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Rooted maps on orientable surfaces, Riccati's equation and continued fractions

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Abstract

We present a new approach in the study of rooted maps without regard to genus. We prove the existence of a new type of equation for the generating series of these maps enumerated with respect to edges and vertices. This is Riccati's equation. It seems to be the first time that such a differential equation appears in the enumeration of rooted maps. Solving this equation leads to different closed forms of the studied generating series. The most interesting consequence is a development of this generating function in a very nice continued fraction leading to a new equation generalizing the well known Dyck equation for rooted planar trees. In a second part, we also obtain a differential equation for the generating series of rooted trees regardless of the genus, with respect to edges. This also leads to a continued fraction for the generating series of rooted genus independent trees and to an unexpected relation between both previous generating series of trees and rooted maps.

Keywords
rooted map, Dyck's equation, Riccati's equation, continued fraction

1. Introduction

Recent history of maps began with W. T. Tutte in 1962. Three principal approaches characterize this combinatorial domain:

1. The bijective approach was the first used (see [17]). Its principle is to construct a one-to-one correspondence between the family of the studied maps and another family of objects for which are known efficient enumeration methods. This approach was developed by R. Cori, B. Vauquelin and D. Arquès (cf. [6, 7, 14]).
2. The algebraic approach is more recent and was developed by D. M. Jackson and T. I. Visentin in [16]. A rooted map can be seen as a pair of permutations acting transitively on the set of half-edges. This approach applies algebraic combinatorics methods of the symmetric group to these permutations to derive a closed form for the exponential generating series of maps regardless of genus. One of the most interesting results in [16] is the formula:

$$\mathrm{Me}(y,z) = 2z\frac{\partial}{\partial z}\log\sum_{n\geq 0}\frac{1}{2^n n!}y(y+1)\ldots(y+2n-1)z^n$$

for the generating function Me(y,z) enumerating orientable rooted maps regardless of genus with respect to vertices (y) and edges (z). While theoretically possible to go from these equations to those obtained topologically (see 3.), it seems difficult to do so.

3. The topological approach consists in applying to rooted maps a topological operation interpreted in terms of a functional equation for the generating series of the studied maps. Many articles based on this kind of approach can be found in the literature [1-5, 8-12, 18, 20].

Using a topological approach, W. Tutte obtained by contracting the root edge, the first equation (and an explicit enumeration, [18]) for the generating series $M_0(v,z)$ of rooted planar maps with respect to edges (z) and degree of the root vertex (v):

$$M_0(v,z) = 1 + v^2 z M_0(v,z)^2 + vz\frac{vM_0(v,z) - M_0(1,z)}{v-1}$$

This equation can be generalized to any genus g and solved (cf. [3, 10]). It has also been generalized for the maps on locally orientable surfaces [1, 9, 11]. Unfortunately, this set of equations for the generating series $M_g(v,z)$ ($g \geq 0$) contains new unknown generating series for maps with multiple root vertices and cannot be gathered to derive a unique equation for the generating series M(v,z) of genus independent rooted maps.

The aim of this paper is to obtain such an equation. In the second section of this paper we quickly recall the basic notions of map theory. In Section 3 we derive a new functional equation for the generating series of rooted map without regard to genus, with respect to edges and vertices. This equation is obtained by a new interpretation of the classical operation consisting in removing the root edge of the map. We also obtain a functional equation for the generating series of rooted trees, with respect to edges. In Section 4 we enumerate rooted maps and rooted trees. We express the series as continued fractions, that leads to two unexpected relations, among them a generalization for genus independent maps of the well known Dyck equation for rooted planar trees.

Remark 1. Continued fractions never appeared in the literature to express the generating series of maps. However, in [6, 7], using the bijective approach, D. Arquès introduced multi-continued fractions to express the generating series of rooted planar maps and hypermaps.

2. Definitions

For the convenience of the reader we recall quickly some definitions used in the following (for more details about combinatorial maps refer for example to [13]).

