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Reed's Conjecture on hole expansions

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Abstract

In 1998, Reed conjectured that for any graph G , $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$, where $\chi(G)$, $\omega(G)$, and $\Delta(G)$ respectively denote the chromatic number, the clique number and the maximum degree of G . In this paper, we study this conjecture for some *expansions* of graphs, that is graphs obtained with the well known operation *composition* of graphs.

We prove that Reed's Conjecture holds for expansions of bipartite graphs, for expansions of odd holes where the minimum chromatic number of the components is even, when some component of the expansion has chromatic number 1 or when a component induces a bipartite graph. Moreover, Reed's Conjecture holds if all components have the same chromatic number, if the components have chromatic number at most 4 and when the odd hole has length 5. Finally, when G is an odd hole expansion, we prove $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil + 1$.

1 Introduction

We consider here simple and undirected graphs. For terms which are not defined we refer to Bondy and Murty [2].

The chromatic number of a graph G , denoted by $\chi(G)$, is the minimum number of colors required to a proper colouring of the graph, that is to colour the vertices of G so that no two adjacent vertices receive the same colour ; the size of the largest clique (independent set) in G is called the *clique number* (*independence number*) of G , and denoted by $\omega(G)$ ($\alpha(G)$) ; the maximum degree of G , denoted $\Delta(G)$ is the maximum number of neighbours of a vertex over all vertices of G .

Bounding the chromatic number of a graph in terms of others graphs parameters attracted much attention in the past. For example, it is well know that for any graph G we have $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. This upper bound

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was reduced to $\Delta(G)$ by Brooks [3] in 1941 for connected graphs which are not complete graphs neither odd cycles.

In 1998 Reed [9] stated the following Conjecture also known as *Reed's Conjecture*:

Conjecture 1 [9] *For any graph G , $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$.*

This conjecture has been stated true for some restrictions of the graph parameters. Hence Conjecture 1 holds true when $\chi(G) > \lceil \frac{|V(G)|}{2} \rceil$ (see [8]), when $\chi(G) \leq \omega(G) + 2$ [5], when $\alpha(G) = 2$ [6, 7] or when $\Delta(G) \geq |V(G)| - \alpha(G) - 4$ (see [7]).

Some classes of graphs also verify Conjecture 1. That's trivially the case for perfect graphs (a graph G is said to be perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G), for graphs with disconnected complement [8] for almost split graphs (an almost-split graph is a graph that can be partitioned into a maximum independent set and a graph having independence number at most 2) or particular classes of triangle free graphs [7] and for classes defined with forbidden configurations such that $(2K_2, C_4)$ -free graphs, odd hole free graphs [1] or some particular classe of P_5 -free graphs [1].

The well known operation *composition* of graphs, also called *expansion* in [1] is defined as follows :

Given a graph H on n vertices $v_0 \dots v_{n-1}$ and a family of graphs $G_0 \dots G_{n-1}$, an *expansion* of H , denoted $H(G_0 \dots G_{n-1})$ is obtained from H by replacing each vertex v_i of H with G_i for $i = 0 \dots n - 1$ and joining a vertex x in G_i to a vertex y of G_j if and only if v_i and v_j are adjacent in H . The graph G_i , $i = 0 \dots n - 1$ is said to be the *component* of the expansion associated to v_i .

In [1], Aravind et al proved that Conjecture 1 holds true for *full* expansions and *independent* expansions of odd holes, that is expansions $H(G_1 \dots G_n)$ of odd holes where all the G_i 's are either complete graphs or edgeless graphs. Moreover, they ask for proving Conjecture 1 for graph expansions whenever every component of the expansion statisfies Conjecture 1.

In this paper, we consider Conjecture 1 for expansion of bipartite graphs, namely bipartite expansions and odd hole expansions. We use for this a colouring algorithm of bipartite expansions that we extend to odd hole expansions, this allows us to compute the chromatic number of those graphs. We prove that Conjecture 1 holds for a bipartite expansion (Theorem 9).

Moreover, Conjecture 1 holds for odd hole expansions when the minimum chromatic number of the components is even (Corollary 17), when some component of the expansion has chromatic number 1 (Theorem 18), or when a component induces a bipartite graph (Theorem 19). It is also the case if all components have the same chromatic number (Theorem 20), if the components have chromatic number at most 4 (Theorem 23), and when the odd hole has length 5 (Theorem 25). In addition, if G is an odd hole expansion we have $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil + 1$ (Theorem 26).

These results improve the result of Aravind et al on full and independent expansions of odd holes.

The present section ends with some notations and preliminary results. Section 2 is devoted to the colouring of bipartite expansions and its consequences

on Conjecture 1 for such graphs while in Section 3 we consider the colouring of odd hole expansions and its implications on Conjecture 1 are considered in Section 4.

1.1 Notations and preliminary results

Given a graph G and X a subset of its vertex set, we denote $G[X]$ the subgraph of G induced by X . The degree of a vertex v in the graph G is denoted $d_G(v)$ or $d(v)$ when no confusion is possible. For an expansion $H(G_0 \dots G_{n-1})$ of some graph H , we will assume in the following that the vertices of H are weighted with the chromatic number of their associated component while an edge of H is weighted with the sum of the weights of its endpoints. Moreover, for $i = 0, \dots, n-1$, we will denote χ_i as the chromatic number of G_i , while V_i is for the vertex set of G_i , Δ_i is the maximum degree of G_i , and ω_i its clique number.

Lemma 2 *Let H be an induced subgraph of some graph G such that $\chi(H) = \chi(G)$. If $\chi(H) \leq \lceil \frac{\omega(H) + \Delta(H) + 1}{2} \rceil$ then $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$.*

Proof Since H is an induced subgraph of G , $\omega(H) \leq \omega(G)$ and $\Delta(H) \leq \Delta(G)$. Thus $\chi(G) = \chi(H) \leq \lceil \frac{\omega(H) + \Delta(H) + 1}{2} \rceil \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$. \square

Theorem 3 [8] *If \overline{G} is disconnected then $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$*

Lemma 4 *Let $G = H(G_0 \dots G_{n-1})$ be an odd hole expansion that is a minimum counter-example of Conjecture 1 (if any). For $i \in \{0 \dots n\}$, G_i is connected.*

Proof Without loss of generality assume that the subgraph induced by G_0 is not connected. Let X_1 and X_2 be two subset of $V(G_0)$ inducing a connected component and suppose that we need at most χ_j colors ($j = 1, 2$) to color X_j with $\chi_1 \leq \chi_2$. Let G' be the subgraph obtained from G by deleting X_1 . Since G' satisfies Conjecture 1 by hypothesis, we have $\chi(G') \leq \lceil \frac{\Delta(G') + \omega(G') + 1}{2} \rceil$. We can then color the vertices of X_1 by using the colors appearing in X_2 since $\chi_1 \leq \chi_2$. Since $\omega(G) \geq \omega(G')$ and $\Delta(G) \geq \Delta(G')$, we have $\chi(G) = \chi(G') \leq \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$, a contradiction. \square

2 Coloring of bipartite expansion

Notations 5 *Let H be a bipartite graph with n vertices: v_0, \dots, v_{n-1} and $H(G_0 \dots G_{n-1})$ be an expansion of H . Without loss of generality we assume that v_0 and v_1 are adjacent and are such that the edge v_0v_1 has maximum weight in H .*

Let Γ_0 be a set of χ_0 colors and Γ_1 be a set of χ_1 other colors.

