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► **To cite this version:**

Emmanuel Hanert, Eva Schumacher, Eric Deleersnijder. Front dynamics in fractional-order epidemic models. *Journal of Theoretical Biology*, 2011, 279 (1), pp.9. 10.1016/j.jtbi.2011.03.012 . hal-00694284

HAL Id: hal-00694284

<https://hal.science/hal-00694284>

Submitted on 4 May 2012

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Author's Accepted Manuscript

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PII: S0022-5193(11)00152-4
DOI: doi:10.1016/j.jtbi.2011.03.012
Reference: YJTBI6405

To appear in: *Journal of Theoretical Biology*

Received date: 7 October 2010
Revised date: 21 February 2011
Accepted date: 14 March 2011

Cite this article as: Emmanuel Hanert, Eva Schumacher and Eric Deleersnijder, Front dynamics in fractional-order epidemic models, *Journal of Theoretical Biology*, doi:[10.1016/j.jtbi.2011.03.012](https://doi.org/10.1016/j.jtbi.2011.03.012)

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Front dynamics in fractional-order epidemic models

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Abstract

A number of recent studies suggest that human and animal mobility patterns exhibit scale-free, Lévy-flight dynamics. However, current reaction-diffusion epidemics models do not account for the superdiffusive spread of modern epidemics due to Lévy flights. We have developed a SIR model to simulate the spatial spread of a hypothetical epidemic driven by long-range displacements in the infective and susceptible populations. The model has been obtained by replacing the second-order diffusion operator by a fractional-order operator. Theoretical developments and numerical simulations show that fractional-order diffusion leads to an exponential acceleration of the epidemic's front and a power-law decay of the front's leading tail. Our results indicate the potential of fractional-order reaction-diffusion models to represent modern epidemics.

Keywords: Fractional-order diffusion, Lévy flights, epidemics spatial spread, front dynamics

1. Introduction

2 The spread of epidemics caused by directly-transmitted pathogens is re-
 3 lated to the interactions between susceptible and infective individuals. The
 4 occurrence of such interactions is a direct consequence of the mobility pat-
 5 terns of individuals in their home range. Recent studies have shown that

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6 mobility patterns for both humans and animals can be quite complex and
7 exhibit a scale-free dynamics, characteristic of Lévy flights (see [3] for a re-
8 view). Unlike ordinary Gaussian dispersion processes, Lévy flights are drawn
9 from a probability distribution function with heavier tails than a Normal dis-
10 tribution. Such an asymptotic behaviour means that large displacements are
11 more likely. Lévy flights are therefore superdiffusive as they disperse par-
12 ticles faster than a Gaussian random-walk. Lévy-flight patterns have been
13 observed in the dispersion of bank notes [2], human mobility patterns de-
14 rived from mobile phone data [18] as well as in the foraging patterns of a
15 numbers of animal species [39, 40, 36, 23] although questions remain about
16 the empirical evidence of some biological Lévy flights [15].

17 The existence of Lévy-flight mobility patterns suggests that modern epi-
18 demics cannot be represented by second-order reaction-diffusion models that
19 implicitly assume a Gaussian dispersion process. Such models are only appli-
20 cable when the infective and susceptible individuals travel short distances as
21 compared to geographical distances. They lead to epidemic fronts that travel
22 at a constant velocity and have been used, for instance, to model the spread
23 of the Black Death in Europe in the 14th century [30]. Modern epidemics
24 such as SARS or avian influenza can spread around the world in a few weeks
25 and seem to follow a non-Gaussian, scale-free dynamics [22, 37].

26 It has been shown that for Lévy-flight dispersion, the density function
27 representing the population is the solution of a fractional-order diffusion
28 equation [6, 28]. Unlike integer-order derivatives that are local operators,
29 fractional-order derivatives are non-local, integro-differential operators [31,
30 33]. As such, they can be used to represent memory effects and long-range
31 dispersion processes. In the last decade, fractional-order diffusion models
32 have been an active field of research both from a theoretical and applied
33 perspective. They have been proposed to model a wide range of problems
34 in surface and subsurface hydrology [32, 11, 24], plasma turbulence [9, 10],
35 finance [35, 27, 19, 4] and biology [13, 12].

36 In epidemiology, fractional-order models have just been mentioned in a
37 couple of studies. Brockmann et al. [2] have shown that the density of bank
38 notes originating from a given city is solution of a space-time fractional diffu-
39 sion equation and have suggested that an epidemic spread could be modelled
40 by a similar equation. In a subsequent study [1], Brockmann has proposed a
41 SIR model that includes fractional-order diffusion. Some preliminary results
42 are presented in that study but the properties of the model solutions are not
43 discussed in detail. The goal of the present study is therefore to discuss in

44 greater detail the properties of fractional-order epidemics model solutions.
 45 In particular, we consider the propagation of epidemic fronts and show that
 46 the use of a fractional-order diffusion term can lead to an acceleration of the
 47 front and thus a rapid spread of epidemics.

48 2. Model description

49 We shall consider a SIR model representing the spatial spread of an epi-
 50 demic with non-local diffusion. The model takes into account 3 population
 51 densities, the susceptibles $S(x, t)$, the infectives $I(x, t)$ and a removed class
 52 $R(x, t)$. The spatial dynamics is assumed to be one-dimensional and is repre-
 53 sented by a Riesz/Weyl, fractional-order differential operator denoted ${}_{-\infty}D_x^\alpha$,
 54 where $1 < \alpha \leq 2$ is the order of the spatial derivative. The fractional-order
 55 diffusivity is denoted K_α and has units of $\text{m}^\alpha \text{s}^{-1}$. The susceptible and infec-
 56 tive populations are assumed to exhibit the same spatial dispersion patterns
 57 and are thus characterised by the same values of α and K_α . The trans-
 58 mission from susceptibles to infectives and the disease-induced mortality are
 59 represented by a transmission-efficiency parameter r and a mortality-rate pa-
 60 rameter a , respectively. With these assumptions, the model equations read:

$$\frac{\partial S}{\partial t} = -rIS + K_\alpha({}_{-\infty}D_x^\alpha S), \quad (1)$$

$$\frac{\partial I}{\partial t} = rIS - aI + K_\alpha({}_{-\infty}D_x^\alpha I), \quad (2)$$

$$\frac{\partial R}{\partial t} = aI, \quad (3)$$

61 where r , K_α and a are positive constants. The dynamics of the removed
 62 class being entirely driven by the infectives and having no influence on the
 63 other two classes, it will not be further considered in the remainder of this
 64 study. The model equations have to be supplemented by a set of initial and
 65 boundary conditions to obtain a unique solution. The precise expression of
 66 the initial and boundary conditions is not required at this stage. We will just
 67 assume that the epidemic wave is advancing into a uniform population with
 68 an initially homogeneous susceptibles density \mathcal{S} .

