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# Algorithm to calculate the Minkowski sums of 3-polytopes : application to tolerance analysis

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## Abstract

In tolerance analysis, it is necessary to check that the cumulative defect limits specified for the component parts of a product are compliant with the functional requirements expected of the product. Cumulative defect limits can be modelled using a calculated polytope, the result of a set of intersections and Minkowski sums of polytopes. This article presents a method to be used to determine from which vertices of the operands the vertices of the Minkowski sum derive and to which facets of the operands each facet of the Minkowski sum is oriented.

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*Keywords:* Tolerancing analysis ; Minkowski sum ; polytope ; normal fan

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## 1. Introduction

Minkowski sums can be used in many applications. Some of the most important ones that should be mentioned are for determining the envelope volume generated by displacement between two solids, whether in geometric modelling, robotics or for simulating shapes obtained by digitally controlled machining [1].

In geometric tolerancing Fleming established in 1988 [2] the correlation between cumulative defect limits on parts in contact and the Minkowski sum of finite sets of geometric constraints. For examples of modelling dimension chains using Minkowski sums of finite sets of constraints, see [3], [4], [5] and [6]. In tolerance analysis, it is necessary to check that the cumulative defect limits specified for the component parts of a product are compliant with the functional requirements expected of the product. Defect limits can be modelled by tolerance zones constructed by offsets on nominal models of parts [7]. Cumulative defect limits can be modelled using a calculated polytope, the result of a set of intersections and Minkowski sums of polytopes. A functional requirement can be qualified by a functional polytope, in other words a target polytope. It is then necessary to verify whether the calculated polytope is included in the functional polytope [8], see fig. 1.

To optimise the filling of the functional polytope (see fig. 1), it is crucial to know.

- from which vertices of the four operands the vertices of the calculated polytope derive,
- from which facets of the four operands the normals of the facets of the calculated polytope derive.

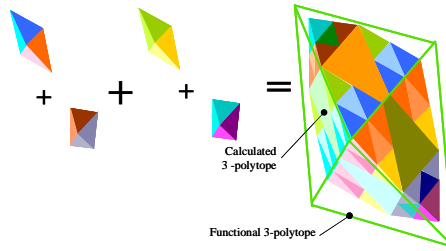


Fig. 1. Verification of inclusion of 3-polytope [13]

The purpose of this article is to determine the Minkowski sum of 3-dimension polytopes and apply this effectively in order to optimise the filling of the functional polytope. Our approach is based on polytope properties, most of which are described in [9] and [10].

Several different approaches have been proposed in the literature to determine the Minkowski sum, most of which relate to 3-dimension geometrical applications.

For example, [11] improves the concept of slope diagrams introduced by [12] to determine facets of connection. The same principle is used by [13] who proposes an algorithm called a Contributing Vertices-based Minkowski Sum.

## 2. Some properties of polytopes

### 2.1. Two dual definitions for polytopes

A polytope  $\mathcal{P}$  is a bounded intersection of many finitely closed half-spaces in some  $\mathbb{R}^n$  (see fig.2) [10]. This is the h-representation of a polytope [14].

In this article, a system of inequalities for  $m$  half-spaces  $\bar{H}^-$  has been chosen to define a polytope  $\mathcal{P}$  as eq. 1:

$$\mathcal{P} = \mathcal{P}(A, b) = \{x \in \mathbb{R}^n : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\} \text{ with } \mathbf{A} \in \mathbb{R}^{m \times n} \text{ and } \mathbf{b} \in \mathbb{R}^m \tag{1}$$

A polytope  $\mathcal{P}$  is a convex hull of a finite set of points in some  $\mathbb{R}^n$  (see fig. 2) [10], [14].

Let us consider  $\mathcal{V}$ , a finite set of points in some  $\mathbb{R}^n$  (see fig. 2):  $\mathcal{P} = \text{conv}(\mathcal{V})$ .

This is the v-representation of a polytope [14].

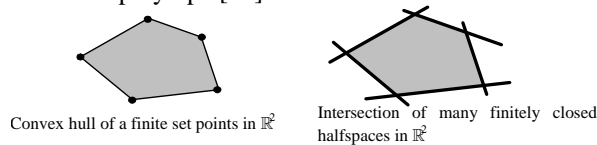


Fig. 2 Definitions of a 2-polytope in  $\mathbb{R}^2$  [10].

A polytope of dimension  $k$  is denoted a  $k$ -polytope in  $\mathbb{R}^n$  with  $(n \geq k)$ .

