# Dispersive limit from the Kawahara to the KdV equation

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# DISPERSIVE LIMIT FROM THE KAWAHARA TO THE KDV EQUATION

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ABSTRACT. We investigate the limit behavior of the solutions to the Kawahara equation

$$u_t + u_{3x} + \varepsilon u_{5x} + uu_x = 0 \quad , \varepsilon > 0$$

as $\varepsilon \to 0$. In this equation, the terms $u_{3x}$ and $\varepsilon u_{5x}$ do compete together and do cancel each other at frequencies of order $1/\sqrt{\varepsilon}$. This prohibits the use of a standard dispersive approach for this problem. Nervertheless, by combining different dispersive approaches according to the range of spaces frequencies, we succeed in proving that the solutions to this equation converges in $C([0,T];H^1(\mathbb{R}))$ towards the solutions of the KdV equation for any fixed $T > 0$.

## 1. Introduction and main results

1.1. **Introduction.** In this paper we are interested in the limit behavior of the solutions to the Kawahara equation

$$(K_\varepsilon) \qquad u_t + u_{3x} + \varepsilon u_{5x} + uu_x = 0, \quad (t,x) \in \mathbb{R}^2,\ \varepsilon > 0,$$

as the positive coefficient $\varepsilon \to 0$.
Our goal is to prove that they converge in a strong sense towards the solutions of the KdV equation

$$u_t + u_{3x} + uu_x = 0, \quad (t,x) \in \mathbb{R}^2. \tag{1.1}$$

This study can be seen as a peculiar case of the following class of limit behavior problems :

$$\partial_t u + \partial_x\Big(L_1 - \varepsilon L_2\Big)u + N_1(u) + \varepsilon N_2(u) = 0\,, \tag{1.2}$$

where $u : \mathbb{R} \to \mathbb{R}$, $L_1$ and $L_2$ are speudo-differential operators with Fourier symbols $|\xi|^{\alpha_1}$ and $|\xi|^{\alpha_2}$ with $0 < \alpha_1 < \alpha_2$ and $N_1$ and $N_2$ are polynomial functions that depends on $u$, its derivatives and possibly on the image of $u$ by some speudo-differential operator (as for instance the Hilbert transform) . Note that the dispersive limits from the Benjamin equation or some higher-order BO equations derived in [2] towards the Benjamin-Ono equation enter this class.

In this class of limit behavior problems, the main difficulty comes from the fact that the dispersive terms $\partial_x L_1 u$ and $\varepsilon \partial_x L_2 u$ do compete together. As one can easily check, the derivatives of the associated phase function $\phi(\xi) = \xi|\xi|^{\alpha_1}(1 - \varepsilon|\xi|^{\alpha_2-\alpha_1})$ does vanish at frequencies of order $\varepsilon^{-\frac{1}{\alpha_2-\alpha_1}}$. This will make classical dispersive estimates as Strichartz estimates, global Kato smoothing effect or maximal in time

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estimate, not uniform in $\varepsilon$. Therefore it is not clear to get even boundedness uniformly in $\varepsilon$ of the solutions to (1.2) by classical dispersive resolution methods.

On the other hand, by using only energy estimates that do not take into account the dispersive terms, we can see immediately that the solutions to $(K_\varepsilon)$ will stay bounded in $H^s(\mathbb{R})$, uniformly in $\varepsilon$, providing we work in Sobolev spaces $H^s(\mathbb{R})$ with index $s > 3/2$. Moreover, using for instance Bona-Smith argument, we could prove the convergence of the solution of $(K_\varepsilon)$ to the ones of (1.1) in $C([0,T];H^s(\mathbb{R}))$ with $T = T(\|u(0)\|_{H^s}$ and $s > 3/2$. However this approach is far to be satisfactory since it does not use at all the dispersive effects. Moreover, the KdV and Kawahara equations are known to be well-posed in low indices Sobolev spaces (see for instance [1], [6], [5] ) and one can ask wether such convergence result does hold in those spaces. In this work we make a first step in this direction by proving that this convergence result holds in $H^s(\mathbb{R})$ with $s \geq 1$. Note that $H^1(\mathbb{R})$ is a natural space for this problem since it is the energy space for the KdV equation. Our main idea is to combine different dispersive method according to the area of frequencies we consider. More precisely, we will use a Bourgain's approach (cf. [1], [3]) outside the area $D_\varepsilon$ where the first derivative of the phase function $\phi'$ does vanish whereas we will use Koch-Tzvetkov approach (cf. [8]) in $D_\varepsilon$. Indeed, noticing that $\phi''$ does not vanish in this area, the Strichartz estimate are valid uniformly in $\varepsilon$ on $D_\varepsilon$ so that we can apply Koch-Tzvetkov approach. On the other hand, outside $D_\varepsilon$ one can easily see that one has a strong resonance relation at least for the worst interactions, namely the high-low interactions. Indeed, assuming that $|\xi_1| >> |\xi_2|$, by the mean-value theroem, it holds

$$|\phi_\varepsilon(\xi_1+\xi_2)-\phi_\varepsilon(\xi_1)-\phi_\varepsilon(\xi_2)| \sim |\phi'_\varepsilon(\xi_1)\xi_2-\phi_\varepsilon(\xi_2)| \sim |\phi'_\varepsilon(\xi_1)\xi_2| \sim |\xi^2(3-5\varepsilon\xi^2)\xi_2| \gtrsim \xi^2|\xi_2|,$$

where $\xi = \xi_1+\xi_2$ is the output frequency and $\phi_\varepsilon(\xi) = \xi^3 - \varepsilon\xi^5$ is the phase function associated with the $(K_\varepsilon)$. It is worth noticing that this resonance relation is similar to the one of the KdV equation that reads $(\xi_1+\xi_2)^3-(\xi_1)^3-(\xi_2)^3 = 3\xi\xi_1\xi_2$. To rely on this strong resonance relation even when one of the input frequency belongs to $D_\varepsilon$ we will make use of the fact that any $H^1$-solution to $(K_\varepsilon)$ must belong to some Bourgain's space with time regularity one.

### 1.2. **Main results.**

**Theorem 1.1.** *Let $s \geq 1$, $\varphi \in H^s(\mathbb{R})$, $T > 0$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers converging to 0. The sequence $u_n \in C(\mathbb{R};H^s(\mathbb{R}))$ of solutions to $(K_\varepsilon)$ emanating from $\varphi$ satisfies*

$$u_n \to u \text{ in } C([0,T];H^s(\mathbb{R})) \tag{1.3}$$

*where $u \in C(\mathbb{R};H^s(\mathbb{R}))$ is the unique solution to the KdV equation (1.1) emanating from $\varphi$.*

Theorem 1 is actually a consequence of the fact that the Cauchy problem associated with $(K_\varepsilon)$ is well-posed in $H^s(\mathbb{R})$, $s \geq 1$, uniformly in $\varepsilon \in ]0,1[$ in the following sense

**Theorem 1.2.** *Let $s \geq 1$ and $\varphi \in H^s(\mathbb{R})$. There exists $T = T(\|\varphi\|_{H^1}) \in ]0,1[$ and $C > 0$ such that for any $\varepsilon \in ]0,1[$ the solution $u_\varepsilon \in C(\mathbb{R};H^1(\mathbb{R}))$ to $(K_\varepsilon)$ satisfies*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{H^s} \leq C\|\varphi\|_{H^s} \tag{1.4}$$

*Moreover, for any $R>0$, the solution-map $\varphi \mapsto u_\varepsilon$ from $B(0,R)_{H^s}$ into $C([0,T(R)];H^s(\mathbb{R}))$ is Lipschitz uniformly in $\varepsilon \in ]0,1[$, i.e. there exists $C=C(R)>0$ such that for any $\varepsilon \in ]0,1[$ and any couple of initial data $(\varphi_1,\varphi_2)\in B(0,R)^2_{H^s}$ it holds*

$$\|u_{1,\varepsilon}-u_{2,\varepsilon}\|_{L^\infty(0,T(R);H^s(\mathbb{R}))} \leq C\,\|\varphi_1-\varphi_2\|_{H^s} \tag{1.5}$$

*where $u_{i,\varepsilon}$, $i=1,2$, are the solution to $(K_\varepsilon)$ emanating from $\varphi_i$.*

1.3. **Notation.** For any positive numbers $a$ and $b$, the notation $a \lesssim b$ means that there exists a positive constant $c$ such that $a \leq cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, $\alpha_+$, respectively $\alpha_-$, will denote a number slightly greater, respectively lesser, than $\alpha$.