69 The Riesz/Weyl fractional-order operator ${}_{-\infty}D_x^\alpha$ is an integro-differential
 70 operator defined as follows:

$${}_{-\infty}D_x^\alpha f(x) = \mathcal{F}_k^{-1} \left[(ik)^\alpha \hat{f}(k) \right] = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^x \frac{f(y)}{(x-y)^{\alpha-1}} dy, \quad (4)$$

71 where $\Gamma(\cdot)$ is Euler's gamma function and \mathcal{F} denotes the Fourier transform.
 72 When $\alpha = 2$, the fractional-order derivative (4) reduces to a standard second-
 73 order derivative. Eq. (4) defines a "left-sided" operator as it only takes into
 74 account the values of the function f at the left-hand side of x . A right-sided
 75 operator (denoted ${}_x D_\infty^\alpha$) can also be defined in a similar fashion (see for
 76 instance [33] for details). Combining left- and right-sided operators allows
 77 the definition of fractional-order derivative with arbitrary skewness. In this
 78 work, we only consider an asymmetric diffusion term in order to highlight the
 79 resulting different dynamics for left- and right-propagating epidemic waves.

80 The origin of anomalous, non-local diffusion lies in the random displace-
 81 ments of the individuals constituting the population. If one assumes that
 82 these individuals follow a Gaussian diffusive process, *i.e.* the distribution
 83 of random displacements has a finite variance, then the density of individ-
 84 uals is solution of a second-order diffusion equation. That result is a direct
 85 consequence of the central limit theorem (CLT). However, if one does not
 86 assume that the distribution of random displacements has a finite variance,
 87 the standard version of the CLT cannot be applied anymore. Instead, the
 88 generalization due to Lévy-Gnedenko [17, 25] can be used. That version of
 89 the theorem does not rely on the assumption that the sum of all the fluctu-
 90 ations has a finite variance but instead assumes that it has a power-law tail
 91 distribution decreasing as $|x|^{-(\alpha+1)}$ with $0 < \alpha \leq 2$. In that case, the density
 92 of individuals tends towards a stable Lévy distribution with exponent α [16],
 93 which is solution of a diffusion equation of fractional order α (see [28] for
 94 details). Here we restrict ourselves to $1 < \alpha \leq 2$. It is important to note
 95 that when $\alpha = 2$, the Lévy distribution reduces to a Normal distribution and
 96 the corresponding differential equation is the classical second-order diffusion
 97 equation. Eqs. (1)-(3) are therefore a generalization of a standard SIR model
 98 describing the spatial spread of an epidemic, as described for instance in [29],
 99 and reduce to such a model when $\alpha = 2$. With power-law tail distribution de-
 100 creasing as $|x|^{-(\alpha+1)}$, one can see that the probability of large displacements
 101 (called Lévy flights) increases as the value of α decreases. Fig. 1 illustrates
 102 the random displacements of a single individual following a Lévy motion for
 103 different values of the exponent α .

104 Before moving to the analysis of the solutions of Eqs. (1)-(2), let us first
 105 recast them in non-dimensional form. Following Murray [29], we introduce
 106 the following dimensionless variables:

$$I^* = I/S, \quad S^* = S/S, \quad x^* = \left(\frac{rS}{K_\alpha}\right)^{1/\alpha} x, \quad t^* = rSt, \quad \lambda = \frac{a}{rS},$$

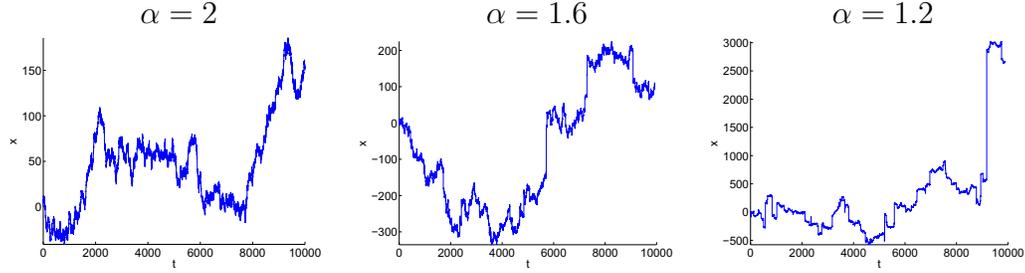


Figure 1: Trajectories of a random walker following a Lévy motion with different values of the exponent α . Brownian motion is recovered when $\alpha = 2$. Note the changes in the range of the vertical axis (position) while the horizontal axis (time) remains the same.

107 which allow us to rewrite Eqs. (1)-(2) as follows:

$$\frac{\partial S}{\partial t} = -IS + {}_{-\infty}D_x^\alpha S, \quad (5)$$

$$\frac{\partial I}{\partial t} = IS - \lambda I + {}_{-\infty}D_x^\alpha I, \quad (6)$$

108 where we have dropped the “*”. As mentioned by Murray [29], the dimen-
 109 sionless parameter λ is the ratio between the contagious time of the disease
 110 and the life expectancy. It is a key parameter that controls the development
 111 of the epidemic wave.

112 3. Theoretical Analysis

113 In this section, we investigate the spatial spread of an epidemic wave of
 114 infectives into a uniform population of susceptibles. In order to highlight the
 115 impact of the fractional-order diffusion operator ${}_{-\infty}D_x^\alpha$ on the dynamics of
 116 the epidemic wave, we consider both left- and right-propagating fronts. The
 117 following analysis shares similarities with the one performed by del Castillo
 118 Negrete et al. [8] for the fractional-order Fisher-Kolmogorov equation. For
 119 that equation and by using the same left-sided diffusion operator as here, they
 120 found that left-propagating fronts have a self-similar profile and move at a
 121 constant speed, while right-propagating fronts are accelerated and exhibit a
 122 algebraic (power-law) decay of the tail.

123 *3.1. Left-propagating front*

124 For such a front, we look for a self-similar travelling-wave solution by
125 setting

$$I(x, t) = I(z), \quad S(x, t) = S(z), \quad z = x + ct,$$

126 where c is the unknown positive wave speed. Note that the "+" in the
127 definition of z indicates that the wave is travelling to the left. Furthermore,
128 we replace $S(z)$ by $1 - s(z)$, where $s(z)$ is the deviation with respect to the
129 initial susceptibles density. Eqs. (5)-(6) can then be expressed as:

$$-c \frac{ds}{dz} = -I(1 - s) - {}_{-\infty}D_z^\alpha s, \quad (7)$$

$$c \frac{dI}{dz} = I(1 - s) - \lambda I + {}_{-\infty}D_z^\alpha I. \quad (8)$$

130 It can easily be seen that the Jacobian of the non-linear reaction term in
131 Eqs. (7)-(8) only has real eigenvalues that reach their maximum value when
132 $s = 0$. The front is thus driven by the region just ahead of it or, in other
133 words, it is pulled by its leading edge. Such a front is thus referred to as a
134 *pulled front* and its speed can be derived by performing a linear analysis in
135 the leading edge region [5]. In our case, the leading edge is where $I \rightarrow 0$ and
136 $s \rightarrow 0$ (see Fig. 2a for an illustration) and linearizing the equations in that
137 region results in the following system:

$$-c \frac{ds}{dz} = -I - {}_{-\infty}D_z^\alpha s, \quad (9)$$

$$c \frac{dI}{dz} = (1 - \lambda)I + {}_{-\infty}D_z^\alpha I. \quad (10)$$

138 Assuming a solution of the form $(S(z), I(z)) = (\hat{S}, \hat{I})e^{\kappa z}$, where κ is a
139 parameter depending on the wave speed, and using the fact that ${}_{-\infty}D_z^\alpha e^{\kappa z} =$
140 $\kappa^\alpha e^{\kappa z}$, one obtains the following system:

$$\begin{pmatrix} c\kappa - \kappa^\alpha & -1 \\ 0 & c\kappa - \kappa^\alpha - (1 - \lambda) \end{pmatrix} \begin{pmatrix} \hat{S} \\ \hat{I} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