2.1 Topological map

A *topological map* C on an orientable surface Σ of $\mathbb{R}^3$ is a partition of Σ in three finite sets of *cells*:

i. The set of the vertices of C, that is a finite set of points ;
ii. The set of the edges of C, that is a finite set of simple open Jordan arcs, disjoint in pairs, whose extremities are vertices ;
iii. The set of the faces of C. Each face is homeomorphic to an open disc, and its border is a union of vertices and edges.

The *genus of the map C* is the genus of the surface Σ.
A cell is called *incident* to another cell if one of them is in the border of the other. An *isthmus* is an edge incident on both sides to the same face.

2.2 Combinatorial map

We call an oriented edge of the map a *half-edge*. We denote by B the set of all half-edges of the map. To each half-edge, is associated in an evident way its initial vertex, its final vertex, and the underlying edge.

α (respectively σ) is the permutation on B associating each half-edge h with its opposite half-edge (resp. the first half-edge met by turning around the initial vertex of h in the positive way chosen on the orientable surface). α is a fixed point free involution. The cycles of α (resp. σ) represent the edges (resp. vertices) of the map.

$\overline{\sigma}$ is the permutation $\sigma o \alpha$ on B. The cycles of $\overline{\sigma}$ represent the oriented borders of the faces of the map.

In the following, a vertex (resp. edge, face) will be, depending on the context, either the topological object defined at 2.1, or the cycle of σ (resp. α, $\overline{\sigma}$) according to the previous definitions.

The triplet (B, σ, α) is called the combinatorial definition of the associated topological map C.

2.3 Rooted map

A map is called a *rooted map* if a half-edge $\tilde{h}$ is chosen. The half-edge $\tilde{h}$ is called the *root half-edge* of the map, and its initial vertex is called the *root vertex* of the map.

We call *external face* (or root face), the face $\overline{\sigma}^*(\tilde{h})$ generated by the root half-edge $\tilde{h}$. The planar map with only a vertex and no edge is also regarded as rooted, even though it contains no half-edge.

Two rooted maps with the same genus are isomorphic if there exists a homeomorphism of the associated surface, preserving its orientation, mapping the vertices, edges, faces and the root half-edge of the first map respectively on those of the second one.

An isomorphic class of rooted maps of genus g will simply be called a *rooted map*. Our goal is to enumerate these equivalence classes of rooted maps independently of their genus.

3. A differential equation for the series of rooted maps

We present here the first topological equation for the generating series of orientable rooted maps regardless of the genus, with respect to vertices and edges.

3.1 An equation for maps

We denote by M(y,z) the generating series of orientable rooted maps of any genus (we simply call them rooted maps in the following), where the exponent of y (resp. z) refers to the number of vertices (resp. edges) in the map.

Theorem 1. *The generating series M(y,z) of rooted maps is the solution of the following Riccati's differential equation:*

$$M(y,z) = y + zM(y,z)^2 + zM(y,z) + 2z^2 \frac{\partial}{\partial z}\left[M(y,z)\right]. \tag{1}$$

Proof. Let C denote a rooted map with root half-edge $\tilde{h}$. The proof is based on the topological operation of deleting the root half-edge $\tilde{h}$ as introduced by W. Tutte [18] in the study of planar maps. Four cases are possible. The first two terms (in the right part of the equation) look like the first two terms of Tutte's equation recalled in the introduction, and are obtained in the same way (but generalized to maps enumerated without regard to genus). The last two terms are of a new type.

First case. If C is the rooted planar map reduced to a vertex, the contribution in Eq. (1) is $y^1z^0=y$.