A 0-polytope is a vertex, a 1-polytope is an edge and a 2-polytope is a 2-face.

### 2.2. Primal cone and dual cone

A cone is a non-empty set of vectors that with any finite set of vectors also contains all their linear combinations with non-negative coefficients [10].

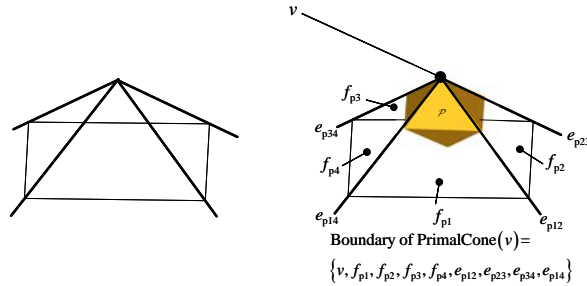


Fig. 3 Cone  $\mathbb{R}^3$  and primal cone of a  $\mathbb{R}^3$  polytope.

Let us consider  $Y = \{y_i\}$  a finite set of  $m$  points in some  $\mathbb{R}^n$ . The cone associated to  $Y$  is [10]:

$$\text{Cone}(Y) = \{t_1 \cdot y_1 + \dots + t_i \cdot y_i + \dots + t_m \cdot y_m : t_i \geq 0\} \tag{2}$$

There is an equivalent definition with half-spaces such that the border contains the origin (see fig. 3):

$$\text{Cone}(Y) = \bigcap_{i=1}^m \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n a_{ij} x_j \leq 0 \right\} \tag{3}$$

Every vertex  $v$  of a polytope  $\mathcal{P}$  has an associated primal cone and dual cone.

In 3-dimension, the boundary of the primal cone  $\text{PrimalCone}(v)$  consists of the vertex  $v$ , the facets  $f_{pi}$  of  $\mathcal{P}$  converging at the vertex  $v$  and the edges  $e_{pij}$  converging at the vertex  $v$  so that an edge  $e_{pij}$  forms a common boundary between adjacent facets  $f_{pi}$  and  $f_{pj}$  (see fig. 3).

Let us consider an objective function of the shape:  $p(x, y, z) = \alpha \cdot x + \beta \cdot y + \gamma \cdot z + \lambda$  and polytope  $\mathcal{P}$  together define the function  $p(x, y, z)$ . The objective function is maximal on one face  $F$  (2-face, edge or vertex) of  $\mathcal{P}$ . Let  $\mathbf{d}$  be a normal to the objective function oriented towards the exterior of polytope  $\mathcal{P}$  (see fig. 4a). The set of objective functions that reach their maximum in  $F$  is characterised by a polyhedral cone  $\text{DualCone}(F)$  defined by the set of normals  $\mathbf{d}$ , thus in  $\mathbb{R}^3$  [10]:

$$\text{DualCone}(F) = \left\{ \mathbf{d} \in \mathbb{R}^3 : F \subseteq \left\{ \mathbf{x} \in \mathcal{P} : \mathbf{d} \cdot \mathbf{x} = \max_{\mathbf{y} \in \mathcal{P}} \mathbf{d} \cdot \mathbf{y} \right\} \right\} \tag{4}$$

$\text{DualCone}(F)$  is called the dual cone of polytope  $\mathcal{P}$  in  $F$ . It is called the normal cone of polytope  $\mathcal{P}$  in  $F$  [10], [14]. The dual cone associated with a face of dimension  $i$  has a dimension  $(n - i)$  in  $\mathbb{R}^n$  [9].

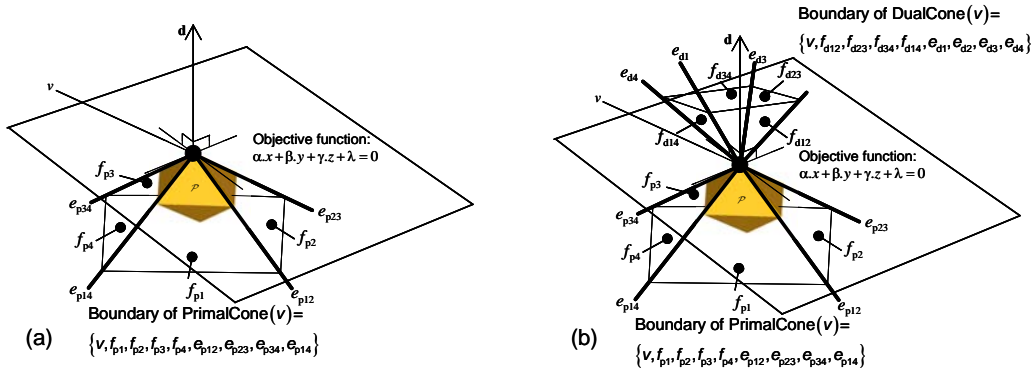


Fig. 4 Definition of a dual cone - Dual cone and primal cone attached to a vertex.

