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MODELLING ADHESION BY ASYMPTOTIC TECHNIQUES

F. Lebon¹ and R. Rizzoni²

¹Aix-Marseille University and LMNA-GNRS, France

²Ferrara University, Italy

Abstract

In this chapter, a review of theoretical and numerical asymptotic studies on thin adhesive layers is proposed. A general mathematical method is presented for modelling the mechanical behavior of bonding and interfaces. This method is based on a simple idea that the adhesive film is supposed to be very thin; the mechanical problem depends strongly on the thinness of the adhesive. It is quite natural, mathematically and mechanically, to consider the limit problem, that is, the asymptotic problem obtained when the thickness and, possibly, the mechanical characteristics of the adhesive thin layer tend to zero. This asymptotic analysis leads to a limit problem with a mechanical constraint on the surface, to which the layer shrinks. The formulation of the limit problem includes the mechanical and geometrical properties of the layer. This limit problem is usually easier to solve numerically by using finite elements software. Theoretical results (i.e. limit problems) can be usually obtained by using at least four mathematical techniques: gamma-convergence, variational analysis, asymptotic expansions and numerical studies. In the chapter, some examples will be presented: comparable rigidity between the adhesive and the adherents, soft interfaces, adhesive governed by a non convex energy and imperfect adhesion between adhesive and adherents. Some numerical examples will also be given and, finally, an example of a numerical algorithm will be presented.

Keywords: Thin layer, interface, asymptotic theory.

1. Introduction

It is now widely admitted that “interphases” (the small volume between two solids) play a crucial role in the analysis of structure assemblies. Nevertheless, due to their small thickness (typically in the 1 μm to 1 mm range), it is difficult in a complete finite element analysis of a structure to take them directly into account. A simplified theory is crucial because of

the large number of degrees of freedom and the ill-conditioned numerical problem, which it is very hard to solve even in the linear case. On the other hand, from the numerical point of view, it is of great importance to obtain conditions on the parameters, indicating whenever it is possible to replace the real problem with a simplified one. There exist various strategies to overcome these difficulties. The most classical one is to introduce phenomenological models, taking into account the macroscopic behavior of the interface and eliminating it geometrically. The most famous law of this kind is the classical Coulomb's law [8, 15]. One other possible strategy consists in undertaking an asymptotic analysis (where the small parameter is the thickness of the interphase) to eliminate the interphase geometrically and to obtain an equivalent interface model, which will be simpler to implement in numerical simulations. This idea has been used in many studies and applied, under linear elastic, finite strains, and viscoelastic conditions, to obtain interface laws similar to the phenomenological laws described in the literature [1, 4, 11, 14, 20–27, 31, 33–35, 37–42, 44, 45, 48–50]. Within this approach, the layer no longer exists from the geometrical point of view, but is replaced by a constraint, taking the asymptotic behavior of the parameters into account. In many of these studies, a soft interphase has often been assumed to exist, with a much smaller stiffness than that of the adherents (the stiffness is another small parameter). Fewer studies have focused on joints consisting of adherents and an interphase with a comparable level of rigidity. The present chapter is devoted to a general strategy to analyze this kind of problems.

To summarize, the analysis of thin adhesive bonded joints between deformable bodies involves problems with several parameters. At least two kinds of these parameters are essential because they link the stress vector to the jump in the displacement vector at the interface:

- the thickness of the joint, which is small with respect to those of the adherents,
- possibly, the mechanical characteristics (stiffness, viscosity, etc.) of the joint, which can be smaller than those of the adherents.

To study these asymptotic problems, several mathematical methods have been introduced: matched asymptotic expansions [18, 47], Γ -convergence and variational theory [9, 16], and numerical procedures. After introducing some general notations and the mechanical problem in section 2, we summarize these theories in section 3 and then we apply them to several cases of interphases. In particular, in section 4 we present the asymptotic analysis of a joint made of materials with comparable elastic moduli. Section 5 is devoted to the study of Signorini-Coulomb's conditions at the interfaces between the adherents and the adhesive. In section 6, we report on the study of an interphase characterized by a non convex deformation energy, which is traditionally associated to martensitic phase transformations. An example of the numerical implementation of the laws obtained in section 5 is presented in section 7.

2. General Notations

Let us consider a body occupying an open bounded set Ω of \mathbb{R}^3 with a smooth boundary $\partial\Omega$, where the three dimensional space is referred to the orthonormal frame (O, x_1, x_2, x_3) .

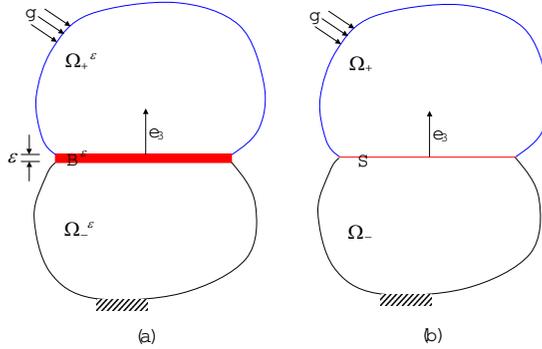


Figure 1. (a) Reference and (b) limit configurations of the joint.

This set Ω is assumed to have a non-empty intersection S with the plane $\{x_3 = 0\}$. Let $\varepsilon > 0$ be a small parameter, and let us define the following sets:

$$\begin{aligned}
 B^\varepsilon &= \{x = (x_1, x_2, x_3) \in \Omega : |x_3| < \frac{\varepsilon}{2}\}, \\
 \Omega_\pm^\varepsilon &= \{x = (x_1, x_2, x_3) \in \Omega : \pm x_3 > \frac{\varepsilon}{2}\}, \\
 \Omega^\varepsilon &= \Omega_+^\varepsilon \cup \Omega_-^\varepsilon, \\
 S_\pm^\varepsilon &= \{x = (x_1, x_2, x_3) \in \Omega : \pm x_3 = \frac{\varepsilon}{2}\}, \\
 \Omega_\pm &= \{x = (x_1, x_2, x_3) \in \Omega : \pm x_3 > 0\}, \\
 \Omega_0 &= \Omega_+ \cup \Omega_-, \\
 S &= \{x = (x_1, x_2, x_3) \in \Omega : x_3 = 0\}.
 \end{aligned} \tag{1}$$

B^ε and Ω^ε are the domains occupied by the adhesive and the adherents, respectively (see fig. 1). The structure is subjected to body force density φ and to surface force density g acting on the part Γ_1 of the boundary, whereas it is clamped on the remaining part Γ_0 of the boundary. The adherents and the adhesive are assumed to be linear elastic. We take σ^ε and u^ε to denote the stress tensor and the displacement field, respectively. Under the small perturbation hypothesis, the strain tensor is

$$e_{kh}(u^\varepsilon) = \frac{1}{2} \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right). \tag{2}$$

We take a_{ijkl} to denote the elasticity coefficients of the adherents and λ and μ to stand for the Lamé's coefficients of the glue. We have therefore to solve the following problem

Problem 1

$$(P_\varepsilon) \left\{ \begin{array}{ll} \text{Find } (u^\varepsilon, \sigma^\varepsilon) \text{ such that :} & \\ \sigma_{ij,j}^\varepsilon = -\varphi_i & \text{in } \Omega \\ \sigma_{ij}^\varepsilon = a_{ijkh} e_{kh}(u^\varepsilon) & \text{in } \Omega^\varepsilon \\ \sigma_{ij}^\varepsilon = \lambda e_{kk}(u^\varepsilon) \delta_{ij} + 2\mu e_{ij}(u^\varepsilon) & \text{in } B^\varepsilon \\ u^\varepsilon = 0 & \text{on } \Gamma_0 \\ \sigma^\varepsilon n = g & \text{on } \Gamma_1 \\ + \text{ interface laws on } S_\pm^\varepsilon \end{array} \right.$$

where n denotes the external unit normal vector to Ω . The interface laws between the adherents and the adhesive (perfect interface, Signorini's law or Coulomb's law) will be specified later on.

For a given function $f : \Omega \mapsto R^3$, we define the restrictions of f to the adherents by f_ε^\pm and to the adhesive by f_ε^m . Denoting $\hat{x} = (x_1, x_2)$ the in-plane coordinates of the adhesive, we define the following jumps of f :

$$[f]_\varepsilon^+(\hat{x}) := f_\varepsilon^+(x_1, x_2, (\frac{\varepsilon}{2})^+) - f_\varepsilon^m(x_1, x_2, (\frac{\varepsilon}{2})^-), \quad (3)$$

$$[f]_\varepsilon^-(\hat{x}) := f_\varepsilon^-(x_1, x_2, (-\frac{\varepsilon}{2})^-) - f_\varepsilon^m(x_1, x_2, (-\frac{\varepsilon}{2})^+), \quad (4)$$

$$[f]_\varepsilon(\hat{x}) := f_\varepsilon^m(x_1, x_2, (\frac{\varepsilon}{2})^-) - f_\varepsilon^m(x_1, x_2, (-\frac{\varepsilon}{2})^+). \quad (5)$$

For a given function $f : \Omega_0 \mapsto R^3$, we define the restrictions of f to Ω_\pm by f^\pm and we also define the following jump of f on S :

$$[f](\hat{x}) := f^+(x_1, x_2, 0^+) - f^-(x_1, x_2, 0^-). \quad (6)$$

Finally, we also introduce the following notations:

$$[[f]]_{N\varepsilon}^\pm := \pm [f]_\varepsilon^\pm e_3, \quad (7)$$

$$[[f]]_{T\varepsilon}^\pm := [f]_\varepsilon^\pm \mp [[f]]_{N\varepsilon}^\pm e_3, \quad (8)$$

$$\sigma_N^{\varepsilon\pm} := \pm (\sigma^\varepsilon e_3)_\varepsilon^\pm e_3, \quad (9)$$

$$\sigma_T^{\varepsilon\pm} := \pm \sigma^\varepsilon e_3 \mp \sigma_N^{\varepsilon\pm} e_3. \quad (10)$$

3. Mathematical Methods

3.1. Matched Asymptotic Expansion Method

The idea underlying matched asymptotic expansions is to find two expansions of the displacement u^ε and the stress σ^ε in powers of ε , i.e., an external expansion for the bodies and an internal one for the joint, and to connect these two expansions to obtain the same limit [18, 47].

