$  when  $u$  satisfies some conditions (Theorem 1.1) and infer the concavity of the functions  $F_k = \ln \sigma_k \lambda$  and more generally  $\ln \sigma_k \lambda_B$  (Corollary 1.3) as well as the convexity of their sets of definition (Corollary 1.4). Our proof is an alternative to the one sketched in [3] page 277. It is based on the spectral functions theory and the Legendre–Fenchel conjugation, it generalizes the proof of a problem on the space of real symmetric matrices given in [4] page 300.

Let  $\mathcal{H}_m(\mathbb{C})$  be the set of Hermitian complex square matrices of order  $m$ ,  $\mathcal{H}_m(\mathbb{C})$  has a structure of **Euclidian space** thanks to the following scalar product  $\ll A, B \gg = \text{tr}({}^t \bar{A}B) = \text{tr}(AB)$ , called **the Schur product**. Let us denote by  $\Gamma_0(\mathbb{R}^m)$  the set of functions  $u : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  that are convex, lower semi-continuous on  $\mathbb{R}^m$ , and finite in at least one point. Given  $u \in \Gamma_0(\mathbb{R}^m)$  symmetric and  $B \in \mathcal{H}_m(\mathbb{C})$  **positive definite**, we define :

$$V_u^B : \mathcal{H}_m(\mathbb{C}) \rightarrow \mathbb{R} \cup \{+\infty\}, \text{ by } C \mapsto V_u^B(C) := u(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))$$

where  $\lambda_{B,1}(C) \geq \lambda_{B,2}(C) \geq \dots \geq \lambda_{B,m}(C)$  denote the  **$B$ -eigenvalues** of  $C$  repeated with their multiplicity. Such functions  $V_u^B$  are called **functions of  $B$ -eigenvalues** or  **$B$ -spectral functions**. Our first aim is to determine the conjugation for such a function  $V_u^B$  using the conjugate function of  $u$ . Let us remind that the conjugation or the Legendre–Fenchel transform of  $u$  is the function  $u^* : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by :

$$\forall s \in \mathbb{R}^m, \quad u^*(s) = \sup_{x \in \mathbb{R}^m} \left\{ \langle s, x \rangle - u(x) \right\}$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product on  $\mathbb{R}^m$ . Our main result is the following :

**Theorem 1.1 (Conjugation of functions of  $B$ -eigenvalues)** *Let  $u \in \Gamma_0(\mathbb{R}^m)$  be symmetric, then :*

- (i) *The conjugate  $u^*$  ( $\in \Gamma_0(\mathbb{R}^m)$ ) is also symmetric.*
- (ii) *The functions of  $B$ -eigenvalues  $V_u^B$  and  $V_{u^*}^B$  (defined as above) belong to  $\Gamma_0(\mathcal{H}_m(\mathbb{C}))$  with  $V_{u^*}^{B^{-1}} = (V_u^B)^*$ , so that in particular the function of  $B$ -eigenvalues  $V_u^B$  is convex and lower semi-continuous.*

Our proof requires the following lemma (named after Ky Fan like similar inequalities established in [1,2]):

**Lemma 1.2 (Generalized Ky Fan Inequality)** *Let  $B$  be a fixed Hermitian positive definite matrix of order  $m$ . Then the following inequality is satisfied :*

$$\forall C, S \in \mathcal{H}_m(\mathbb{C}), \quad \text{tr}(CS) = \sum_{i=1}^m \lambda_i(CS) \leq \sum_{i=1}^m \lambda_{B,i}(C) \lambda_{B^{-1},i}(S)$$

where  $\lambda_{B,1}(C) \geq \dots \geq \lambda_{B,m}(C)$  denote the  $B$ -eigenvalues of  $C$  and  $\lambda_{B^{-1},1}(S) \geq \dots \geq \lambda_{B^{-1},m}(S)$  the  $B^{-1}$ -eigenvalues of  $S$ . Moreover, the equality case holds if and only if there exists  $U \in U_m(\mathbb{C})$  such that  $C = B^{\frac{1}{2}} \bar{U} \left( \text{diag}(\lambda_B(C)) \right) U B^{\frac{1}{2}}$  and  $S = B^{-\frac{1}{2}} \bar{U} \left( \text{diag}(\lambda_{B^{-1}}(S)) \right) U B^{-\frac{1}{2}}$  with  $\lambda_B(C) := (\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))$  and  $\lambda_{B^{-1}}(S) := (\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S))$ .