Let us consider the dual cone of polytope  $\mathcal{P}$   $\text{DualCone}(v)$  associated with the vertex  $v$ . In  $\mathbb{R}^3$  this cone is 3-dimension given that the dimension of vertex  $v$  is 0. It consists of the vertex  $v$ , facets  $f_{dij}$  that converge at  $v$  with their normals being respectively edges  $e_{pij}$  and edges  $e_{di}$  converging in  $v$  and these are in turn normal to facets  $f_{pi}$ .

Fig. 4b shows the primal cone and the dual cone associated with the vertex  $v$  of polytope  $\mathcal{P}$ .

### 2.3. Fan and normal fan

A fan in  $\mathbb{R}^n$  is a family  $\Phi = \{C_1, \dots, C_k\}$  of polyhedral cones with the following properties:

- each non-empty face of a cone in  $\Phi$  is also a cone in  $\Phi$ ,
- the intersection of two cones in  $\Phi$  is a face common to the two cones.

The fan  $\Phi$  is complete if and only if:  $\bigcup_{i=1}^k C_i = \mathbb{R}^n$  [10].

For any facet  $F$  of polytope  $\mathcal{P}$ , the set of dual cones  $\text{DualCone}(F)$  partitions  $\mathbb{R}^n$ . The set of dual cones defines a fan, which we will call the normal fan [9], [10].

The normal fan associated with polytope  $\mathcal{P}$  is:  $N(\mathcal{P})$ .

Let  $\Phi_1$  and  $\Phi_2$  be two fans of  $\mathbb{R}^n$ . Then the common refinement of  $\Phi_1$  and  $\Phi_2$  [10] is:

$$\Phi_1 \wedge \Phi_2 = \{C_1 \cap C_2 : C_1 \in \Phi_1, C_2 \in \Phi_2\} \tag{5}$$

To determine the common refinement of two fans  $\Phi_1$  and  $\Phi_2$  a normal fan has to be determined which consists of the set of all the intersections of the dual cones of the two fans  $\Phi_1$  and  $\Phi_2$  considered two by two.

## 3. Minkowski sum by operations on dual cone

### 3.1. Problems in determining the Minkowski sum for two polytopes

#### 3.1.1. Properties of dual cones

The definition of a Minkowski sum is given in equation (6) [14].

$$\mathcal{A} + \mathcal{B} = \mathcal{C} = \{ \mathbf{c} \in \mathbb{R}^n \mid \exists \mathbf{a} \in \mathcal{A}, \exists \mathbf{b} \in \mathcal{B} : \mathbf{c} = \mathbf{a} + \mathbf{b} \} \tag{6}$$

In 1-dimension, this consists of adding together variables with boundaries at certain intervals.

For 2 and 3 dimensions, the Minkowski sum consists of carrying out a sweep from a reference point on one operand at the boundary of the other operand [11], [13].

In  $\mathbb{R}^2$ , the edges of the polytope sum are translations of the edges of the two operand polytopes [11], [13]. In  $\mathbb{R}^3$ , certain facets of the polytope sum are translations of the facets of the two operand polytopes. However, other facets are created, which we will call facets of connection. Thus it is not possible to deduce the facets of the polytope sum knowing only the facets of the operand polytopes. This property is illustrated in fig. 5 and discussed in [11], [13].

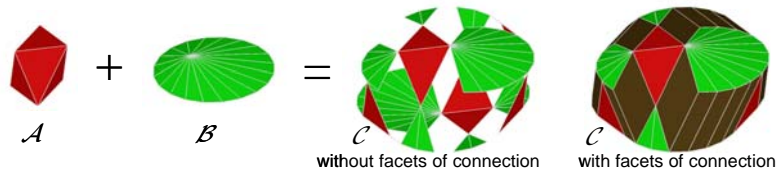


Fig. 5 Minkowski sum of  $\mathbb{R}^3$  polytopes.