3.1.1. External Expansions

The external expansion is a classical expansion in powers of ε

$$\begin{aligned}
 u^\varepsilon(\hat{x}, x_3) &= u^0(\hat{x}, x_3) + \varepsilon u^1(\hat{x}, x_3) + \dots, \\
 e_{ij}(u^\varepsilon)(\hat{x}, x_3) &= e_{ij}^0(\hat{x}, x_3) + \varepsilon e_{ij}^1(\hat{x}, x_3) + \dots, \\
 e_{ij}^l &= \frac{1}{2} \left(\frac{\partial u_i^l}{\partial x_j} + \frac{\partial u_j^l}{\partial x_i} \right), \\
 \sigma_{ij}^\varepsilon(\hat{x}, x_3) &= \sigma_{ij}^0(\hat{x}, x_3) + \varepsilon \sigma_{ij}^1(\hat{x}, x_3) + \dots
 \end{aligned} \tag{11}$$

3.1.2. Internal Expansions

In the internal expansions, we perform a blow-up of the second variable. Let $y_3 = \frac{x_3}{\varepsilon}$. The internal expansion gives

$$\begin{aligned}
 u^\varepsilon(\hat{x}, y_3) &= v^0(\hat{x}, y_3) + \varepsilon v^1(\hat{x}, y_3) + \dots, \\
 e_{ij}(u^\varepsilon)(\hat{x}, y_3) &= \varepsilon^{-1} e_{ij}^{-1}(\hat{x}, y_3) + e_{ij}^0(\hat{x}, y_3) + \varepsilon e_{ij}^1(\hat{x}, y_3) + \dots, \\
 e_{\alpha\alpha}^l &= \frac{\partial v_\alpha^l}{\partial x_\alpha}, \quad \alpha = 1, 2, \\
 e_{33}^l &= \frac{\partial v_3^{l+1}}{\partial y_3}, \\
 e_{\alpha\beta}^l &= \frac{1}{2} \left(\frac{\partial v_\beta^l}{\partial x_\alpha} + \frac{\partial v_\alpha^l}{\partial y_\beta} \right), \\
 e_{\alpha 3}^l &= \frac{1}{2} \left(\frac{\partial v_3^l}{\partial x_\alpha} + \frac{\partial v_\alpha^{l+1}}{\partial y_3} \right), \\
 \sigma_{ij}^\varepsilon(\hat{x}, y_3) &= \varepsilon^{-1} \tau_{ij}^{-1}(\hat{x}, y_3) + \tau_{ij}^0(\hat{x}, y_3) + \varepsilon \tau_{ij}^1(\hat{x}, y_3) + \dots, \\
 \sigma_{ij,j}^\varepsilon &= \sum_{l=-2}^{\infty} \varepsilon^l \left(\frac{\partial \tau_{i\alpha}^l}{\partial x_\alpha} + \frac{\partial \tau_{i3}^{l+1}}{\partial y_3} \right).
 \end{aligned} \tag{12}$$

In the last relation, we use the convention that $v^l = 0$ when $l < 0$ and $\tau^l = 0$ when $l < -1$.

3.1.3. Continuity Conditions

The third step of the method consists in connecting the two expansions. We choose some intermediate planes defined by $x_3 = \pm \zeta \varepsilon^t$, $0 < t < 1$, $\zeta \in]0, +\infty[$. When ε tends to zero, x_3 tends to 0^\pm and $y_3 = x_3/\varepsilon$ tends to $\pm\infty$. The principle of the method [18, 47] consists in assuming that the two expansions give both the same asymptotic limits, that is

$$\begin{aligned}
 (i) \quad v^0(\hat{x}, \pm\infty) &= u^0(\hat{x}, 0^\pm), \\
 (ii) \quad \tau^{-1}(\hat{x}, \pm\infty) &= 0, \\
 (iii) \quad \tau^0(\hat{x}, \pm\infty) &= \sigma^0(\hat{x}, 0^\pm).
 \end{aligned} \tag{13}$$

3.1.4. Identification

The fourth step of the method consists in introducing the expansions in the equilibrium equations and in the constitutive equations. We obtain systems of equations in the following form:

$$\sum \varepsilon^n B^n(v^l, \tau^l) = 0 \quad (14)$$

We now impose that these relations are satisfied at each order n :

$$B^n(\cdot) = 0, \forall n. \quad (15)$$

These equations together with (13) identify an interface law [33, 34].

3.2. Γ -convergence Theory

When Problem 1 admits an energy form, it is natural to study the variational convergence of this energy F^ε , as the small parameter ε tends to zero. A now classical theory to analyze the convergence is provided by the notion of Γ -convergence [9, 16].

We recall the following definition:

Definition 1 *Let X be a topological space and $F^\varepsilon : X \rightarrow [0, \infty]$ a sequence of functionals on X . The sequence F^ε is said to Γ -converge to the Γ -limit $F^0 : X \rightarrow [0, \infty]$ if the following two conditions hold:*

- *For every sequence u^ε in X such that $u^\varepsilon \rightarrow u^0$ as $\varepsilon \rightarrow 0$,
 $F^0(u^0) \leq \liminf_{\varepsilon \rightarrow 0} F^\varepsilon(u^\varepsilon)$ (Lower bound inequality);*
- *For every $u^0 \in X$, there exists a sequence u^ε converging to u^0 as $\varepsilon \rightarrow 0$, such that
 $F^0(u^0) \geq \limsup_{\varepsilon \rightarrow 0} F^\varepsilon(u^\varepsilon)$ (Upper bound inequality).*

F^0 is an optimal asymptotic lower bound for the sequence F^ε in the sense specified by the following proposition.

Proposition 2 *If F^ε Γ -converges to F^0 , then*

$$(i) \quad F^\varepsilon(u^\varepsilon) := \min_{u \in X} F^\varepsilon(u) \rightarrow F^0(u^0) := \min_{u \in X} F^0(u), \quad (16)$$

$$(ii) \quad u^\varepsilon \rightarrow u^0. \quad (17)$$

This theory, applied to the total energy of the system, allows to obtain an interface law at order zero [36, 42].

3.3. Energy Asymptotical Method

This method has been introduced in [36]. As in the Γ -convergence theory, the equilibrium problem is written as a minimization problem of the total energy. As in the asymptotic expansion method, the energy asymptotical method is based on an (external) expansion of u in power of ε . Moreover, in order to reformulate the equilibrium problem in an interphase domain independent of ε , a change of variables is introduced:

$$(\hat{z}, z_3) = (\hat{x}, x_3 \varepsilon^{-1}), \quad (\hat{x}, x_3) \in B_\varepsilon, \quad (18)$$

$$(\hat{z}, z_3) = (\hat{x}, x_3 \pm \frac{\varepsilon}{2} \mp \frac{1}{2}), \quad (\hat{x}, x_3) \in \Omega_\pm. \quad (19)$$

In particular, B_ε is rescaled by a factor ε^{-1} along the interphase thickness and the bodies Ω_\pm are shifted by $\pm 1/2(1 - \varepsilon)$ in the same direction.

Let $\bar{u}^\varepsilon(z, z_3)$ be the rescaled displacement. The rescaled equilibrium problem can be formulated as follows: find the \bar{u}^ε minimizing the rescaled energy, \mathcal{F}^ε , in the set of rescaled displacements. Substituting the expansion (11)₁ into the rescaled energy, we obtain

$$\mathcal{F}^\varepsilon(u^l) = \frac{1}{\varepsilon} \mathcal{F}^{-1}(u^l) + \mathcal{F}^0(u^l) + \varepsilon \mathcal{F}^1(u^l) + \varepsilon^2 \mathcal{F}^2(u^l) + o(\varepsilon^2) \quad (20)$$

The main assumption of this method is that we can obtain the fields which are stationary points of the energy \mathcal{F}^ε by finding the stationary points of the energies \mathcal{F}^l obtained at each level of the expansion.

3.4. Numerical Methods

In this part, we present a method used in previous studies. Examples, usually academic, are studied numerically (long bar, dovetail assembly, etc.). The aim of this numerical study by finite elements is to confirm that the numerical results are coherent with a theoretical approach, to check the existence of the theory (validation) and to determine the thickness of the layer at which the limit interface law can be taken to be a valid approximation of the original constitutive relation of the adhesive (quantification). The initial problem is therefore solved numerically in the case of a thin layer with decreasing thickness (and decreasing moduli in the case of a soft interphase). The numerical results obtained are then compared with the theoretical results. For example, in the case of a soft elastic isotropic interphase, in order to simplify the computations, the Lamé's coefficients are assumed to be given by

$$\lambda_\varepsilon = \lambda^\delta, \mu_\varepsilon = \mu^\gamma$$

where λ, μ are given and the non negative coefficients δ, γ correspond to cases of soft or rigid interfaces. We give an example in section 7, where the problem is solved with the following decreasing values of ε : $1.10^{-2}, 5.10^{-3}, 1.10^{-3}, 5.10^{-4}, 1.10^{-4}$. This method is used in [31, 39].

4. The Case Involving Similar Rigidity: Perfect Interface?

In this section, we consider:

- that the adhesive is linear elastic,
- that the link between the adhesive and the adherents is perfect,
- that the stiffness of the adhesive is similar to those of the adherents.

We present the results obtained in [35], where we have used Γ -convergence techniques to analyze the asymptotic behavior of the total energy up to the second order of the expansion in ε .