For $u=u(x,t)\in \mathcal{S}(\mathbb{R}^2)$, $\mathcal{F}u=\widehat{u}$ will denote its space-time Fourier transform, whereas $\mathcal{F}_x u=(u)^{\wedge_x}$, respectively $\mathcal{F}_t u=(u)^{\wedge_t}$, will denote its Fourier transform in space, respectively in time. For $s\in\mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, $J^s_x$ and $D^s_x$, by

$$J^s_x u=\mathcal{F}_x^{-1}\big((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}_x u\big) \quad \text{and} \quad D^s_x u=\mathcal{F}_x^{-1}\big(|\xi|^s\mathcal{F}_x u\big).$$

We will need a Littlewood-Paley analysis. Let $\psi\in C^\infty_0(\mathbb{R})$ be an even function such that $\psi\geq 0$, $\operatorname{supp}\psi\subset[-3/2,3/2]$, $\psi\equiv 1$ on $[-5/4,5/4]$. We set $\eta_0:=\psi$ and for all $k\in\mathbb{N}^*$, $\eta_{2^k}(\xi):=\psi(2^{-k}\xi)-\psi(2^{-k+1}\xi)$, $\eta_{\leq 2^k}:=\psi(2^{-k}\cdot)=\sum_{j=0}^k\eta_{2^j}$ and $\eta_{\geq 2^k}:=1-\psi(2^{k-1}\cdot)=1-\eta_{\leq 2^{k-1}}$. The Fourier multiplicator operators by $\eta_{2^j}$, $\eta_{\leq 2^j}$ and $\eta_{\geq 2^j}$ will be denoted respectively by $P_{2^j}$, $P_{\leq 2^j}$ and $P_{\geq 2^j}$, i.e. for any $u\in L^2(\mathbb{R})$

$$\widehat{P_{2^j}u}:=\eta_{2^j}\widehat{u}, \quad \widehat{P_{\leq 2^j}u}:=\eta_{\leq 2^j}\hat{u} \quad \text{and } \widehat{P_{\geq 2^j}u}:=\eta_{\geq 2^j}\hat{u}\ .$$

Note that, to simplify the notations, any summations over capitalized variables such as $N$ are presumed to be dyadic with $N\geq 1$, *i.e.*, these variables range over numbers of the form $2^k$, $k\in\mathbb{Z}_+$. $P_+$ and $P_-$ will denote the projection on respectively the positive and the negative Fourier frequencies.

Finally, we denote by $U_\varepsilon(t):=e^{-t(\partial_x^3+\varepsilon\partial_x^5)}$ the free evolution associated with the linear part of $(K_\varepsilon)$.

1.4. **Function spaces.** For $1\leq p\leq\infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$, and for $s\in\mathbb{R}$ , the real-valued Sobolev spaces $H^s(\mathbb{R})$ denote the spaces of all real-valued functions with the usual norms

$$\|\varphi\|_{H^s}=\|J^s_x\varphi\|_{L^2}\ .$$

If $f=f(x,t)$ is a function defined for $x\in\mathbb{R}$ and $t$ in the time interval $[0,T]$, with $T>0$, if $B$ is one of the spaces defined above, $1\leq p\leq\infty$ and $1\leq q\leq\infty$, we will define the mixed space-time spaces $L^p_T B_x$, $L^p_t B_x$, $L^q_x L^p_T$ by the norms

$$\|f\|_{L^p_T B_x}=\Big(\int_0^T\|f(\cdot,t)\|_B^p dt\Big)^{\frac{1}{p}} \quad , \quad \|f\|_{L^p_t B_x}=\Big(\int_{\mathbb{R}}\|f(\cdot,t)\|_B^p dt\Big)^{\frac{1}{p}},$$

and

$$\|f\|_{L^q_x L^p_T}=\left(\int_{\mathbb{R}}\Big(\int_0^T|f(x,t)|^p dt\Big)^{\frac{q}{p}}dx\right)^{\frac{1}{q}}.$$

For $s$, $b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}_\epsilon$ related to the linear part of $(K_\varepsilon)$ as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|v\|_{X^{s,b}_\epsilon} := \left( \int_{\mathbb{R}^2} \langle \tau - \phi_\varepsilon(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{v}(\xi,\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}, \tag{1.6}$$

where $\langle x \rangle := 1 + |x|$. We will also use a dyadic version of those spaces introduced in [9] in the context of wave maps. For $s$, $b \in \mathbb{R}$, $1 \leq q \leq \infty$, $X^{s,b,q}_\epsilon$ will denote the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|v\|_{X^{s,b,q}_\epsilon} := \left( \sum_{k \geq 0} \Big( \sum_{j \geq 0} \langle 2^k \rangle^{sq} \langle 2^j \rangle^{bq} \|P_{2^k}(\xi) P_{2^j}(\tau - \phi_\varepsilon(\xi)) \widehat{v}(\xi,\tau)\|^q_{L^2_{\tau,\xi}} \Big)^{\frac{2}{q}} \right)^{\frac{1}{2}}. \tag{1.7}$$

Moreover, we define a localized (in time) version of these spaces. Let $T > 0$ be a positive time and $Y = X^{s,b}_\epsilon$ or $Y = X^{s,b,q}_\epsilon$. Then, if $v : \mathbb{R} \times ]0,T[ \to \mathbb{R}$, we have that

$$\|v\|_{Y_T} := \inf\{ \|\tilde{v}\|_Y \mid \tilde{v} : \mathbb{R} \times \mathbb{R} \to \mathbb{C},\ \tilde{v}|_{\mathbb{R} \times ]0,T[} = v \}.$$

## 2. Uniform estimates far from the stationary point of the phase function

As we explained in the introduction, it is crucial that the first and the second derivatives of the phase function $\phi_\varepsilon(\xi) = \xi^3 - \varepsilon \xi^5$ do not cancel exactly at the same point. Indeed, $\phi'_\varepsilon(\xi) = 0 \Leftrightarrow |\xi| = \sqrt{\frac{3}{5\varepsilon}}$ while , $\phi''_\varepsilon(\xi) = 0 \Leftrightarrow |\xi| = \sqrt{\frac{3}{10\varepsilon}}$. Consequently, we introduce the following smooth Fourier projectors

$$\widehat{P_{A_\varepsilon} f} = \Big[ 1 - \eta_0 \Big[ 20\sqrt{\varepsilon} \Big( |\xi| - \sqrt{\frac{3}{5\varepsilon}} \Big) \Big] \Big] \widehat{f}$$

and

$$\widehat{P_{B_\varepsilon} f} = \Big[ 1 - \eta_0 \Big[ 20\sqrt{\varepsilon} \Big( |\xi| - \sqrt{\frac{3}{10\varepsilon}} \Big) \Big] \Big] \widehat{f}$$

Clearly, $\widehat{P_{A_\varepsilon} f}$ cancels in a region of order $\varepsilon^{-1/2}$ around $\sqrt{\frac{3}{5\varepsilon}}$ whereas $\widehat{P_{B_\varepsilon} f}$ cancels in a region of order $\varepsilon^{-1/2}$ around $\sqrt{\frac{3}{10\varepsilon}}$. We are now in position to state the main proposition of this section :

**Proposition 2.1.** *Let $u_{i,\varepsilon} \in C([0,T]; H^s(\mathbb{R}))$, $i = 1, 2$, be two solutions to $(K_\varepsilon)$ with $0 < \varepsilon << 1$ and initial data $\varphi_i$. Then it holds*

$$\|P_{A_\varepsilon} u_{i,\varepsilon}\|_{X^{1,1/2,1}_{\varepsilon,T}} \lesssim \|\varphi_i\|_{H^1} + \|u_{i,\varepsilon}\|^2_{Y_{\varepsilon,T}} (1 + \|u_{i,\varepsilon}\|^2_{Y_{\varepsilon,T}}) \tag{2.1}$$

*and, setting $w = u_{1,\varepsilon} - u_{2,\varepsilon}$,*

$$\|P_{A_\varepsilon} w\|_{X^{1,1/2,1}_{\varepsilon,T}} \lesssim \|\varphi_1 - \varphi_2\|_{H^1} + \|w\|_{Y_{\varepsilon,T}} \sum_{i=1}^{2} \|u_{i,\varepsilon}\|_{Y_{\varepsilon,T}} (1 + \|u_{i,\varepsilon}\|^2_{Y_{\varepsilon,T}}) \tag{2.2}$$

*where*

$$\|u\|_{Y_{\varepsilon,T}} := \|P_{A_\varepsilon} u\|_{X^{1,1/2,1}_{\varepsilon,T}} + \|u\|_{L^\infty_T H^1} \tag{2.3}$$

We will make a frequent use of the following linear estimates

**Lemma 2.1.** *Let $\varphi \in S(\mathbb{R})$ and $T \in ]0,1]$ then $\forall 0 < \varepsilon \ll 1$,*

$$\|P_{A_\varepsilon}\partial_x U_\varepsilon(t)\varphi\|_{L^\infty_x L^2_t} \lesssim \|\varphi\|_{L^2} \tag{2.4}$$

$$\|D^{1/4}P_{\complement A_\varepsilon}U_\varepsilon(t)\varphi\|_{L^4_t L^\infty_x} + \|D^{1/4}P_{B_\varepsilon}U_\varepsilon(t)\varphi\|_{L^4_t L^\infty_x} \lesssim \|\varphi\|_{L^2} \tag{2.5}$$

$$\|P_{\leq 2}U_\varepsilon(t)\varphi\|_{L^2_x L^\infty_T} \lesssim \|\varphi\|_{L^2}\,, \tag{2.6}$$

*where $\mathcal{F}_x(P_{\complement A_\varepsilon}\varphi) = (1-\eta_{A_\varepsilon})\mathcal{F}_x\varphi$ and the implicit constants are independent of $\varepsilon > 0$.*

*Proof.* First, (2.4) follows from the classical proof of the local Kato smoothing effect, by using that $|\phi'_\varepsilon(\xi)| \gtrsim |\xi|^2$ on the Fourier support of $P_{A_\varepsilon}$.