The work of [11], [13] can be further justified. The following property developed in [9], and also mentioned in [10] and [14], shows that the normal fan  $N(\mathcal{C})$  of polytope  $\mathcal{C}$ , the Minkowski sum of polytopes  $\mathcal{A}$  and  $\mathcal{B}$ , is the common refinement of the two normal fans of polytopes  $\mathcal{A}$  and  $\mathcal{B}$ :

$$N(\mathcal{C}) = N(\mathcal{A} + \mathcal{B}) = N(\mathcal{A}) \wedge N(\mathcal{B}) \tag{7}$$

According to (5), determining the common refinement of two normal fans is based only on intersections of dual cones considered two by two. It is therefore not possible to create new edges in 2-dimension when determining the normal fan  $N(\mathcal{C})$ . This is the reason why there is no facet of connection in the Minkowski sum for two polytopes. In 3-dimension, new edges can be created. According to (4) and the properties of the dual cones cited in §2.2, these new edges are normals to the facets of connection of the polytope sum.

In this article we propose a method to construct facets of the polytope sum  $\mathcal{C}$  based solely on intersections of dual cones on operands  $\mathcal{A}$  and  $\mathcal{B}$ .

We can then determine: the Minkowski vertices of  $\mathcal{C}$ , the dual cones associated with the vertices of  $\mathcal{C}$  and the normal fan  $N(\mathcal{C})$ .

From the vertices of  $\mathcal{C}$  and the respective dual cones, the facets of  $\mathcal{C}$  can be defined.

Finally, using the proximity of the dual cones in the normal fan  $N(\mathcal{C})$ , the ordered edges defining the limits of the support hyperplane will be determined in order to define each facet of  $\mathcal{C}$ .

### 3.2. Determining the vertices of the Minkowski sum of polytopes

Let the points of a face of polytope  $\mathcal{A}$  maximising the objective function characterised by the vector be  $\mathbf{a}$ :

$$S(\mathcal{A}, \mathbf{a}) = \left\{ \mathbf{x} \in \mathcal{A} : \mathbf{a} \cdot \mathbf{x} = \max_{\mathbf{y} \in \mathcal{P}} \mathbf{a} \cdot \mathbf{y} \right\} \tag{8}$$

Let us consider the following property:

Let  $a$  be a vertex of  $\mathcal{A}$  and  $\text{DualCone}(a)$  the associated dual cone.

$$\text{We have: } \mathbf{a} \in \text{DualCone}(a) \Leftrightarrow S(\mathcal{A}, \mathbf{a}) = \{\mathbf{a}\} \tag{9}$$

Let  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  be three polytopes such that:  $\mathcal{A} + \mathcal{B} = \mathcal{C}$ .

Let  $a$  and  $b$  be two vertices of  $\mathcal{A}$  and  $\mathcal{B}$  respectively. Let us consider the following property:

$$(\mathbf{a} + \mathbf{b}) \text{ is a vertex of } \mathcal{C} \Leftrightarrow \exists \gamma \neq \mathbf{0} : S(\mathcal{A}, \gamma) = \{\mathbf{a}\} \text{ et } S(\mathcal{B}, \gamma) = \{\mathbf{b}\} \tag{10}$$

This property expresses the fact that if the same objective function reaches its maximum at  $\mathcal{A}$  in a single vertex  $a$  and at  $\mathcal{B}$  in a single vertex  $b$  then  $(\mathbf{a} + \mathbf{b})$  is a vertex of  $\mathcal{C}$ .

Any vertex of  $\mathcal{C} = \mathcal{A} + \mathcal{B}$  is the sum of a vertex of  $\mathcal{A}$  and a vertex of  $\mathcal{B}$  according to Ewald's 1.5 theorem [15].

Using the previous properties, we can deduce eq. 11:

$$\begin{aligned} \text{Let us consider } a \text{ and } b \text{ two vertices of } \mathcal{A} \text{ and } \mathcal{B} : \\ \mathbf{c} = \mathbf{a} + \mathbf{b} \text{ vertex of the Minkowski sum } \mathcal{A} + \mathcal{B} = \mathcal{C} \Leftrightarrow \exists \gamma \neq \mathbf{0} \text{ such } S(\mathcal{A}, \gamma) = \{\mathbf{a}\} \text{ and } S(\mathcal{B}, \gamma) = \{\mathbf{b}\} \tag{11} \\ \Leftrightarrow \exists \gamma \neq \mathbf{0} \text{ such } S(\mathcal{C}, \gamma) = \{\mathbf{c}\} \Leftrightarrow \gamma \in \text{DualCone}(\mathbf{c}) \Leftrightarrow \text{dimension of DualCone}(\mathbf{c}) = n \end{aligned}$$