4.1. First Order Study

The local formulation of the equilibrium problem corresponds to find the fields of displacement u^ε , strain $e(u^\varepsilon)$ and stress σ^ε which solve the following problem:

$$(P_\varepsilon^1) \begin{cases} \text{Find } (u^\varepsilon, \sigma^\varepsilon) \text{ such that :} \\ \sigma_{ij,j}^\varepsilon = -\varphi_i & \text{in } \Omega \\ \sigma_{ij}^\varepsilon = a_{ij}^\pm e_{kh}(u^\varepsilon) & \text{in } \Omega_\pm^\varepsilon \\ \sigma_{ij}^\varepsilon = a_{ij}^m e_{kh}(u^\varepsilon) & \text{in } B^\varepsilon \\ u^\varepsilon = 0 & \text{on } \Gamma_0 \\ \sigma_\varepsilon n = g & \text{on } \Gamma_1 \\ [u^\varepsilon]_\varepsilon^\pm = 0, [\sigma^\varepsilon e_3]_\varepsilon^\pm = 0 \end{cases}$$

We introduce the space of admissible displacements (the Sobolev spaces are denoted by usual notations)

$$V^\varepsilon = \{u \in (W^{1,2}(\Omega))^3 : u = 0 \text{ on } \Gamma_0\}, \quad (21)$$

and the works of the internal and external loads, respectively,

$$\begin{aligned} A^\varepsilon(u, v) &= \int_{\Omega_\pm^\varepsilon} a^\pm e(u) : e(v) dx + \int_{B^\varepsilon} a^m e(u) : e(v) dx, \\ l^\varepsilon(v) &= \int_{\Omega} \phi \cdot v dx + \int_{\Gamma_1} g \cdot v ds, \end{aligned}$$

we reformulate (P_ε^1) in the following way

$$(\bar{P}_\varepsilon^1) \begin{cases} \text{Find } u^\varepsilon \in V_\varepsilon : \\ A(u^\varepsilon, v) = l^\varepsilon(v) \quad \forall v \in V^\varepsilon. \end{cases} \quad (22)$$

Under the three following regularity assumptions

$$H1) \begin{cases} a_{ijkl} \in L^\infty(\Omega), \\ a_{ijkl} = a_{klij} = a_{jilk} \\ \exists \eta > 0 : a_{ijkl} e_{ij} e_{kl} \geq \eta e_{ij} e_{ij} \quad \forall e_{ij} = e_{ji}, \end{cases} \quad (23)$$

$$H2) \quad \exists \varepsilon_0 : B_\varepsilon \cap (\Gamma_1 \cup \text{supp}(\phi)) = \emptyset, \quad \forall \varepsilon < \varepsilon_0, \quad (24)$$

$$H3) \quad \phi \in (L^2(\Omega))^3, g \in (L^2(\Gamma_1))^3.$$

the variational problem (\bar{P}_ε^1) admits a unique solution (due to the Lax-Milgram theorem), which is also the unique solution of the following problem of minimization:

$$(\tilde{P}_\varepsilon^1) \begin{cases} \text{Find } u^\varepsilon \in V^\varepsilon : \\ J^\varepsilon(u^\varepsilon) \leq J^\varepsilon(v) \quad \forall v \in V^\varepsilon, \end{cases} \quad (25)$$

where

$$J^\varepsilon(v) = \frac{1}{2} A^\varepsilon(v, v) - l^\varepsilon(v) \quad (26)$$

is the potential energy associated with the displacement field v . To apply the notion of Γ -convergence, we introduce the strain energy functional

$$F^\varepsilon(v) = \begin{cases} \frac{1}{2}A^\varepsilon(v, v) & \text{if } v \in V_\varepsilon, \\ +\infty & \text{if } v \in X \setminus V_\varepsilon, \end{cases} \quad (27)$$

where $X = (L^2(\Omega))^3$. In [35], it is introduced the following limit functional:

$$F^0(v) = \begin{cases} \int_{\Omega_0} a^\pm e(v) : e(v) \, dx & \text{if } v \in V^0, \\ +\infty & \text{if } v \in X \setminus V^0, \end{cases} \quad (28)$$

where

$$V^0 = \{u \in (W^{1,2}(\Omega_0))^3 : u = 0 \text{ on } \Gamma_0, [v] = 0 \text{ on } S\}. \quad (29)$$

We now recall the following result proved in [35].

Theorem 3 *The sequence of functionals F^ε Γ -converges to F^0 in the strong topology of X .*

Note that the limit problem, i. e. the minimization problem of F^0 , involves only the adherents. The adhesive has vanished geometrically (as expected) as well as mechanically. We also observe that, because the displacement vector on S has to be continuous (see (29)), the mechanical implication of the theorem is that a very thin interphase behaves like a perfect interface.

In some cases, it may become necessary to improve the model, and it turns out that the most natural solution is to go to a higher order in the asymptotic analysis. This is done in the next section.

4.2. Second Order Study

In this section, we recall the results obtained in [35] at the second order and we focus on the case of isotropic adhesive with Lamé's coefficients λ and μ . We introduce the displacement field u^1 and the stress field σ^1 as the limit, in a sense made precise by the following lemma, of suitable subsequences of $\frac{u^\varepsilon - u^0}{\varepsilon}$ and $\frac{\sigma^\varepsilon - \sigma^0}{\varepsilon}$ respectively. We denote $D(A)$ the space of the C^∞ functions with compact support on the open set A and $D'(A)$ its dual space.

Lemma 1 *Let u^ε , σ^ε and u^0 , and σ^0 be the displacement and stress fields corresponding to minimizers of the energies F^ε and F^0 , respectively. Then there exist subsequences, not relabeled, such that*

$$\begin{aligned} \frac{u^\varepsilon - u^0}{\varepsilon} &\rightharpoonup u^1 \text{ in } L^2(\Omega) \text{ (weak)}, \\ \frac{\sigma^\varepsilon - \sigma^0}{\varepsilon} &\rightharpoonup \sigma^1 \text{ in } D'(\Omega) \text{ (weak)}. \end{aligned} \quad (30)$$

This Lemma is proved in [35]. Contrarily to the fields at order zero u^0 and $\sigma^0 e_3$ which are continuous on the surface S , the fields u^1 and $\sigma^1 e_3$ suffer discontinuity on S . The following Theorem gives a relationship between their jumps, involving the restrictions of the traction and the displacement vectors u^0 and $\sigma^0 e_3$ on S .

Theorem 4 *The fields u^1 and σ^1 satisfy (in a weak sense) the following equilibrium equations:*

$$\begin{aligned}\sigma_{ij,j}^1 &= 0 & \text{in } \Omega_{\pm} \\ \sigma_{ij}^1 &= a_{ijkh}^{\pm} e_{kh}(u^1) & \text{in } \Omega_{\pm} \\ u^1 &= 0 & \text{on } \Gamma_0 \\ \sigma^1 n &= 0 & \text{on } \Gamma_1\end{aligned}$$

and the following relations at the interface S :

$$\begin{aligned}[u_{\alpha}^1] &= \frac{1}{\mu} \sigma_{\alpha 3}^0(\hat{x}, 0) - u_{3,\alpha}^0(\hat{x}, 0) - \frac{1}{2}(u_{\alpha,3}^0(\hat{x}, 0^+) + u_{\alpha,3}^0(\hat{x}, 0^-)), \quad \alpha = 1, 2, \\ [u_3^1] &= \frac{1}{\lambda + 2\mu} \sigma_{33}^0(\hat{x}, 0) - \frac{\lambda}{\lambda + 2\mu} (u_{1,1}^0(\hat{x}, 0) + u_{2,2}^0(\hat{x}, 0)) \\ &\quad - \frac{1}{2} (u_{3,3}^0(\hat{x}, 0^+) + u_{3,3}^0(\hat{x}, 0^-)), \\ [\sigma_{13}^1] &= -\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_{1,11}^0(\hat{x}, 0) - \mu u_{1,22}^0(\hat{x}, 0) - \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} u_{2,21}^0 \\ &\quad - \frac{\lambda}{\lambda + 2\mu} \sigma_{33,1}^0(\hat{x}, 0) - \frac{1}{2} (\sigma_{13,3}^0(\hat{x}, 0^+) + \sigma_{13,3}^0(\hat{x}, 0^-)), \\ [\sigma_{23}^1] &= -\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_{2,22}^0(\hat{x}, 0) - \mu u_{2,11}^0(\hat{x}, 0) - \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} u_{1,12}^0(\hat{x}, 0) \\ &\quad - \frac{\lambda}{\lambda + 2\mu} \sigma_{33,2}^0(\hat{x}, 0) - \frac{1}{2} (\sigma_{23,3}^0(\hat{x}, 0^+) + \sigma_{23,3}^0(\hat{x}, 0^-)), \\ [\sigma_{33}^1] &= -\sigma_{13,1}^0(\hat{x}, 0) - \sigma_{23,2}^0(\hat{x}, 0) - \frac{1}{2} (\sigma_{33,3}^0(\hat{x}, 0^+) + \sigma_{33,3}^0(\hat{x}, 0^-)).\end{aligned}\tag{31}$$

As for the minimization problem of the energy (28), the limit problem defined by theorem 4 involves only the adherents, since the adhesive has vanished geometrically. However, the adhesive has not disappeared from the mechanical point of view: it has been replaced by the mechanical constraints (31), linking the jump in the displacement and traction vectors to the displacement and traction vectors (and their derivatives) at order zero. In particular, the presence of the tangential derivatives of u_0 indicates a non-local character of the interface law (31). Relations (31) also suggest that at higher orders, a thin interphase behaves like an imperfect interface, which keeps trace of the elastic behavior of the interphase.

4.3. An One Dimensional Example

In this section, we propose a simple example to illustrate, in the one dimensional setting, the origin of the terms entering the interface law (31). We consider a bar, AB , fixed at one of its extremities, A , as represented in Figure 2. The part AB is composed of the adhesive AC and of the adherent CB , made by two different materials with elastic moduli E_1 and E_2 , respectively. A force density $\phi(x) = g_1 x$ is applied on AC and a force density $\phi(x) = g_2 x$ is applied on CB . We have

$$AC = \varepsilon, \quad CB = L - \varepsilon,$$

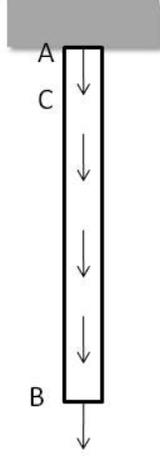


Figure 2. A composite bar subject to body forces.

$$u(0) = 0, \quad E_2 \frac{du}{dx}(L) = a.$$

At order zero, we obtain:

$$u^0(x) = -\frac{g_2 x^3}{6E_2} + \frac{(a - 1/2g_2 L^2)x}{E_2} \quad (32)$$

$$\sigma^0(x) = -\frac{g_2}{2}x^2 + a - 1/2g_2 L^2 \quad (33)$$

We verify easily that $[u^0] = [\sigma^0] = 0$. At order one, we obtain:

$$u^1(x) = \left(\frac{1}{E_1} - \frac{1}{E_2}\right)(a - 1/2g_2 L^2) \quad (34)$$

$$\sigma^1(x) = 0. \quad (35)$$

We can verify that, $[u^1] = \frac{1}{E_1}\sigma^0(0) - u_{,x}^0(0)$, and $[\sigma^1] = -\frac{1}{2}(\sigma_{,x}^0(0^+) + \sigma_{,x}^0(0^-))$.

5. Non Linear Imperfect Interface: Taking into Account Frictional Contact

In this section, we consider:

- that the thin layer is elastic,
- that the stiffness of the glue is lower than that of the bodies,
- that the link between the glue and the bodies is imperfect.