141 A non-trivial solution can only be obtained if

$$(c\kappa - \kappa^\alpha)^2 - (1 - \lambda)(c\kappa - \kappa^\alpha) = 0,$$

142 which results in the following dispersion relations:

$$c(\kappa) = \kappa^{\alpha-1}, \quad (11)$$

$$c(\kappa) = \kappa^{\alpha-1} + \frac{1-\lambda}{\kappa}. \quad (12)$$

143 As for the Fisher-Kolmogorov equation, we expect the front to propagate
 144 at the minimum wave speed for a "sufficiently steep" initial condition. Re-
 145 lation (11) is thus discarded as it would result in a non-moving front. By
 146 minimizing relation (12) with respect to κ , one finds the minimum value
 147 of the left-propagating front speed c_{min} and the corresponding exponential
 148 decay rate κ_{min} to be:

$$c_{min} = \alpha \left(\frac{1-\lambda}{\alpha-1} \right)^{\frac{\alpha-1}{\alpha}}, \quad \kappa_{min} = \left(\frac{1-\lambda}{\alpha-1} \right)^{1/\alpha}. \quad (13)$$

149 The former reduces to the classical result, $c = 2\sqrt{1-\lambda}$, when $\alpha = 2$. Since
 150 we are assuming that $1 < \alpha \leq 2$, c_{min} and κ_{min} are both well defined. The
 151 existence of a left-propagating front in the fractional diffusion case is still
 152 conditional on $\lambda < 1$. Note that the more general method proposed by van
 153 Saarloos et al. [14, 38] leads to the same results (see Appendix A for details).

154 The value of the front speed can also be obtained by using a more heuristic
 155 argument. Indeed, at the leading edge, the linearization of Eq. (6) is similar
 156 to the linearization of the fractional-order Fisher-Kolmogorov equation. The
 157 same approach as in [8] can thus be used to derive the speed of the I front,
 158 which leads to Eq. (13). Moreover, it is easily seen that without infectives
 159 there is no susceptibles front as the reaction term in Eq. (5) would vanish.
 160 This means that the susceptibles solution is driven by the infectives solution
 161 and thus both fronts travel at the same speed.

162 3.2. Right-propagating front

163 That case is a bit more complex as the velocity of the traveling wave is
 164 no more constant. One can thus not look for a self-similar solution. In what
 165 follows, we pursue the same approach as the one used by del Castillo Negrete
 166 et al. [8] for the fractional-order Fisher-Kolmogorov equation. Starting from
 167 Eq. (10), we can again linearize it at the leading edge, *i.e.* around $S = 1$
 168 (see Fig. 4a for an illustration), and obtain:

$$\frac{\partial I}{\partial t} = (1-\lambda)I + {}_{-\infty}D_x^\alpha I. \quad (14)$$

169 Since the term $(1 - \lambda)I$ is responsible for the exponential growth or decay of
 170 the solution, one can assume a solution of the form $I(x, t) = e^{(1-\lambda)t}\psi(x, t)$.
 171 Substituting this solution into (20), one finds a differential equation for ψ :

$$\frac{\partial \psi}{\partial t} = -{}_{-\infty}D_x^\alpha \psi, \quad (15)$$

172 with $\psi(x, t = 0) = I(x, t = 0) = I_0(x)$, the initial condition for the infectives
 173 population.

174 The general solution of Eq. (15) can be expressed as

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x - x_0, t) I_0(x_0) dx_0,$$

175 where $G(x, t) = \frac{1}{t^{1/\alpha}} p_\alpha\left(\frac{x}{t^{1/\alpha}}\right)$ is the Green function of Eq. (15) and $p_\alpha(\eta) =$
 176 $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\eta} e^{(ik)^\alpha} dk$ is a skewed Lévy distribution with exponent α [26, 16].
 177 By considering a *localized* initial condition of the form $I_0(x < -l) = 0$;
 178 $I_0(-l \leq x < 0) = A$ and $I_0(x \geq 0) = Ae^{-\kappa x}$ – where l and A are arbitrary
 179 positive and non-zero constants, respectively – one finds the relation

$$\psi(x, t) = A \int_{xt^{-1/\alpha}}^{\infty} p_\alpha(\eta) d\eta + Ae^{-\kappa x} \int_{-\infty}^{xt^{-1/\alpha}} p_\alpha(\eta) e^{\kappa t^{1/\alpha} \eta} d\eta. \quad (16)$$

180 As we look for the asymptotic behaviour, we consider solutions for large,
 181 fixed t and $x \rightarrow \infty$. In that case, we can use the asymptotic behaviour of
 182 the Lévy distributions, *i.e.* $p_\alpha(\eta) \sim \eta^{-(\alpha+1)}$ as $\eta \rightarrow \infty$. After some alge-
 183 braic manipulations (see Appendix B for details), one finds the asymptotic
 184 behaviour of I :

$$I(x, t) = Ate^{(1-\lambda)t} \left(\left(l + \frac{1}{\kappa} \right) x^{-(\alpha+1)} + \frac{1 + \alpha}{\kappa t^{1+1/\alpha}} \int_{-\infty}^{xt^{-1/\alpha}} \frac{e^{\kappa(t^{-1/\alpha}\eta - x)}}{\eta^{\alpha+2}} d\eta + \dots \right).$$

185 The right-propagating front thus asymptotically decays as

$$I(x, t) \sim te^{(1-\lambda)t} x^{-(\alpha+1)}, \quad (17)$$

186 which highlights an algebraic (power-law) decaying tail that totally differs
 187 from the exponential decay observed for the left-moving front and from clas-
 188 sical results obtained with second-order diffusion operators.