A given index $i \in \{0, \dots, n-1\}$ will have a preferred index in $\{0, 1\}$, say $p(i)$, defined as follows : $p(i) = 0$ whenever v_i and v_0 are vertices of the same class of the bipartition otherwise $p(i)$ will be defined to be 1. Moreover we define the index $p'(i)$ such that $\{p(i), p'(i)\} = \{0, 1\}$.

When $H(G_0 \dots G_{n-1})$ is a bipartite expansion, according to the above notations, G_i ($0 \leq i \leq n-1$) will be colored by using preferably the set of colors $\Gamma_{p(i)}$.

More precisely G_i will be colored by using $Min(\chi_i, \chi_{p(i)})$ colors of $\Gamma_{p(i)}$ and $Max(0, \chi_i - \chi_{p(i)})$ colors of $\Gamma_{p'(i)}$ (see Theorem 6).

Theorem 6 *Let $H(G_0 \dots G_{n-1})$ be a bipartite expansion.*

For $i \in \{0 \dots n-1\}$,

if $\chi_i \leq \chi_{p(i)}$ then G_i can be colored by using χ_i colors of $\Gamma_{p(i)}$

otherwise G_i can be colored by using the $\chi_{p(i)}$ colors of $\Gamma_{p(i)}$ together with $\chi_i - \chi_{p(i)}$ colors of $\Gamma_{p'(i)}$.

Proof Let us colour the vertices of G_0 with the χ_0 colors of Γ_0 . In the same way we colour the vertices of G_1 by using the χ_1 colors of Γ_1 (recall that $\Gamma_0 \cap \Gamma_1 = \emptyset$).

For $i \in \{2 \dots n-1\}$, we color the graph G_i as follows : when $\chi_i \leq \chi_{p(i)}$ we can use the χ_i first colors in $\Gamma_{p(i)}$ to colour G_i ; and when $\chi_i > \chi_{p(i)}$ we color the vertices of G_i by using the $\chi_{p(i)}$ colors of $\Gamma_{p(i)}$ and the $\chi_i - \chi_{p(i)}$ last colors of $\Gamma_{p'(i)}$.

We claim that the resulting coloring is a proper coloring of $H(G_0, \dots, G_{n-1})$.

Indeed let $v_i v_j$ be an edge of H . Let us remark first that we do not have $\chi_i > \chi_{p(i)}$ and $\chi_j > \chi_{p(j)}$ since $\chi_i + \chi_j \leq \chi_0 + \chi_1$ by hypothesis, moreover, since v_i and v_j are adjacent we have $p(i) = p'(j)$ and $p(j) = p'(i)$.

case 1 $\chi_i \leq \chi_{p(i)}$ and $\chi_j \leq \chi_{p(j)}$

The colors used in G_i are only colors of $\Gamma_{p(i)}$ and those of G_j are only colors of $\Gamma_{p(j)} = \Gamma_{p'(i)}$ and these two sets of colours are disjoint.

case 2 $\chi_i \leq \chi_{p(i)}$ and $\chi_j > \chi_{p(j)}$

The colours used in the coloring of G_i are only the χ_i first colors of $\Gamma_{p(i)}$. In order to color G_j , we use all the $\chi_{p(j)}$ colours of $\Gamma_{p(j)}$ and we need to use the last $\chi_j - \chi_{p(j)}$ colors of $\Gamma_{p'(j)}$. Since $\chi_i + \chi_j \leq \chi_0 + \chi_1 = \chi_{p(j)} + \chi_{p(i)}$, we have $\chi_j - \chi_{p(j)} \leq \chi_{p(i)} - \chi_i$. Hence the set of colors of $\Gamma_{p(i)}$ used in order to achieve the colouring of G_j is disjoint from the set of colors used in G_i .

case 3 $\chi_i > \chi_{p(i)}$ and $\chi_j \leq \chi_{p(j)}$ The same argument works. □

From Theorem 6 and according to Notations 5, since $\chi(H) \geq \chi_0 + \chi_1$, we have:

Corollary 7 *Let $G = H(G_0 \dots G_{n-1})$ be a bipartite expansion, $\chi(G) = \chi_0 + \chi_1$.*

Remark 8 Let us remark that the coloring given in Theorem 6 has the following property: $|\Gamma_i| = \chi_i$ for $i \in \{0 \dots n-1\}$.

Theorem 9 *Any expansion of a bipartite graph satisfies Conjecture 1.*

Proof Let $H(G_0 \dots G_{n-1})$ be an expansion of a bipartite graph H . According to Notations 5 and by Theorem 3, the subgraph induced by $V(G_0) \cup V(G_1)$, say G' , verifies Conjecture 1. Moreover $\chi(G') = \chi_0 + \chi_1$ and by Corollary 7 $\chi(G) = \chi(G')$. The result follows from Lemma 2. □

3 Odd hole expansions coloring

By Theorem 3, an expansion of triangle verifies Conjecture 1. In what follows C_{2k+1} denotes an odd hole of length $2k + 1$ ($k \geq 2$) and all indexes are taken modulo $2k + 1$. Moreover, the vertex set of C_{2k+1} is $\{v_0, \dots, v_{2k}\}$ and $v_i v_j$ is an edge if and only if $j = i + 1$.

Theorem 10 below provides a proper coloring for odd hole expansions.

Theorem 10 *Let $G = C_{2k+1}(G_0 \dots G_{2k})$ be an expansion of an odd hole. Assume that the edge $v_0 v_1$ has maximum weight in H .*

Let i be an index in $\{3 \dots 2k - 1\}$.

If $\chi_0 + \chi_1 \geq \chi_{i-1} + \chi_i + \chi_{i+1}$ then $\chi(G) \leq \chi_0 + \chi_1$

else if $\chi_{i-1} > \chi_{p(i-1)}$ and $\chi_{i+1} > \chi_{p(i+1)}$ then $\chi(G) \leq \chi_0 + \chi_1 + \lfloor \frac{\chi_{i+1}}{2} \rfloor$

else $\chi(G) \leq \chi_0 + \chi_1 + \lfloor \frac{\chi_{i-1} + \chi_i + \chi_{i+1} - \chi_0 - \chi_1 + 1}{2} \rfloor$.

Proof

Let H' be the bipartite graph whose vertex set is $V(C_{2k+1}) - \{v_i\}$. Assume that the coloring described in Theorem 6 has been applied to the expansion $H'(G_0, G_1 \dots G_{i-1}, G_{i+1} \dots G_{2k})$. Observe that the notations $p(i)$ and $p'(i)$ are not defined in $C_{2k+1}(G_0 \dots G_{2k})$, however in the following we will use this notations as meant in $H'(G_0, G_1 \dots G_{i-1}, G_{i+1} \dots G_{2k})$, thus we have $p(i-1) = p'(i+1)$ and $p'(i-1) = p(i+1)$.