We assume the contact to involve dry friction conditions between the bodies and the thin layer. The Signorini's law of unilateral contact and the Coulomb's law of dry friction are written in the case of monotonous quasi-static loading as:

$$\begin{aligned}
[\sigma^\varepsilon e_3]_\varepsilon^\pm &= 0, \\
\sigma_N^{\varepsilon\pm} &\leq 0, \\
[[u^\varepsilon]]_{N\varepsilon}^\pm &\leq 0, \\
\sigma_N^{\varepsilon\pm} [[u^\varepsilon]]_{N\varepsilon}^\pm &= 0, \\
|\sigma_T^{\varepsilon\pm}| &\leq f|\sigma_N^{\varepsilon\pm}|, \\
\text{If } |\sigma_T^{\varepsilon\pm}| < f|\sigma_N^{\varepsilon\pm}| &\text{ then } [[u^\varepsilon]]_{T\varepsilon}^\pm = 0, \\
\text{If } |\sigma_T^{\varepsilon\pm}| = f|\sigma_N^{\varepsilon\pm}| &\text{ then } [[u^\varepsilon]]_{T\varepsilon}^\pm = -\zeta\sigma_T, \text{ with } \zeta \geq 0,
\end{aligned} \tag{36}$$

where f is the friction coefficient. To analyze the asymptotic behavior of the interphase, we use two of the methods presented in section 3: matched asymptotic expansion and a numerical method.

5.1. Asymptotic Expansions

In the following, we summarize the results obtained in [39]. Given the constitutive equations in the thin layer, we have

$$\begin{aligned}
\varepsilon^{-1}\tau_{ij}^{-1} + \tau_{ij}^0 + \varepsilon\tau_{ij}^1 + \dots &= \lambda(\varepsilon^{-1}e_{kk}^{-1} + e_{kk}^0 + \varepsilon e_{kk}^1 + \dots)\delta_{ij} \\
&+ 2\mu(\varepsilon^{-1}e_{ij}^{-1} + e_{ij}^0 + \varepsilon e_{ij}^1 + \dots).
\end{aligned} \tag{37}$$

The identification of the various orders depends on the behavior of the Lamé coefficients λ and μ with respect to ε . We obtain nine cases corresponding to the values of the limits $\bar{\lambda}$ and $\bar{\mu}$ of the ratios $\frac{\lambda}{\varepsilon}$ and $\frac{\mu}{\varepsilon}$:

- (a) $\bar{\lambda} = \bar{\mu} = 0, \tau_{ij}^{-1} = 0, \tau_{ij}^0 = 0.$
- (b) $\bar{\lambda} = 0, 0 < \bar{\mu} < \infty, \tau_{ij}^{-1} = 0, \tau_{ij}^0 = 2\bar{\mu}e_{ij}^{-1}.$
- (c) $0 < \bar{\lambda} < \infty, \bar{\mu} = 0, \tau_{ij}^{-1} = 0, \tau_{ij}^0 = \bar{\lambda}e_{kk}^{-1}\delta_{ij}.$
- (d) $0 < \bar{\lambda} < \infty, 0 < \bar{\mu} < \infty, \tau_{ij}^{-1} = 0, \tau_{ij}^0 = \bar{\lambda}e_{kk}^{-1}\delta_{ij} + 2\bar{\mu}e_{ij}^{-1},$
- (e) $\bar{\lambda} = \infty, \bar{\mu} = 0, e_{kk}^{-1} = 0, \tau_{ij}^0 = 0.$
- (f) $\bar{\lambda} = \infty, 0 < \bar{\mu} < \infty, e_{kk}^{-1} = 0, \tau_{ij}^0 = 2\bar{\mu}e_{ij}^{-1}.$
- (g) $\bar{\lambda} = 0, \bar{\mu} = \infty, e_{ij}^{-1} = 0, e_{ij}^0 = 0.$
- (h) $0 < \bar{\lambda} < \infty, \bar{\mu} = \infty, e_{ij}^{-1} = 0, e_{ij}^0 = 0.$
- (i) $\bar{\lambda} = \infty, \bar{\mu} = \infty, e_{ij}^{-1} = 0, e_{ij}^0 = 0.$

Let $K_1 = K_T = \bar{\mu}$ and $K_2 = K_N = \bar{\lambda} + 2\bar{\mu}$. f^+ is the positive part of a function f .

- (a) $\bar{\lambda} = \bar{\mu} = 0$
In this case, $\tau_{ij}^0 = 0$. Because of the continuity conditions, we obtain

$$\sigma^0(x_1, 0) = 0 \quad (38)$$

- (b) $\bar{\lambda} = 0, 0 < \bar{\mu} < \infty$

In this case, $\tau_{11}^0 = 0, \tau_{12}^0 = \bar{\mu} \frac{\partial v_1^0}{\partial y_2}, \tau_{22}^0 = 2\bar{\mu} \frac{\partial v_2^0}{\partial y_2}$.

Using standard arguments, we obtain

$$\begin{aligned} \sigma_N^0 &= -2\bar{\mu}[[u^0]]_N^+, \\ |\sigma_T^0| &\leq f|\sigma_N^0|, \\ \text{If } |\sigma_T^0| < f|\sigma_N^0| \text{ then } \sigma_T^0 &= -\bar{\mu}[[u^0]]_T, \\ \text{If } |\sigma_T^0| = f|\sigma_N^0| \text{ then } [[u^0]]_T &= -\zeta\sigma_T^0, \zeta \geq 0. \end{aligned} \quad (39)$$

- (c) $0 < \bar{\lambda} < \infty, \bar{\mu} = 0$

In this case $\tau_{11}^0 = \tau_{22}^0 = \bar{\lambda} \frac{\partial v_2^0}{\partial y_2}, \tau_{12}^0 = 0$. For the normal part, we proceed as in (b), by replacing $2\bar{\mu}$ by $\bar{\lambda}$. The tangential part corresponds to (a). We obtain

$$\begin{aligned} \sigma_N^0 &= -\bar{\lambda}[[u^0]]_N^+, \\ \sigma_T^0 &= 0. \end{aligned} \quad (40)$$

- (d) $0 < \bar{\lambda} < \infty, 0 < \bar{\mu} < \infty$

In this case, $\tau_{11}^0 = \bar{\lambda} \frac{\partial v_2^0}{\partial y_2}, \tau_{i2}^0 = K_i \frac{\partial v_i^0}{\partial y_2}$. Here, we again proceed as in (b). We obtain

$$\begin{aligned} \sigma_N^0 &= -(\bar{\lambda} + 2\bar{\mu})[[u^0]]_N^+, \\ |\sigma_T^0| &\leq f|\sigma_N^0|, \\ \text{If } |\sigma_T^0| < f|\sigma_N^0| \text{ then } \sigma_T^0 &= -\bar{\mu}[[u^0]]_T, \\ \text{If } |\sigma_T^0| = f|\sigma_N^0| \text{ then } [[u^0]]_T &= -\zeta\sigma_T^0, \zeta \geq 0. \end{aligned} \quad (41)$$

- (e) $\bar{\lambda} = \infty, \bar{\mu} = 0$

We have $\frac{\partial v_2^0}{\partial y_2} = 0, \tau_{12}^0 = 0$. For the normal tangential part, we proceed as in (a).

Using standard arguments, we have

$$\begin{aligned} [[u^0]]_N \leq 0, \quad \sigma_N^0 \leq 0, \quad [[u^0]]_N \sigma_N^0 &= 0, \\ \sigma_T^0 &= 0. \end{aligned} \quad (42)$$

- (f) $\bar{\lambda} = \infty, 0 < \bar{\mu} < \infty$

We have $\frac{\partial v_2^0}{\partial y_2} = 0, \tau_{12}^0 = \bar{\mu} \frac{\partial v_1^0}{\partial y_2}$. The treatment of the tangential part is similar to (b) and the treatment of the normal part is similar to (e). To summarize, we obtain

$$\begin{aligned} [[u^0]]_N \leq 0, \quad \sigma_N^0 \leq 0, \quad [[u^0]]_N \sigma_N^0 &= 0, \\ |\sigma_T^0| &\leq f|\sigma_N^0|, \\ \text{If } |\sigma_T^0| < f|\sigma_N^0| \text{ then } \sigma_T^0 &= -\bar{\mu}[[u^0]]_T, \\ \text{If } |\sigma_T^0| = f|\sigma_N^0| \text{ then } [[u^0]]_T &= -\zeta\sigma_T^0, \zeta \geq 0. \end{aligned} \quad (43)$$

The three last cases call for a special treatment.

- (g)(h)(i) $\bar{\mu} = \infty$

In these cases, we have $\frac{\partial v_i^0}{\partial y_2} = 0$. We proceed as in (e) on the normal and tangential parts. This gives

$$\begin{aligned}
 & [[u^0]]_N \leq 0, \quad \sigma_N^0 \leq 0, \quad [[u^0]]_N \sigma_N^0 = 0, \\
 & \text{If } |\sigma_T^0| < f|\sigma_N^0| \quad \text{then} \quad [[u^0]]_T = 0, \\
 & \text{If } |\sigma_T^0| = f|\sigma_N^0| \quad \text{then} \quad [[u^0]]_T = -\zeta \sigma_T^0, \zeta \geq 0.
 \end{aligned} \tag{44}$$

5.2. Numerical Validations

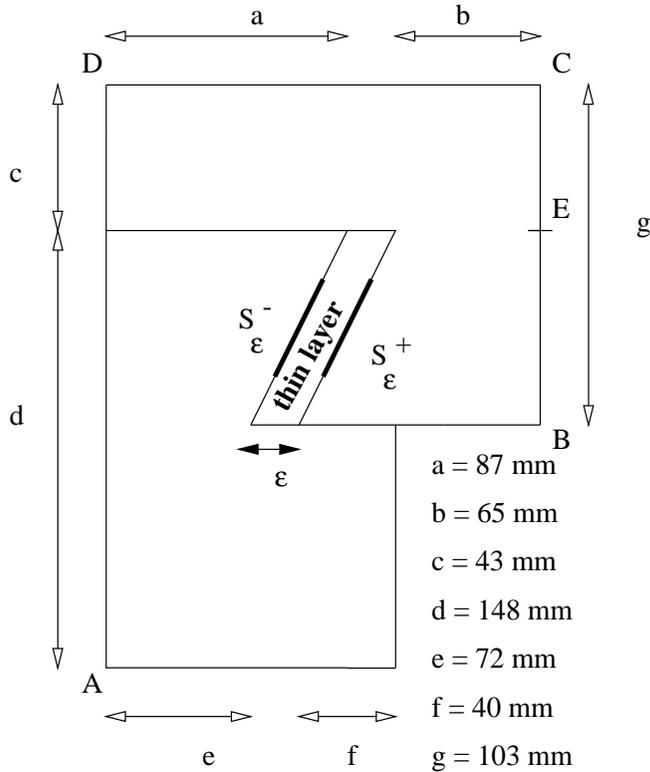


Figure 3. The dovetail assembly.