189 The solution for the susceptibles population is obtained by assuming again
 190 that $S(x, t) = 1 - s(x, t)$ where $s(x, t) \ll 1$. The following expression can

191 then be derived: $s(x, t) = \frac{1}{1-\lambda} e^{(1-\lambda)t} \psi(x, t)$, which leads to the asymptotic
 192 behaviour:

$$1 - S(x, t) \sim t e^{(1-\lambda)t} x^{-(\alpha+1)}. \quad (18)$$

193 Just as for the left-propagating front, the susceptibles solution is entirely
 194 driven by the infectives solution, *i.e.* in the absence of infectives, there is no
 195 susceptibles front. As a result, the decay rate of the S front is the same as
 196 for the I front whatever the initial condition for S . In other words, even if S
 197 is initially nonzero over the entire domain, its decay rate will be of the form
 198 $x^{-(\alpha+1)}$ rather than $x^{-\alpha}$, provided that I is initially confined. However, if
 199 I is not initially confined but greater than zero over the entire domain, the
 200 decay rate for both solutions will be of the form $x^{-\alpha}$ (see Appendix B).

201 The asymptotic front speed can be derived from either Eq. (17) or (18)
 202 by computing the Lagrangian trajectory of a point at the leading edge of
 203 the front. For instance, if we consider a point with a fixed value $\hat{S} \approx 1$, its
 204 position $\hat{x} = x(t, \hat{S})$ can be expressed as

$$\hat{x} \sim (1 - \hat{S})^{-1/(\alpha+1)} t^{1/(\alpha+1)} e^{\frac{1-\lambda}{\alpha+1}t},$$

205 and the front speed, $c_R(t) = \frac{d\hat{x}}{dt}$, therefore reads

$$\begin{aligned} c_R(t) &\sim (1 - \hat{S})^{-1/(\alpha+1)} \frac{t^{1/(\alpha+1)}}{\alpha + 1} e^{\frac{1-\lambda}{\alpha+1}t} \left(\frac{1}{t} + 1 - \lambda \right), \\ &\sim (1 - \lambda) e^{\frac{1-\lambda}{\alpha+1}t}, \end{aligned}$$

206 for large values of t . This highlights the exponential acceleration of right-
 207 propagating fronts.

208 4. Numerical examples

209 In this section, we present some numerical simulations that illustrate the
 210 theoretical results derived in the previous section. The model Eqs. (1)-(2)
 211 are solved on a finite domain $[0, L]$, where $L > 0$, and the fractional-order
 212 derivative is thus defined as follows:

$${}_0D_x^\alpha f(x) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_0^x \frac{f(y)}{(x-y)^{\alpha-1}} dy,$$

213 where $1 < \alpha \leq 2$ and $0 \leq x \leq L$, since model variables are now only defined
 214 on $[0, L]$. Solving the model equations on a finite domain has an impact on

215 the behavior of the solution as fractional-order derivatives take into account
 216 the global aspect of the solution, which is obviously modified when truncating
 217 the domain. One can still take into account the effect of an infinite domain
 218 by using the following equations:

$$\frac{\partial S}{\partial t} = -IS + {}_0D_x^\alpha(S - S(0) - S'(0)), \quad (19)$$

$$\frac{\partial I}{\partial t} = IS - \lambda I + {}_0D_x^\alpha(I - I(0) - I'(0)), \quad (20)$$

219 where $S(0) = S(x = 0, t)$ and $S'(0) = S'(x = 0, t)$, and the same for I .
 220 In Eqs. (19)-(20), the fractional-order derivative is now a so-called Caputo
 221 fractional derivative that can be defined as follows:

$$\begin{aligned} {}_0^C D_x^\alpha f(x) &= \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{f''(y)}{(x-y)^{\alpha-1}} dy, \\ &= {}_0D_x^\alpha(f(x) - f(0) - f'(0)). \end{aligned}$$

222 By using some standard properties of the Caputo derivative, it can be shown
 223 that Eqs. (19)-(20) are equivalent to Eqs. (5)-(6) if one assumes constant
 224 density values for $x \leq 0$, *i.e.* $I(x < 0, t) = I(0)$ and $S(x < 0, t) = S(0)$. One
 225 can then represent the effect of an infinite reservoir of susceptibles and/or
 226 infectives at the left-hand side of the domain by selecting an initial condition
 227 which is such that $S(0) > 0$ and/or $I(0) > 0$.

228 Eqs. (19)-(20) are discretized with a continuous piecewise-linear finite-
 229 element scheme on a uniform grid whose resolution is equal to $L/500$. The
 230 finite element scheme is based on a Galerkin formulation that allows one to
 231 integrate the fractional-order diffusion term by parts and impose a vanishing
 232 fractional-order flux on the left boundary (see [20] for details). A zero-slope
 233 boundary condition is imposed on the right boundary. The diffusion term
 234 being entirely left-sided, the right boundary condition only has a local im-
 235 pact on the solution. The use of a zero-slope boundary condition allows the
 236 right-propagating front to smoothly leave the domain. Time integration is
 237 computed with a third-order Adams-Bashforth scheme.

238 The following initial conditions are used for both left- and right-propagating
 239 fronts:

$$S_0(x) = 1 - h_S \left(1 \pm \tanh \left(\frac{x - x_0}{w_S L} \right) \right), \quad (21)$$

$$I_0(x) = h_I \exp \left(-\frac{(x - x_0)^2}{w_I L^2} \right), \quad (22)$$

240 where $w_S = 0.009$, $h_S = 0.37$, $w_I = 0.0003$ and $h_I = 0.14$. For a left-
 241 propagating front it takes the “+” sign and $x_0 = 9L/10$, whereas for a
 242 right-propagating front, $S_0(x)$ takes the “-” sign and $x_0 = L/10$ or 0. The
 243 dimensionless domain length is set to $L = 10^5$ for a right-propagating front
 244 and $L = 60$ for a left-propagating front. For all simulations, $\lambda = 0.5$.

245 The time evolution of a left-propagating front is shown in Fig. 2. The top
 246 panel shows the evolution of both the susceptibles and infectives densities.
 247 The middle and bottom panels show a close-up view on the susceptibles and
 248 infectives density tail decay. The duration of the simulation is set to 30 di-
 249 mensionless time units and $\alpha = 1.2$. As expected, both the susceptibles and
 250 infectives fronts propagate at the same speed and exhibit the same exponen-
 251 tially decaying tail, *i.e.* $1 - S$ and $I \sim \exp\left[\left(\frac{1-\lambda}{\alpha-1}\right)^{1/\alpha} x\right]$. The front speed
 252 has been numerically estimated by computing the instantaneous speed of a
 253 point with density $S = 0.99$ (Fig. 3). The resulting time series highlights
 254 the convergence of the front speed towards the minimum speed $c_{min} = 1.398$,
 255 obtained from Eq. (13).

