

# Numerical Analysis of Two Non-linear Soft Thin Layers

Frédéric Lebon, Raffaella Rizzoni, and Sylvie Ronel-Idrissi

**Abstract.** In a first part, we consider a bar with extremities subject to a given displacement and made by two elastic bodies with linear stress-strain relation separated by an adhesive layer of thickness  $h$ . The material of the adhesive is characterized by a non convex (piecewise quadratic) strain energy density with elastic modulus  $k$ . After considering the equilibrium problem of the bar and determining the stable and metastable solutions, we let  $(h, k)$  tending to zero and we obtain the corresponding asymptotic contact laws, linking the stress to the jump of the displacement at the adhesive interface. The second part of the paper is devoted to the bi-dimensional problem of two elastic bodies separated by a thin soft adhesive. The behaviour of the adhesive is non associated elastic-plastic. As in the first part, we study the asymptotic contact laws.

## 1 Introduction

The modelling of adhesive bonding of elastic bodies leads to problems taking into account several parameters, typically the thickness and the rigidity of the adhesive layer. It comes natural to study the limit problem when both these

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Frédéric Lebon

Laboratoire de Mécanique et d'Acoustique, Université de Provence,  
31, Chemin Joseph Aiguier, 13402 Marseille Cedex 20 - France  
e-mail: lebon@lma.cnrs-mrs.fr

Raffaella Rizzoni

Dipartimento di Ingegneria, Università di Ferrara,  
Via Saragat 1, 44100 Ferrara - Italy  
e-mail: raffaella.rizzoni@unife.it

Sylvie Ronel-Idrissi

Laboratoire Mécanique, Matériaux, Structures, Université Claude Bernard,  
81, Boulevard Niels Bohr, 69622 Villeurbanne Cedex - France

parameters tend to zero. In the limit, the thin layer is replaced by a sharp interface and an asymptotic contact law is obtained, linking the stress to the jump of the displacement at the interface [2,1]. Different types of material behaviours have been considered for the adherent and the adhesive bodies. In this paper, two kind of material are considered. We present the first results of a study still in progress in which we consider an elastic bar composed by two adherent parts with linear stress-strain curve separated by an adhesive thin layer of thickness  $h$  with non convex energy density and rigidity  $k$ . We first discuss the stability and the metastability of the equilibrium configurations when a given relative displacement is imposed to the ends of the bar. Due to the non convexity of the energy, there are multiple metastable configurations made by a mixture of two phases, each phase corresponding to one ascending branch of the stress-strain curve. We then study the limit problem when the pair  $(h, k)$  tends to zero and discuss the asymptotic contact laws corresponding to the stable and metastable equilibrium configurations of the bar. The second part is devoted to some numerical results obtained considering an adhesive obeying to a non associated elastic plastic behaviour.

One of the motivations of this analysis is the complexity of the finite element modelling due to the strongly non-linear behaviour of the thin layer. This behaviour implies a very large number of computations. The cost is amplified by the thickness and the stiffness of the layer which are very small compared with the substrata characteristics. These low values increase the ill-conditioning of the tangent matrices and thus, the precision of the computations.

## 2 First Part: Non-convex Behaviour

### 2.1 *The Equilibrium Problem*

Consider a one-dimensional bar occupying a reference unstressed configuration  $0 \leq x \leq L$  from which we consider displacement fields  $u = u(x)$  continuous with piecewise continuous derivative. The bar is fixed at the extremity  $x = 0$ , and subject to a prescribed displacement  $\delta > 0$  at  $x = L$ , so the following boundary conditions are assigned  $u(0) = 0$ ,  $u(L) = \delta$ . Let  $\mathcal{U}$  be the class of displacements continuous with piecewise continuous derivative, and satisfying the boundary conditions. The bar is assumed to be made by two different elastic materials. In interval  $0 \leq x \leq \epsilon L$  it is composed by an adhesive layer characterized by the piecewise quadratic stored energy density

$$w_a(\xi) = \min_{\xi > 0} \left\{ \frac{k}{2}\xi^2; \frac{k}{2}\xi^2 - \alpha\xi + \alpha\Delta \right\} \quad (1)$$

where the modulus  $k$ , and the material parameters  $\alpha, \Delta$ , are positive constants. The derivative of the energy density  $w_a$  has two ascending branches, each one corresponding to a different phase. In the interval  $\epsilon L \leq x \leq L$  the bar is composed by a material with quadratic stored energy density

$$w_b(\xi) = \frac{K}{2}\xi^2 \quad (2)$$

The total energy of the bar is given by

$$E(u) = \int_0^{\epsilon L} w_a(u') dx + \int_{\epsilon L}^L w_b(u') dx \quad (3)$$

with  $u \in \mathcal{U}$ . We say that  $\bar{u} \in \mathcal{U}$  corresponds to