To prove (2.5), we first notice that the Fourier supports of $P_{B_\varepsilon}$ and $P_{\complement A_\varepsilon}$ do not intersect the region $\{\xi \in \mathbb{R},\, |\xi| \in [\sqrt{\frac{1}{4\varepsilon}}, \sqrt{\frac{7}{20\varepsilon}}]\}$. By the $TT^*$ argument it suffices to prove that

$$\|U_\varepsilon(t)D_x^{1/2}P_{\complement B_\epsilon}\varphi\|_{L^\infty_x} + \|U_\varepsilon(t)D_x^{1/2}P_{A_\epsilon}\varphi\|_{L^\infty_x} \lesssim t^{-1/2}\|\varphi\|_{L^1} \tag{2.7}$$

By classical arguments, (1.3) will be proven if we show

$$\Big\| \int_{\mathbb{R}} \chi_{\{|\xi| \notin [\sqrt{\frac{1}{4\varepsilon}}, \sqrt{\frac{7}{20\varepsilon}}]\}} |\xi|^{1/2} e^{i[x\xi + (\xi^3 - \varepsilon\xi^5)t]}\, d\xi \Big\|_{L^\infty_x} \lesssim t^{-1/2}\,.$$

Setting $\theta := \xi|t|^{1/3}$ this is equivalent to prove

$$I_\epsilon := \sup_{t\in\mathbb{R}, X\in\mathbb{R}} \Big| \int_{\mathbb{R}} \chi_{\{|\theta| \notin [\sqrt{\frac{|t|^{2/3}}{4\varepsilon}}, \sqrt{\frac{7|t|^{2/3}}{20\varepsilon}}]\}} |\theta|^{1/2}\, e^{i[X\theta + \theta^3 - \frac{\varepsilon}{|t|^{2/3}}\theta^5]}\, d\theta \Big| \lesssim 1 \tag{2.8}$$

We set $\Phi(\theta) = \Phi_{t,\varepsilon}(\theta) := \theta^3 - \frac{\varepsilon}{|t|^{2/3}}\theta^5$ and notice that

$$\Phi'(\theta) := 3\theta^2 - \frac{5\varepsilon}{|t|^{2/3}}\theta^4 \;\text{ and }\; \Phi''(\theta) = 2\theta\Big(3 - \frac{10\varepsilon}{|t|^{2/3}}\theta^2\Big)\,.$$

(2.8) is obvious when restricted on $|\theta| \leq 100$. Now, it is worth noticing that

$$|\Phi''(\theta)| \gtrsim 1 + \max\Big(|\theta|, \frac{\varepsilon}{|t|^{2/3}}\theta^3\Big)$$

whenever $\theta \in \{|z| \geq 100 \,/\, |z| \notin [\sqrt{\frac{|t|^{2/3}}{4\varepsilon}}, \sqrt{\frac{7|t|^{2/3}}{20\varepsilon}}]\}$. Therefore, in the region $|\theta| \in [\sqrt{\frac{|t|^{2/3}}{10\varepsilon}}, \sqrt{\frac{2|t|^{2/3}}{\varepsilon}}]$, (2.8) follows from Van der Corput lemma since $|\Phi''(\theta)| \gtrsim 1 + \frac{|t|^{1/3}}{\sqrt{\varepsilon}}$ and $|\theta|^{1/2} \sim \frac{|t|^{1/6}}{\varepsilon^{1/4}}$. It thus remains to consider the region $|\theta| \notin [\sqrt{\frac{|t|^{2/3}}{10\varepsilon}}, \sqrt{\frac{2|t|^{2/3}}{\varepsilon}}]$. We notice that, in this region, it holds

$$|\Phi'(\theta)| \sim |\theta|^2 \text{ for } |\theta| \leq \sqrt{\frac{|t|^{2/3}}{10\varepsilon}} \;\text{ and }\; |\Phi'(\theta)| \sim \frac{\varepsilon|\theta|^4}{|t|^{2/3}} \text{ for } |\theta| \geq \sqrt{\frac{2|t|^{2/3}}{\varepsilon}} \tag{2.9}$$

and divide this region into two subregions.

• The subregion $|\Phi'(\theta) - X| \leq |X|/2$. Then $|\Phi'(\theta)| \sim |X|$. Assuming we are in the region $100 < |\theta| \leq \sqrt{\frac{|t|^{2/3}}{10\varepsilon}}$, we have $|\Phi'(\theta)| \sim |\theta|^2$ and thus $|\theta| \sim \sqrt{|X|}$. Then (2.8) follows from Van der Corput lemma since $|\Phi''(\theta)| \gtrsim |\theta| \sim \sqrt{|X|}$. On the other hand, assuming that $|\theta| \geq \sqrt{\frac{2|t|^{2/3}}{\varepsilon}} \geq 100$ then $|\Phi'(\theta)| \sim \varepsilon|\theta|^4|t|^{-2/3}$ and thus $|\theta| \sim \varepsilon^{-1/4}|X|^{1/4}|t|^{1/6}$. (2.8) follows again from Van der Corput lemma since $|\Phi''(\theta)| \gtrsim |\theta| \sim \varepsilon^{-1/4}|X|^{1/4}|t|^{1/6}$.

• The subregion $|\Phi'(\theta)-X|>|X|/2$. Then $|\Phi'(\theta)-X|\sim|\Phi'(\theta)|$ and (2.8) is obtained by integrating by parts and using (2.9). This completes the proof of (2.5).

Finally, to show (2.6) we notice that it suffices to prove that for $|x|\geq 10^4$,

$$\sup_{t\in[0,1]}\Big|\int_{\mathbb{R}}\eta_{\leq 2}(\xi)\,e^{i[x\xi+\phi_\varepsilon(\xi)t]}\,d\xi\Big|\lesssim|x|^{-2}\,,$$

where $\phi_\varepsilon(\xi)=\xi^3-\varepsilon\xi^5$. But this follows directly by integrating by parts twice since $|x-\phi'_\varepsilon(\xi)t|\gtrsim|x|$ for any $|t|\leq 1$ and $|\xi|\leq 4$. □

To prove Proposition 2.1 we will have to put the whole solution $u_\varepsilon$ of $(K_\varepsilon)$ and not only $P_{A_\varepsilon}u_\varepsilon$ in some Bourgain's space with regularity 1 in time. This will be done in the next lemma by noticing that any solution to $(K_\varepsilon)$ that belongs to $C([0,T];H^1(\mathbb{R}))$ automatically belongs to $X^{0,1}_{\varepsilon,T}$.