256 The time evolution of a right-propagating front is shown in Fig. 4. We
 257 first consider a localized initial condition for the infectives by taking $x_0 =$
 258 $L/10$. In that case, $I(0) = 0$ and there is no reservoir of infectives at the left-
 259 had side of the computational domain. However, $S(0) > 0$ and there is thus
 260 an infinite reservoir of susceptibles at the left-hand side. Such a situation
 261 is quite realistic as an epidemic usually starts with a confined population
 262 of infective individuals in contact with a broad population of susceptible
 263 individuals. The simulation duration and the value of α are the same as for
 264 the left-propagating front. As expected, both the susceptibles and infectives
 265 fronts accelerate and develop the same algebraic decaying tail, *i.e.* $1 - S$
 266 and $I \sim x^{-(\alpha+1)}$. After an initial adjustment, the acceleration of the front
 267 leads to a rapid spread of the epidemic over the entire domain. It should be
 268 noted that the length of the domain is equal to 10^5 as compared to 60 for the
 269 left-propagating front simulation whereas the simulation duration remains
 270 the same. It should also be noted that infinite reservoir of susceptibles does
 271 not influence the rate of decay. The epidemic being driven by the infectives
 272 and not by the susceptibles density, the front dynamics entirely depends on
 273 the infectives density. Actually, even if there was no diffusion term in the
 274 S equation and thus if susceptible individuals were not moving, there would
 275 still be the same epidemic front.

276 The acceleration of the right-propagating front is highlighted in the space-

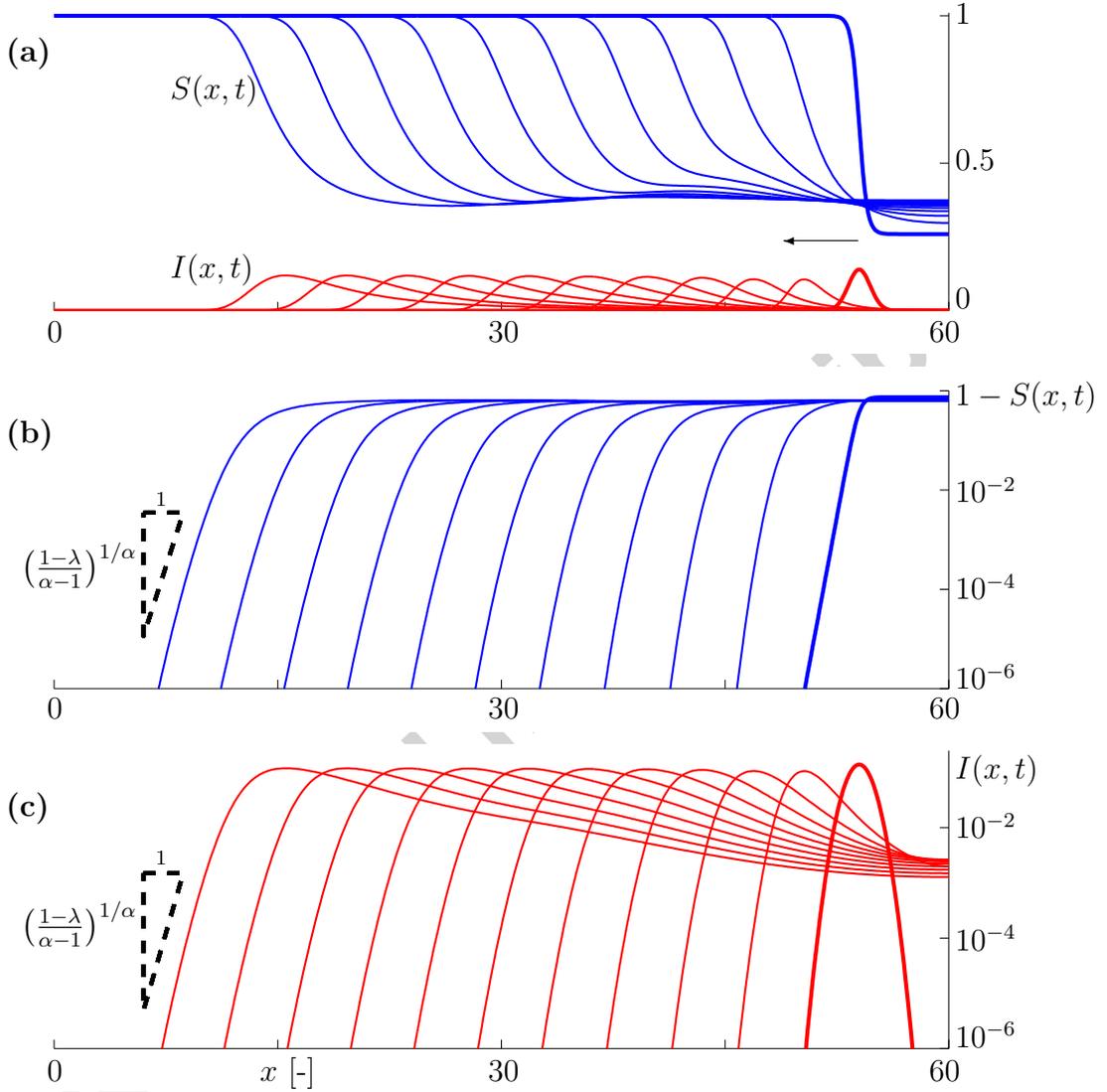


Figure 2: (a) Left-propagating front profiles for the susceptibles and infectives densities at different time instants obtained by solving Eqs. (19)-(20) with $\alpha = 1.2$. The arrow indicates the direction of propagation of the front. (b) and (c) Close-up views of the susceptibles and infectives densities, respectively, highlighting the same exponentially decaying tail for both fronts, *i.e.* $1 - S$ and $I \sim \exp\left[\left(\frac{1-\lambda}{\alpha-1}\right)^{1/\alpha} x\right]$, and the same front speeds. The simulation duration is set to 30 and the time interval between front profiles is set to 3.

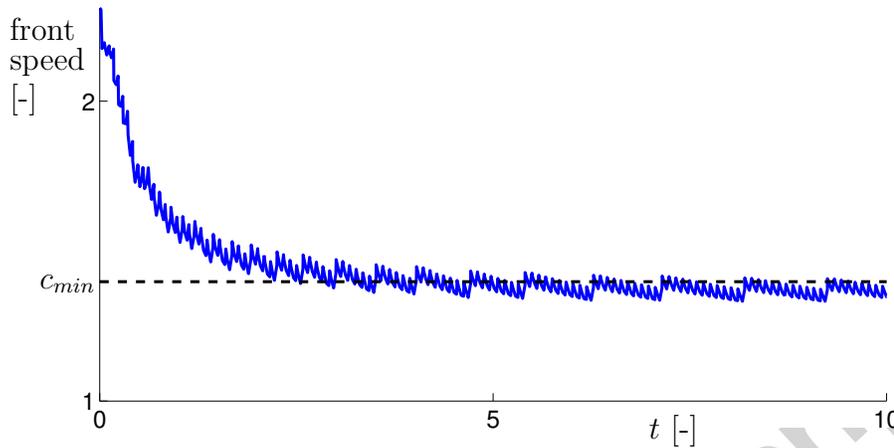


Figure 3: Time evolution of the left-propagating front instantaneous speed, taken as the time derivative of the Lagrangian trajectory $x_f(t; S = 0.99)$, for $\alpha = 1.2$. As expected, the front speed converges towards the minimum front speed value $c_{min} \approx 1.398$.