Second case. If the edge supporting $\tilde{h}$ is an isthmus whose deletion disconnects the map into two maps C_1 and C_2, the contribution of this case in Eq. (1) is

$$zM(y,z)^2 .$$

The contribution of the first map C_1 is $M(y_1,z_1)$. The contribution of C_2 is $M(y_2,z_2)$. During the reconstruction of the map C, all the edges add up ($z_1=z_2=z$), and we do not forget the added root isthmus (multiplication by z). The vertices add up too ($y_1=y_2=y$).

Third case. If $\tilde{h}$ is a planar loop and is the oriented border of the root face, we remove the loop and root the resulting map C_1 with $\sigma(\tilde{h})$ (Fig. 1). C_1 is a general rooted map to which we add a loop during the inverse step of reconstruction. So the contribution of this case in (1) is

$$zM(y,z)$$

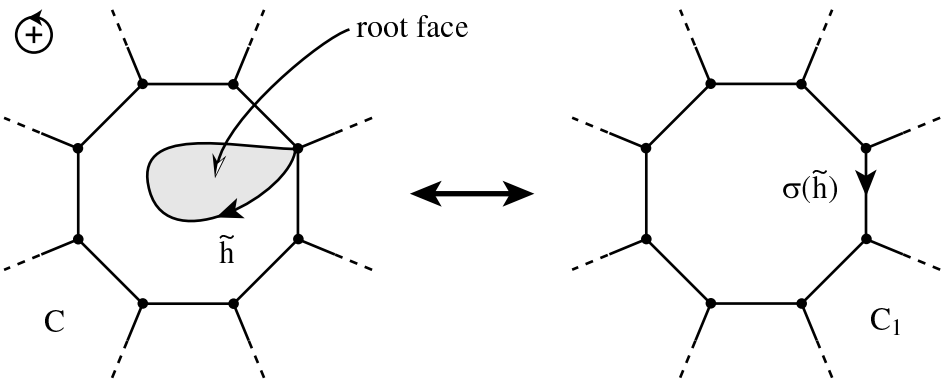

Fig. 1: The root half-edge $\tilde{h}$ is a planar loop, and the root face is "inside" the loop.

Fourth case. Here we group together all the cases which have not been studied above. Two sub-cases may be encountered:

Subcase 1. The root half-edge $\tilde{h}$ is not an isthmus and is not a planar loop border of the root face (Fig. 2) ;

Subcase 2. The root half-edge $\tilde{h}$ is an isthmus whose removal does not disconnect the map C (Fig. 3).

a) Removal of the root half-edge $\tilde{h}$.

We remove the root half-edge $\tilde{h}$. It gives the map C_1 (cf. points 1 and 2). We root C_1 with both root half-edges $h_1 = \sigma_{C_1}(\tilde{h})$ (where $\sigma_{C_1}(\tilde{h})$ is the first half-edge in C_1 among $\sigma(\tilde{h})$ and

$\sigma^2(\tilde{h})$) and $h_2 = \bar{\sigma}(\tilde{h})$ to be able to reconstruct C from C_1. We obtain a map C_1 with two root half-edges, possibly equal.

b) Addition of a root half-edge in a double rooted map: the inverse operation.
Let C_1 be a map with two root half-edges h_1 and h_2 (with initial vertices v_1 and v_2 respectively). We reconstruct the rooted map C from C_1 by adding an edge from v_1 to v_2, in the angular sectors a_1 and a_2 defined by h_1 and h_2 (more precisely for k=1,2 we consider the sector between $\sigma^{-1}(h_k)$ and h_k) and we root the half-edge oriented from v_1 to v_2.
If the angular sectors a_1 and a_2 are in the same face of C_1, the root half-edge $\tilde{h}$ of C is embedded in this face, splitting it in two new faces of C: $\tilde{h}$ is not an isthmus (Fig. 2). It could be a planar loop, if $a_1=a_2$, but in this case the root face is "outside" the loop. If a_1 and a_2 are not in the same face of C_1, the root half-edge $\tilde{h}$ of C is embedded on an added handle that collects both faces generated by h_1 and h_2 into one face: $\tilde{h}$ is an isthmus of C whose removal does not disconnect the map (Fig. 3).