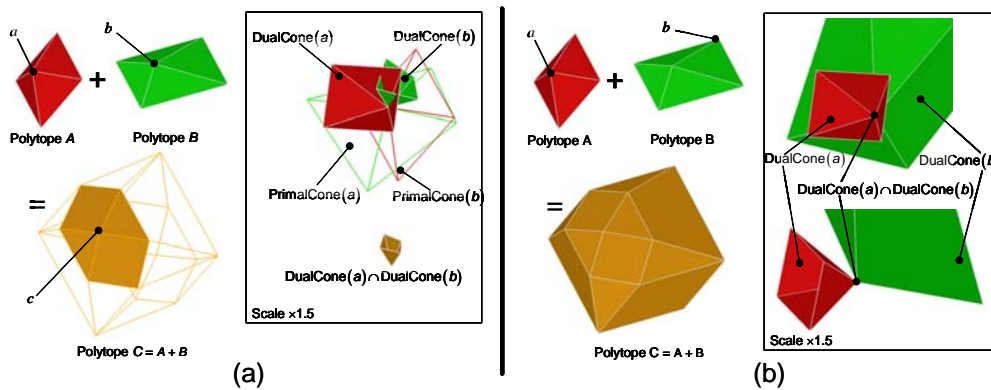


Fig. 6 Intersection of dual cones generating a vertex. Fig. 11 Intersection of dual cones not generating vertex.

Let us consider  $\text{DualCone}(a)$  and  $\text{DualCone}(b)$  the respective dual cones of polytopes  $\mathcal{A}$  at vertex  $a$  and  $\mathcal{B}$  at vertex  $b$ .

If  $(\text{DualCone}(a) \cap \text{DualCone}(b))$  is 3-dimension, then  $(\mathbf{a} + \mathbf{b}) = \mathbf{c}$  where  $c$  is the vertex of  $\mathcal{C}$ : see fig. 6a.

If  $(\text{DualCone}(a) \cap \text{DualCone}(b))$  is clearly less than 3-dimension, then  $(\mathbf{a} + \mathbf{b})$  is not a vertex of  $\mathcal{C}$ : see fig. 6b.

There is a corollary which consists in determining the number of non-coplanar edges in  $(\text{DualCone}(a) \cap \text{DualCone}(b))$  which should be at least equal to 3.

### 3.2.1. Proposal for an algorithm

From eq. 11, we are able to formulate an algorithm to determine the vertices of the Minkowski sum. In addition, determining  $(\text{DualCone}(a) \cap \text{DualCone}(b))$  for all the vertices of  $\mathcal{A}$  and  $\mathcal{B}$  gives the common refinement of  $\mathcal{A}$  and  $\mathcal{B}$  according to eq. 5 and hence we can deduce the normal fan  $N(\mathcal{C})$ .

**Require:** two 3-polytopes  $\mathcal{A}$  and  $\mathcal{B}$

**Ensure:** determination of  $L_{v,C}$ ,  $L_{\text{DualCone},C}$  and  $N(\mathcal{C})$  with  $\mathcal{A} + \mathcal{B} = \mathcal{C}$

```

1:   k = 0
2:   for each vertex  $a_i$  of  $A$  with  $L_{f,A}$  do DualCone( $a_i$ )
3:       for each vertex  $b_j$  of  $B$  with  $L_{f,B}$  do DualCone( $b_j$ ) ; compute  $I_{ij} = \text{DualCone}(a_i) \cap \text{DualCone}(b_j)$ 
4:           if dimension of  $I_{ij} = 3$  then  $k = k + 1$  ; compute  $c_k = a_i + b_j$  ; add  $c_k$  in  $L_{v,C}$  ; add  $I_{ij} = \text{DualCone}(c_k)$  in  $L_{\text{DualCone},C}$ 
5:           end if
6:       end for
7:   end for
8:   end for
9:  $n_{v,C} = k$  ;  $N(\mathcal{C}) = \{\text{DualCone}(c_k)\}$  with  $\text{DualCone}(c_k) \in L_{\text{DualCone},C}$ 

```

Fig. 7 Determining the vertices of the Minkowski sum of 3-polytopes.

Polytope  $\mathcal{A}$  is characterised by its list of vertices  $L_{v,A}$  and its list of facets  $L_{f,A}$ .

Let  $a_i$  be the  $i^{\text{th}}$  vertex of  $L_{v,A}$ . We have:  $1 \leq i \leq n_{v,A}$  where  $n_{v,A}$  is the number of vertices of  $\mathcal{A}$ .