277 time diagram shown in Fig. 5. It shows the evolution of the position of the
 278 front leading edge defined as the Lagrangian trajectory $x_f(t) = x(t; S = 0.99)$
 279 that corresponds to the position of a point in the front with density $S = 0.99$.
 280 The simulation duration is set to 60. It can be seen that the front eventually
 281 accelerates as soon as $\alpha < 2$. The smaller the value of α , the sooner the
 282 acceleration takes place. Fig. 5 also shows that the numerical results are in
 283 good agreement with the asymptotic expansion, *i.e.* $1 - S \sim x^{-(\alpha+1)}e^{(1-\lambda)t}$.

284 Although that might not be very realistic, one can still represent the effect
 285 of an infinite reservoir of infective individuals located at the left-hand side of
 286 the domain by considering an initial solution that does not vanish at $x = 0$.
 287 This is achieved by taking $x_0 = 0$ in the initial conditions (21)-(22) such
 288 that $I(0) > 0$ and $S(0) > 0$. In that case, the front is expected to exhibit an
 289 algebraic decay rate of order $-\alpha$ rather than $-(\alpha+1)$ (see Appendix B). Fig.
 290 6 shows that such an asymptotic behavior is indeed observed as $1 - S$ and
 291 $I \sim x^{-\alpha}$. These results are similar to those obtained by del Castillo Negrete
 292 et al. [8] for the fractional-order Fisher-Kolmogorov model for which they
 293 had also considered an initial condition that did not vanish at $x = 0$.

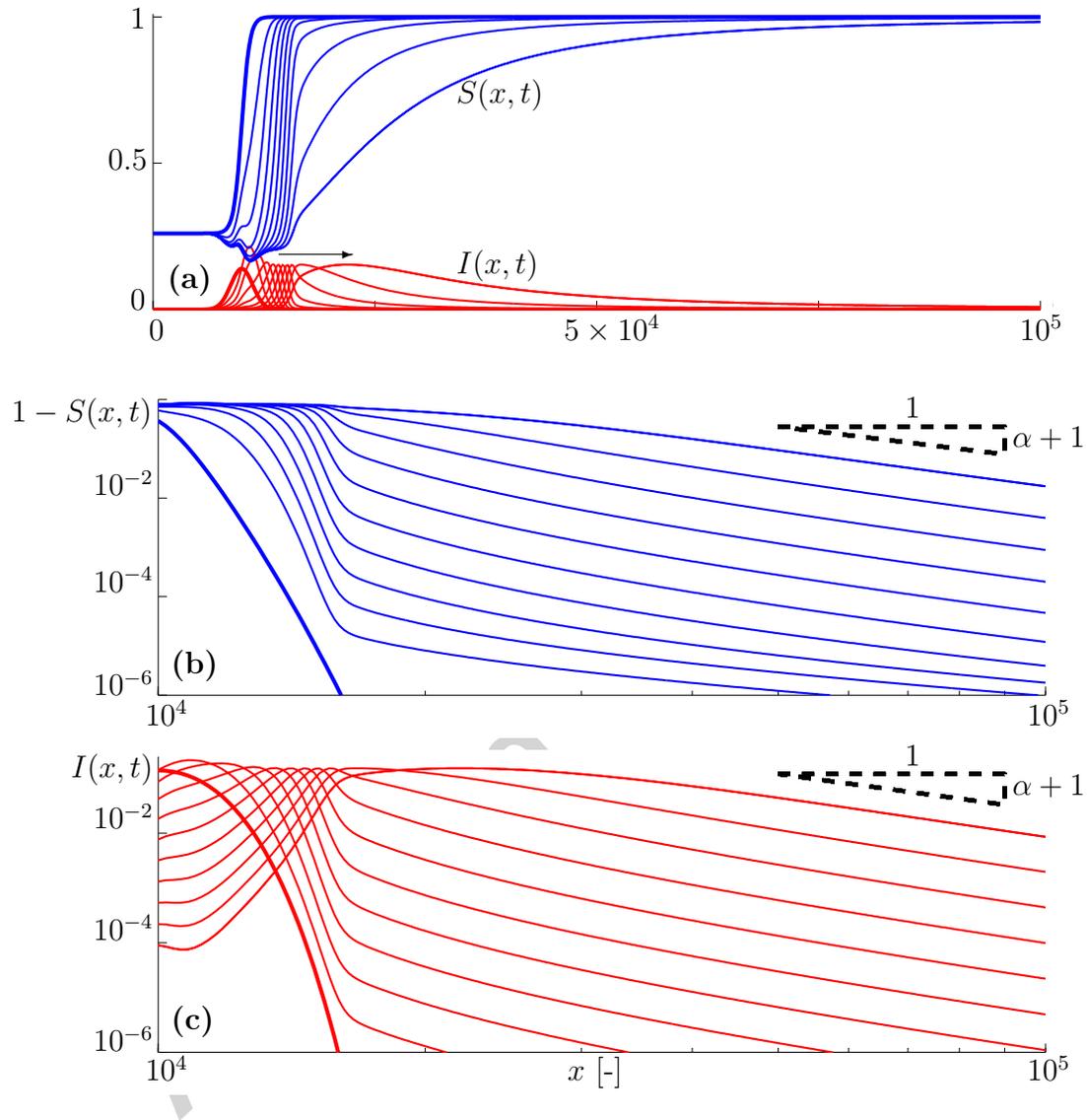


Figure 4: (a) Right-propagating front profiles for the susceptibles and infectives densities at different time instants obtained by solving Eqs. (19)-(20) with $\alpha = 1.2$. The arrow indicates the direction of propagation of the front. (b) and (c) Close-up views of the susceptibles and infectives densities, respectively, highlighting the same power-law decaying tail for both fronts, *i.e.* $1 - S$ and $I \sim x^{-(\alpha+1)}$. The simulation duration is set to 30 and the time interval between front profiles is set to 3.

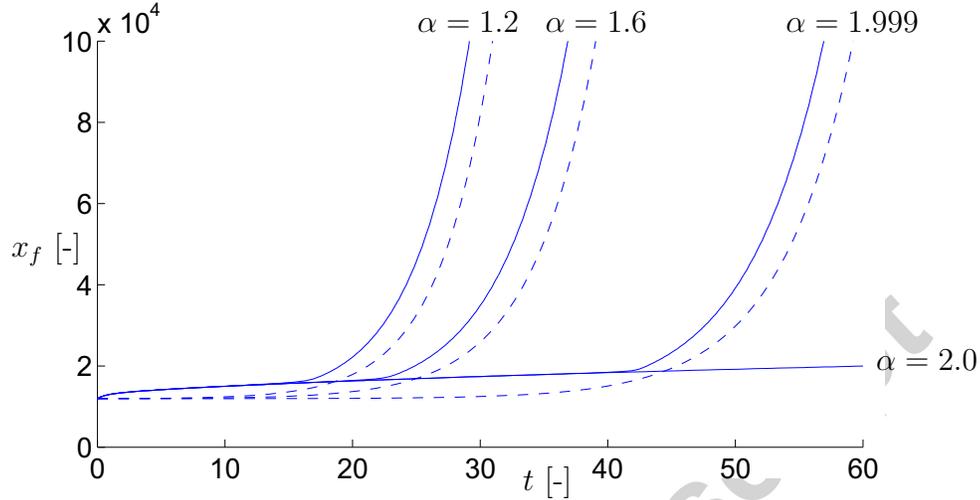


Figure 5: Time evolution of the epidemic front leading edge position, taken as the Lagrangian trajectory $x_f(t; S = 0.99)$, for different values of α . As soon as $\alpha < 2$, the front eventually accelerates leading to long-range spreading of the epidemic. The dashed line corresponds to the analytical scaling result $1 - S \sim x^{-(\alpha+1)} e^{(1-\lambda)t}$.