- a *stable configuration* of the bar, if it is a global minimizer of  $E(u)$  in  $\mathcal{U}$ , i.e. if  $E(\bar{u}) \leq E(u) \quad \forall u \in \mathcal{U}$ .
- a *metastable configuration*, if it corresponds to a relative minimum with respect to the norm of the uniform convergence, i.e.  $E(\bar{u}) \leq E(u)$  for all those  $u \in \mathcal{U}$  such that

$$\sup_{x \in (0, L)} |u(x) - \bar{u}(x)| + \sup_{x \in (0, L)} |u'(x) - \bar{u}'(x)| < \eta \quad \text{for some } \eta > 0$$

For  $\bar{u} \in \mathcal{U}$  to be a global or a relative minimizer, it is necessary that the first variation of  $E(u)$  vanishes at  $\bar{u}$  :

$$\left[ \frac{d}{dh} E(\bar{u} + hv) \right]_{h=0} = 0 \quad (4)$$

for all admissible perturbations  $v : [0, L] \mapsto \mathbb{R}$  continuous with piecewise continuous gradient and such that  $v(L) = 0 = v(0)$ . This leads to the system of equilibrium equations:

$$\begin{aligned} w_{a,\xi}(\bar{u}') &= \sigma \quad x \in (0, \epsilon L) \\ w_{b,\xi}(\bar{u}') &= \sigma \quad x \in (\epsilon L, L) \end{aligned} \quad (5)$$

with  $\sigma$  the (constant) stress in the bar. In view of (1) and (2), this system admits the following solutions [3]:

i) If  $0 \leq \sigma < k\Delta - \alpha$ , then, (5) are satisfied by

$$\bar{u} = \begin{cases} \sigma x / k & x \in [0, \epsilon L] \\ \sigma x / K + \sigma \epsilon L (1/k - 1/K) & x \in (\epsilon L, L) \end{cases} \quad (6)$$

with  $\sigma = \hat{k}_\epsilon \delta L^{-1}$ , and  $\hat{k}_\epsilon^{-1} = \epsilon k^{-1} + (1 - \epsilon) K^{-1}$ . By using the expression of  $\sigma$ , it is easy to see that this solution is possible whenever  $0 \leq \delta < (k\Delta - \alpha) L \hat{k}_\epsilon^{-1}$ . This solution is said to correspond to a *low-strain monophase*

configuration because the strain in the adhesive layer  $(0, \epsilon L)$  lies in the first ascending branch of the stress-strain curve  $w'_a$ .

ii) If  $k\Delta - \alpha \leq \sigma < k\Delta$ , then, the adhesive layer is a *mixture of two phases*; a low-strain phase with strain on the first ascending branch of  $w'_a$ , and a high-strain phase with strain on the second ascending branch. Let  $\lambda \in (0, 1)$  denote the volume fraction of the low-strain solution. As  $\lambda$  varies in  $[0, 1]$ , we obtain a family of equilibrium solutions:

$$\bar{u} = \begin{cases} \sigma x/k & x \in [0, \lambda \epsilon L] \\ [(\sigma + \alpha)x]/k - \alpha \epsilon \lambda L/k & x \in (\lambda \epsilon L, \epsilon L] \\ \sigma x/K + \epsilon L \{[\sigma + (1 - \lambda)\alpha]/k - \sigma/K\} & x \in (\epsilon L, L] \end{cases} \quad (7)$$

with

$$\sigma = \hat{k}_\epsilon \left[ \frac{\delta}{L} - \epsilon(1 - \lambda) \frac{\alpha}{k} \right] \quad (8)$$

By using this expression for  $\sigma$ , we find that the two-phase solution exists whenever

$$\left[ (k\Delta - \alpha)\hat{k}_\epsilon^{-1} + \epsilon\alpha(1 - \lambda)k^{-1} \right] L \leq \delta \leq \left[ k\Delta\hat{k}_\epsilon^{-1} + \epsilon\alpha(1 - \lambda)k^{-1} \right] L \quad (9)$$