Let us now consider the coloring of G_i .

According to the coloring of G_{i-1} and those of G_{i+1} four cases may occur.

Case 1 : $\chi_{i-1} \leq \chi_{p(i-1)}$ and $\chi_{i+1} \leq \chi_{p(i+1)}$. The coloring of G_{i-1} uses χ_{i-1} colors of $\Gamma_{p(i-1)}$ and none in the set $\Gamma_{p'(i-1)}$ while the coloring of G_{i+1} needs only χ_{i+1} colors in $\Gamma_{p(i+1)}$; consequently there are $\chi_0 + \chi_1 - \chi_{i-1} - \chi_{i+1}$ colors free in $\Gamma_0 \cup \Gamma_1$ for the coloring of G_i .

Case 2 : $\chi_{i-1} \leq \chi_{p(i-1)}$ and $\chi_{i+1} > \chi_{p(i+1)}$. We have the same coloring for G_{i-1} as in *Case 1*. But the subgraph G_{i+1} is colored with all the colors of $\Gamma_{p(i+1)}$ together with $\chi_{i+1} - \chi_{p(i+1)}$ colors of $\Gamma_{p'(i+1)}$, once again there are in $\Gamma_{p'(i+1)}$ at least $\chi_{p'(i+1)} - \chi_{i-1} - (\chi_{i+1} - \chi_{p(i+1)})$ free colors for the coloring of G_i .

Case 3 : $\chi_{i-1} > \chi_{p(i-1)}$ and $\chi_{i+1} \leq \chi_{p(i+1)}$. We color G_{i-1} with the $\chi_{p(i-1)}$ colors of $\Gamma_{p(i-1)}$ and with $\chi_{i-1} - \chi_{p(i-1)}$ colors of $\Gamma_{p'(i-1)}$. The subgraph G_{i+1} being colored with χ_{i+1} colors in $\Gamma_{p(i+1)}$. Thus there are $\chi_{p'(i-1)} - \chi_{i+1} - (\chi_{i-1} - \chi_{p(i-1)})$ unused colors in $\Gamma_{p'(i-1)}$.

Case 4 : $\chi_{i-1} > \chi_{p(i-1)}$ and $\chi_{i+1} > \chi_{p(i+1)}$. In this case G_{i-1} can be colored with all the colors in $\Gamma_{p(i-1)}$ and $\chi_{i-1} - \chi_{p(i-1)}$ colors of $\Gamma_{p'(i-1)}$. Moreover the coloring of G_{i+1} is done with the colors of $\Gamma_{p(i+1)}$ and $\chi_{i+1} - \chi_{p(i+1)}$ additional colors of $\Gamma_{p'(i+1)}$. All colors of $\Gamma_0 \cup \Gamma_1$ are used in this colorings, but just observe that $\chi_i < \text{Min}(\chi_0, \chi_1)$.

Suppose first $\chi_0 + \chi_1 \geq \chi_{i-1} + \chi_i + \chi_{i+1}$.

In this situation *Case 4* cannot occur and there are enough free colors in $\Gamma_0 \cup \Gamma_1$ for the coloring of G_i . Hence $\chi(G) \leq \chi_0 + \chi_1$.

From now on $\chi_0 + \chi_1 < \chi_{i-1} + \chi_i + \chi_{i+1}$.

Assume now $\chi_{i-1} > \chi_{p(i-1)}$ and $\chi_{i+1} > \chi_{p(i+1)}$. Recall that $\chi_i < \chi_0$ and $\chi_i < \chi_1$. Let $a = \lfloor \frac{\chi_i}{2} \rfloor$ and Γ be a set of a additional colors. The coloring of G_{i-1} uses $\chi_{p(i-1)}$ colors of $\Gamma_{p(i-1)}$, let us replace a of those colors with the colors of Γ . We also replace a colors of $\Gamma_{p(i+1)}$ with the same colors of Γ . Thus $2a$ colors are left for the coloring of G_i , that is χ_i or $\chi_i - 1$ according to the parity of χ_i . Hence in this case the whole graph can be colored with at most $|\Gamma_0| + |\Gamma_1| + a + 1$ colors, that is $\chi(G) \leq \chi_0 + \chi_1 + \lfloor \frac{\chi_i + 1}{2} \rfloor$.

Finally, assume $\chi_{i-1} \leq \chi_{p(i-1)}$ or $\chi_{i+1} \leq \chi_{p(i+1)}$. Recall that there are $\chi_0 + \chi_1 - \chi_{i-1} - \chi_{i+1}$ colors free in $\Gamma_0 \cup \Gamma_1$ for the coloring of G_i . Since $\chi_0 + \chi_1 \geq \chi_i + \chi_{i-1}$ it is clear that $\chi_{i+1} \geq \frac{\chi_{i-1} + \chi_i + \chi_{i+1} - \chi_0 - \chi_1}{2}$. Similarly $\chi_{i-1} \geq \frac{\chi_{i-1} + \chi_i + \chi_{i+1} - \chi_0 - \chi_1}{2}$. Let us state $a = \lfloor \frac{\chi_{i-1} + \chi_i + \chi_{i+1} - \chi_0 - \chi_1}{2} \rfloor$ and Γ be a set of a additional colors. We replace, in the coloring of G_{i-1} , a colors of $\Gamma_{p(i-1)}$ with the colors of Γ as well as a colors of $\Gamma_{p(i+1)}$ in the coloring of G_{i+1} . Hence we have $2a$ more colors for the coloring of G_i . It follows that the whole graph can be colored with the colors of $\Gamma_0 \cup \Gamma_1 \cup \Gamma$ and possibly an additional color according to the parity of $\chi_{i+1} + \chi_i + \chi_{i-1} - \chi_0 - \chi_1$. Thus in this case $\chi(G) \leq \chi_0 + \chi_1 + \lfloor \frac{\chi_{i-1} + \chi_i + \chi_{i+1} - \chi_0 - \chi_1 + 1}{2} \rfloor$. \square

Theorem 11 gives the chromatic number for odd hole expansions.

Theorem 11 *Let $G = C_{2k+1}(G_0 \dots G_{2k})$ be an expansion of an odd hole. We assume that the edge v_0v_1 has maximum weight in C_{2k+1} .*

Let l be an index such that $\chi_{l-1} + \chi_l + \chi_{l+1} = \text{Min}_{3 \leq i \leq 2k-1} \{\chi_{i-1} + \chi_i + \chi_{i+1}\}$

If $\chi_0 + \chi_1 \geq \chi_{l-1} + \chi_l + \chi_{l+1}$ then $\chi(G) = \chi_0 + \chi_1$

else $\chi(G) = \chi_0 + \chi_1 + \lfloor \frac{\chi_{l-1} + \chi_l + \chi_{l+1} - \chi_0 - \chi_1 + 1}{2} \rfloor$.