Two corollaries follow from Theorem 1.1, namely :

**Corollary 1.3** For each  $k \in \{1, \dots, m\}$ , the function :

$$F_k^B : \mathcal{H}_m(\mathbb{C}) \rightarrow \mathbb{R} \cup \{+\infty\}, \quad C \mapsto F_k^B(C) = \begin{cases} -\ln \sigma_k \left( \lambda_B(C) \right) & \text{if } C \in \lambda_B^{-1}(\Gamma_k) \\ +\infty & \text{otherwise,} \end{cases} \quad (1)$$

$$\text{where } \lambda_B^{-1}(\Gamma_k) := \{C \in \mathcal{H}_m(\mathbb{C}) / \lambda_B(C) \in \Gamma_k\}, \text{ with } \Gamma_k := \{\lambda \in \mathbb{R}^m / \forall 1 \leq j \leq k, \sigma_j(\lambda) > 0\} \quad (2)$$

is convex. Here  $\sigma_j$  denotes the  $j$ -th elementary symmetric function.

**Corollary 1.4** If  $\Gamma$  is a (non empty) symmetric convex closed set of  $\mathbb{R}^m$ , then  $\lambda_B^{-1}(\Gamma) := \{C \in \mathcal{H}_m(\mathbb{C}) / \lambda_B(C) \in \Gamma\}$  is a convex closed set of  $\mathcal{H}_m(\mathbb{C})$ . In particular,  $\lambda_B^{-1}(\bar{\Gamma}_k)$  is a convex closed set of  $\mathcal{H}_m(\mathbb{C})$  so that the set of  $k$ -admissible functions  $\mathcal{A}_k := \{\varphi \in C^2(M, \mathbb{R}) / \lambda_\omega(\omega + i\partial\bar{\partial}\varphi) \in \Gamma_k\}$  (in the sense of [5,6] where  $(M, J, g, \omega)$  is a Kähler manifold) is convex.

In the sequel of the note, we will successively prove Lemma 1.2 (Section 2), Theorem 1.1 (Section 3) and the corollaries (Section 4).

## 2 Proof of the lemma

We present here a generalisation (Lemma 1.2) in terms of  $B$ -eigenvalues of Hermitian matrices of an inequality called Ky Fan inequality (see [1,2]), since it is required for the above proof of the Theorem 1.1. Here, we just indicate the proof (for details, see [5]). We first prove the theorem with  $B = I$  (classical eigenvalues) by mimicking the proof of [1] (pages 17-18 and exercise 12 page 20) of a similar theorem for real symmetric matrices; the main ingredients are a Birkhoff theorem and a Hardy-Littlewood-Polya inequality. Our theorem, with general  $B$ , follows by application of the previous result to the matrices  $B^{-\frac{1}{2}}CB^{-\frac{1}{2}}$  and  $B^{\frac{1}{2}}SB^{\frac{1}{2}}$  with  $C, S \in \mathcal{H}_m(\mathbb{C})$ .