Because the volume fraction  $\lambda$  ranges over the interval  $0 < \lambda < 1$ , this condition can be further extended as follows

$$(k\Delta - \alpha)\hat{k}_\epsilon^{-1}L \leq \delta \leq (k\Delta\hat{k}_\epsilon^{-1} + \epsilon\alpha k^{-1})L \quad (10)$$

Thus, given any  $\lambda \in (0, 1)$ , if the prescribed elongation  $\delta$  lies in the above range, then in the adhesive there exists a two-phase solution involving a mixture of both phases.

iii) If  $\sigma \geq k\Delta$  then, the solution is

$$\bar{u} = \begin{cases} (\sigma + \alpha)x/k & x \in [0, \epsilon L] \\ \sigma x/K + \epsilon L [(\sigma + \alpha)/k - \sigma/K] & x \in (\epsilon L, L] \end{cases} \quad (11)$$

with  $\sigma = \hat{k}_\epsilon(\delta L^{-1} - \alpha \epsilon k^{-1})$ . This solution, possible whenever  $\delta > k\Delta L \hat{k}_\epsilon^{-1} + \alpha \epsilon L k^{-1}$ , corresponds to a *high-strain monophase* configuration because the strain in  $(0, \epsilon L)$  lies in the second ascending branch of the stress-strain curve of the adhesive.

Note that a discussion about the *metastability* of the equilibrium solutions is given in [3]. As shown in [3] a study of the global *stability* of the equilibrium solutions restricts the solutions to those corresponding to stable configurations of the bar. This has the effect to select the following configurations:

- a) If  $0 \leq \delta < (k\Delta - \alpha/2)L\hat{k}_\epsilon^{-1}$ , then the low strain monophase solution (6) is energy minimizer.
- b) If  $(k\Delta - \alpha/2)L\hat{k}_\epsilon^{-1} \leq \delta < (k\Delta - \alpha/2)L\hat{k}_\epsilon^{-1} + \epsilon\alpha L k^{-1}$ , then it is minimizer the mixture of low and high strain phases (7) associated with the Maxwell

stress  $\sigma_M = k\Delta - \alpha/2$ . By substituting  $\sigma_M$  into (8), we obtain that in stable configurations the volume fraction of the low strain phase is uniquely determined for the given  $\delta$ :

$$\lambda_{sta} = 1 + \frac{K}{\epsilon\alpha} \left( \frac{k}{\hat{k}_\epsilon} \Delta - \frac{\alpha}{2\hat{k}_\epsilon} - \frac{\delta}{L} \right) \quad (12)$$

Note that when  $\delta = (k\Delta - \alpha/2)L\hat{k}_\epsilon^{-1}$  we have  $\lambda_{sta} = 1$ : this means that the adhesive is still in the low strain phase. When  $\delta = (k\Delta - \alpha/2)L\hat{k}_\epsilon^{-1} + \epsilon\alpha Lk^{-1}$ , then  $\lambda_{sta} = 0$  and the material of the adhesive has completed the transformation from the low strain phase to the high strain phase.

- c) If  $\delta \geq (k\Delta - \alpha/2)L\hat{k}_\epsilon^{-1} + \epsilon\alpha Lk^{-1}$ , then the high strain monophase solution (11) is energy minimizer.

## 2.2 Contact Laws

We now consider the case in which both the thickness and the elastic modulus of the adhesive are very small compared to the corresponding quantities of the adherent elastic bodies. We would then lead to study the preceding equilibrium solutions when both the parameters  $(\epsilon, k)$  tend to zero and the thin adhesive layer is replaced by a sharp interface. Accordingly, we set

$$k = k_0\epsilon \quad (13)$$

and we rescale the material parameters so as

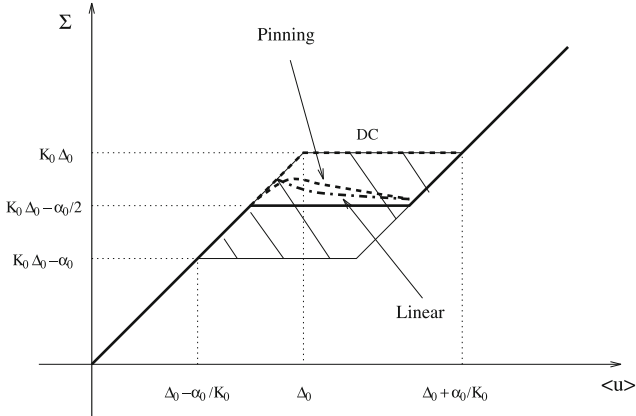
$$\begin{aligned} \Delta &= \Delta_0\epsilon^{-1} \\ \alpha &= \alpha_0 \end{aligned} \quad (14)$$