This section illustrates some numerical results obtained in [31, 44]. The example of a dovetail assembly is treated. The aim of this numerical study is to confirm that the numerical results are coherent with the theory (validation) and to determine the thickness of the layer at which the limit interface law can be taken to be valid (quantification). The initial problem is therefore solved numerically in the case of a thin layer with decreasing thickness and stiffness values, as done by [44]. A relaxation procedure is used for this purpose, as in previous studies [30, 46].

The numerical results obtained are then compared with the theoretical results. In order to

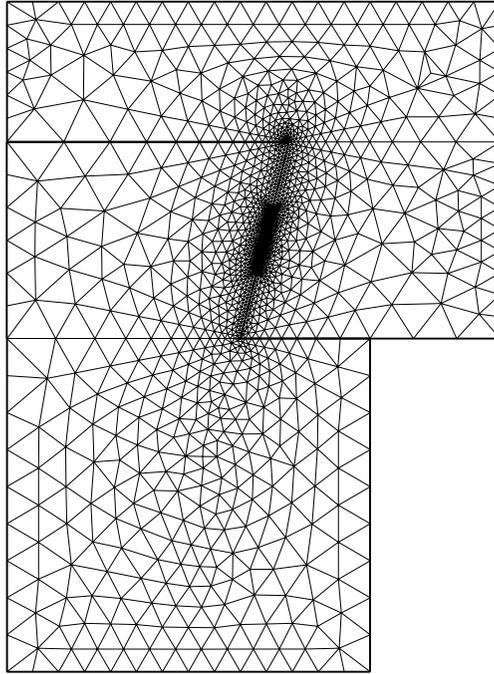


Figure 4. Mesh of the structure.

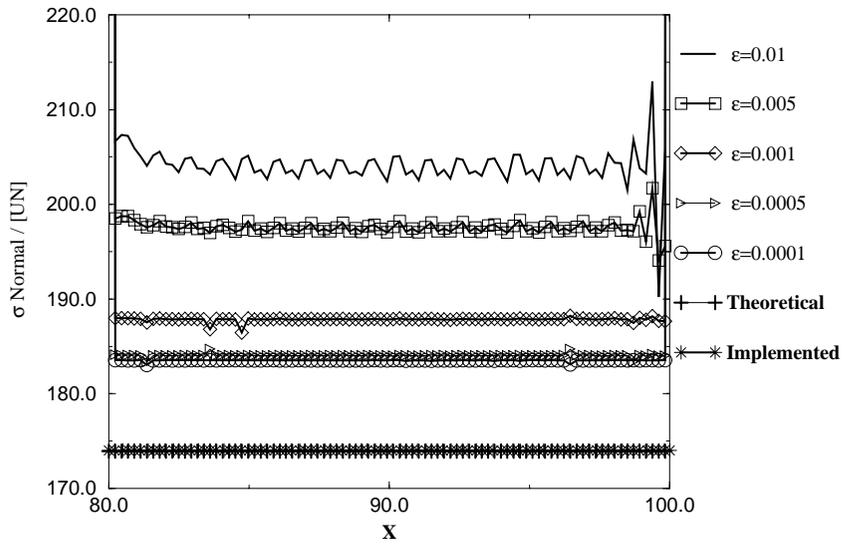


Figure 5. Ratio between normal stress and normal displacement (case b).

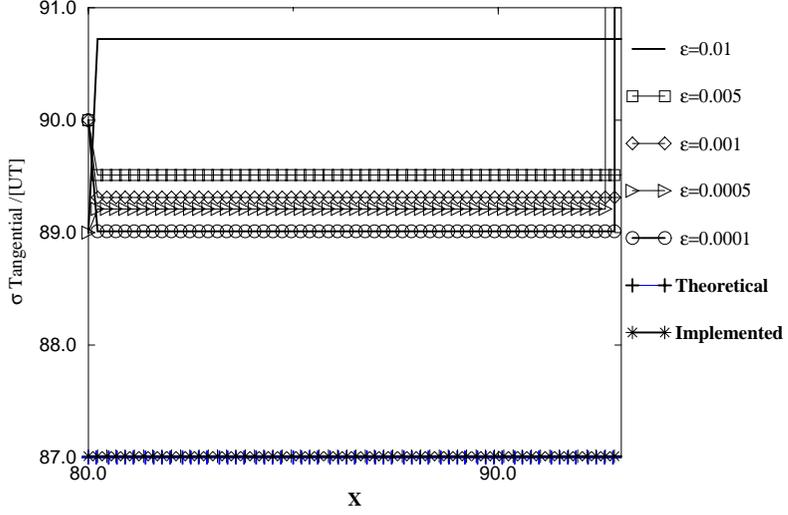


Figure 6. Ratio between tangential stress and tangential displacement (case d).

simplify the computations, the Lamé's coefficients of the joint are assumed to be given by

$$\begin{cases} \lambda_\epsilon = \epsilon^\gamma \lambda, & \gamma \geq 0, \\ \mu_\epsilon = \epsilon^\delta \mu, & 0 \leq \delta < 2. \end{cases}$$

The problem is solved with the following decreasing values of ϵ :

$$1.10^{-2}, 5.10^{-3}, 1.10^{-3}, 5.10^{-4}, 1.10^{-4}.$$

In order to analyze the results obtained, the displacement fields (u_N, u_T) and the stress vector (σ_N, σ_T) were computed on S_ϵ^\pm in the case of the initial problem (and on S in that of the limit problem) in the nine cases studied. The theoretical and numerical curves of the stick and sliding nodes are then compared. The friction problem is solved by means of a fixed point algorithm coupled with a relaxation procedure [46]. Similar results were obtained (see fig. 5 and 6). As can be seen from these figures, the limit law can be taken to be valid for at thin layer thickness up to 10^{-3} mm. This value is suitable for a large class of problems (for example in glue-bonding processes).

5.3. Predictive Nature of the Model

In this section, we focus on a particular characteristic of the model, its predictive nature which depends on the mechanical and geometrical parameters of the glue and the adherents. The model presented in the previous paragraphs involves four parameters: the thickness of the glue ϵ , the stiffness coefficients of the glue λ and μ and the friction coefficient f . We observe that the various cases depend both on the limits of λ/ϵ and of μ/ϵ . We stress

the fact that the nine laws that appear depend only on the relative values of the parameters. If mathematically, these limits are perfectly defined, mechanically, in a non academic problem, they are a priori unknown. The engineer only knows (or has the possibility of knowing) the thickness and the stiffness of the glue. The limits are approximated in a satisfactory way by the ratio between the stiffness and the thickness. In other words, the values of $\bar{\lambda}$ and $\bar{\mu}$ can be replaced by λ/ε and μ/ε respectively. But these approximations are never equal to zero or to infinity. So when shall we consider that these limit values are obtained ? Simply by comparison to a reference value. For example in the case of isotropic adherents, the reference values, would be λ_m/L or μ_m/L , where λ_m and μ_m are the moduli and L the characteristic length of the adherents (chosen equal to 1 in the numerical applications), with $\lambda \ll \lambda_m$, $\mu \ll \mu_m$ and $\varepsilon \ll L$, which expresses the fact that the interface is both soft and thin.

More precisely, when

- λ/ε and μ/ε are very small by comparison to the reference value, i.e. the glue is relatively more soft than thin, $\bar{\lambda}$ and $\bar{\mu}$ shall be taken equal to zero;
- λ/ε and μ/ε are very large by comparison to the reference value, i.e. the glue is relatively more soft than thin, $\bar{\lambda}$ and $\bar{\mu}$ shall be taken equal to infinity;
- λ/ε and μ/ε are finite by comparison to the reference value, i.e. the glue is as thin as soft, $\bar{\lambda}$ and $\bar{\mu}$ shall be taken equal to λ/ε and μ/ε respectively.

Numerically, we can observe that "very small" corresponds to less than 3% of the reference value and that "very large" corresponds to more than 30 times the reference value. The "finite case" corresponds to the range between these two extreme values.

6. Non Linear Imperfect Interface: Non Convex Energy

In this section, we consider:

- that the thin layer is governed by a non convex energy,
- that the stiffness of the glue is lower than that of the bodies,
- that the link between the glue and the bodies is perfect.

Non convex energies are usually associated to phase transforming materials [10, 19]

6.1. Constitutive Relation of a Phase Transforming Adhesive

To model the constitutive behavior of the adhesive, we adopt the Frémond constitutive relation, which involves three phases, the austenite and two variants of martensite [19]:

$$\begin{cases} \sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} + \alpha \chi_{ij}, & \text{if } \alpha \chi_{ij} e_{ij} \leq -c \\ \sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}, & \text{if } |\alpha \chi_{ij} e_{ij}| \leq c \\ \sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} - \alpha \chi_{ij}, & \text{if } \alpha \chi_{ij} e_{ij} \geq c, \end{cases} \quad \text{in } B^\varepsilon \quad (45)$$

and where constants α and c are two material parameters. The constant c can be related to a critical temperature, below which the stress-strain curve has only two ascending branches, each one corresponding to a different variant of martensite (fig. 7.1-a) Above the transformation temperature, a third phase exists, called austenite and corresponding to an intermediate ascending branch in the stress-strain curve (fig. 7.1-b).

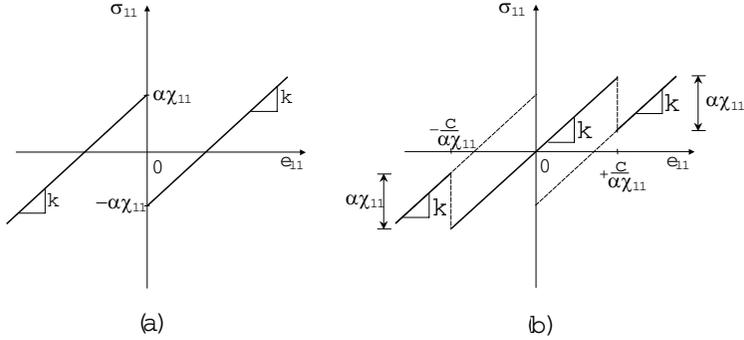


Figure 7. Stress-strain diagrams simulating an extension test on a phase transforming adhesive: (a) $c \leq 0$, (b) $c > 0$.