## 3 Proof of the theorem

Let  $u \in \Gamma_0(\mathbb{R}^m)$  be symmetric. Let  $\sigma$  be a permutation of  $\{1, \dots, m\}$  and  $s \in \mathbb{R}^m$ , we have :

$$u^*(s_{\sigma(1)}, \dots, s_{\sigma(m)}) = \sup_{x \in \mathbb{R}^m} \left\{ \langle (s_{\sigma(1)}, \dots, s_{\sigma(m)}), x \rangle - u(x) \right\} = \sup_{x \in \mathbb{R}^m} \left\{ \langle s, x \rangle - u(x) \right\} = u^*(s)$$

which proves that  $u^*$  ( $\in \Gamma_0(\mathbb{R}^m)$ ) is also symmetric. It is then possible, as done for  $u$ , to define the function of  $B$ -eigenvalues  $V_{u^*}^B$ . In order to show the second point of Theorem 1.1, we first prove the following equality (setting  $E = \mathcal{H}_m(\mathbb{C})$  for short) :

$$\forall C \in E, \quad V_u^B(C) = \sup_{S \in E} \left\{ \text{tr}(CS) - u^*(\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S)) \right\} \quad (3)$$

where  $tr(C)$  stands for the trace of  $C$ .

**Step 1** : We claim that the right-hand side of (3) is always  $\geq V_u^B(C)$ . Indeed, let  $C \in E$  and  $U \in U_m(\mathbb{C})$  be such that :

$$U(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})U^{-1} = \text{diag}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))$$

so that  $C = B^{\frac{1}{2}}U^{-1}\text{diag}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))UB^{\frac{1}{2}}$ . For each  $S \in E$ , we have :

$$tr(CS) = tr(B^{\frac{1}{2}}U^{-1}\text{diag}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))UB^{\frac{1}{2}}S) = tr(\text{diag}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))UB^{\frac{1}{2}}SB^{\frac{1}{2}}U^{-1})$$

consequently :

$$\begin{aligned} & \sup_{S \in E} \left\{ tr(CS) - u^*(\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S)) \right\} \\ &= \sup_{S \in E} \left\{ tr(\text{diag}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))UB^{\frac{1}{2}}SB^{\frac{1}{2}}U^{-1}) - u^*(\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S)) \right\}. \end{aligned}$$

Since the map :  $S \in E \mapsto UB^{\frac{1}{2}}SB^{\frac{1}{2}}U^{-1} \in E$  is bijective, setting  $\tilde{S} = UB^{\frac{1}{2}}SB^{\frac{1}{2}}U^{-1}$ , we obtain :

$$\begin{aligned} \sup_{S \in E} \left\{ tr(CS) - u^*(\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S)) \right\} &= \sup_{\tilde{S} \in E} \left\{ tr(\text{diag}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))\tilde{S}) \right. \\ &\quad \left. - u^*(\lambda_{B^{-1},1}(B^{-\frac{1}{2}}U^{-1}\tilde{S}UB^{-\frac{1}{2}}), \dots, \lambda_{B^{-1},m}(B^{-\frac{1}{2}}U^{-1}\tilde{S}UB^{-\frac{1}{2}})) \right\}. \end{aligned}$$

Now, the  $B^{-1}$ -eigenvalues of  $B^{-\frac{1}{2}}U^{-1}\tilde{S}UB^{-\frac{1}{2}}$  are nothing but the eigenvalues of  $U^{-1}\tilde{S}U$ , hence those of  $\tilde{S}$  as well, therefore :

$$\sup_{S \in E} \left\{ tr(CS) - u^*(\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S)) \right\} = \sup_{\tilde{S} \in E} \left\{ tr(\text{diag}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C))\tilde{S}) - u^*(\lambda_1(\tilde{S}), \dots, \lambda_m(\tilde{S})) \right\}.$$

Restricting to diagonal matrices, we get :

$$\begin{aligned} & \sup_{S \in E} \left\{ tr(CS) - u^*(\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S)) \right\} \\ & \geq \sup_{(s_1, \dots, s_m) \in \mathbb{R}^m} \left\{ \sum_{i=1}^m \lambda_{B,i}(C) s_i - u^*(s_1, \dots, s_m) \right\} = u^{**}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C)) \end{aligned}$$

But  $u^{**}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C)) = u(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C)) = V_u^B(C)$  since  $u \in \Gamma_0(\mathbb{R}^m)$ , so step 1 is proved.