Proof Since $\chi(G) \geq \chi_0 + \chi_1$, by Theorem 10 we can suppose that $\chi_0 + \chi_1 < \chi_{l-1} + \chi_l + \chi_{l+1}$.

In addition $\chi_{l-1} \leq \chi_{p(l-1)}$ or $\chi_{l+1} > \chi_{p(l+1)}$. Otherwise, since $\chi_{l-1} > \chi_{p(l-1)}$ we have $l - 1 > 2$ and then $\chi_{l-2} < \chi_{p(l-2)} = \chi_{p(l+1)} < \chi_{l+1}$. It follows that $\chi_l + \chi_{l-1} + \chi_{l-2} < \chi_{l+1} + \chi_l + \chi_{l-1}$, a contradiction with the choice of the index l .

Hence by Theorem 10 we have $\chi(G) \leq \chi_0 + \chi_1 + \lfloor \frac{\chi_{l-1} + \chi_l + \chi_{l+1} - \chi_0 - \chi_1 + 1}{2} \rfloor$ and there is a coloring of G using colors in $\Gamma_0 \cup \Gamma_1 \cup \Gamma$ where Γ_0 , Γ_1 and Γ are disjoint sets of colors such that $|\Gamma_0| = \chi_0$, $|\Gamma_1| = \chi_1$ and $|\Gamma| = \lfloor \frac{\chi_{l-1} + \chi_l + \chi_{l+1} - \chi_0 - \chi_1 + 1}{2} \rfloor$. Since the sum $\chi_{l+1} + \chi_l + \chi_{l-1}$ is minimum, Theorem 10 cannot provide a coloring using less colors.

Assume now $\chi(G) < |\Gamma_0| + |\Gamma_1| + |\Gamma|$. We can suppose that an optimal coloring of G uses the set $\Gamma_0 \cup \Gamma_1 \cup \Gamma'$ as set of colors where $\Gamma' \cap (\Gamma_0 \cup \Gamma_1) = \emptyset$ and $|\Gamma'| < |\Gamma|$. In a such coloring the number of unused colors for the coloring of X_{l+1} and X_{l-1} is at most $\chi_0 + \chi_1 + |\Gamma'| - \chi_{l+1} - \chi_{l-1}$.

Thus $\chi_l \leq \chi_0 + \chi_1 + |\Gamma'| - \chi_{l+1} - \chi_{l-1}$ and then

$$\chi_l + \chi_{l+1} + \chi_{l-1} - \chi_0 - \chi_1 \leq |\Gamma'| < \lfloor \frac{\chi_l + \chi_{l+1} + \chi_{l-1} - \chi_0 - \chi_1 + 1}{2} \rfloor,$$

a contradiction with the fact that $\chi_l + \chi_{l+1} + \chi_{l-1} - \chi_0 - \chi_1$ is a positive integer. \square

4 Applications

In [1] Aravind et al observed that the complete or independent expansions of an odd hole satisfy Conjecture 1. We give below improvements of this results.

Corollary 12 *Conjecture 1 holds for an odd hole expansion when, in the conditions of Theorem 10, we have $\chi(A) = \omega(A)$ for $A \in \{G_0, G_1, G_l\}$.*

Proof By Theorem 11 we know that $\chi(G) \leq \frac{\chi_0 + \chi_1 + \chi_{l-1} + \chi_l + \chi_{l+1} + 1}{2}$. By assumption we have $\chi_0 + \chi_1 = \omega(G_0) + \omega(G_1) \leq \omega(G)$, moreover if v is a vertex of a maximum clique in G_l , $d(v) \geq \omega(G_l) - 1 + |V_{l+1}| + |V_{l-1}|$ then $\Delta \geq \chi_l + \chi_{l+1} + \chi_{l-1} - 1$. The result follows. \square

Corollary 13 *Let $G = C_{2k+1}(G_0 \dots G_{2k})$ be an expansion of an odd hole. Let $p = \min_{0 \leq i \leq 2k} \chi_i$. Assume that the edge $v_i v_{i+1}$ has maximum weight in C_{2k+1} for some $i \in \{0, \dots, 2k\}$. Then $\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{p+1}{2} \rfloor$.*

Proof

By Theorem 10, we may assume for $j \in \{i+3, i+4 \dots i-2\}$

$$\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{\chi_{j-1} + \chi_j + \chi_{j+1} - \chi_i - \chi_{i+1} + 1}{2} \rfloor. \quad (1)$$

Moreover, there is an index $l \in \{i+2, \dots, i-1\}$ such that $\chi_l = p$, otherwise $\chi_i = p$ or $\chi_{i+1} = p$. Suppose without loss of generality $\chi_{i+1} = p$. But now, since $\chi_{i-1} > \chi_{i+1}$ we have $\chi_{i-1} + \chi_i > \chi_i + \chi_{i+1}$, a contradiction, since the edge $v_i v_{i+1}$ has maximum weight in C_{2k+1} .

If $l \geq i+4$, we apply (1) with $j = l-1$, since $\chi_{l-1} + \chi_{l-2} \leq \chi_i + \chi_{i+1}$ we get $\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{\chi_{l-2} + \chi_{l-1} + \chi_l - \chi_i - \chi_{i+1} + 1}{2} \rfloor \leq \chi_i + \chi_{i+1} + \lfloor \frac{\chi_l + 1}{2} \rfloor$.

If $l = i+2$ or $l = i+3$, we apply (1) with $j = l+1$, since $\chi_{l+1} + \chi_{l+2} \leq \chi_i + \chi_{i+1}$ we get $\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{\chi_{l+2} + \chi_{l+1} + \chi_l - \chi_i - \chi_{i+1} + 1}{2} \rfloor \leq \chi_i + \chi_{i+1} + \lfloor \frac{\chi_l + 1}{2} \rfloor$. In both cases, it follows $\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{p+1}{2} \rfloor$. \square

Corollary 14 *Let $G = C_{2k+1}(G_0 \dots G_{2k})$ be an expansion of an odd hole. Let $v_i v_{i+1}$ be an edge of maximal weight in C_{2k+1} . Assume that $\chi(G) = \chi_i + \chi_{i+1} + q + 1$ for some integer $q \geq 0$. If $G[V_i \cup V_{i+1}]$ has a vertex of maximum degree in V_i (resp. V_{i+1}) then either Conjecture 1 holds for G or V_{i-1} (resp. V_{i+2}) induces a graph on at most $2q + 1$ vertices.*

Proof Assume that G is a counter-example to Conjecture 1. For convenience we note $G' = G[V_i \cup V_{i+1}]$.

Let v be a vertex of maximum degree in G' , suppose that $v \in V_i$ and V_{i-1} has at least $2q + 2$ vertices.

We have $\Delta(G) \geq d_{G'}(v) + |V_{i-1}| \geq \Delta(G') + 2q + 2$ and $\omega(G) \geq \omega(G')$. Thus $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + 1 + 2q + 2}{2} \rceil = \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil + q + 1$.