The problem is studied by means of matched asymptotic expansions. Let S, T denote the diagonal 2×2 matrices with $S_{11} = K_T$, $S_{22} = K_N$ and $T_{11} = T_{22} = K_{PT}$. Let x be the vector with $x_1 = \chi_{12}$ and $x_2 = \chi_{22}$. Note that the coefficients of S (resp. T) can be equal to infinity, zero or a non-zero bounded value (resp. infinity or a non-zero bounded value). Using these notations, we obtain the following family of contact laws

- $\bar{c} \leq 0$

$$\begin{aligned} \sigma n &= S[u] + Tx & \text{if } \alpha x_i[u_i] \leq 0, \\ \sigma n &= S[u] - Tx & \text{if } \alpha x_i[u_i] \geq 0, \end{aligned} \quad (46)$$

- $\bar{c} > 0$

$$\begin{aligned} \sigma n &= S[u] + Tx & \text{if } \alpha x_i[u_i] \leq -\bar{c}, \\ \sigma n &= S[u] & \text{if } |\alpha x_i[u_i]| \leq \bar{c}, \\ \sigma n &= S[u] - Tx & \text{if } \alpha x_i[u_i] \geq \bar{c}, \end{aligned} \quad (47)$$

6.2. Interface Laws Arising from Energy Minimization

In a one dimensional setting, it is possible to solve the equilibrium problem and discuss the stability of the equilibrium solutions for an adhesive obeying the constitutive relation

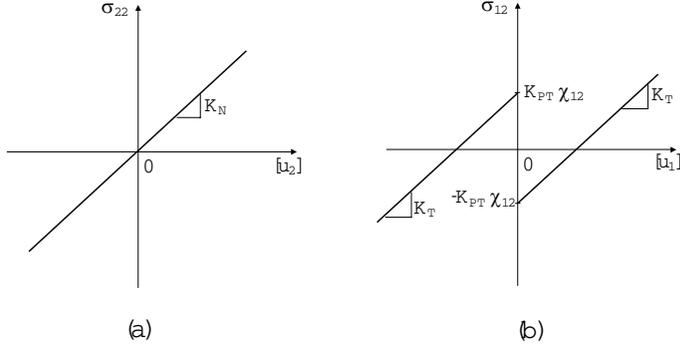


Figure 8. Contact laws for $c \leq 0$: (a) normal contact law for $\chi_{22} = 0$; (b) tangential contact law.

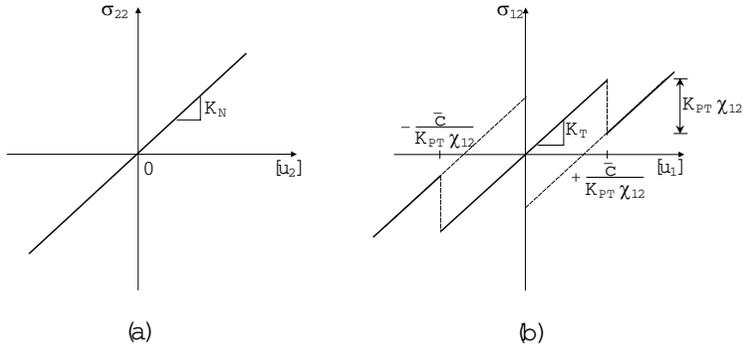


Figure 9. Contact laws for $c > 0$: (a) normal contact law for $\chi_{22} = 0$; (b) tangential contact law.

(45). This is done in the next subsection. The closed form solutions allow us to explicitly calculate the corresponding interface laws when the thickness of the adhesive tends to zero (see subsection 6.2.2).

6.2.1. Metastable States for a One Dimensional Phase Transforming Adhesive

Let us take a one-dimensional bar occupying a reference unstressed space configuration $\Omega = (0, l)$, starting from which there are displacement fields $u = u(x)$, $x \in (0, l)$ which are continuous, with the piecewise continuous derivative $u' \geq 0$. The bar is fixed at the extremity $x = 0$, and is subjected to a prescribed displacement $\delta > 0$ at $x = L$.

The bar is assumed to be made of two different elastic materials. In the range $0 \leq x \leq \varepsilon L$, the bar is composed of an adhesive layer characterized by the piecewise quadratic stored energy density

$$w_a(e) = \min_{e>0} \left\{ \frac{k}{2}e^2; \frac{k}{2}e^2 - \alpha\chi e + c \right\}, \quad (48)$$

where $k > 0$ is the elasticity of the adhesive.

In the range $\varepsilon L \leq x \leq L$, the bar is composed of a material with quadratic stored energy density having an elastic modulus K

$$w_b(e) = \frac{K}{2}e^2. \quad (49)$$

This system admits the following possible solutions:

i) If $0 \leq \sigma_\varepsilon < kc(\alpha\chi)^{-1} - \alpha\chi$, then,

$$\bar{u} = \begin{cases} \frac{\sigma_\varepsilon}{k}x & x \in [0, \varepsilon L], \\ \frac{\sigma_\varepsilon}{K}x + \sigma_\varepsilon \varepsilon L \left(\frac{1}{k} - \frac{1}{K} \right) & x \in (\varepsilon L, L], \end{cases} \quad (50)$$

with $\sigma_\varepsilon = \hat{k}_\varepsilon \delta L^{-1}$, and $\hat{k}_\varepsilon^{-1} := \varepsilon k^{-1} + (1 - \varepsilon)K^{-1}$. Using the expression for σ_ε , it is easy to see that this solution is possible whenever $0 \leq \delta < (kc(\alpha\chi)^{-1} - \alpha\chi)L\hat{k}_\varepsilon^{-1}$. Since δ is positive, this solution exists only if $c > 0$ and corresponds to the case of adhesive consisting of only austenite.

ii) If $c > 0$ and $kc(\alpha\chi)^{-1} - \alpha\chi \leq \sigma_\varepsilon < kc(\alpha\chi)^{-1}$, then the adhesive is a mixture of austenite and martensite. Let $\lambda \in (0, 1)$ denote the austenite volume fraction. As λ varies in $[0, 1]$, we obtain a set of equilibrium solutions:

$$\bar{u} = \begin{cases} \frac{\sigma_\varepsilon}{k}x & x \in [0, \lambda\varepsilon L], \\ \frac{\sigma_\varepsilon + \alpha\chi}{k}x - \frac{\alpha\chi\varepsilon\lambda L}{k} & x \in (\lambda\varepsilon L, \varepsilon L], \\ \frac{\sigma_\varepsilon}{K}x + \varepsilon L \left(\frac{\sigma_\varepsilon + (1 - \lambda)\alpha\chi}{k} - \frac{\sigma_\varepsilon}{K} \right) & x \in (\varepsilon L, L], \end{cases} \quad (51)$$

with

$$\sigma_\varepsilon = \hat{k}_\varepsilon \left(\frac{\delta}{L} - \varepsilon(1 - \lambda) \frac{\alpha\chi}{k} \right). \quad (52)$$

Using this expression for σ_ε , it turns out that the two-phase solution exists whenever

$$\left((kc(\alpha\chi)^{-1} - \alpha\chi)\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi(1 - \lambda)k^{-1} \right)L \leq \delta \leq \left(kc(\alpha\chi)^{-1}\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi(1 - \lambda)k^{-1} \right)L. \quad (53)$$

Since the value of the volume fraction λ is within the $0 < \lambda < 1$ range, this condition can be further extended as follows

$$(kc(\alpha\chi)^{-1} - \alpha\chi)\hat{k}_\varepsilon^{-1}L \leq \delta \leq (kc(\alpha\chi)^{-1} + \varepsilon\alpha\chi k^{-1})L. \quad (54)$$

Therefore, given any $\lambda \in (0, 1)$, if the prescribed elongation δ lies in the above range, then there will exist in the adhesive a two-phase solution involving a mixture of both phases.

iii) If $\sigma_\varepsilon \geq kc(\alpha\chi)^{-1}$, then the solution is

$$\bar{u} = \begin{cases} \frac{\sigma_\varepsilon + \alpha\chi}{k}x & x \in [0, \varepsilon L], \\ \frac{\sigma_\varepsilon}{K}x + \varepsilon L \left(\frac{\sigma_\varepsilon + \alpha\chi}{k} - \frac{\sigma_\varepsilon}{K} \right) & x \in (\varepsilon L, L], \end{cases} \quad (55)$$

with $\sigma_\varepsilon = \hat{k}_\varepsilon(\delta L^{-1} - \alpha\chi\varepsilon k^{-1})$. This solution, which is possible for $\delta > kc(\alpha\chi)^{-1}L\hat{k}_\varepsilon^{-1} + \alpha\chi\varepsilon Lk^{-1}$ when $c > 0$ and for any positive δ when $c \leq 0$, describes the case of adhesive consisting entirely of martensite.

It is established that the configurations described by the solutions (50), (51) and (55) correspond to weak local minimizers [34]. It is also established that the following equilibrium configurations correspond to a global minimizer:

- a) if $0 \leq \delta < (kc(\alpha\chi)^{-1} - \alpha\chi/2)Lke^{-1}$, then solution (50) is a global minimizer;
- b) if $(kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1} \leq \delta < (kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi Lk^{-1}$, then the solution (51) with σ_ε given by the Maxwell stress $\sigma_M = kc(\alpha\chi)^{-1} - \alpha\chi/2$ is a global minimizer. Substituting σ_M into (52), we find the austenite volume fraction determined solely for the given δ :

$$\lambda_{glo} = 1 + \frac{K}{\varepsilon\alpha\chi} \left(\frac{k}{\hat{k}_\varepsilon}c(\alpha\chi)^{-1} - \frac{\alpha\chi}{2\hat{k}_\varepsilon} - \frac{\delta}{L} \right). \quad (56)$$

Note that when $\delta = (kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1}$ we have $\lambda_{glo} = 1$, i.e., the adhesive is still in the austenite phase. If δ is continuously increased, the austenite volume fraction decreases and the stress in the bar remains constant and equal to the Maxwell stress. When $\delta = (kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi Lk^{-1}$, we have $\lambda_{glo} = 0$, and the adhesive has completed the transformation from austenite to martensite.