**Lemma 2.2.** *Let $T\in]0,1[$ and $u\in C([0,T];H^1(\mathbb{R}))$ be a solution to $(K_\varepsilon)$. Then,*

$$\|u\|_{X^{0,1}_{\varepsilon,T}}\lesssim\|u\|_{L^\infty_T H^1}+\|u\|^2_{L^\infty_T H^1}\,, \tag{2.10}$$

*where the implicit constant is independent of $\varepsilon$.*

*Proof.* First, we consider $v(t)=U_\varepsilon(-t)u(t)$ on the time interval $]0,T]$ and extend $v$ on $]-2,2[$ by setting $\partial_t v=0$ on $[-2,2]\setminus[0,T]$. Then, it is pretty clear that

$$\|\partial_t v\|_{L^2(]-2,2[\times\mathbb{R})}=\|\partial_t v\|_{L^2(]-T,T[\times\mathbb{R})},\quad\text{and}\quad\|v\|_{L^2(]-2,2[\times\mathbb{R})}\lesssim\|v\|_{L^\infty_T L^2_x}\,.$$

Now, we define $\tilde{u}(x,t)=\eta(t)U(t)v(t)$. Obviously, $\tilde{u}$ is an extension of $u$ outside $]-T,T[$ and it holds

$$\|\tilde{u}\|_{X^{0,1}_\varepsilon}\lesssim\|\partial_t v\|_{L^2(]-2,2[\times\mathbb{R})}+\|v\|_{L^2(]-2,2[\times\mathbb{R})}\lesssim\|\partial_t v\|_{L^2(]-2,2[\times\mathbb{R})}+\|v\|_{L^\infty_T L^2_x}\,. \tag{2.11}$$

Therefore (2.10) follows from the identity

$$\partial_t v=U_\varepsilon(-t)\Big[u_t+u_{xxx}+\varepsilon u_{5x}\Big]$$

together with the facts that $u$ is a solution to $(K_\varepsilon)$ and that $H^1(\mathbb{R})$ is an algebra. □

Now, according to the Duhamel formula and to classical linear estimates in Bourgain's spaces (cf. [1], [3]), Proposition 2.1 is a direct consequence of the following bilinear estimate

$$\|P_{A_\varepsilon}\partial_x(u_1u_2)\|_{X^{1,-1/2,1}_\varepsilon}\lesssim\prod_{i=1}^{2}\Big(\|P_{A_\varepsilon}u_i\|_{X^{1,1/2,1}_\varepsilon}+\|u_i\|_{X^{0,1}_\varepsilon}+\|u_i\|_{L^\infty_t H^1_x}\Big) \tag{2.12}$$

where the functions $u_i$ are supported in time in $]-T,T[$ with $0<T\leq 1$. To prove this bilinear estimate we first note that by symmetry it suffices to consider $\partial_x\Lambda(u,v)$ where $\Lambda(\cdot,\cdot)$ is defined by

$$\mathcal{F}_x(\Lambda(u,v)):=\int_{\mathbb{R}}\chi_{|\xi_1|\leq|\xi-\xi_1|}(\mathcal{F}_x u)(\xi_1)(\mathcal{F}_x v)(\xi-\xi_1)\,d\xi_1\,.$$

As mentioned in the introduction, the following resonance relation is crucial for our analysis in this frequency area :

$$\Theta(\xi,\xi_1):=\sigma-\sigma_1-\sigma_2=\xi\xi_1(\xi-\xi_1)\Big[3-5\varepsilon\Big((\xi_1+\xi_2)^2-\xi_1\xi_2\Big)\Big] \tag{2.13}$$

where

$$\sigma:=\sigma(\tau,\xi):=\tau-\xi^3-\varepsilon\xi^5,\quad\sigma_1:=\sigma(\tau_1,\xi_1)\ \text{ and }\sigma_2:=\sigma(\tau-\tau_1,\xi-\xi_1)\,.$$

We start by noticing that the case of ouput frequencies of order less or equal to one is harmless. Indeed, it is easy to check that for any couple $u_i$, $i=1,2$, of smooth functions supported in time in $]-T,T[$ with $0<T\leq 1$ it holds

$$\|\partial_x P_{A_\varepsilon} P_{\leq 8}\Lambda(u_1,u_2)\|_{X^{1,-1/2,1}_\varepsilon} \lesssim \|\Lambda(u_1,u_2)\|_{L^2} \lesssim \|u_1\|_{L^\infty_t H^1}\|u_2\|_{L^\infty_t H^1}\ . \tag{2.14}$$

Let us continue by deriving an estimate for the interactions of high frequencies with frequencies of order less or equal to 1.

**Lemma 2.3.** *Let $u_i$, $i=1,2$, be two smooth functions supported in time in $]-T,T[$ with $0<T\leq 1$. Then it holds*

$$\|\partial_x P_{A_\varepsilon}\Lambda(P_{\leq 8}u_1,u_2)\|_{X^{1,-1/2,1}_\varepsilon} \lesssim \|u_1\|_{X^{0,1}_\varepsilon}\Big(\|P_{A_\varepsilon}u_2\|_{X^{1,1/2,1}_\varepsilon}+\|u_2\|_{X^{0,1}_\varepsilon}+\|\partial_x u_2\|_{L^2_{tx}}\Big)\ . \tag{2.15}$$

*Proof.* Since the norms in the right-hand side of (2.15) only see the size of the modulus of the Fourier transform, we can assume that all our functions have non negative Fourier transform. We set $\eta_{A_\varepsilon}=1-\eta_0\Big[20\sqrt{\varepsilon}\Big(|\xi|-\sqrt{\frac{3}{5\varepsilon}}\Big)\Big]$ so that $\widehat{P_{A_\varepsilon}f}=\eta_{A_\varepsilon}\widehat{f}$. Rewriting $\eta_{A_\varepsilon}(\xi)$ as $\eta_{A_\varepsilon}(\xi-\xi_1)+(\eta_{A_\varepsilon}(\xi)-\eta_{A_\varepsilon}(\xi-\xi_1))$, it suffices to estimate the two following terms

$$I_1:=\Big\|\mathcal{F}_x^{-1}\Big(\partial_x\Lambda(\eta_{\leq 8}\mathcal{F}_x(u_1),\eta_{A_\varepsilon}\mathcal{F}_x(u_2)\Big)\Big\|_{X^{1,-1/2,1}}$$

and

$$I_2:=\Big\|\mathcal{F}_x^{-1}\Big(\xi\int_{\mathbb{R}}\eta_{\leq 8}(\xi_1)\mathcal{F}_x(u_1)(\xi_1)(\eta_{A_\varepsilon}(\xi)-\eta_{A_\varepsilon}(\xi-\xi_1))\mathcal{F}_x(u_2)(\xi-\xi_1)\,d\xi_1\Big)\Big\|_{X^{1,-1/2,1}}$$

$I_1$ is easily estimate thanks to (2.6) by

$$\begin{aligned} I_1^2 &\lesssim \sum_{N\geq 1}\Big\|(\eta_{\leq 8}\widehat{u_1})*(\eta_N\eta_{A_\varepsilon}\widehat{\partial_x^2 u_2})\Big\|_{L^2}^2\\ &\lesssim \sum_{N\geq 1}\|P_{\leq 8}u_1\|^2_{L^2_xL^\infty_t}\|\partial_x^2 P_N P_{A_\varepsilon}u_2\|^2_{L^\infty_xL^2_t}\\ &\lesssim \|u_1\|^2_{X^{0,1}}\|P_{A_\varepsilon}u_2\|^2_{X^{1,1/2,1}}\quad . \end{aligned}$$

To estimate $I_2$ we first notice that for $|\xi_1|\leq 4$ and $0<\varepsilon<10^{-8}$,

$$\eta_{A_\varepsilon}(\xi)-\eta_{A_\varepsilon}(\xi-\xi_1)=0 \text{ whenever } |\xi|\in\Big[\frac{15}{16}\sqrt{\frac{3}{5\varepsilon}},\frac{17}{16}\sqrt{\frac{3}{5\varepsilon}}\Big]\cup\complement\Big[\frac{2^{-3}}{\sqrt{\varepsilon}},\frac{2^3}{\sqrt{\varepsilon}}\Big]\ . \tag{2.16}$$

and for any $(\xi,\xi_1)\in\mathbb{R}^2$,

$$|\eta_{A_\varepsilon}(\xi)-\eta_{A_\varepsilon}(\xi-\xi_1)|\lesssim \min\Big(1,\sqrt{\varepsilon}|\xi_1|\Big)\ . \tag{2.17}$$

Moreover, in the region $|\xi_1|\leq 4$ and $|\xi|\notin[\frac{15}{16}\sqrt{\frac{3}{5\varepsilon}},\frac{17}{16}\sqrt{\frac{3}{5\varepsilon}}]$ the resonance relation (2.13) ensures that

$$|\sigma_{max}|:=\max(|\sigma|,|\sigma_1|,|\sigma_2|)\gtrsim|\xi\xi_1(\xi-\xi_1)| \tag{2.18}$$

where $\sigma(\tau,\xi):=\tau-\phi_\varepsilon(\xi)$, $\sigma_1=\sigma(\tau_1,\xi_1)$ and $\sigma_2=\sigma(\tau-\tau_1,\xi-\xi_1)$. We separate three regions