To summarize we must choose a half-edge anywhere (eventually equal to the first root half-edge) in any map. To convey this operation on the generating series, we first choose an edge by applying the operator z.∂/∂z to the series M(y,z). There are two possible half-edges associated to the chosen underlying edge, so we multiply by 2, and the contribution of the added edge is z. Thus the contribution of this case in (1) is

$$2z^2 \frac{\partial}{\partial z}\left[M(y,z)\right]$$

This concludes the proof of Theorem 1.

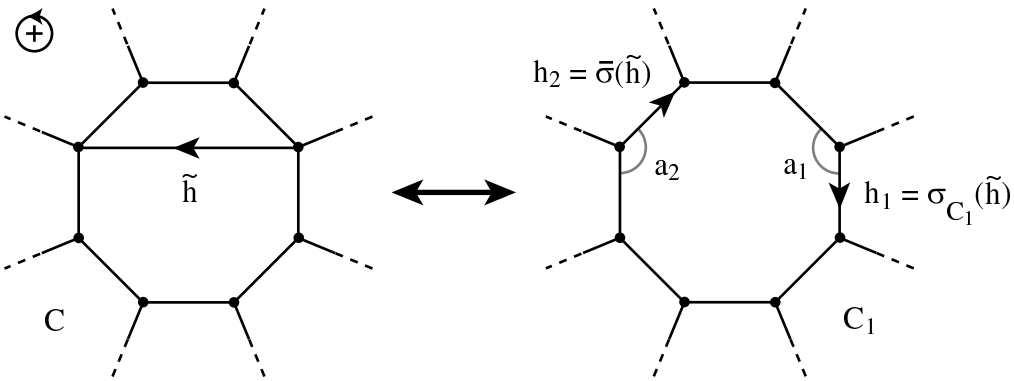

Fig. 2: (Subcase 1) The second root half-edge h_2 of C_1 is in the face generated by h_1: the root half-edge $\tilde{h}$ of the map C is not an isthmus.

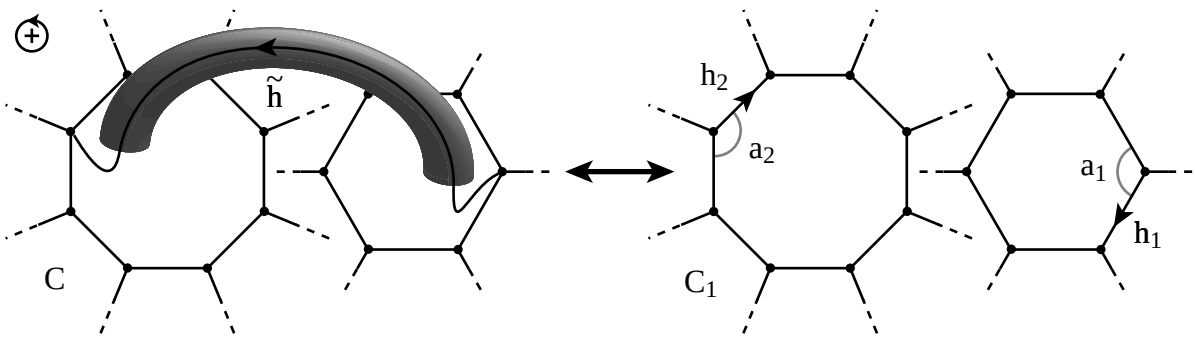

Fig. 3: (Subcase 2) The second root half-edge h_2 of C_1 is not in the face generated by h_1: the root half-edge $\tilde{h}$ of the map C is an isthmus whose removal does not disconnect the map.

Corollary 1. *The generating series M(z) of rooted maps with respect to the number of edges is solution of the differential equation:*

$$M(z) = 1 + zM(z)^2 + zM(z) + 2z^2 \frac{\partial}{\partial z}\left[M(z)\right]. \tag{2}$$

Proof. Immediate from (1) by substituting y by 1.