- c) if $\delta \geq (kc(\alpha\chi)^{-1} - \alpha\chi/2)L\hat{k}_\varepsilon^{-1} + \varepsilon\alpha\chi Lk^{-1}$, then the solution (55) is a global minimizer.

6.2.2. Interface Laws for a Thin Phase Transforming Adhesive

To obtain the interface law we study the asymptotic behavior of the above equilibrium solutions when both the parameters (ε, k) tend to zero and the thin adhesive layer is replaced by a point. To study the asymptotic behavior of the adhesive, we set

$$k = \bar{k} \varepsilon. \quad (57)$$

Our aim here is to study the relations between the limits

$$\sigma = \lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon, \quad [u] = \lim_{\varepsilon \rightarrow 0} (\bar{u}(\varepsilon L) - \bar{u}(0)), \quad (58)$$

where σ_ε and \bar{u} correspond to the equilibrium configurations determined in the previous subsection. This study leads to defining an asymptotic contact law linking the limit stress

in the bar, σ_ε , to the jump in the displacement occurring at the adhesive interface, $[u]$. This contact law describes the limit behavior of the adhesive.

In addition to (57), we need to specify the scaling of the material parameters with ε . To make the limits (58) finite, we take

$$c = \bar{c} \varepsilon^{-1}, \quad \alpha = \bar{\alpha}, \quad \chi = \bar{\chi}, \quad (59)$$

where \bar{k} , \bar{c} , $\bar{\alpha}$ and $\bar{\chi}$ are independent of ε .

Substituting (57), (59) into the expressions for σ_ε and \bar{u} listed in section 2 in (a), (b) and (c), taking the limit $\varepsilon \rightarrow 0^+$ and eliminating δ between σ_ε and $[u]$, we obtain the following contact law:

$$\sigma_{glo} = \begin{cases} \bar{k} \frac{[u]}{L}, & 0 \leq [u] < L(\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \frac{\bar{\alpha}\bar{\chi}}{2\bar{k}}), \\ (\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \frac{\bar{\alpha}\bar{\chi}}{2}), & L(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \frac{\bar{\alpha}\bar{\chi}}{2\bar{k}}) \leq [u] < L(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \frac{\bar{\alpha}\bar{\chi}}{2\bar{k}}), \\ \bar{k} \frac{[u]}{L} - \bar{\alpha}\bar{\chi}, & [u] \geq L(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \frac{\bar{\alpha}\bar{\chi}}{2\bar{k}}), \end{cases} \quad (60)$$

corresponding to global minimizers of the original equilibrium problem. In the same way, taking σ_ε and \bar{u} as in (i), (ii) and (iii), we can calculate the limit contact law corresponding to local minimizers. Note that this law turns out to be undefined, because of the lack of information available due to the non uniqueness of the local minimizers. Indeed, we obtain

$$\sigma_{loc} = \begin{cases} \bar{k} \frac{[u]}{L}, & 0 \leq [u] < L(\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \frac{\bar{\alpha}\bar{\chi}}{2\bar{k}}), \\ \bar{\Sigma}, & L(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \frac{\bar{\alpha}\bar{\chi}}{2\bar{k}}) \leq [u] < L(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \frac{\bar{\alpha}\bar{\chi}}{2\bar{k}}), \\ \bar{k} \frac{[u]}{L} - \bar{\alpha}, & [u] \geq L(\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \frac{\bar{\alpha}\bar{\chi}}{2\bar{k}}), \end{cases} \quad (61)$$

where $\bar{\Sigma}$ can take any value in $[\bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} - \bar{\alpha}\bar{\chi}, \bar{k}\bar{c}(\bar{\alpha}\bar{\chi})^{-1} + \bar{\alpha}\bar{\chi}]$. Therefore, local minimizers give rise to multiple contact laws, all of which are included in the dashed parallelogram depicted in Figure 10. Note the non uniqueness of the equilibrium solutions, due to the non convexity of the deformation energy associated to (45). To rule out non uniqueness, one can postulate a nucleation condition and a kinetic relation, specifying the path inside the parallelogram of the possible states [2].

7. An Example of Numerical Implementation

This section is devoted to the numerical solution of the problem presented in the previous section. Note that there exist several studies to solve this kind of problems [3, 5–7, 12, 13, 28–30, 32, 46]. In what follows, we will adopt the two-dimensional context, focusing on the contact between a deformable solid A occupying Ω and a rigid body. In this case,

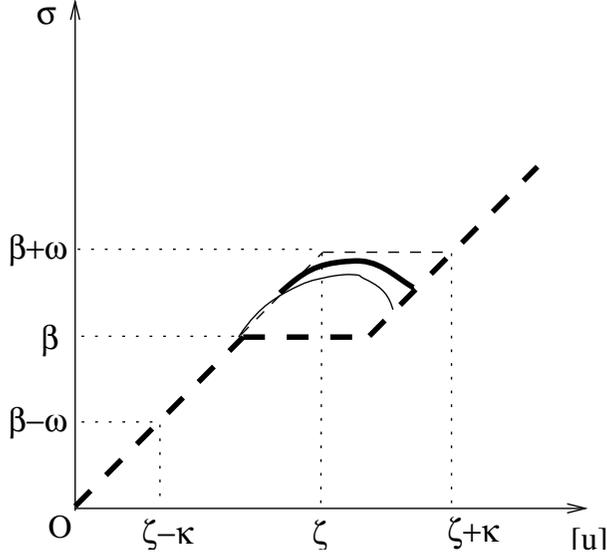


Figure 10. Contact law obtained by energy minimization. To make the figure clearer, we have defined $\zeta = c(\bar{\alpha}\bar{\chi})^{-1}$, $\kappa = \bar{\alpha}\bar{\chi}\bar{k}^{-1}$, $\beta = \bar{k}(\bar{\alpha}\bar{\chi})^{-1} - \bar{\alpha}\bar{\chi}/2$, $\omega = \bar{\alpha}\bar{\chi}/2$. The shaded region is the contact law domain corresponding to local minimizers. Contact law corresponding to global minimizers (Maxwell line): - -. Contact laws corresponding to local minimizers: — linear kinetic, — pinning kinetic, - - convex decomposition.

Problem 2

$$(P) \left\{ \begin{array}{ll} \text{Find } (u, \sigma) \text{ such that :} & \\ \sigma_{ij,j} = -\varphi_i & \text{in } \Omega \\ \sigma_{ij} = a_{ijkh}e_{kh}(u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \sigma n = g & \text{on } \Gamma_1 \\ \sigma_N \leq 0 & \text{on } S \\ \sigma_N = -K_N u_N^+ & \text{on } S \\ |\sigma_T| \leq f|\sigma_N| & \text{on } S \\ \text{If } |\sigma_T| < f|\sigma_N| \text{ then } \sigma_T = -K_T u_T & \\ \text{If } |\sigma_T| = f|\sigma_N| \text{ then } u_T = -\zeta\sigma_T & \text{with } \zeta \geq 0 \end{array} \right.$$

The contact laws are shown in figs. 11 and 12. In lines with previous studies, the above problem is equivalent to:

Problem 3

Find ρ , the fixed point of the application $\rho \longrightarrow f|\sigma_N(u(\rho))|$, where $u = u(\rho)$ is the solution of

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ such that :} \\ J(u) \leq J(v), \quad \forall v \in V, \end{array} \right.$$

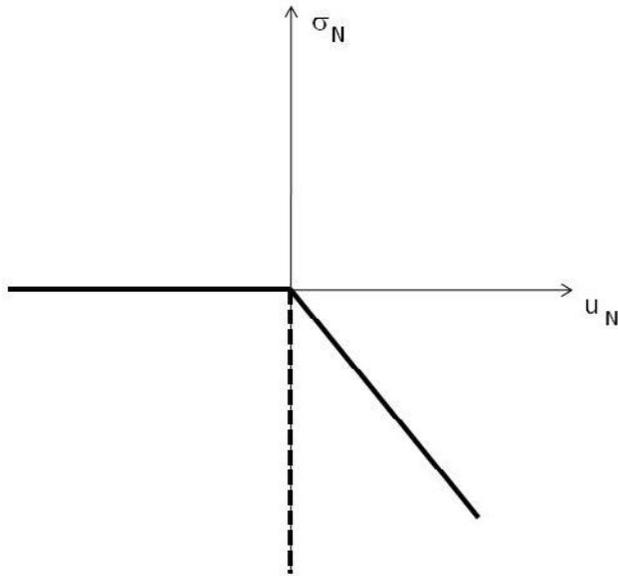


Figure 11. Contact law: normal component (the dotted line corresponds to an infinite value of K_N).

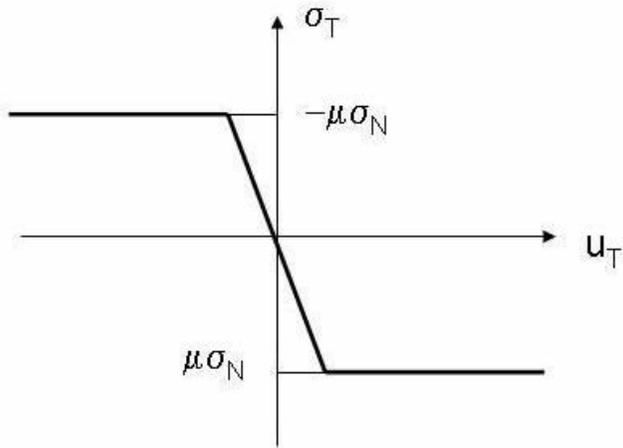


Figure 12. Contact law: tangential component (the dotted line corresponds to an infinite value of K_T).

with

$$J(v) = \frac{1}{2} \int_{\Omega} a e(v) e(v) \, dx - \int_{\Omega} \varphi \cdot u \, dx + \int_{\Gamma_1} g \cdot u \, dl + \int_S \phi(v_N) \, ds + \int_S \psi(v_T) \, ds,$$

where ϕ is given by

$$\phi(v_N) = \frac{K_N}{2}(v_N^+)^2,$$

and ψ is given by

$$\psi(v_T) = \frac{1}{2}K_T \cdot (v_T)^2 + \rho(|v_T| - \frac{\rho}{K_T})^+.$$

Problem 3 is discretized using a finite element method formulated in terms of the displacements. We usually adopt $P1$ finite elements (triangles with three nodes and six degrees of freedom) or $Q1$ finite elements (quadrangles with four nodes and eight degrees of freedom). We have to minimize a functional still denoted J for v in R^{2NP} such that

$$\mathcal{J}(v) = \frac{1}{2}v^T A v + \frac{1}{2}(Pv)^T B(Pv) - v^T L.$$

The following notations are used:

NP : total number of nodes,

NC : number of contact nodes,

I_N : indices of normal components of contact nodes,

I_T : indices of tangential components of contact nodes,

A : stiffness matrix associated with volume terms with coefficient a_{ij} ,

B : stiffness matrix to the surface terms with coefficient b_{ij} ,

P : projection from R^{2NP} to R^{2NC} ,

L : generalized loading vector with coefficients L_i .