• $\sigma_{max}=\sigma_2$. Then according to (2.16)-(2.18),

$$\begin{aligned} I_2 &\lesssim \Big\| \int_{\mathbb{R}} (\eta_{\leq 8}\widehat{u_1})(\xi_1)\sqrt{\varepsilon}\frac{|\xi_1|\langle\xi\rangle^2}{|\xi_1||\xi-\xi_1|^2}\langle\sigma_2\rangle\chi_{\{|\xi-\xi_1|\sim\frac{1}{\sqrt{\varepsilon}}\}}\widehat{u_2}(\xi-\xi_1)\,d\xi_1\Big\|_{L^2(|\xi|\sim\frac{1}{\sqrt{\varepsilon}})} \\ &\lesssim \|P_{\leq 8}u_1\|_{L^\infty_{tx}}\|u_2\|_{X^{-1/2,1}} \\ &\lesssim \|u_1\|_{X^{0,1}}\|u_2\|_{X^{0,1}} \end{aligned}$$

• $\sigma_{max}=\sigma_1$. Then according to (2.16)-(2.18),

$$\begin{aligned} I_2 &\lesssim \Big\| \langle\xi\rangle^2 \int_{\mathbb{R}} \frac{\langle\sigma_1\rangle}{|\xi_1||\xi-\xi_1|^2}(\eta_{\leq 8}\widehat{u_1})(\xi_1)\sqrt{\varepsilon}|\xi_1|\chi_{\{|\xi-\xi_1|\sim\frac{1}{\sqrt{\varepsilon}}\}}\widehat{u_2}(\xi-\xi_1)\,d\xi_1\Big\|_{L^2(|\xi|\sim\frac{1}{\sqrt{\varepsilon}})} \\ &\lesssim \|u_1\|_{X^{0,1}}\|D_x^{-1/2}\mathcal{F}^{-1}(\chi_{\{|\xi|\sim\frac{1}{\sqrt{\varepsilon}}\}}\widehat{u_2})\|_{L^\infty_{tx}} \\ &\lesssim \|u_1\|_{X^{0,1}}\|\mathcal{F}^{-1}(\chi_{\{|\xi|\sim\frac{1}{\sqrt{\varepsilon}}\}}\widehat{u_2}\|_{L^\infty_t L^2_x} \\ &\lesssim \|u_1\|_{X^{0,1}}\|u_2\|_{X^{0,1}} \end{aligned}$$

• $\sigma_{max}=\sigma$. Then according to (2.16)-(2.18),

$$\begin{aligned} I_2 &\lesssim \Big\| \langle\xi\rangle^2 \int_{\mathbb{R}} \frac{\sqrt{\varepsilon}|\xi_1|}{|\xi_1|^{3/8}|\xi-\xi_1|^{3/4}}(\eta_{\leq 8}\widehat{u_1})(\xi_1)\chi_{\{|\xi-\xi_1|\sim\frac{1}{\sqrt{\varepsilon}}\}}\widehat{u_2}(\xi-\xi_1)\,d\xi_1\Big\|_{L^2(|\xi|\sim\frac{1}{\sqrt{\varepsilon}})} \\ &\lesssim \sqrt{\varepsilon}\|P_{\leq 8}u_1\|_{L^\infty_{tx}}\|D_x^{5/4}\mathcal{F}^{-1}(\chi_{\{|\xi|\sim\frac{1}{\sqrt{\varepsilon}}\}}\widehat{u_2})\|_{L^2_{tx}} \\ &\lesssim \|u_1\|_{X^{0,1}}\|\partial_x u_2\|_{L^2_{tx}} \end{aligned}$$

This completes the proof of the lemma. □

The next lemma ensures that the restriction of the left-side member of (2.12) on the region $|\xi|\gtrsim 1$, $|\xi_1|\gtrsim 1$ and $|\sigma_{max}|\geq 2^{-5}|\xi\xi_1(\xi-\xi_1)|$ can be easily controlled.

**Lemma 2.4.** *Under the same hypotheses as in Lemma 2.3, in the region where the following strong resonance relation holds*

$$|\sigma_{max}|\geq 2^{-5}|\xi\xi_1(\xi-\xi_1)|\,, \tag{2.19}$$

*we have*

$$\|\partial_x P_{A_\varepsilon}P_{\geq 8}\Lambda(P_{\geq 8}u_1,u_2)\|_{X^{1,-1/2,1}_\varepsilon}\lesssim\prod_{i=1}^{2}\Big(\|u_i\|_{X^{0,1}}+\|\partial_x u_i\|_{L^2_{tx}}\Big)\,. \tag{2.20}$$

*Proof.* Again we notice that the norms in the right-hand side of (2.4) only see the size of the modulus of the Fourier transforms. We can thus assume that all our functions have non-negative Fourier transforms. We set $I:=\|\partial_x P_{A_\varepsilon}P_{\geq 8}\Lambda(P_{\geq 8}u_1,u_2)\|_{X^{1,-1/2,1}_\varepsilon}$ and separate different subregions .

• $|\sigma_1|\geq 2^{-5}|\xi\xi_1(\xi-\xi_1)|$. Then direct gives

$$\begin{aligned} I &\lesssim \|u_1\|_{X^{0,1}}\|D_x^{-1}P_{\geq 2}u_2\|_{L^\infty_{tx}} \\ &\lesssim \|u_1\|_{X^{0,1}}\|u_2\|_{X^{0,1}}\,. \end{aligned}$$

• $|\sigma_2|\geq 2^{-5}|\xi\xi_1(\xi-\xi_1)|$. This case can be treated exactly in the same way by exchanging the role of $u_1$ and $u_2$.

• $|\sigma|\geq 2^{-5}|\xi\xi_1(\xi-\xi_1)|$ and $\max(|\sigma_1|,\sigma_2|)\leq 2^{-5}|\xi\xi_1(\xi-\xi_1)|$.

Then we separate two subregions.
1. $|\xi_1| \geq 2^{-7}|\xi|$. Then $|\xi_1| \gtrsim |\xi_{max}|$ and taking $\delta > 0$ close enough to 0 we get

$$\begin{aligned} I &\lesssim \|\partial_x P_{A_\varepsilon} P_{\geq 8}\Lambda(P_{\geq 8}u_1, u_2)\|_{X_\varepsilon^{1,-1/2+\delta}} \\ &\lesssim \Big\|\partial_x u_2\, D_x^{-1/2+3\delta} P_{\geq 8}u_1\Big\|_{L^2} \\ &\lesssim \|D_x^{-1/2+3\delta}P_{\geq 8}u_1\|_{L^\infty_{tx}}\|\partial_x u_2\|_{L^2_{tx}} \\ &\lesssim \|u_1\|_{X^{1/4,3/4}}\|\partial_x u_2\|_{L^2_{tx}} \\ &\lesssim (\|u_1\|_{X^{0,1}} + \|\partial_x u_1\|_{L^2_{tx}})\|\partial_x u_2\|_{L^2_{tx}}\ . \end{aligned}$$

2. $|\xi_1| \leq 2^{-7}|\xi|$. Then, we notice that in this region $\frac{1}{2}|\xi| \leq |\xi - \xi_1| \leq 2|\xi|$ and thus

$$(1-2^{-6})\xi^2 \leq \xi^2 - \xi_1(\xi-\xi_1) \leq (1+2^{-6})\xi^2\ .$$

Since $\eta_{A_\varepsilon}$ does vanish on $\Big\{|\xi| \in \Big[\frac{15}{16}\sqrt{\frac{3}{5\varepsilon}}, \frac{17}{16}\sqrt{\frac{3}{5\varepsilon}}\Big]\Big\}$, we deduce from (2.13) that

$$|\sigma| \sim \max\Big(|\xi\xi_1(\xi-\xi_1)|, \varepsilon|\xi^3\xi_1(\xi-\xi_1)|\Big)$$

on the support of $\eta_{A_\varepsilon}$. We thus can write

$$\begin{aligned} I^2 &\lesssim \sum_{N\geq 4}\Big(\sum_{4\leq N_1\leq 2^{-5}N}\Big\|\eta_N(\xi)\eta_{A_\varepsilon}(\xi)|\xi|\chi_{\{|\sigma|\sim\max(N_1N^2,\varepsilon N^4N_1)\}}\mathcal{F}_x\Big(\Lambda(P_{N_1}u, u_2)\Big)\Big\|_{X_\varepsilon^{1,-1/2,1}}\Big)^2 \\ &\lesssim \sum_{N\geq 4}\Big(\sum_{4\leq N_1\leq 2^{-5}N}\|P_{N_1}D_x^{-1/2}u_1\|_{L^\infty_{tx}}\|\chi_{\{|\xi|\sim N\}}\xi\,\widehat{u_2}\|_{L^2_{\tau,\xi}}\Big)^2 \\ &\lesssim \sum_{N\geq 4}\|\chi_{\{|\xi|\sim N\}}\xi\,\widehat{u_2}\|^2_{L^2_{\tau,\xi}}\Big(\sum_{4\leq N_1\leq 2^{-5}N}N_1^{-1/4}\|P_{N_1}D_x^{1/4}u_1\|_{L^\infty_t L^2_x}\Big)^2 \\ &\lesssim \|u_1\|^2_{X^{1/4,3/4}}\|\partial_x u_2\|^2_{L^2_{tx}} \\ &\lesssim (\|u_1\|_{X^{0,1}} + \|\partial_x u_1\|_{L^2_{tx}})^2\|\partial_x u_2\|^2_{L^2_{tx}}\ . \end{aligned}$$