with  $k_0$ ,  $\alpha_0$  and  $\Delta_0$  independent on  $\epsilon$ . Our aim here is to study the relations between the limits

$$\begin{aligned} \Sigma &= \lim_{\epsilon \rightarrow 0^+} \sigma \\ [u] &= \lim_{\epsilon \rightarrow 0^+} \bar{u}(\epsilon L) \end{aligned} \quad (15)$$

where  $\sigma$  and  $\bar{u}$  correspond to the stable equilibrium configurations determined in the previous section. Our aim here is to identify an asymptotic contact law, linking the limit stress in the bar ( $\Sigma$ ) to the jump of the displacement at the adhesive interface ( $[u]$ ). This contact law describes the limit behaviour of the adhesive. By substituting (13) and (14) into the expressions of  $\sigma$  and  $\bar{u}$  listed in Section 2.1 in (a), (b) and (c), taking the limit  $\epsilon \rightarrow 0^+$  and eliminating  $\delta$  between  $\Sigma$  and  $[u]$  we obtain the following contact law

$$\Sigma_{sta} = \begin{cases} k_0 \frac{[u]}{L} & 0 \leq [u] < L(\Delta_0 - \frac{\alpha_0}{2k_0}) \\ (k_0\Delta_0 - \frac{\alpha_0}{2}) & L(k_0\Delta_0 - \frac{\alpha_0}{2k_0}) \leq [u] < L(k_0\Delta_0 + \frac{\alpha_0}{2k_0}) \\ k_0 \frac{[u]}{L} - \alpha_0 & [u] \geq L(k_0\Delta_0 + \frac{\alpha_0}{2k_0}) \end{cases} \quad (16)$$



**Fig. 1** Contact law corresponding to stable configurations (bold line), domain of contact laws corresponding to metastable configurations (shaded region) and three metastable configurations (linear, pinning, DC)

corresponding to stable configurations of the original equilibrium problem. In the same way, considering the  $\sigma$  and  $\bar{u}$  as in (i), (ii) and (iii), we can calculate the limit contact law corresponding to metastable configurations. Note that this law turns out to be undefined, due to the lack of information associated to non uniqueness of metastable equilibrium solutions. Indeed, we find

$$\Sigma_{meta} = \begin{cases} k_0 \frac{[u]}{L} & 0 \leq [u] < L(\Delta_0 - \frac{\alpha_0}{k_0}) \\ \bar{\Sigma} & (k_0 \Delta_0 - \frac{\alpha_0}{k_0}) \leq [u] < L(k_0 \Delta_0 + \frac{\alpha_0}{k_0}) \\ k_0 \frac{[u]}{L} - \alpha_0 & [u] \geq L(\Delta_0 + \frac{\alpha_0}{k_0}) \end{cases} \quad (17)$$

with  $\bar{\Sigma}$  taking a value in  $[k_0 \Delta_0 - \alpha_0, k_0 \Delta_0 + \alpha_0]$ . Therefore, the metastable configurations give rise to a multiplicity of contact laws, all contained in the interior of the parallelogram shown in Fig. 1 for  $(k_0 \Delta_0 - \alpha_0/k_0) \leq [u] < L(k_0 \Delta_0 + \alpha_0/k_0)$ .