Since by Theorem 3, G' verifies Conjecture 1, $\lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil \geq \chi_i + \chi_{i+1}$. Hence by Corollary 13, $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \chi_i + \chi_{i+1} + q + 1 = \chi(G)$, a contradiction. \square

Corollary 15 *Let $G = C_{2k+1}(G_0 \dots G_{2k})$ be an expansion of an odd hole and let $p = \min_{0 \leq i \leq 2k} \chi_i$. If the edge $v_i v_{i+1}$ has maximum weight in C_{2k+1} then Conjecture 1 holds for G or $\chi(G) = \chi_i + \chi_{i+1} + \lfloor \frac{p+1}{2} \rfloor$.*

Proof We know by Corollary 13 that $\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{p+1}{2} \rfloor$. Assume that G is a counter-example to Conjecture 1 and $\chi(G) \neq \chi_i + \chi_{i+1} + \lfloor \frac{p+1}{2} \rfloor$. Thus, we have $\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{p}{2} \rfloor$.

Assume without loss of generality that $v \in V_{i+1}$ is a vertex with maximum degree in $G' = G[V_i \cup V_{i+1}]$. By Theorem 3, G' satisfies Conjecture 1.

Hence $\lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil \geq \chi_i + \chi_{i+1} = \chi(G')$. Since G_{i+2} has at least p vertices, we have $\Delta(G) \geq d(v) \geq |V_i| + \Delta_{i+1} + p \geq \Delta(G') + p$, which leads to $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + p + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil + \lfloor \frac{p}{2} \rfloor$.

Hence $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \chi(G)$, a contradiction. \square

Theorem 16 *Let $G = C_{2k+1}(G_0 \dots G_{2k})$ be an expansion of an odd hole of length $2k + 1$ and let $p = \min_{0 \leq i \leq 2k} \chi_i$. Let $v_i v_{i+1}$ be an edge of maximal weight in C_{2k+1} and assume that $v \in V_{i+1}$ is a vertex of maximum degree in $G' = G[V_i \cup V_{i+1}]$. If G does not satisfy Conjecture 1 then V_{i+2} induces a complete graph on p vertices and $v_{i+3} v_{i+4}$ is an edge of maximal weight in C_{2k+1} .*

Proof Assume that G does not satisfy Conjecture 1 and V_{i+2} does not induce a complete graph on p vertices. By Corollary 15, we have $\chi(G) = \chi_i + \chi_{i+1} + \lfloor \frac{p+1}{2} \rfloor$.

We may assume that $|V_{i+2}| \geq p + 1$ otherwise V_{i+2} would induce a complete graph on p vertices, a contradiction.

We have $\Delta(G) \geq d_{G'}(v) + |V_{i+2}| \geq \Delta(G') + p + 1$ and $\omega(G) \geq \omega(G')$.

Hence $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + p + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil + \lfloor \frac{p+1}{2} \rfloor = \chi(G)$, a contradiction.

Assume now that $\chi_{i+3} + \chi_{i+4} \leq \chi_i + \chi_{i+1} - 1$. By Theorem 11 we have $\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{\chi_{i+2} + \chi_{i+3} + \chi_{i+4} - \chi_i - \chi_{i+1} + 1}{2} \rfloor$ which leads to $\chi(G) \leq \chi_i + \chi_{i+1} + \lfloor \frac{p}{2} \rfloor$. Moreover, $\Delta(G) \geq d_{G'}(v) + |V_{i+2}| \geq \Delta(G') + p$ and $\omega(G) \geq \omega(G')$. Hence, $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + p + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil + \lfloor \frac{p}{2} \rfloor \geq \chi(G)$, a contradiction. Henceforth $v_{i+3} v_{i+4}$ is an edge of maximum weight in C_{2k+1} as claimed. \square

Corollary 17 *Let $G = C_{2k+1}(G_0 \dots G_{2k})$ be an expansion of an odd hole. Let $p = \min_{0 \leq i \leq 2k} \chi_i$. If p is even then Conjecture 1 holds for G .*

Proof Let us write $C_{2k+1} = v_0 \dots v_{2k}$. Suppose the edge $v_i v_{i+1}$ has maximum weight in C_{2k+1} . Let $G' = G[V_i \cup V_{i+1}]$ and v be a vertex of maximum degree in G' . Assume without loss of generality $v \in V_{i+1}$. Since p is even, $\lfloor \frac{p+1}{2} \rfloor = \lfloor \frac{p}{2} \rfloor$ and from Corollary 15 we have: $\chi(G) = \chi_i + \chi_{i+1} + \lfloor \frac{p}{2} \rfloor$. In addition, by Theorem 16, V_{i+2} induces a complete graph on p vertices. Thus, $\Delta(G) \geq d_{G'}(v) + |V_{i+2}| \geq \Delta(G') + p$. Consequently, $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + p}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil + \lfloor \frac{p}{2} \rfloor = \chi(G)$, a contradiction. \square

Theorem 18 *If $G = C_{2k+1}(G_0 \dots G_{2k})$ is an expansion of an odd hole such that $\chi_i = 1$ for some $i \in \{0 \dots 2k\}$ then Conjecture 1 holds for G .*

Proof Suppose that G is a counter-example to Conjecture 1.

Assume, without loss of generality that $v_0 v_1$ has maximum weight. By Corollary 13 we have $\chi(G) \leq \chi_0 + \chi_1 + 1$. If $\chi(G) = \chi_0 + \chi_1$ then G satisfies Conjecture 1 by Lemma 2, a contradiction. Hence $\chi(G) = \chi_0 + \chi_1 + 1$ and by Theorem 16 we can suppose that V_{2k} is reduced to a single vertex v .

We consider an optimal coloring of the bipartite expansion $G - v$, such a coloring requires precisely $\chi_0 + \chi_1$ colors and we can assume that this optimal coloring have been obtained via the algorithm described in the previous section (expansion of bipartite graphs). We denote Γ_i the set of colors used for the coloring of G_i , $i = 0 \dots 2k - 1$. When i is even, 0 is the preferred index for the coloring of G_i and, 1 is its preferred index when i is odd. Let us remark that, for this coloring, when $i \in \{0 \dots 2k\}$, $\Gamma_i \cap \Gamma_{i+1} = \emptyset$, $\Gamma_i \subseteq \Gamma_0 \cup \Gamma_1$, and $|\Gamma_i| = \chi_i$ (see Remark 8). We get an optimal coloring of the whole graph G by giving a new color to the vertex v .

CLAIM 1 *$v_{2k-1} v_{2k-2}$ is an edge of maximum weight, moreover $\Gamma_1 \subseteq \Gamma_{2k-1}$ and $\Gamma_{2k-2} \subseteq \Gamma_0$.*

Proof Suppose $\chi_{2k-1} < \chi_1$. Since $|\Gamma_{2k-1}| = \chi_{2k-1}$, some color a of Γ_1 does not appear in Γ_{2k-1} . This color could be given to v , a contradiction. Hence, $\chi_{2k-1} \geq \chi_1$, $\Gamma_1 \subseteq \Gamma_{2k-1}$ and, consequently, $\Gamma_{2k-2} \subseteq \Gamma_0$.