□

**Proof of the bilinear estimate** (2.12)
First, according to (2.14) and Lemma 2.3 and to the support of $\eta_{A_\varepsilon}$ it suffices to consider

$$I := \Big[\sum_{N\geq 4}N^2\Big(\sum_L L^{-1/2}\Big\|\eta_L(\sigma)\eta_N(\xi)\int_{\mathbb{R}^2}\widehat{P_{\geq 8}u_1}(\xi_1,\tau_1)\widehat{P_{\geq 8}u_2}(\xi_2,\tau_2)\,d\tau_1\,d\xi_1\Big\|_{L^2_{\tau,\xi}(|\xi|\notin[\frac{15}{16}\sqrt{\frac{3}{5\varepsilon}},\frac{17}{16}\sqrt{\frac{3}{5\varepsilon}}])}\Big)^2\Big]^{1/2}.$$

where $\tau_2 = \tau - \tau_1$ and $\xi_2 = \xi - \xi_1$. Now we will decompose the region of integration into different regions and we will check that in most of these regions the strong resonance relation (2.19) holds. For the remaining it is convenient to introduce the function

$$\Gamma(\xi,\xi_1) := \Big|3 - 5\varepsilon\Big(\xi^2 - \xi_1(\xi-\xi_1)\Big)\Big|$$

which is related to the resonance relation (2.13).

1. The region $|\xi| \geq \sqrt{\frac{1}{\varepsilon}}$. Then since $\xi^2 - \xi_1\xi_2 \geq \frac{3}{4}\xi^2$ we get $\Gamma(\xi,\xi_1) \geq 3/4$ which, according to (2.13) ensures that (2.19) holds.
2. The region $|\xi| \leq \sqrt{\frac{2}{5\varepsilon}}$.

2.1. The subregion $\xi_1\xi_2 \geq 0$. Then $\xi^2 - \xi_1\xi_2 \leq \xi^2 \leq \frac{2}{5\varepsilon}$ and thus $\Gamma(\xi,\xi_1) \geq 1$ which again ensures that (2.19) holds.

2.2. The subregion $\xi_1\xi_2 \leq 0$. Then $P_{\geq 8}u_1 P_{\geq 8}u_2$ can be decomposed as

$$\sum_{8\leq N_1\leq N_2} P_{N_1}P_+u_1 P_{N_2}P_-u_2 + \sum_{8\leq N_1\leq N_2} P_{N_1}P_-u_1 P_{N_2}P_+u_2$$

where, by symmetry, we assumed $N_1 \leq N_2$. Since these two terms can be treated in exactly the same way, we will only consider the first one. We decompose the sum into different parts.

2.2.1. $8 \leq N_1 \leq 2^{-4}N_2$. Then it holds $|\xi| \geq \frac{3}{4}|\xi_2|$ and thus

$$\xi^2 - \xi_1\xi_2 \leq \xi^2 + \frac{1}{4}\xi_2^2 \leq \xi^2 + \frac{4}{9}\xi^2 = \frac{13}{9}\xi^2 \ .$$

This forces $\Gamma(\xi,\xi_1) \geq \frac{1}{9}$ and ensures that (2.19) holds.

2.2.2. $N_1 > 2^{-4}N_2$.

2.2.2.1. $|\xi| \notin \left[\sqrt{\frac{17}{80\varepsilon}}, \sqrt{\frac{2}{5\varepsilon}}\right]$. In this region, by (2.5) of Lemma 2.1 and duality, we get

$$\begin{aligned} I &\lesssim \sum_{\min(4,2^{-4}N_2)<N_1\leq N_2} \|D_x^{-\frac{1}{4}+}\partial_x^2(P_{N_1}u_1 P_{N_2}u_2)\|_{L_t^{\frac{4}{3}+}L_x^{1+}} \\ &\lesssim \sum_{\min(4,2^{-4}N_2)<N_1\leq N_2} N_2^{-\frac{1}{4}+}\|\partial_x P_{N_1}u_1\|_{L_t^\infty L_x^2}\|\partial_x P_{N_2}u_2\|_{L_t^\infty L_x^{2+}} \\ &\lesssim \|u_1\|_{L_t^\infty H^1}\|u_2\|_{L_t^\infty H^1} \ . \end{aligned}$$

2.2.2.2. $|\xi| \in \left[\sqrt{\frac{17}{80\varepsilon}}, \sqrt{\frac{2}{5\varepsilon}}\right]$. We separate this last region into two subregions : We first consider the subregion $|\xi_1| \leq \sqrt{\frac{17}{80\varepsilon}}$. Then, according to (2.5) and the support of $\eta_{A_\varepsilon}$ and $\eta_{B_\varepsilon}$, we get

$$\begin{aligned} I &\lesssim \sum_{\min(4,2^{-4}N_2)<N_1\leq N_2} \|\partial_x^2(P_{B_\varepsilon}P_{A_\varepsilon}P_{N_1}u_1 P_{N_2}u_2)\|_{L_{tx}^2} \\ &\lesssim \sum_{\min(4,2^{-4}N_2)<N_1\leq N_2} \|P_{B_\varepsilon}P_{A_\varepsilon}\partial_x P_{N_1}u_1\|_{L_t^4 L_x^\infty}\|\partial_x P_{N_2}u_2\|_{L_t^\infty L_x^2} \\ &\lesssim \sum_{\min(4,2^{-4}N_2)<N_1\leq N_2} N_1^{-1/4}\|P_{A_\varepsilon}\partial_x P_{N_1}u_1\|_{X^{1,1/2,1}}\|\partial_x P_{N_2}u_2\|_{L_t^\infty L_x^2} \\ &\lesssim \|P_{A_\varepsilon}u_1\|_{X^{1,1/2,1}}\|u_2\|_{L_t^\infty H^1} \ . \end{aligned}$$

Finally in the subregion $|\xi_1| \geq \sqrt{\frac{17}{80\varepsilon}}$, we notice that, since $\xi_1\xi_2 \leq 0$ and $|\xi| \geq \sqrt{\frac{17}{80\varepsilon}}$, we must have $|\xi_2| \geq 2\sqrt{\frac{17}{80\varepsilon}}$. Therefore, $\xi^2 - \xi_1\xi_2 \geq 3\frac{17}{80\varepsilon}$ and thus $\Gamma(\xi,\xi_1) \geq \frac{3}{16}$ which ensures that (2.19) holds and completes the proof of (2.12).

## 3. Uniform estimate close to the stationary point of the phase function

As announced in the introduction, close the the stationary point of the phase function we will apply the approach developed by Koch and Tzvetkov in [8]. Note that, in [7], Kenig and Koenig improved this approach by adding the use of the nonlinear local Kato smoothing effect. However, this improvement can not be used here since this smoothing effect is not uniform in $\varepsilon$ close to the stationary point.

**Proposition 3.1.** *Let $u_{i,\varepsilon} \in C([0,T];H^1(\mathbb{R}))$, $i=1,2$, be two solutions to $(K_\varepsilon)$ with initial data $\varphi_i$. Then it holds*

$$\|\|P_{\mathfrak{C}A_\varepsilon}u_{i,\varepsilon}\|^2_{L^\infty_T H^1_x} \lesssim \|P_{\mathfrak{C}A_\varepsilon}\varphi_i\|^2_{H^1} + (1+T)\|u_{i,\varepsilon}\|^2_{Y_{\varepsilon,T}}\Big(\|u_{i,\varepsilon}\|_{Y_{\varepsilon,T}} + \|u_{i,\varepsilon}\|^2_{Y_{\varepsilon,T}}\Big) \tag{3.1}$$

*and, setting $w = u_{1,\varepsilon} - u_{2,\varepsilon}$,*

$$\|\|P_{\mathfrak{C}A_\varepsilon}w\|^2_{L^\infty_T H^1_x} \lesssim \|P_{\mathfrak{C}A_\varepsilon}(\varphi_1-\varphi_2)\|^2_{H^1} + (1+T)\|w\|^2_{Y_{\varepsilon,T}} \sum_{i=1}^{2}\Big(\|u_{i,\varepsilon}\|_{Y_{\varepsilon,T}} + \|u_{i,\varepsilon}\|^2_{Y_{\varepsilon,T}}\Big)\,, \tag{3.2}$$

*where $Y_{\varepsilon,T}$ is defined in* (2.3) *and $\mathcal{F}_x(P_{\mathfrak{C}A_\varepsilon}\varphi) = (1-\eta_{A_\varepsilon})\mathcal{F}_x\varphi$.*

First we establish an estimate, uniform in $\varepsilon$, on the solution to the associated non homogenous linear problem.