Relaxation method with constraints The relaxation method consists in finding the solution to problem 3 by solving a sequence of minimization problems in R^{2NP}

$$\left\{ \begin{array}{l} \text{Find } u_i^{n+1} \text{ such that } \forall v \in R^{2NP} \\ \mathcal{J}(u_1^{n+1}, \dots, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^n, \dots, u_{2NP}^n) \leq \mathcal{J}(u_1^{n+1}, \dots, u_{i-1}^{n+1}, v, u_{i+1}^n, \dots, u_{2NP}^n). \end{array} \right.$$

ω is taken to denote the relaxation coefficient.

First we deal with the normal components. When $i \in I_N$, the algorithm is written in

the following form:

$$\left\{ \begin{array}{l} u_i^{n+\frac{1}{2}} = \frac{1}{d_{ii}^{n+\frac{1}{2}}} (L_i - \sum_{j=1}^{i-1} d_{ij}^{n+1} u_j^{n+1} - \sum_{j=i+1}^{2NP} d_{ij}^n u_j^n) \\ \text{with} \\ d_{ij}^n = \begin{cases} a_{ij} + \gamma(u_j^n) b_{ij} & \text{if } j \in I_N \\ a_{ij} + \eta(u_j^n) b_{ij} & \text{if } j \in I_T \\ a_{ij} & \text{otherwise} \end{cases} , \\ \gamma(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ 1 & \text{if } u > 0 \end{cases} , \\ \text{and } \eta(u) = \begin{cases} 0 & \text{if } |u| > \frac{\rho}{K_T} \\ 1 & \text{if } |u| \leq \frac{\rho}{K_T} \end{cases} , \\ u_i^{n+1} = (1 - \omega) u_i^n + \omega u_i^{n+\frac{1}{2}} \end{array} \right.$$

As regards the tangential components, when $i \in I_T$ we first take the fixed point problem $\rho_i^{l+1} = f |\sigma_N(u(\rho_i^l))|$ and then write

$$\left\{ \begin{array}{l} u_i^{n+\frac{1}{2}} = \frac{1}{d_{ii}^{n+\frac{1}{2}}} (L_i - \sum_{j=1}^{i-1} d_{ij}^{n+1} u_j^{n+1} - \sum_{j=i+1}^{2NP} d_{ij}^n u_j^n - \theta(u_i^{n+\frac{1}{2}}) \cdot \rho_i^l) \\ \text{with} \\ \theta(u) = \begin{cases} -1 & \text{if } u < \frac{-\rho}{K_T} \\ 1 & \text{if } u > \frac{\rho}{K_T} \\ 0 & \text{if } |u| \leq \frac{\rho}{K_T} \end{cases} , \\ u_i^{n+1} = (1 - \omega) u_i^n + \omega u_i^{n+\frac{1}{2}} \end{array} \right.$$

Testing the validity of the algorithm: compression of a bar In this paragraph the algorithm is tested and its validity is confirmed. We used a benchmark test developed by the group working on "Validation of computer codes" at the French Research Group "Large Deformations and Damage" [46].

Here we adopt the context of plane strains and take the case of a long bar with a square section (fig. 13) and Lamé's coefficients $\lambda = 45GPa$ and $\mu = 54GPa$. The contact zone (interface law) corresponds to the part AD with a friction coefficient equal to $f = 1$. $u_1 = 0$ on DE and $u_1 = u_2 = 0$ at point D . The loading $F1 = 10 \text{ daN/mm}^2$ is imposed on AG and $F2 = -5 \text{ daN/mm}^2$ on GE .

Using the finite element method, the contact zone is discretized by 32 nodes. The changes in contact status are given in table 1 at different values of $\bar{\mu}$ and $\bar{\lambda}$. If we compare the re-

Table 1. Behavior of the interface for some stiffness

Stiffness	Gap AB	Sliding $ \sigma_T = f \sigma_N $ BC	Stick $ \sigma_T < f \sigma_N $ CD
(1) $\bar{\mu} = 5.410^{+8}; \bar{\lambda} = 4.510^{+8}$	3 nodes	16 nodes	13 nodes
(2) $\bar{\mu} = 5.410^{+2}; \bar{\lambda} = 4.510^{+8}$	3 nodes	14 nodes	15 nodes
(3) $\bar{\mu} = 5.410^{+2}; \bar{\lambda} = 4.510^{+2}$	3 nodes	13 nodes	16 nodes
(4) $\bar{\mu} = 5.410^{-2}; \bar{\lambda} = 4.510^{+8}$	0 nodes	00 nodes	32 nodes
(5) $\bar{\mu} = 5.410^{-2}; \bar{\lambda} = 4.510^{+2}$	0 nodes	00 nodes	32 nodes
(6) $\bar{\mu} = 5.410^{-2}; \bar{\lambda} = 4.510^{-2}$	0 nodes	00 nodes	32 nodes

sults obtained with those published in [46] (Signorini-Coulomb laws), it can be seen that at $\bar{\lambda} = 4.510^{+8}$ and $\bar{\mu} = 5.410^{+8}$ the present results are similar to those obtained by [46]. These coefficients, which are very large (i.e. "equal" to infinity), correspond to the limit case (Signorini-Coulomb). The decrease in the values of $\bar{\mu}$ corresponds to the increase in the stick zone (13, 15 and 32 nodes). Note that for this example and for cases presented in figure 14, the normal displacements are slightly changed by the variations of K_N and K_T . In particular, the penetration is negligible. However, as observed on figure 15, for this problem the coefficients variations strongly affect the tangential displacements. A low value of the coefficient K_T reflects in an increase of the tangential displacement. The nodes slide without reaching the value of Coulomb's sliding limit (fig. 15). Note that numerical tests have shown the robustness of this algorithm [49].

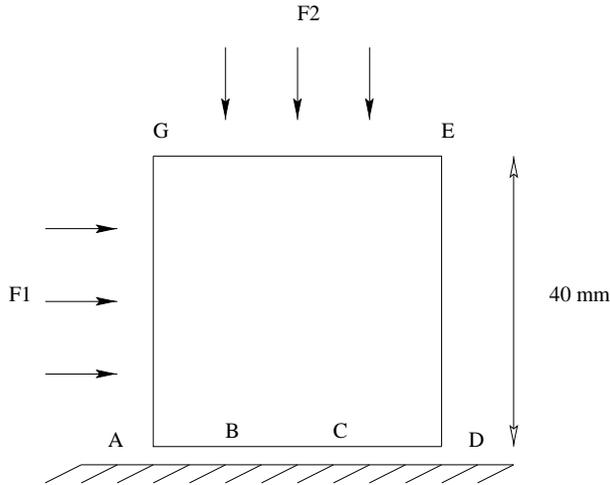


Figure 13. The problem of the long bar.

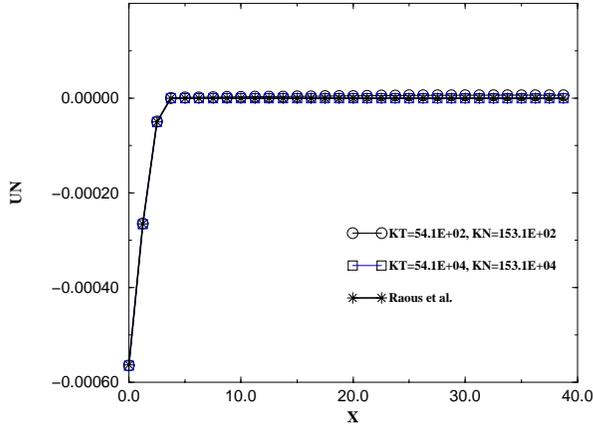


Figure 14. Normal displacement with respect to X.

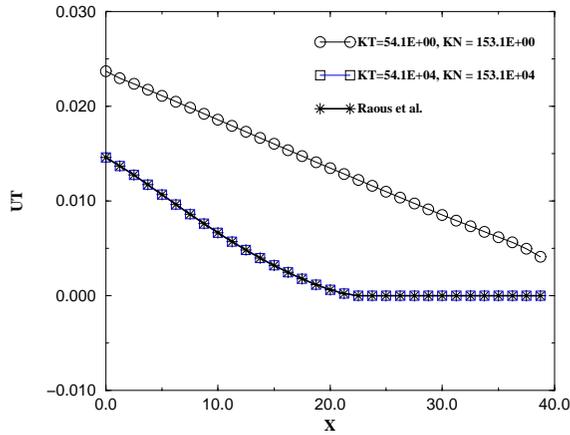


Figure 15. Tangential displacement with respect to X.

8. Conclusion

In this chapter, we analyze the asymptotic behavior of a thin adhesive. To obtain the interface law, which describes the behavior of an adhesive of vanishing thickness, different approaches have been tested. First, we have dealt with adherents and adhesive having similar rigidity and performed a Γ -convergence analysis to obtain the relation between the stress vector and the displacement fields. Next, we have used matched asymptotic expansions to analyze the asymptotic behavior of a two-dimensional equilibrium problem including frictional contact at the interface between the thin adhesive and the adherents. Lastly, we have

studied the problem of a thin layer with a non convex energy, which is used to model phase transformation of the adhesive. To conclude, we have presented a numerical algorithm able to solve the problem with frictional contact.

The asymptotic methods presented in the chapter allow to obtain a precise description of the limit behavior of a vanishing adhesive in terms of an interface law. The cases presented show that different constitutive behaviors of the adhesive give rise to different interface laws. The mathematical methods presented in the chapter provide a precise limit behavior if the constitutive relation of the adhesive is known, an occurrence which is rarely satisfied in practice. Another delicate point is that the methods illustrated in the chapter can be used under the assumption that the model of continuum is valid at the scale of the adhesive. There are cases in which the adhesive behavior is better described by means of molecular dynamics and other techniques should then be applied [43, 51].

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