**Step 2** : let us prove the reversed inequality. Given  $C, S \in E$ , the generalized Ky Fan inequality (Lemma 1.2) implies :

$$tr(CS) = \sum_{i=1}^m \lambda_i(CS) \leq \sum_{i=1}^m \lambda_{B,i}(C) \lambda_{B^{-1},i}(S)$$

hence :

$$\begin{aligned} tr(CS) - u^*(\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S)) &\leq \sum_{i=1}^m \lambda_{B,i}(C) \lambda_{B^{-1},i}(S) - u^*(\lambda_{B^{-1},1}(S), \dots, \lambda_{B^{-1},m}(S)) \\ &\leq \sup_{(s_1, \dots, s_m) \in \mathbb{R}^m} \left\{ \sum_{i=1}^m \lambda_{B,i}(C) s_i - u^*(s_1, \dots, s_m) \right\} = u^{**}(\lambda_{B,1}(C), \dots, \lambda_{B,m}(C)) = V_u^B(C) \end{aligned}$$

so the proof of the equality (3) is complete. It says exactly that :

$$V_u^B(C) = \sup_{S \in E} \left\{ \ll C, S \gg - V_{u^*}^{B^{-1}}(S) \right\} = \left( V_{u^*}^{B^{-1}} \right)^*(C) \quad (4)$$

Accordingly  $V_u^B = \left( V_{u^*}^{B^{-1}} \right)^*$ , thus  $V_u^B \in \Gamma_0(E)$  (it is convex lower semi-continuous as a conjugate, finite in at least one point because  $u$  is so). The conjugate  $u^*$  being also a symmetric function of  $\Gamma_0(\mathbb{R}^m)$ , the previous result applied to  $u^*$  shows that  $V_{u^*}^B \in \Gamma_0(E)$  and  $V_{u^*}^B = \left( V_u^{B^{-1}} \right)^*$ . The proof of Theorem 1.1 is complete.

#### 4 Proof of the corollaries

The proof of Corollary 1.3 is a direct application of Theorem 1.1 to the function :

$$u : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}, x = (x_1, \dots, x_m) \mapsto u(x) = \begin{cases} -\ln \sigma_k(x_1, \dots, x_m) & \text{if } x \in \Gamma_k \\ +\infty & \text{otherwise} \end{cases} \quad (5)$$

The proof of Corollary 1.4 goes by considering the indicatrix function  $f_0 := I_\Gamma$  of the set  $\Gamma$ , namely :

$$f_0 := I_\Gamma : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}, x = (x_1, \dots, x_m) \mapsto I_\Gamma(x) = \begin{cases} 0 & \text{if } x \in \Gamma \\ +\infty & \text{otherwise} \end{cases} \quad (6)$$

From the assumptions made on  $\Gamma$ ,  $f_0$  lies in  $\Gamma_0(\mathbb{R}^m)$  and is symmetric. So Theorem 1.1 implies that the function of  $B$ -eigenvalues  $V_{I_\Gamma}^B$  lies in  $\Gamma_0(\mathbb{R}^m)$ ; in particular it is, convex lower semi-continuous. But this function is given by :

$$V_{I_\Gamma}^B : \mathcal{H}_m(\mathbb{C}) \rightarrow \mathbb{R} \cup \{+\infty\}, C \mapsto V_{I_\Gamma}^B(C) = \begin{cases} 0 & \text{if } C \in \lambda_B^{-1}(\Gamma) \\ +\infty & \text{otherwise} \end{cases} \quad (7)$$

in other words, it coincides with  $I_{\lambda_B^{-1}(\Gamma)}$ , the indicatrix function of  $\lambda_B^{-1}(\Gamma)$ . So the latter must itself be convex lower semi-continuous. As a consequence,  $\lambda_B^{-1}(\Gamma)$  is a convex closed (non empty) set of  $\mathcal{H}_m(\mathbb{C})$ .

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