**Lemma 3.1.** *Let $v \in C([0,T];H^\infty(\mathbb{R}))$ be a solution of*

$$v_t + v_{xxx} + \varepsilon u_{5x} = -F_x \quad . \tag{3.3}$$

*Then*

$$\|P_{\mathfrak{C}A_\varepsilon}v\|_{L^1_T L^\infty_x} \lesssim (1+T)\|P_{\mathfrak{C}A_\varepsilon}v\|_{L^\infty_T L^2_x} + \|P_{\mathfrak{C}A_\varepsilon}F\|_{L^1_T L^2_x} \quad . \tag{3.4}$$

*Proof.* For $0<\varepsilon<<1$ fixed, we write a natural splitting

$$[0,T] = \cup I_j$$

of $[0,T]$ where $I_j=[a_j,b_j]$ are with disjoint interiors and $|I_j| \le \varepsilon^{1/2}$. Clearly, we can suppose that the number of the intervals $I_j$ is bounded by $C(1+T)\varepsilon^{-1/2}$. Using the Hölder inequality in time, we can write

$$\|v\|_{L^1_T L^\infty_x} \lesssim \sum_j \|v\|_{L^1_{I_j} L^\infty_x} \lesssim \varepsilon^{\frac{3}{8}} \sum_j \|v\|_{L^4_{I_j} L^\infty_x}\,.$$

Next, we apply the Duhamel formula on each $I_j$ to obtain

$$P_{\mathfrak{C}A_\varepsilon}v(t) = U_\varepsilon(t-a_j)P_{\mathfrak{C}A_\varepsilon}v(a_j) - \int_{a_j}^{t} U_\varepsilon(t-t')P_{\mathfrak{C}A_\varepsilon}\partial_x F(t')\,dt'\,.$$

Using the uniform in $\varepsilon$ Strichartz estimate (2.5) and classical $TT^*$ arguments, it yields

$$\begin{aligned}\|P_{\mathfrak{C}A_\varepsilon}v\|_{L^4_{I_j}L^\infty_x} &\lesssim \|D_x^{-1/4}P_{\mathfrak{C}A_\varepsilon}v(a_j)\|_{L^2} + \|D_x^{3/4}P_{\mathfrak{C}A_\varepsilon}F\|_{L^1_{I_j}L^2_x}\\ &\lesssim \varepsilon^{1/8}\|P_{\mathfrak{C}A_\varepsilon}v(a_j)\|_{L^2} + \varepsilon^{-3/8}\|P_{\mathfrak{C}A_\varepsilon}F\|_{L^1_{I_j}L^2_x}\,.\end{aligned}$$

Therefore, we get

$$\|P_{\mathfrak{C}A_\varepsilon}v\|_{L^1_{I_j}L^\infty_x} \lesssim \varepsilon^{1/2}\|P_{\mathfrak{C}A_\varepsilon}v(a_j)\|_{L^2} + \|P_{\mathfrak{C}A_\varepsilon}F\|_{L^1_{I_j}L^2_x}$$

and summing over $j$,

$$\begin{aligned}\|P_{\mathfrak{C}A_\varepsilon} v\|_{L^1_T L^\infty_{xy}} &\lesssim \varepsilon^{1/2}\sum_j \|P_{\mathfrak{C}A_\varepsilon} v\|_{L^\infty_T L^2_x} + \|P_{\mathfrak{C}A_\varepsilon} F\|_{L^1_T L^2_x}\,.\\ &\lesssim (1+T)\|P_{\mathfrak{C}A_\varepsilon} v\|_{L^\infty_T L^2_x} + \|P_{\mathfrak{C}A_\varepsilon} F\|_{L^1_T L^2_x}\,.\end{aligned}$$

□

We now need the following energy estimate

**Lemma 3.2.** *Let $s \geq 1$. There exists $C > 0$ such that all $0 < \varepsilon << 1$ the solution $u \in C(0,T;H^s)$ of $(K_\varepsilon)$ satisfies*

(3.5)
$$\|P_{\mathfrak{C}A_\varepsilon} u\|^2_{L^\infty_T H^s} \leq \|P_{\mathfrak{C}A_\varepsilon} u_0\|^2_{H^s} + C\left(\|P_{B_\varepsilon} u_x\|_{L^1_T L^\infty_x} + \|P_{\mathfrak{C}A_\varepsilon} u_x\|_{L^1_T L^\infty_x}\right)\|u\|^2_{L^\infty_T H^s}\,.$$

*Proof.* Applying the operator $P_{\mathfrak{C}A_\varepsilon}$ on $(K_\varepsilon)$ and taking the $H^s$-scalar product with $P_{\mathfrak{C}A_\varepsilon} u$ we get

$$\frac{d}{dt}\|P_{\mathfrak{C}A_\varepsilon} u(t)\|^2_{H^s} = \int_{\mathbb{R}} J^s_x P_{\mathfrak{C}A_\varepsilon}\partial_x(u^2) J^s_x P_{\mathfrak{C}A_\varepsilon} u\,.$$

Decomposing $u$ as $u = P_{B_\varepsilon} u + P_{\mathfrak{C}B_\varepsilon} u$ we can rewrite the right-hand side member of the above equality as

$$\int_{\mathbb{R}} J^s_x P_{\mathfrak{C}A_\varepsilon}\partial_x(P_{B_\varepsilon} u)^2 J^s_x P_{\mathfrak{C}A_\varepsilon} u + \int_{\mathbb{R}} J^s_x P_{\mathfrak{C}A_\varepsilon}\partial_x\Big((P_{\mathfrak{C}B_\varepsilon} u)^2 + 2P_{B_\varepsilon} u P_{\mathfrak{C}B_\varepsilon} u\Big) J^s_x P_{\mathfrak{C}A_\varepsilon} u := I_1 + I_2\,.$$

By integration by parts and Kato-Ponce commutator estimates we easily estimate the first term by

$$\begin{aligned} I_1 &= 2\int_{\mathbb{R}} P_{B_\varepsilon} u\,\partial_x\Big(J^s_x P_{\mathfrak{C}A_\varepsilon} P_{B_\varepsilon} u\Big) J^s_x P_{\mathfrak{C}A_\varepsilon} u + 2\int_{\mathbb{R}}\Big[J^s_x P_{\mathfrak{C}A_\varepsilon}, P_{B_\varepsilon} u\Big] P_{B_\varepsilon} u_x\, J^s_x P_{\mathfrak{C}A_\varepsilon} u\\ &\lesssim \|P_{B_\varepsilon} u_x\|_{L^\infty_x}\|u\|^2_{H^s}\,.\end{aligned}$$

For the second term, we notice that by the frequency projections, all the functions in the integral are supported in frequencies of order $1/\sqrt{\varepsilon}$. Therefore, using Bernstein inequalities and the fact that $H^s(\mathbb{R})$, $s \geq 1$, is an algebra, we get

$$\begin{aligned} I_2 &\lesssim \varepsilon^{-s-1/2}\Big\|\int_{\mathbb{R}} \chi_{\{|\xi_1|\sim\frac{1}{\sqrt{\varepsilon}}\}}|\widehat{u}(\xi_1)|\chi_{\{|\xi-\xi_1|\sim\frac{1}{\sqrt{\varepsilon}}\}}|\widehat{u}(\xi-\xi_1)|\,d\xi_1\Big\|_{L^2_\xi}\|P_{\mathfrak{C}A_\varepsilon} u\|_{L^\infty_x}\\ &\lesssim \|P_{\mathfrak{C}A_\varepsilon} u_x\|_{L^\infty_x}\|u\|^2_{H^s}\,.\end{aligned}$$

(3.5) then follows by integration in time. □

**Proof of Proposition 3.1** Applying (3.4) to $u_x$ with $u$ solving $(K_\varepsilon)$ we get

$$\begin{aligned}\|P_{\mathfrak{C}A_\varepsilon} u_x\|_{L^1_T L^\infty_x} &\lesssim (1+T)\|P_{\mathfrak{C}A_\varepsilon} u_x\|_{L^\infty_T L^2_x} + \|P_{\mathfrak{C}A_\varepsilon}\partial_x(u^2)\|_{L^1_T L^2_x}\\ &\lesssim (1+T)\|u\|_{L^\infty_T H^1_x} + T\|u\|^2_{L^\infty_T H^1_x}\,. \qquad (3.6)\end{aligned}$$