If $\chi_{2k-2} < \chi_0$ then some color $a \in \Gamma_0 \setminus \Gamma_{2k-2}$ does not appear in Γ_{2k-1} . Choose any color $b \in \Gamma_1$ and change the color of the vertices of G_{2k-1} , with that color, in a . Hence b is now available to color v , a contradiction.

It follows $\chi_{2k-1} + \chi_{2k-2} \geq \chi_0 + \chi_1$, that is the edge $v_{2k-1} v_{2k-2}$ has maximum weight. \blacksquare

CLAIM 2 *Let a be a color in $\Gamma_{2k-1} \cap \Gamma_1$ and b be a color in Γ_{2k-2} . Then the subgraph G_{ab} of G induced by these two colors is connected.*

Proof Let us remark that, by the definition of the expansion of an hole, it is sufficient to prove that G_{ab} contains a vertex of color b of G_0 . Assume to the contrary that G_{ab} is not connected. That is, the set of vertices colored with b in G_0 is not contained in the connected component of G_{ab} containing

the vertices of color a in G_{2k-1} . We can thus exchange the two colors a and b on the component containing the vertices of color a in G_{2k-1} . Since a does no longer appear in the neighborhood of v , we can give this color to v and we get a $\chi_0 + \chi_1$ coloring of G , a contradiction. ■

CLAIM 3 For any i ($0 \leq i \leq 2k-1$), $\Gamma_i \subseteq \Gamma_0$ when i is even and $\Gamma_1 \subseteq \Gamma_i$ when i is odd.

Proof Let a be any color in $\Gamma_{2k-1} \cap \Gamma_1$ and b any color in Γ_{2k-2} . Since by Claim 2, G_{ab} is connected, a shortest path in this subgraph joining a vertex in G_0 to a vertex in G_{2k-1} must contain an edge between G_i and G_{i+1} for any index i ($0 \leq i \leq 2k-2$). Hence, when i is even G_i contains a vertex colored with b ($0 \leq i \leq 2k-2$) while for i odd G_i contains a vertex colored with a ($1 \leq i \leq 2k-1$). Since, by Claim 1, $\Gamma_1 \subseteq \Gamma_{2k-1}$ and $\Gamma_{2k-2} \subseteq \Gamma_0$, the claim follows. ■

CLAIM 4 For any even index i ($2 \leq i \leq 2k-2$), $\Gamma_i \subseteq \Gamma_{i-2}$.

Proof Assume that some color a of Γ_i does not appear in Γ_{i-2} and let b be any color in $\Gamma_1 \cap \Gamma_{2k-1}$. Let G_{ab} be the subgraph of G induced by these two colors and let Q be the connected component of G_{ab} containing the vertices colored with b in G_{2k-1} . Since $\Gamma_{i-2} \subseteq \Gamma_0$ by Claim 3 and $a \notin \Gamma_{i-2}$, Q does not contain any vertex in Γ_{i-2} . Hence Q does not contain any vertex colored with a in G_0 and G_{ab} is not connected, a contradiction with Claim 2. ■

CLAIM 5 For an odd index i ($1 \leq i \leq 2k-1$), $v_{i-1}v_i$ is an edge with maximum weight.

Proof Since $\Gamma_1 \subseteq \Gamma_i$ and $\Gamma_{i-1} \subseteq \Gamma_0$ by Claim 3, let us prove that $\Gamma_0 - \Gamma_{i-1} \subseteq \Gamma_i$. Assume that some color $a \in \Gamma_0 - \Gamma_{i-1}$ does not appear in Γ_i . Let b be any color in Γ_{2k-2} (recall that $\Gamma_{2k-2} \subseteq \Gamma_0$ by Claim 1) and let G_{ab} be the subgraph induced by these two colors. Since a does not appear in $\Gamma_i \cup \Gamma_{i-1}$ but appears in Γ_{2k-1} by Claim 1, the connected component Q of G_{ab} containing the vertices of color a in G_{2k-1} is distinct from the component containing the vertices of color a in G_0 .

Let us now exchange the colors a and b on Q . In this new coloring of G , let Q' be the connected component of the subgraph induced by the colors a and c where c is any color in Γ_1 . Since a is always lacking in the sets of color Γ_i as well as in Γ_{i-1} , Q' does not contain any vertex colored with a in G_0 . We can thus proceed to a new exchange of colors a and c on Q' . The color c is now available to coloring v , a contradiction.

But now, since $\chi_i = |\Gamma_i| = |\Gamma_1| + |\Gamma_0| - |\Gamma_{i-1}|$ and $\chi_{i-1} = |\Gamma_{i-1}|$, we have $\chi_i + \chi_{i-1} = \chi_0 + \chi_1$, in other words $v_{i-1}v_i$ is an edge with maximum weight. ■

CLAIM 6 For any odd index i ($1 \leq i \leq 2k-3$), $\Gamma_i \subseteq \Gamma_{i+2}$.

Proof Obvious by virtue of Claims 5 and 4. ■

CLAIM 7 For any index i ($0 \leq i \leq 2k - 1$), G_i has at least two vertices

Proof Assume to the contrary that G_i is reduced to a single vertex for some $i \in \{0, \dots, 2k - 1\}$.

If i is even then, by Claim 5, $v_i v_{i+1}$ has maximum weight and the unique vertex in G_i has maximum degree in $G[V_i \cup V_{i+1}]$. Consequently, by Theorem 16, G_{i-1} is reduced to a single vertex. But now, by Claim 6, $\Gamma_1 \subseteq \Gamma_{i-1}$, that means $\chi_1 = 1$ since $|\Gamma_{i-1}| = 1$. By Claim 4, $|\Gamma_{i+2}| = |\Gamma_{i+4}| = \dots = |\Gamma_{2k-2}| = 1$. In addition, $v_0 v_{2k}$ has maximum weight, it follows $|V_{2k-1}| = 1$. Let us set $\Gamma_{2k-2} = \{a\}$ and $\Gamma_{2k-1} = \Gamma_1 = \{b\}$, of course $a \in \Gamma_0$.

We claim that $\Gamma_0 = \{a\}$. Assume, on the contrary, that in Γ_0 there is a color, say c , distinct from a . The subgraph G_{bc} induced by the vertices of G colored with b and c is not connected since $c \notin \Gamma_{2k-2}$. In this conditions, we could exchange the colors b and c on the component of G_{bc} which contains vertices of V_0 and use the color c for the coloring of the vertex v , a contradiction.

Hence, $|\Gamma_0| = 1 = \chi_0$ and $\chi_0 + \chi_1 = 2$. In other words for $0 \leq i \leq 2k$, V_i is a stable set and G is an empty expansion of an odd hole, a contradiction (see [1]).