In the same way, polytope  $\mathcal{B}$  is characterised by  $L_{v,B}$  and  $L_{f,B}$ .

Let  $b_j$  be the  $j^{\text{th}}$  vertex of  $L_{v,B}$ . We have:  $1 \leq j \leq n_{v,B}$  where  $n_{v,B}$  is the number of vertices of  $\mathcal{B}$ .

Polytope  $\mathcal{C}$  is characterised by its list of vertices  $L_{v,C}$ , its list of dual cones  $L_{\text{DualCone},C}$  and its normal fan  $N(\mathcal{C})$ .

Let  $c_k$  be the  $k^{\text{th}}$  vertex of  $L_{v,C}$ . We have:  $1 \leq k \leq n_{v,C}$  where  $n_{v,C}$  is the number of vertices of  $\mathcal{C}$ .

Let  $\text{DualCone}(c_k)$  be the  $k^{\text{th}}$  dual cone of  $\mathcal{C}$  associated with  $c_k$  of  $L_{\text{DualCone},C}$ .

The fig. 7 presents the algorithm to determine the vertices of the Minkowski sum of 3-polytopes.

### 3.3. Determining the facets of the Minkowski sum of polytopes

#### 3.3.1. Properties of dual cones

We shall go straight to eq. 4 developed in [10].

Property 1: In  $\mathbb{R}^3$ , dual cones associated with  $k$  vertices  $v_i$  of the same facet of a polytope share one and the same edge in the polytope's normal fan. (12)

Fig. 8a shows the 4 dual cones  $\text{DualCone}(c_k)$  associated respectively with vertices  $c_1, c_2, c_3$  and  $c_4$  on the same facet  $f$  of polytope  $\mathcal{C}$ . The 4 dual cones  $\text{DualCone}(c_k)$  are translated on vertex  $c_1$ . They define a sub-set of the normal fan  $N(\mathcal{C})$  translated into  $c_1$ . The edge common to the 4 dual cones

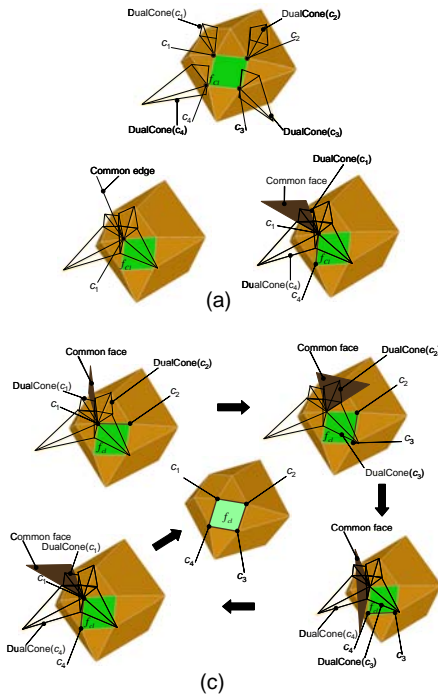


DualCone( $c_k$ ) associated respectively with vertices  $c_k$  (where  $1 \leq k \leq 4$ ) is normal to facet  $f$  : see fig. 8a.

Property 2: the two dual cones associated with the two vertices of the same edge of a polytope share a single face in the polytope's normal fan (13)

Fig. 8a illustrates property 2 for the edge of polytope  $\mathcal{C}$  bounded by vertices  $c_1$  and  $c_2$ .