Therefore, gathering (3.5), (3.6) and (2.5) we obtain

$$\begin{aligned}\|P_{\mathfrak{C}A_\varepsilon} u\|^2_{L^\infty_T H^1_x} &\lesssim \|P_{\mathfrak{C}A_\varepsilon} u_0\|^2_{H^1} + C\,\|u\|^2_{L^\infty_T H^1_x}\Big(T^{1/4}\|P_{B_\varepsilon} P_{A_\varepsilon} u_x\|_{L^4_T L^\infty_x} + \|P_{\mathfrak{C}A_\varepsilon} u_x\|_{L^1_T L^\infty_x}\Big)\\ &\lesssim \|P_{\mathfrak{C}A_\varepsilon} u_0\|^2_{H^1} + C\,(1+T)\|u\|^2_{L^\infty_T H^1_x}\Big(\|u\|_{Y_{\varepsilon,T}} + \|u\|^2_{Y_{\varepsilon,T}}\Big)\,,\end{aligned}$$

which completes the proof of (3.1). Finally (3.2) follows in the same way by writing the equation for the difference of two solutions. □

## 4. Proofs of Theorem 1.1 and 1.2

4.1. **Proof of Theorem 1.2.** We can restrict ourselves to the case $\varepsilon \in ]0, \varepsilon_0[$ with $0 < \varepsilon_0 << 1$ since the result for $\varepsilon \in [\varepsilon_0, 1]$ follows directly from the local well-posedness of the Kawahara equation with $\varepsilon = 1$ (see for instance [5]). Also to simplify the expository, we only consider the worst case that is $s = 1$. We first treat the case of small initial data. Let $u \in C^\infty(\mathbb{R}; H^\infty(\mathbb{R}))$ be a solution of $(K_\varepsilon)$. Combining Propositions 2.1 and 3.1 we infer that

$$\|u\|^2_{Y_{\varepsilon,T}} \lesssim \|\varphi\|^2_{L^2} + \|u\|^2_{Y_{\varepsilon,T}} \Big( \|u\|_{Y_{\varepsilon,T}} + \|u\|^2_{Y_{\varepsilon,T}} \Big) .$$

Since $u$ is smooth, $t \mapsto \|u\|_{Y_{\varepsilon,T}}$ is continuous and $\|u\|_{Y_{\varepsilon,T}} \to \|\varphi\|_{H^1}$ as $T \searrow 0$. Therefore a classical continuity argument ensures that there exists $\alpha_0 > 0$ such that $\|u\|_{Y_{\varepsilon,1}} \lesssim \|\varphi\|_{H^1}$ provided

$$\|\varphi\|_{H^1} \leq \alpha_0 . \tag{4.1}$$

By continuity with respect to initial data (for any fixed $\varepsilon > 0$) it follows that for any solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ of $(K_\varepsilon)$ with initial data $\varphi$ satisfying (4.1), it holds

$$\|u\|_{Y_{\varepsilon,1}} \lesssim \|\varphi\|_{H^1} . \tag{4.2}$$

Moreover, (2.2) together with (3.2) ensure that for any couple of solutions $u_i \in C(\mathbb{R}; H^1(\mathbb{R}))$, $i = 1, 2$, of $(K_\varepsilon)$ with initial data $\varphi_i$ satisfying (4.1), it holds

$$\|u_1 - u_2\|_{Y_{\varepsilon,1}} \lesssim \|\varphi_1 - \varphi_2\|_{H^1} . \tag{4.3}$$

Now the case of general initial data follows by a classical dilation argument. Indeed, it is easy to check that $u$ is a solution of $(K_\varepsilon)$ with initial data $\varphi$ if and only if $v_\lambda = v_\lambda(t, x) = \lambda^{-2} u(\lambda^{-3} t, \lambda^{-1} x)$ is a solution of $(K_{\lambda^{-2}\varepsilon})$ with initial data $\varphi_\lambda = \lambda^{-2} \varphi(\lambda^{-1} x)$. Fixing $\varphi \in H^1(\mathbb{R})$ and noticing that $\|\varphi_\lambda\|_{H^1} \lesssim \lambda^{-3/2} \|\varphi\|_{H^1}$ we deduce $\varphi_\lambda$ satisfies (4.1) as soon as $\lambda \geq (\|\varphi\|_{H^1}/\alpha_0)^{2/3}$. Taking

$$\lambda = \min(1, (\|\varphi\|_{H^1}/\alpha_0)^{2/3}) ,$$

this ensures that (4.2) holds for the solution $v_\lambda$ of $(K_{\lambda^{-2}\varepsilon})$ emanating from $\varphi_\lambda$. Coming back to $u$ we deduce that $u$ satisfies (2.8) with $T \sim \min(1, (\frac{\|\varphi\|_{H^1}}{\alpha_0})^{-4/3})$. Finally, (1.5) follows from (4.3) by similar arguments.

4.2. **Proof of Theorem 1.1.** We follow general arguments (see for instance [4]). Let us denote by $S_{K_\varepsilon}$ and $S_{KdV}$ the nonlinear group associated with respectively $(K_\varepsilon)$ and KdV. Let $\varphi \in H^s(\mathbb{R})$, $s \geq 1$ and let $T = T(\|\varphi\|_{H^1}) > 0$ be given by Theorem 1.1. For any $N > 0$ we can rewrite $S_{K_\varepsilon}(\varphi) - S_{KdV}(\varphi)$ as

$$\begin{aligned} S_{K_\varepsilon}(\varphi) - S_{KdV}(\varphi) &= \Big( S_{K_\varepsilon}(\varphi) - S_{K_\varepsilon}(P_{\leq N}\varphi) \Big) + \Big( S_{K_\varepsilon}(P_{\leq N}\varphi) - S_{KdV}(P_{\leq N}\varphi) \Big) \\ &\quad + \Big( S_{KdV}(P_{\leq N}\varphi) - S_{KdV}(\varphi) \Big) = I_{\varepsilon,N} + J_{\varepsilon,N} + K_N . \end{aligned}$$

By continuity with respect to initial data in $H^s(\mathbb{R})$ of the solution map associated with the KdV equation, we have $\lim_{N\to\infty} \|K_N\|_{L^\infty(0,T;H^s)} = 0$. On the other hand, (1.5) ensures that

$$\lim_{N\to\infty} \sup_{\varepsilon\in]0,1[} \|I_{\varepsilon,N}\|_{L^\infty(0,T;H^s)} = 0 .$$

It thus remains to check that for any fixed $N > 0$, $\lim_{\varepsilon\to 0} \|J_{\varepsilon,N}\|_{L^\infty(0,T;H^s)} = 0$. Since $P_{\leq N}\varphi \in H^\infty(\mathbb{R})$, it is worth noticing that $S_{K_\varepsilon}(P_{\leq N}\varphi)$ and $S_{KdV}(P_{\leq N}\varphi)$ belong

to $C^\infty(\mathbb{R};H^\infty(\mathbb{R}))$. Setting $v_\varepsilon := S_{K_\varepsilon}(P_{\leq N}\varphi)$ and $v := S_{KdV}(P_{\leq N}\varphi)$, we observe that $w := v_\varepsilon - v$ satisfies

$$w_t + w_{xxx} + \varepsilon w_{5x} = -\frac{1}{2}\partial_x\Big(w(v_\varepsilon + v)\Big) - \varepsilon v_{5x}$$

with initial data $w(0) = 0$. Therefore, proceeding as in the proof of Theorem 1.2 we eventually obtain

$$\|w\|_{L^\infty(0,T;H^s)} \lesssim \varepsilon \|v_{5x}\|_{L^\infty_T L^2} \lesssim \varepsilon N^5 \|v\|_{L^\infty_T L^2} \lesssim \varepsilon N^5 \|\varphi\|_{L^2} \, .$$

This proves that $\lim_{\varepsilon\to 0} \|w\|_{L^\infty(0,T;H^s)} = 0$ and completes the proof of Theorem 1.1 with $T = T(\|\varphi\|_{H^1})$. Finally, recalling that the energy conservation of the KdV equation ensures that for any $\varphi \in H^1(\mathbb{R})$ it holds,

$$\sup_{t\in\mathbb{R}} \|S_{KdV}(\varphi)(t)\|_{H^1} \lesssim \|\varphi\|_{H^1} + \|\varphi\|^5_{L^2} \, ,$$

we obtain the same convergence result on any time interval $[0, T_0]$ with $T_0 > T(\|\varphi\|_{H^1})$ by reiterating the convergence result on $[0, T(\|\varphi\|_{H^1})]$ about $T_0/T(\|\varphi\|_{H^1})$ times.

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