When i is odd, the edge $v_i v_{i-1}$ having maximum weight in Γ_{2k+1} by Claim 5, G_{i+1} is reduced to a single vertex by Theorem 16 and the above reasoning holds. ■

To end our proof assume first that $k \geq 3$. An edge $v_i v_{i-1}$ with i odd being of maximum weight in H by Claim 5, one of G_{i+1} or G_{i-2} must be reduced to a single vertex by Theorem 16, a contradiction with Claim 7.

Hence from now on $k = 2$. Let $G' = G[V_0 \cup V_1]$. By Claim 7, $|V_i| \geq 2$ for $i = 0 \dots 4$. Moreover, $\Delta_1 \geq 1$, otherwise the edge $v_0 v_4$ would have maximum weight in C_{2k+1} and $|V_3| = 1$ by Theorem 16, a contradiction with Claim 7.

Assume that $|V_2| \geq |V_1|$ and let w be a vertex of maximum degree in G_1 . We have

$$\Delta \geq d(w) \geq |V_0| + |V_2| + \Delta_1 \geq |V_0| + |V_2| + 1 \geq \Delta_0 + |V_1| + 2 = \Delta(G') + 2.$$

Consequently $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil + 1$ and by Theorem 3, $\lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil + 1 \geq \chi_0 + \chi_1$. Hence $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \chi_0 + \chi_1 + 1$, a contradiction since $\chi_0 + \chi_1 + 1$ is precisely the chromatic number of G .

Hence we must suppose that $|V_2| < |V_1|$. Since $v_2 v_3$ is an edge of maximum weight in C_{2k+1} with v in the neighborhood of G_4 in the expansion, we could have chosen this edge as the edge $v_0 v_1$. With the same reasoning we should obtain that $|V_1| < |V_2|$, a contradiction. □

Theorem 19 If $G = C_{2k+1}(G_0 \dots G_{2k})$ is an expansion of an odd hole such that G_i induces a bipartite graph for some $i \in \{0 \dots 2k\}$ then Conjecture 1 holds for G .

Proof Assume that G is a counter-example to Conjecture 1. By Corollary 13, $\chi(G) \leq \chi_i + \chi_{i+1} + 1$ when $v_i v_{i+1}$ is an edge with maximum weight. When $\chi(G) = \chi_i + \chi_{i+1}$, we have a contradiction with Lemma 2. When $\chi(G) = \chi_i + \chi_{i+1} + 1$, one component of G must be reduced to a single vertex by Corollary 14, a contradiction with Theorem 18. \square

Theorem 20 *If $G = C_{2k+1}(G_0 \dots G_{2k})$ is an expansion of an odd hole such that $\chi_i = q \geq 1$ for all $i \in \{0 \dots 2k\}$ then Conjecture 1 holds for G .*

Proof Assume to the contrary that G is a counter-example to Conjecture 1. Since every edge of H has maximum weight, for every $i \in \{0 \dots 2k\}$ V_{i-2} or V_{i+1} induces a complete graph on exactly q vertices, by the hypothesis and Theorem 16. Hence, it is not difficult to see that at least two components, say V_0 and V_1 , are isomorphic to K_q . We have thus $\omega \geq 2q$ and $\Delta \geq 3q - 1$ (a vertex in V_1 has q neighbors in V_0 , $q - 1$ in V_1 and at least q neighbors in V_2) which leads to

$$\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{5q}{2} \rceil.$$

By Theorem 11 we have $\chi(G) \leq \lceil \frac{5q}{2} \rceil$, a contradiction. \square

Theorem 21 *If $G = C_{2k+1}(G_0 \dots G_{2k})$ is an expansion of an odd hole such that $\chi_i \leq 3$ for all $i \in \{0 \dots 2k\}$ then Conjecture 1 holds for G .*

Proof Assume that G is a counter-example to Conjecture 1. If some component has chromatic number at most 2, we have a contradiction with Theorem 19. Hence we must suppose that each component has chromatic number 3, a contradiction with Theorem 20 \square

The following lemma will be useful in the next theorem. Its proof is standard and left to the reader.

Lemma 22 *Let K be a graph with chromatic number 4.*

- *if K has 5 vertices then K contains a K_4*
- *if $\omega(K) = 2$ then K has at least 8 vertices.*

Theorem 23 *If $G = C_{2k+1}(G_0 \dots G_{2k})$ is an expansion of an odd hole such that $\chi_i \leq 4$ for all $i \in \{0 \dots 2k\}$ then Conjecture 1 holds for G .*

Proof Assume that G is a counter-example to Conjecture 1. If some component has chromatic number at most 2, we have a contradiction with Theorem 19. Hence we must suppose that each component has chromatic number 3 or 4. If no component has chromatic number 4, we have a contradiction with Theorem 20 as well as if every component has chromatic number 4. Hence we can suppose that at least one component has chromatic number 3 and at least one component has chromatic number 4. This forces immediately $\chi_0 + \chi_1 = 7$ or 8. Let us remark also that $\omega \geq 4$.

We have $\chi(G) = 9$ or $\chi(G) = 10$ and, obviously, $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq 9$ as soon as $\omega(G) + \Delta(G) \geq 16$ and $\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq 10$ as soon as $\omega(G) + \Delta(G) \geq 18$.

CLAIM 1 *Every component has at most 7 vertices*

Proof Assume to the contrary that some component V_i has at least 8 vertices. If $\Delta_{i+1} \geq 3$ then $\Delta \geq 14$. Hence $\omega(G) + \Delta(G) \geq 18$ and Reed's conjecture holds for G , a contradiction. If $\Delta_{i+1} \leq 2$ then V_{i+1} must be isomorphic to a triangle by Brook's Theorem. We have thus $\omega(G) \geq 5$ and $\Delta(G) \geq 13$ and Reed's conjecture holds for G , a contradiction. ■

From now on, we can consider that any component has at most 7 vertices and hence, by Lemma 22, any 4-chromatic component contains a triangle.

CLAIM 2 *No two components with chromatic number 4 are consecutive*

Proof Assume to the contrary that for two consecutive components, V_i and V_{i+1} , are such that $\chi_i = 4$ and $\chi_{i+1} = 4$. If these two components are isomorphic to a K_4 then any vertex in these components has degree at least 10. Since a maximum clique of G in this case has at least 8 vertices, we have $\omega + \Delta \geq 18$.

If only one component is isomorphic to a K_4 (without loss of generality say that V_i induces a K_4), then $\Delta_{i+1} \geq 4$ by Brook's theorem and a vertex of maximum degree in V_{i+1} has at least 11 neighbors. Since a maximum clique of G in this case has at least 7 vertices, we have $\omega + \Delta \geq 18$.

If no component is isomorphic to a K_4 then Δ_i and Δ_{i+1} are greater than 4 by Brook's theorem. Moreover V_i and V_{i+1} contain at least 5 vertices each. A vertex of maximum degree in X_i has hence at least 12 neighbors. Since a maximum clique of G in this case has at least 6 vertices, we have $\omega + \Delta \geq 18$.