294 5. Conclusions

295 In this paper, we have proposed a fractional-order reaction-diffusion model
 296 to study the dynamics of epidemics in systems characterized by asymmet-
 297 ric Lévy flights with an exponent $1 < \alpha \leq 2$. We have considered a to-
 298 tally skewed, left-sided diffusion operator and studied its impact on left- and
 299 right-propagating fronts. On the one hand, numerical and analytical results
 300 show that left-propagating fronts move at a constant velocity and have an
 301 exponentially decaying tail. The classical results for the front velocity are
 302 recovered when $\alpha = 2$. On the other hand, right-propagating fronts accel-
 303 erate exponentially and exhibit a power-law decaying tail. Both the infectives
 304 and susceptibles fronts have the same dynamics which is entirely driven by
 305 the infectives density. For an initially localized infectives density profile, the
 306 decay is of the form $\sim x^{-(\alpha+1)}$ while an infinitely-wide initial profile leads
 307 to a decay of the form $\sim x^{-\alpha}$ for both solutions. Obviously, for a symmet-
 308 ric fractional-order diffusion operator, both the left- and right-propagating
 309 fronts accelerate and have a power-law decaying tail.

310 Our results show that a fractional-order reaction-diffusion epidemics model
 311 is able to represent the superdiffusive effect due to Lévy-flight mobility pat-

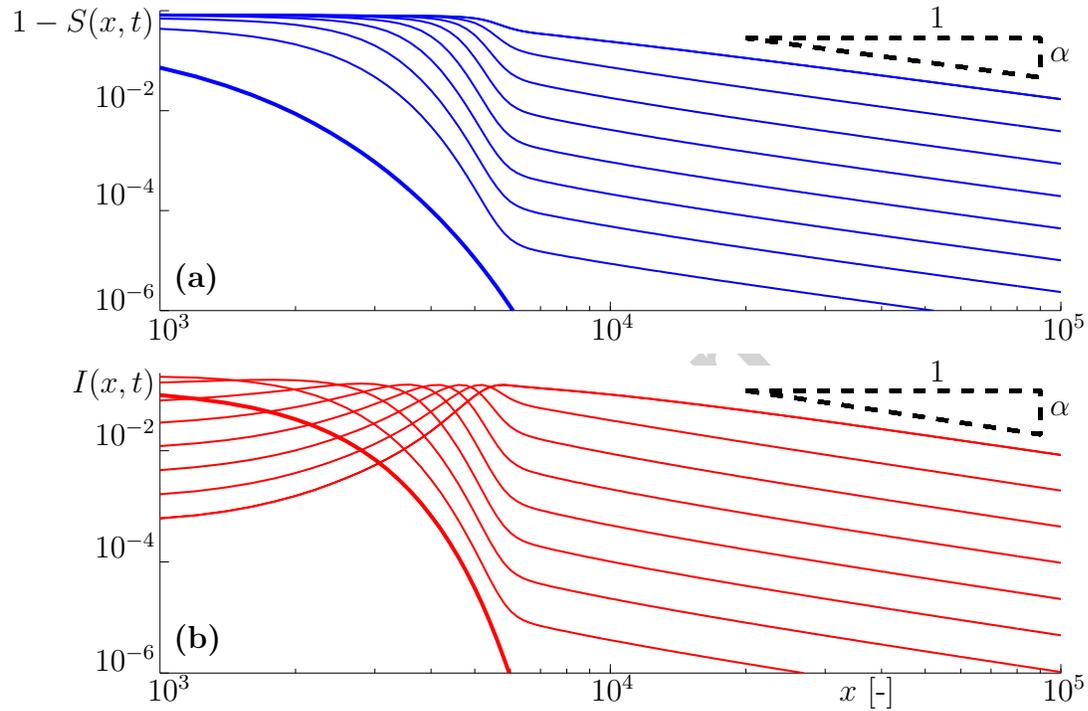


Figure 6: Right-propagating front profiles for the (a) susceptibles and (b) infectives density at different time instants obtained by solving Eqs. (19)-(20) with $\alpha = 1.2$ and taking $x_0 = 0$ in the initial conditions (21)-(22). The power-law decaying tail is the same for both fronts, *i.e.* $1 - S$ and $I \sim x^{-\alpha}$. The simulation duration is set to 24 and the time interval between front profiles is set to 3.

312 terns in the population. The superdiffusion of the infective population leads
 313 to a significantly increased overall reaction rate as both populations meet each
 314 other more often. As a result, an epidemic spreads much faster than predicted
 315 by classical Gaussian models. Our observations suggest that fractional-order
 316 reaction-diffusion models are better suited to represent modern epidemics and
 317 also highlight that the eradication or even containment of such epidemics is
 318 a daunting task.

319 The model could be further improved by taking into account non-Markovian
 320 and truncated Lévy processes. Brockmann et al. [2] have shown that the
 321 dispersion of bank notes is not only non-Gaussian but also non-Markovian.
 322 The non-Markovian effect and the associated subdiffusion can be included
 323 in our model by replacing the first-order time derivative with a fractional-
 324 order time derivative of order less than 1 (see [28] for details). Numerical
 325 methods have recently been proposed to discretize the space-time fractional
 326 diffusion equation [34, 21] and could be applied in a non-Gaussian and non-
 327 Markovian epidemics models. Furthermore, the study by Gonzalez et al. on
 328 human mobility patterns derived from mobile phone data suggests that these
 329 patterns follow a truncated Lévy-flight motion [18]. The effect of truncation
 330 on superdiffusive fronts propagation has been studied by del Castillo Negrete
 331 [7] for a Fisher-Kolmogorov model and should be considered for epidemics
 332 models as well.