### 2.3 The Role of Nucleation and Kinetics

A way rule out the stress indeterminacy of metastable equilibrium solutions is to assign a criterion to select a path between metastable configurations. In [4,3], authors suggest to follow Aberayathne and Knowles [5,6,7] for introducing a nucleation and a kinetic condition into the equilibrium problem of the composite bar. Here we just sketch a description of these conditions. In the context of the dynamical problem of the composite bar, Aberayathne and Knowles assume that the nucleation and the propagation of a phase boundary

is governed by a relation between the normal speed on the phase boundary and the driving force  $f$  acting on it. As discussed in [7], one could consider a quasi-static context in which inertial effects are neglected and the governing problem is still the equilibrium problem for the composite bar except that  $\delta$ , and thus  $\bar{u}$  depend on the time as a parameter. The totality of the solutions are again described by (i), (ii) and (iii). Consider now the dissipation inequality, which states that the rate of change of the total energy (the sum of potential energy and the kinetic energy) minus the power of the work done at the boundary must be non negative for each equilibrium displacement. In the present context, the total energy coincides with the potential energy so the dissipation can be written as

$$D := \frac{dE(\bar{u})}{dt} - \sigma \frac{d\delta}{dt} \geq 0 \quad (18)$$

with  $\bar{u}$  given by (7). By substituting (7) into the dissipation inequality above and differentiating, we obtain

$$D = \dot{s} \left[ w_a \left( \frac{\sigma}{k} \right) - w_a \left( \frac{\sigma + \alpha}{k} \right) + \frac{\sigma \alpha}{k} \right] \quad (19)$$

with  $\dot{s} = (d\lambda/d\delta)(d\delta/dt)\epsilon L$  playing the role of the speed of the phase boundary at  $x = \lambda\epsilon L$ . The driving force  $f$  acting on the phase boundary is defined to be the quantity which multiplies  $\dot{s}$ :

$$f := \left[ w_a \left( \frac{\sigma}{k} \right) - w_a \left( \frac{\sigma + \alpha}{k} \right) + \frac{\sigma \alpha}{k} \right] = \frac{\alpha}{k}(\sigma - \sigma_M) \quad (20)$$

Then, accordingly to Aberayathne and Knowles, a nucleation condition can be stated in the following form: a phase boundary nucleates provided that the driving force on it soon after it nucleates exceeds some critical value  $f_{nuc}$  ( $> 0$ ), i.e.  $f \geq f_{nuc}$ . This condition determines the relative displacement  $\delta$  at which the point  $(\sigma, u'(\epsilon L))$  leaves the first ascending branch of the non monotone stress-strain curve  $w_{a,\xi}$ . Once that a phase boundary has nucleated, it propagates according to the kinetic (or evolution) condition, which relates the driving force with the “speed” of the phase boundary

$$f = \phi(\dot{s}) \quad (21)$$

with  $z\phi(z) \geq 0$  as enforced by the dissipation inequality (18). The condition (21) makes a selection among the metastable solutions and uniquely determines a path in the stress-strain plane between them. Different paths arise from different types of kinetic function  $\phi$ .

Now we consider the classical linear kinetics  $\phi(z) = mz$  and the pinning kinetics  $\phi(z) = m\sqrt{(ax^2 - b)_+}$ , which models the presence of defects that slow down the phase boundary motion [7]. We obtain the evolution of the parameter  $s$

$$\begin{aligned} s(t) &= (s(0) + K_2)e^{-K_3t} + K_1t - K_2 \quad (\text{linear}) \\ m\sqrt{(a\dot{s}^2 - b)} &= K_4 + K_5\delta t/L + K_6s \quad (\text{pinning}) \end{aligned} \quad (22)$$

The constants  $K_i$  are given. We suppose that  $m = m_0\epsilon^{-1}$ . The contact laws obtained in the metastable domain are:

$$\begin{aligned} \Sigma &= k_0[u]/L - s_0\alpha_0 \\ s_0(t) &= (s_0(0) + L_2)e^{-L_3t} + L_1t - L_2 \quad (\text{linear}) \\ m_0\sqrt{(a\dot{s}_0^2 - b)} &= L_4 + L_5\Delta_0t/L + L_6s_0 \quad (\text{pinning}) \end{aligned} \quad (23)$$

The constants  $L_i$  are given. The contact laws are presented in Fig. 1.