3.2 An equation for trees

Let us recall that a *tree* (of any genus) is a map with only one face. By duality it exists a one-to-one correspondence between rooted trees and rooted monopoles (rooted maps with only one vertex). We denote by T(z) the generating series of rooted trees (or rooted monopoles), where the exponent of z refers to the number of edges in the tree.

Theorem 2. *The generating series T(z) of rooted trees—or rooted monopoles—is solution of the following differential equation:*

$$T(z) = 1 + zT(z) + 2z^2 \frac{\partial}{\partial z}\left[T(z)\right]. \tag{3}$$

Proof. The previous proof of Theorem 1 applies to monopoles. Let C denote a rooted monopole with root half-edge $\tilde{h}$. It is easy to see that we can apply the topological operation of removal of the root half-edge in the same way as for Eq. (1). Only the second case, with a deconnecting root isthmus, is impossible here.

Remark 2. Th. 2 can also be obtained by noting that the generating function for rooted trees is the coefficient of y^1 in M(y,z), i.e.

$$T(z) = \left[\frac{M(y,z)}{y}\right]_{|y=0}.$$

4. Enumeration of rooted maps and trees

In this section we solve Eq. (1) and (3) to obtain continued fraction forms of the generating series of rooted maps and trees regardless of genus (Th. 3 and 4). Then we give explicit formulae for the enumeration of the number of rooted maps and trees with n edges (Cor. 3).

4.1 A continued fraction for rooted maps

Equation (1) is a Riccati differential equation. We present in Theorem 3 an iterative solution of Equation (1), which leads to a very nice continued fraction form of the generating series of rooted maps. Well known analytical methods may also solve such equations. In Remark 3, we recall the closed forms of such solutions.

Theorem 3. *The generating series M(y,z) of rooted maps with respect to the number of vertices and edges is the continued fraction:*

$$M(y,z) = \cfrac{y}{1-\cfrac{(y+1)z}{1-\cfrac{(y+2)z}{1-\cfrac{(y+3)z}{1-\dots}}}}. \tag{4}$$

Proof. The proof is by recurrence.
Let us first define the sequence $(M_k(y,z))_{k\geq 0}$ of series by the set of equations:
1) $M_0(y,z)$ is the desired generating function M(y,z) of rooted maps ;
2) For any integer k in $\mathbb{N}$, $M_{k+1}(y,z)$ is obtained from $M_k(y,z)$ by the relation

$$M_k(y,z) = \frac{y+k}{1-zM_{k+1}(y,z)}. \tag{α_k}$$

Then for every k, $M_k(y,z)$ is a solution of the equation

$$M_k(y,z) = (y+k) + zM_k(y,z)^2 + zM_k(y,z) + 2z^2\frac{\partial}{\partial z}\left[M_k(y,z)\right]. \tag{β_k}$$

This result is proved by recurrence:

- For k=0, this is Theorem 1 and Equation (1).
- Let k be a positive integer and let us suppose that $M_k(y,z)$ is a solution of (β_k):

  $$M_k(y,z) = (y+k) + zM_k(y,z)^2 + zM_k(y,z) + 2z^2\frac{\partial}{\partial z}\left[M_k(y,z)\right].$$

  Now we substitute $M_k(y,z)$ by its expression with respect to $M_{k+1}(y,z)$ from (α_k). After little algebra one obtains the Equation (β_{k+1}).

The result is then proved by recurrence and the interpretation of this set of equations $(\alpha_k)_{k\geq 0}$ gives the continued fraction form of M(y,z). In fact,

$$M(y,z) = M_0(y,z) = \frac{y}{1-zM_1(y,z)} = \cfrac{y}{1-z\cfrac{y+1}{1-zM_2(y,z)}} = \dots$$

When iterating the process, one obtains:

$$M(y,z)=\cfrac{y}{1-\cfrac{(y+1)z}{1-\cfrac{(y+2)z}{1-\cfrac{(y+3)z}{1-\ldots}}}}. \tag{5}$$

This concludes the proof of Theorem 3.