### 3.3.2. Proposal for an algorithm



```

Require:  $L_{v,C}$ ,  $N(\mathcal{C})$  and  $n_{v,C}$  DualCone( $c_k$ )
Ensure:  $L_{f,C}$  and  $L_{v,f_C}$  for each  $f_C$ 
1: for each DualCone( $c_i$ ) of  $N(\mathcal{C})$  do
2:   for each  $e_{du}$  of DualCone( $c_i$ ) do
3:     compute  $h_{C_i}$  defined by  $c_i$  and  $e_{du}$ 
4:     if  $h_{C_i}$  is not in  $L_{v,C}$  then
5:       add  $h_{C_i}$  in  $L_{v,C}$ 
6:   find the  $q$  DualCone( $c_m$ ) in  $N(\mathcal{C})$  such that:  $e_{du} \subseteq \text{DualCone}(c_m) \cap \text{DualCone}(c_i)$  with  $m \neq k$ .
7:   find DualCone( $c_{m1}$ ) among the  $q$  DualCone( $c_m$ ) such that:
   DualCone( $c_{m1}$ )  $\cap$  DualCone( $c_i$ ) =  $f_{du1}$ , facet of DualCone( $c_i$ )
8:   find DualCone( $c_{m2}$ ) among the  $q$  DualCone( $c_m$ ) such that:
   DualCone( $c_{m2}$ )  $\cap$  DualCone( $c_i$ ) =  $f_{du2}$ , facet of DualCone( $c_i$ ) with  $f_{du1} \neq f_{du2}$ 
9:    $h_{C_i}$  is the support hyperplane of  $f_{C_i}$  and  $n_{v,f_C} = q + 1$ .
10:   $c_i$  and  $c_{m1}$  limit the first edge  $e_{f_{C_i}}$  of  $f_{C_i}$ , add  $e_{f_{C_i}}$  in  $L_{v,f_C}$ 
11:  DualCone( $c_{mp}$ ) = DualCone( $c_{m1}$ )
12:  DualCone( $c_{mp+1}$ ) = DualCone( $c_i$ )
13:  while DualCone( $c_{mp}$ )  $\neq$  DualCone( $c_{mp+1}$ ) do
14:    find DualCone( $c_{mp+1}$ ) among the  $q$  DualCone( $c_m$ ) such that:
    DualCone( $c_{mp+1}$ )  $\cap$  DualCone( $c_{mp}$ ) =  $f_{dup}$ , facet of DualCone( $c_{mp}$ )
    with DualCone( $c_{mp+1}$ )  $\neq$  DualCone( $c_{mp+1}$ )
15:     $c_{mp}$  and  $c_{mp+1}$  limit the next edge  $e_{f_{C_i}}$  of  $f_{C_i}$ , add  $e_{f_{C_i}}$  in  $L_{v,f_C}$ 
16:    DualCone( $c_{mp-1}$ ) = DualCone( $c_{mp}$ )
17:    DualCone( $c_{mp}$ ) = DualCone( $c_{mp+1}$ )
18:  end while
19:   $c_{mp} = c_{m2}$  and  $c_i$  limit the last edge  $e_{f_{C_i}}$  of  $f_{C_i}$ , add  $e_{f_{C_i}}$  in  $L_{v,f_C}$ 
20:  determine  $f_{C_i}$  with  $h_{C_i}$  and  $L_{v,f_C}$ 
21:  add  $f_{C_i}$  in  $L_{f,C}$ 
22: end if
23: end for
24: end for

```

Fig. 8 Determining the facets of the Minkowski sum of 3-polytopes.

From the two properties described earlier, an algorithm can be formulated to search for the facet edges of a  $\mathbb{R}^3$  polytope when its vertices, its normal fan and the dual cones associated respectively with the polytope vertices are known. From property 1 (12) in the normal fan of a polytope, the edge common to the dual cones associated with the primal vertices of the same facet can be identified. From property 2 (13), we can turn around this common edge and identify the vertices of this facet.

Polytope  $\mathcal{C}$  is characterised by its list of vertices  $L_{v,C}$ , its list of dual cones  $L_{\text{DualCone},C}$  and its normal fan  $N(\mathcal{C})$ .

Let  $c_k$  be the  $k^{\text{th}}$  vertex of  $L_{v,C}$ . We have:  $1 \leq k \leq n_{v,C}$  where  $n_{v,C}$  is the number of vertices of  $\mathcal{C}$ .

Let DualCone( $c_k$ ) be the  $k^{\text{th}}$  dual cone  $\mathcal{C}$  associated with  $c_k$  of  $L_{\text{DualCone},C}$ .

We postulate  $\text{DualCone}(c_k) = \{c_k, e_{du}, f_{dv}\}$  where:

$e_{du}$  is the  $u^{\text{th}}$  edge of  $\text{DualCone}(c_k)$  and  $f_{dv}$  is the  $v^{\text{th}}$  facet of  $\text{DualCone}(c_k)$ .

Let  $L_{f,C}$  be the list of facets of  $\mathcal{C}$ .

We represent as  $f_{Cl}$  the  $l^{\text{th}}$  facet of  $L_{f,C}$  ( $1 \leq l \leq n_{f,C}$  where  $n_{f,C}$  is the number of facets of  $\mathcal{C}$ ).

Let  $L_{h,C}$  be the list of support hyperplanes for  $\mathcal{C}$ .

Let  $h_{Cl}$  be the  $l^{\text{th}}$  hyperplane of  $L_{h,C}$  ( $1 \leq l \leq n_{h,C}$ )

Let  $L_{e,f_{Cl}}$  be the ordered list of the edges of facet  $f_{Cl}$ . Two consecutive edges of  $L_{e,f_{Cl}}$  share a single vertex and the first and last edges.