In each case we have a contradiction since G satisfies Reed's conjecture. ■

We can thus suppose that no two consecutive components have chromatic number 4. In that case we can remark that $\chi(G) = 9$. To end our proof, it is thus sufficient to show that $\omega(G) + \Delta(G) \geq 16$.

Without loss of generality, assume that $\chi_0 = 4$. By Claim 2 we have $\chi_{2p} = 3$ and $\chi_1 = 3$.

If V_0 induces a K_4 then either V_{2p} or V_1 contain a triangle and hence $\omega \geq 7$ or have no triangle and V_{2p} and V_1 contain at least 4 vertices each. In the first case a vertex in V_0 has at least 9 neighbors and $\omega(G) + \Delta(G) \geq 16$. In the second case we have $\omega(G) \geq 5$ and a vertex in V_0 has at least 11 neighbors. We get then $\omega(G) + \Delta(G) \geq 16$.

Assume now that V_0 does not induce a K_4 then $\Delta_0 \geq 4$ by Brook's theorem. If V_{2p} or V_1 contain a triangle then $\omega \geq 6$ and a vertex of maximum degree in V_0 has at least 10 neighbors. We get then $\omega(G) + \Delta(G) \geq 16$.

If V_{2p} and V_1 contain no triangle, these two sets must have at least 4 vertices by Brook's theorem and a vertex of maximum degree in V_0 has at least 12 vertices. Since $\omega \geq 5$ in that case, we get then $\omega(G) + \Delta(G) \geq 17$.

In each case we have a contradiction since G satisfies Reed's conjecture. □

Claim 1 in the proof of Theorem 23 suggests that Reed's conjecture holds asymptotically for expansions of odd cycles.

Theorem 24 For every $k \geq 1$ and every $p \geq 1$, any expansion of an odd cycle C_{2k+1} where each component has chromatic number at most p and with at least $(2k+1)(5p-9)+1$ vertices satisfies Conjecture 1.

Proof By Theorem 23, we can suppose that $p \geq 5$. Moreover, by Theorem 19, we can suppose that each component has chromatic number at least 3 and hence the maximum degree of each component must be at least 2. Let $G = C_{2k+1}(G_0, G_1 \dots G_{2k})$ and assume that $\chi_i \leq p$ ($i = 0 \dots 2k$). By Corollary 13 we have $\chi(G) \leq \lceil \frac{5p}{2} \rceil$.

Suppose that some component V_i ($i = 0 \dots 2k$) contains at least $5p-9$ vertices. Then a vertex in V_{i+1} has degree at least $5p-4$. Since obviously $\omega(G) \geq 4$ we have thus $\lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil \geq \lceil \frac{5p+1}{2} \rceil$. Hence G satisfies Conjecture 1 and the result follows. \square

Theorem 25 If G is a C_5 -expansion then Conjecture 1 holds for G .

Proof Let $G = C_5(G_0, G_1, G_2, G_3, G_4)$ and assume by contradiction that G does not satisfy Conjecture 1. Let $p = \min \chi(G_i)$ $i = 0, \dots, 4$, by Theorem 19 we have $p \geq 3$.

We suppose that $\chi(G_0) + \chi(G_1)$ is maximum among the pairs of consecutive components of G and we denote $G' = G[V_0 \cup V_1]$. By Theorem 16, G_4 or G_2 induce a complete graph on p vertices. We assume that G_4 is this component and there is a vertex in V_0 whose degree in G' is maximum. Moreover, Theorem 16 implies that $\chi_2 + \chi_3 = \chi_0 + \chi_1$.

By Corollary 15 we have $\chi(G) = \chi_0 + \chi_1 + \lfloor \frac{p+1}{2} \rfloor$.

We claim now that $|V_2| < |V_1|$ or G_1 is isomorphic to a C_{2s+1} with $s \geq 2$ (and henceforth $p = 3$). Assume to the contrary that $|V_2| \geq |V_1|$. Let w be a vertex of maximum degree in G_1 . By Theorem 3 we have $\chi(G_0) + \chi(G_1) \leq \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil$. Since $d(w) = \Delta(G_1) + |V_0| + |V_2| \geq \Delta(G') + \Delta(G_1) + 1$ we have $\Delta(G) \geq \Delta(G') + \Delta(G_1) + 1$. By Brook's Theorem [3] we have $\chi(G_1) \leq \Delta(G_1)$ or G_1 is an odd chordless cycle. When $\chi(G_1) \leq \Delta(G_1)$, we get

$$\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{\omega(G') + \Delta(G') + p + 1 + 1}{2} \rceil. \quad (2)$$

Which leads to $\lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil \geq \chi(G_0) + \chi(G_1) + \lfloor \frac{p+1}{2} \rfloor = \chi(G)$, a contradiction.

If G_1 is isomorphic to a C_{2s+1} with $s \geq 2$ we have $\omega(G') = \omega(G_0) + 2$, $\omega(G) \geq \omega(G_0) + 3$ and $\Delta(G) \geq \Delta(G') + 3$. Hence

$$\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil \geq \lceil \frac{\omega(G') + 1 + \Delta(G') + 3 + 1}{2} \rceil. \quad (3)$$

Which leads to $\lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil \geq \chi(G_0) + \chi(G_1) + 2 \geq \chi(G)$, a contradiction.

If $G[V_2 \cup V_3]$ contains a vertex of maximum degree in V_2 , by Theorem 16, G_1 is a complete graph on p vertices, a contradiction with $|V_2| < |V_1|$. Hence a vertex of maximum degree in $G[V_2 \cup V_3]$ must be a vertex of G_3 . By application of the above technique we can thus prove that $|V_1| < |V_2|$ or G_2 is isomorphic to a C_{2s+1} with $s \geq 2$. In the first case, we get a contradiction with $|V_2| < |V_1|$. In the latter case, we can conclude as above.

□

Theorem 26 *If $G = C_{2k+1}(G_0 \dots G_{2k})$ is an expansion of an odd hole then $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil + 1$.*

Proof We consider an optimal colouring of G . Let us denote $p = \min_{0 \leq i \leq 2k} \chi_i$.

If p is even we have $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$ (Corollary 17). Consequently, in the following, we suppose that p is odd.

Let $j \in \{0, \dots, 2k\}$ such that $\chi_j = p$. We choose some colour used for the colouring of G_j , say c_j and we denote S_j as the set of vertices of G_j being coloured with c_j .

We set $G'_j = G[V_j - S_j]$ and for $i \neq j$ we set $G'_i = G_i$.

$G' = C_{2k+1}(G'_0, \dots, G'_{2k})$ is an odd expansion such that the minimum chromatic number of its components is $p - 1$. Since $p - 1$ is even, again by Corollary 17, we have $\chi(G') \leq \lceil \frac{\omega(G') + \Delta(G') + 1}{2} \rceil$ and consequently $\chi(G') \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$.

But now, given an optimal colouring of G' , we can obtain an optimal colouring of G with only one additional colour (for the vertices of S_j). In other words, $\chi(G) \leq \chi(G') + 1$. The result follows. □

In a further paper [4], we will use the above results in order to extend a number of the results given in [1].

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