Remark 3. By solving Riccati's Equation (1) with classical analytical methods, we can obtain an analytical form of the general solution:

$$M(y,z)=\frac{2z^{\frac{-1}{2}}e^{\frac{-1}{2z}}}{\varphi_1(y,z)^2}\frac{1}{C(y)+\int_0^z\frac{u^{\frac{-3}{2}}e^{\frac{-1}{2u}}}{\varphi_1(y,u)^2}du}+2z\frac{\frac{\partial\varphi_1(y,z)}{\partial z}}{\varphi_1(y,z)}.$$

where C is a "constant", depending on y, to be fixed, and $\varphi_1(y,z)$ is the confluent hypergeometric series $\Phi\left(\frac{y}{2},\frac{1}{2};\frac{-1}{2z}\right)$, a particular solution of the second order differential equation (see [15]):

$$\frac{\partial^2\varphi(y,z)}{\partial z^2}+\left(\frac{3}{2z}-\frac{1}{2z^2}\right)\frac{\partial\varphi(y,z)}{\partial z}+\frac{y}{4z^3}\varphi(y,z)=0.$$

Remark 4. As reminded in the introduction, the generating function for rooted maps can be written

$$M(y,z)=y+2z\frac{\partial}{\partial z}\left[\ln\left(\sum_{n\geq 0}\frac{(y)(y+1)\ldots(y+2n-1)}{2^n n!}z^n\right)\right]$$

(we add the vertex map) with an algebraic combinatorics point of view (see [16]). It can be verified that this formula is the solution of Eq. (4).

In Theorem 3 a new relation on maps appears:

Corollary 2. *The generating series M(y,z) of rooted maps with respect to vertices and edges is the solution of the following generalized Dyck equation:*

$$M(y,z)=y+zM(y,z)M(y+1,z). \tag{6}$$

Proof. Straightforward since we note that the continued fraction can be rewritten as

$$M(y,z)=\frac{y}{1-zM(y+1,z)}. \tag{7}$$

4.2 A continued fraction for rooted trees

We now give some similar results for rooted trees.

Theorem 4.

(1) *Both generating series of rooted trees and rooted maps are linked by the relation:*

$$T(z) = \frac{1}{1 - zM(z)}. \tag{8}$$

(2) *The generating series T(z) of rooted trees with respect to edges is the continued fraction:*

$$T(z) = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{2z}{1 - \cfrac{3z}{1 - \cfrac{4z}{1 - \ldots}}}}}. \tag{9}$$

Proof.

(1) From Formula (7) and Remark 2 we obtain

$$T(z) = \left[\frac{1}{1 - zM(y+1, z)}\right]_{|y=0} = \frac{1}{1 - zM(z)}.$$

(2) By Theorem 3, applied with y=1, we deduce

$$T(z) = \frac{1}{1 - zM(z)} = \cfrac{1}{1 - z\cfrac{1}{1 - \cfrac{2z}{1 - \cfrac{3z}{1 - \cfrac{4z}{1 - \ldots}}}}}.$$

This concludes the proof of Theorem 4.

Remark 5. By substituting $-z$ for z in Eq. (3) and solving the new equation with classical analytical methods (see [15]), we obtain

$$T(-z) = \frac{e^{\frac{1}{2z}}}{\sqrt{z}} \int_{\frac{1}{\sqrt{z}}}^{+\infty} e^{\frac{-t^2}{2}} dt.$$

Using well known links between gaussian integrals and continued fraction (see for example [19, eq. 92.15]), we retrieve the result of Theorem 4 after some equivalent transformations of the continued fraction.