333 Acknowledgements

334 Eric Deleersnijder is a Research associate with the Belgian National Fund
 335 for Scientific Research (FNRS).

336 Appendix A. Alternative derivation of the left-moving front speed

337 In this section, we present a more general derivation of Eqs. (13) based on
 338 the approach proposed by van Saarloos et al. [14, 38]. In those studies, the
 339 authors define an asymptotic spreading speed, that they denote v^* , towards
 340 which the front eventually converges if it evolves from a sufficiently steep
 341 initial state. That speed is obtained by linearizing the model equations about
 342 the unstable state ($S = 1$ and $I = 0$ in our case). More specifically, $v^* \in \mathbb{R}$
 343 is given by the largest "dynamically relevant" solution of the saddle points

344 equations

$$\begin{cases} 0 &= \mathcal{S}(k^*, \omega^*), \\ v^* &= \left. \frac{d\omega(k)}{dk} \right|_{k^*}, \\ v^* &= \frac{\text{Im } \omega^*}{\text{Im } k^*}, \end{cases}$$

345 solved for (k^*, ω^*, v^*) where $\mathcal{S}(k, \omega) = 0$ is the characteristic equation ob-
 346 tained by assuming a Fourier mode solution of the form $e^{i(kx - \omega t)}$ (k and
 347 $\omega \in \mathbb{C}$) in the linearization about the unstable state ahead of the front.

348 In our case, the characteristic equation reads:

$$\mathcal{S}(k, \omega) = (-i\omega - (ik)^\alpha)^2 - (1 - \lambda)(-i\omega - (ik)^\alpha) = 0$$

349 and has two solutions: $\omega = i(ik)^\alpha$ and $\omega = i(ik)^\alpha + i(1 - \lambda)$. The former is
 350 rejected as it leads to $v^* = 0$. The latter corresponds to a non-oscillating,
 351 uniformly translating front if k^* and ω^* are purely imaginary, which can only
 352 be achieved if $k^* = -i\kappa^*$ ($\kappa^* \in \mathbb{R}^+$). In that case,

$$\begin{aligned} v^* &= \left. \frac{d\omega(k)}{dk} \right|_{k^*} = -\alpha(\kappa^*)^{\alpha-1}, \\ &= \frac{\text{Im } \omega^*}{\text{Im } k^*} = \frac{(\kappa^*)^\alpha + (1 - \lambda)}{-\kappa^*}, \end{aligned}$$

353 which leads to

$$v^* = -\alpha \left(\frac{1 - \lambda}{\alpha - 1} \right)^{\frac{\alpha-1}{\alpha}}, \quad \kappa^* = \left(\frac{1 - \lambda}{\alpha - 1} \right)^{1/\alpha}.$$

354 As expected, these correspond to Eqs. (13) and highlight that only a left-
 355 propagating front can travel at a constant speed as $v^* = -c_{min} < 0$. More-
 356 over, van Saarloos et al. have shown that the front speed algebraically relaxes
 357 towards that constant speed if the initial state is steeper than $e^{\kappa^* x}$.

358 Appendix B. Asymptotic behaviour of the right-moving front

359 In this section, we provide more details on the derivation of Eq. (17).
 360 Let us first consider the first integral in Eq. (16) and look at the limit for
 361 $xt^{-1/\alpha} \rightarrow \infty$ with a fixed t . In that case, we can use the asymptotic behaviour

362 of the Lévy distribution: $p_\alpha(\eta) \sim \eta^{-(\alpha+1)}$ to obtain:

$$\begin{aligned}
\int_{xt^{-1/\alpha}}^{(x+l)t^{-1/\alpha}} p_\alpha(\eta) \, d\eta, &\sim \int_{xt^{-1/\alpha}}^{(x+l)t^{-1/\alpha}} \frac{1}{\eta^{\alpha+1}} \, d\eta, \\
&= \frac{-1}{\alpha} \left[\frac{1}{\eta^\alpha} \right]_{xt^{-1/\alpha}}^{(x+l)t^{-1/\alpha}}, \\
&= \frac{-t}{\alpha} x^{-\alpha} \left(\left(1 + \frac{l}{x}\right)^{-\alpha} - 1 \right), \\
&\sim tlx^{-(\alpha+1)}, \tag{B.1}
\end{aligned}$$

363 where we have used a first-order binomial-series approximation.

364 For the second integral, one needs to set a cutoff Ω with $1 \ll \Omega < xt^{-1/\alpha}$,
365 so that $p_\alpha(\eta) = \eta^{-(\alpha+1)}$ holds. Integration by parts then leads to

$$\begin{aligned}
&e^{-\kappa x} \int_{-\infty}^{xt^{-1/\alpha}} p_\alpha(\eta) e^{\kappa t^{1/\alpha} \eta} \, d\eta \\
&\sim e^{-\kappa x} \int_{\Omega}^{xt^{-1/\alpha}} \frac{1}{\eta^{\alpha+1}} e^{\kappa t^{1/\alpha} \eta} \, d\eta, \\
&= e^{-\kappa x} \left[\frac{1}{\eta^{\alpha+1}} \frac{e^{\kappa t^{1/\alpha} \eta}}{\kappa t^{1/\alpha}} \right]_{\eta=\Omega}^{\eta=xt^{-1/\alpha}} - e^{-\kappa x} \int_{\Omega}^{xt^{-1/\alpha}} -\frac{\alpha+1}{\eta^{\alpha+2}} \frac{e^{\kappa t^{1/\alpha} \eta}}{\kappa t^{1/\alpha}} \, d\eta, \\
&\sim e^{-\kappa x} \left(\frac{1}{x^{\alpha+1} t^{-1-1/\alpha}} \frac{e^{\kappa t^{1/\alpha} xt^{-1/\alpha}}}{\kappa t^{1/\alpha}} \right) + t \frac{(1+\alpha)}{\kappa t^{1+1/\alpha}} \int_{\Omega}^{xt^{-1/\alpha}} \frac{e^{\kappa(t^{-1/\alpha} \eta - x)}}{\eta^{\alpha+2}} \, d\eta, \\
&= t \left(\frac{x^{-(\alpha+1)}}{\kappa} + \frac{(1+\alpha)}{\kappa t^{1+1/\alpha}} \int_{\Omega}^{xt^{-1/\alpha}} \frac{e^{\kappa(t^{-1/\alpha} \eta - x)}}{\eta^{\alpha+2}} \, d\eta \right). \tag{B.2}
\end{aligned}$$

366 In fact, the integral from $-\infty$ to Ω contains $p_\alpha(\eta)$ that decays exponentially
367 at minus infinity. The exponential in the integrand is bounded on this do-
368 main. We can thus consider that the integral is bounded and not dependant
369 on x . The integrand in the last expression is bounded by $\frac{1}{\eta^{2+\alpha}}$ when $x \rightarrow \infty$
370 and thus, the third term is at most of order $x^{-(\alpha+1)}$.

371 From (B.1) and (B.2), we see that the tail of the right-propagating front
372 behaves like $\psi \sim tx^{-(\alpha+1)}$ for large values of x and t . Note that this behaviour
373 is preserved in the limit where $l \rightarrow 0$. However, in the limit where $l \rightarrow \infty$,
374 the asymptotic behaviour becomes $\psi \sim tx^{-\alpha}$. That case would represent the

375 effect of an infinite initial reservoir of infective individuals on the epidemic
 376 front propagation. In the study by del Castillo Negrete et al. [8] for the
 377 fractional-order Fisher-Kolmogorov equation, non-localized initial conditions
 378 are considered, *i.e.* $l \rightarrow \infty$, and an algebraic decay rate of order $-\alpha$ is
 379 observed.

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