Let  $e_{f_{Cl}w}$  be the  $w^{\text{th}}$  edge of  $f_{Cl}$ . ( $1 \leq w \leq n_{e,f_{Cl}}$  where  $n_{e,f_{Cl}}$  is the number of edges of  $f_{Cl}$ ).

The fig. 8b presents the algorithm to determine the facets of the Minkowski sum of 3-polytopes.

Fig. 8c shows stages 7 to 18 of the algorithm for determining the edges of a facet  $f_{cl}$  of polytope  $\mathcal{C}$  of vertices  $c_1, c_2, c_3$  and  $c_4$ . It corresponds to a case where:  $k = 1 \Rightarrow c_k = c_1$  and  $q = 3 \Rightarrow n_{e,f_{cl}} = 4$ .

To be more precise, fig. 8c shows the determination of the first edge, bounded by  $c_1$  and  $c_2$  and vertex  $c_4$ , the last vertex in the outline of  $f_{cl}$  where:  $m_1 = 2$  and  $m_2 = 4$ .

It shows the determinations of the edges bounded by:

- $c_2$  and  $c_3$  where  $m_p = 2$  and  $m_{p+1} = 3$ ,
- $c_3$  and  $c_4$  where  $m_p = 3$  and  $m_{p+1} = 4$ ,
- $c_4$  and  $c_1$  where  $m_p = m_2 = 4$  and  $k = 1$ .

#### 4. Discussion of the proposed method

The determination method proposed in this article is based solely on intersections of pairs of dual cones associated with searches for common edges and faces of dual cones in a normal fan.

Using this algorithm,  $n_{v,A} \times n_{v,B}$  intersections of pairs of dual cones have to be calculated in order to determine the vertices of polytope  $\mathcal{C}$  and the normal fan  $N(\mathcal{C})$ .

In addition, it is necessary to carry out  $n_{f,c}$  searches for a common edge among the dual cones in the

normal fan  $N(\mathcal{C})$ . Finally,  $\sum_{Cl=1}^{Cl=n_{f,c}} n_{e,f_{cl}}$  searches are needed for a common face between dual cones to determine the edges of polytope  $\mathcal{C}$ . Each search for a common face is carried out in a sub-set of dual cones in the normal fan  $N(\mathcal{C})$  which share an edge.

Each vertex of polytope  $\mathcal{C}$  is the sum of two vectors associated respectively with two vertices of polytopes  $\mathcal{A}$  and  $\mathcal{B}$ . From the  $n_{v,A} \times n_{v,B}$  intersections of pairs of dual cones it is possible to determine from which vertices of operands  $\mathcal{A}$  and  $\mathcal{B}$  the vertices of  $\mathcal{C}$  derived.

In addition, the normal for each facet of  $\mathcal{C}$  is characterised by an edge in the normal fan  $N(\mathcal{C})$ . Thus the translated facets can be differentiated from the facets of connection.

Each normal of the facets of connection is generated by the intersection of two dual cones and more precisely by the intersection of two faces of dual cones. In this way, we can identify the facets of operands  $\mathcal{A}$  and  $\mathcal{B}$  from which the normals of the facets of connection derive.

The normals of facets that are different from the facets of connection derive either from operand  $\mathcal{A}$ , or operand  $\mathcal{B}$ . The method proposed here gives complete traceability of the vertices and facets of polytope  $\mathcal{C}$  from the vertices and facets of operands  $\mathcal{A}$  and  $\mathcal{B}$ . In tolerance analysis, this traceability is used to optimise the filling of a functional polytope by a calculated polytope.

The proofs of properties (11), (12) and (13) are detailed in [16].

These algorithms are currently being developed in the topological structure of an OpenCASCADE distribution [17].

The intersection algorithm used in this work is the OpenCASCADE 6.2 distribution algorithm.

## 5. Conclusion and future research

We have shown how to determine the Minkowski sum for two  $\mathbb{R}^3$  polytopes from intersections of polyhedral cones and using the properties of the common edge and common face between dual cones in a normal fan. The algorithms for determining the vertices and the facets have been described. Ultimately, this method will be applied in a tolerance analysis procedure in an environment that can be multi-physical [6]. Work is currently underway on a method to determine the intersection of two polyhedral cones so that this can be generalised for  $n$  dimension polytopes. It will allow to transpose the algorithms of this article in a  $n$  dimension according to the property computing common refinement (7). This work will be described in a later publication.

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