4.3 Explicit enumeration formulae

From previous expressions, we deduce explicit formulae enumerating rooted maps and trees with a given number of edges.

Corollary 3.

(1) *The number of rooted trees with n edges is:*

$$\frac{(2n)!}{2^n n!}. \tag{10}$$

(2) *The number of rooted maps with n edges is:*

$$\frac{1}{2^{n+1}}\sum_{i=0}^{n}(-1)^i \sum_{\substack{k_1+\ldots+k_{i+1}=n+1\\ k_1,\ldots,k_{i+1}>0}} \prod_{j=1}^{i+1}\frac{(2k_j)!}{k_j!}. \tag{11}$$

Proof.

(1) Assume that T is the series $\sum_{k\geq 0} t_k z^k$. By searching the coefficient of z^n in Eq. (3), we obtain the recurrence relation $t_n = (2n-1)t_{n-1}$. This enumeration has been previously obtained through a very different way by T. Walsh and A. Lehman in [20].

(2) From (8), we express the generating series M(z) with respect to T(z), and we obtain (11) with little algebra.

Remark 6. The number of rooted trees with n edges is equivalent to the number of permutations σ with one cycle over 2n elements (with two chosen ones for the root edge) divided by $2^{n-1}(n-1)!$. We can also note that this is the very classical enumeration formula for fixed-point-free involutions, namely the odd factorial.

Here we present enumerating tables. In the first one, we compute the first terms of the generating series of rooted maps M(y,z) with respect to vertices and edges (see Table 1, where we group together maps, vertically with the same number of vertices, and horizontally with the same number of edges). In the second table, we compute the first terms of the series of rooted trees T(z) and maps M(z) (see Table 2). Note that continued fractions (4) and (9) are used for the computation (faster than the computation from explicit formulae (10) and (11)).

z\y	1	2	3	4	5	6	7	8	9
0	1								
1	1	1							
2	3	5	2						
3	15	32	22	5					
4	105	260	234	93	14				
5	945	2589	2750	1450	386	42			
6	10395	30669	36500	22950	8178	1586	132		
7	135135	422232	546476	388136	166110	43400	6476	429	
8	2027025	6633360	9163236	7123780	3463634	1092560	220708	26333	1430

Table 1: The number of rooted maps regardless of genus, with respect to edges (z) and vertices (y).

z	T(z)	M(z)
0	1	1
1	1	2
2	3	10
3	15	74
4	105	706
5	945	8162
6	10395	110410
7	135135	1708394
8	2027025	29752066
9	34459425	576037442
10	654729075	12277827850
11	13749310575	285764591114
12	316234143225	7213364729026
13	7905853580625	196316804255522
14	213458046676875	5731249477826890
15	6190283353629375	178676789473121834
16	191898783962510625	5925085744543837186
17	6332659870762850625	208256802758892355202
18	221643095476699771875	7734158085942678174730
19	8200794532637891559375	302625067295157128042954

Table 2: The number of rooted trees T(z) and rooted maps M(z), regardless of genus, with respect to edges.

5. Conclusion

In the first part we proved the first functional equation for the generating series of rooted maps regardless of the genus, with respect to vertices and edges (Th. 1). We also derived an analogous equation for the series of rooted trees (Th. 2). On that occasion we proved that we could obtain a functional equation for the generating series of a family of rooted maps without the help of a variable whose exponent counts the valency—the length of the cycle—of the root face (or the root vertex, by duality).

In the last part, by solving these equations we presented the first expression, in terms of continued fractions, of the generating series of rooted maps regardless of genus, with respect to vertices and edges (Th. 3). We also gave a continued fraction form of the generating series of rooted trees regardless of genus, with respect to edges (Th. 4). Then we deduced the first explicit formula for the number of genus independent rooted maps with a given number of edges (Cor. 3).

We are now working on topological interpretations of the generalized Dyck Equation (6), and of the relation between trees and maps (8).

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