### 3 Part 2: Non-associated Elastic Plastic Behaviour

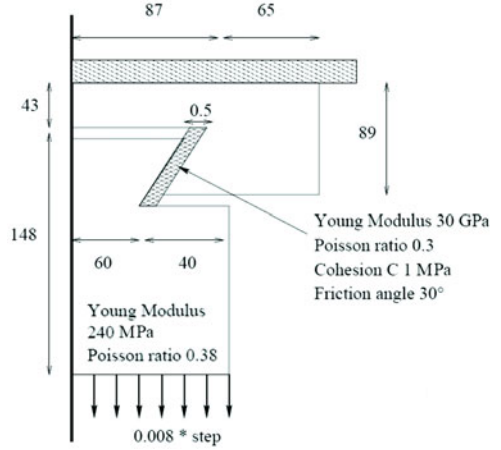
The objective of this part of the paper is to analyse soft thin layers in order to replace them by interface laws. The aim of our work is to study non-linear soft materials; especially, we focus on the case of non-associated elasto-plastic materials of Mohr-Coulomb and Drucker-Prager kind. In [8], we have obtained, using matched asymptotic expansions, contact laws of Coulomb kind. In bidimensional cases we have:

$$\begin{aligned} \Sigma.n &= K_0([u^e] - [u^p]) \\ |\tau| \leq C - \tan(\psi)\sigma_n[\dot{u}]^p &= -k\tau \quad \text{Mohr-Coulomb} \\ |\sigma^d| \leq C - \tan(\psi)\sigma^m[\dot{u}]^p &= -k\sigma^d n \quad \text{Drucker-Prager} \end{aligned} \quad (24)$$

The constants  $C$  and  $\psi$  are material coefficients. We use similar notations than in Section 2. The indices  $e$  and  $p$  correspond to the elastic and the plastic parts of the displacement. The vector  $\tau$  corresponds to the direction of the shear stress,  $\sigma_n$  is the normal stress. The tensor  $\sigma^d$  is the deviatoric part of the stress,  $\sigma^m$  is the pressure. Note that these laws depend on all the stress tensor (non local problem). In this part of the paper we analyse the influence of each component of the stress tensor in order to simplify the previous laws. We have shown in the theoretical part, that in the limit problem it is necessary to solve a local problem coupled with the global one. In the local problem, there are two significant quantities which intervene: the plastic yield and the plastic flow. The aim of this analyse is to quantify the level of each term in the plastic yield. In fact, we want to analyse if it is possible to replace the ‘‘real’’ plastic yield in which all the terms of the



**Fig. 2** The example of a dovetail assembly



stress tensor are considered, by a “simplified” one in which only the terms of the stress vector on the surface are taken into account. The plastic yield is defined by

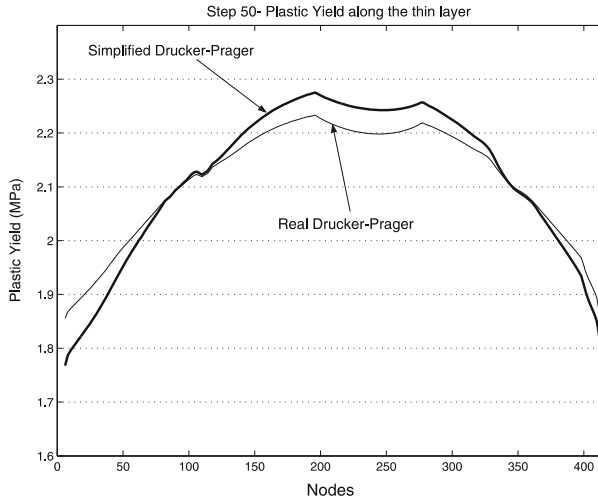
$$|\sigma^d| \leq C - \tan(\psi)\sigma^m \quad (25)$$

and the simplified one is

$$|\sigma_s^d| \leq C - \tan(\psi)\sigma_s^m \quad (26)$$

where  $\sigma_s^m$  and  $\sigma_s^d$  are the normal and the deviatoric parts of the tensor  $\sigma n \otimes_s n$ . The example analysed is a dovetail assembly (Fig. 2). We compare this plastic yield with the simplified one using only the terms corresponding to the stress vector.

Figure 3 shows the differences between the two plastic yields. We observe a very low difference for this example and that this difference does not modify the initiation of the plastic process. The gap is maximum in the elastic zone but generally remains lower than 5%. In the plastic zones, this gap decreases to 2%. As a conclusion of this study, we have shown that our simplification is valid and that it is possible to work only with the stress vector for the computation of the plastic yield. As a conclusion, the numerical results obtained in this section show that the local problem introduced in the theoretical study can be neglected, that is to say that the interface law can be written only in terms of stress vector. We obtain a compliance law (regularized Coulomb law), well known in the literature.



**Fig. 3** Exact and simplified plastic yields

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