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MULTIPLICITY FREE SPACES WITH A ONE DIMENSIONAL QUOTIENT

HUBERT RUBENTHALER

ABSTRACT. The multiplicity free spaces with a one dimensional quotient were introduced by Thierry Levasseur in [11]. Recently, the author has shown that the algebra of differential operators on such spaces which are invariant under the semi-simple part of the group is a Smith algebra ([17]). We give here the classification of these spaces which are indecomposable, up to geometric equivalence. We also investigate whether or not these spaces are regular or of parabolic type as a prehomogeneous vector space.

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1. INTRODUCTION

A multiplicity free space is a representation of a connected reductive group G on a finite dimensional vector space V (everything is defined over \mathbb{C}) such that every irreducible representation of G appears at most once in the associated representation of G on the space $\mathbb{C}[V]$ of polynomials on V (see Section 2 for details). For a survey on multiplicity free spaces we refer to [1]. Multiplicity free spaces, which were introduced by V. Kac in [6], play now an important role in invariant theory and harmonic analysis (see for example [4], [5], [9], the references in [1], see also [8] for a more general concept). Various characterizations of multiplicity free spaces, which are summarized in Theorem 2.3.2 below, were obtained by Vinberg-Kimelfeld ([23]), Howe-Umeda ([5]), Knop ([9]). A corollary of these characterizations is that a multiplicity free space is always a prehomogeneous space (in fact even under a Borel subgroup). This is the reason why prehomogeneous vector spaces occur so often in this paper. The classification of multiplicity free spaces was achieved independently by Benson-Ratcliff ([2]) and Leahy ([10]) after partial classifications by Brion ([3]) and Kac ([6]).

In this paper we are interested in a specific family of multiplicity free spaces, the so-called multiplicity free spaces with a one dimensional quotient which were introduced by T. Levasseur in [11]. This means roughly speaking that the categorical quotient $V//G'$ has dimension one, where G' is the semi-simple part of G (see Definition 2.4.1 below). In his paper T. Levasseur proves that if (G, V) is a multiplicity free space with a one dimensional quotient, then the radial component of the (non-commutative) algebra $D(V)^{G'}$ of G' -invariant differential operators is a Smith algebra over \mathbb{C} . In a recent paper ([17]) we showed that the full algebra $D(V)^{G'}$ is a Smith algebra over its center, which is a polynomial algebra.

The purpose of this paper is to give the complete classification of all multiplicity free spaces with a one dimensional quotient, including irreducible and non irreducible representations. Our classification is obtained up to geometric equivalence, which is the natural equivalence relation among multiplicity free spaces. It is worthwhile noticing that the list of irreducibles already appears in [11]. Moreover our investigations lead in all cases (irreducibles and non irreducibles) to some extra informations like parabolicity, regularity, and explicit fundamental relative invariants of the underlying prehomogeneous vector spaces.

In section 2 we recall general facts about multiplicity free space. We give first a brief account of the theory of prehomogeneous vector spaces (2.1) and also recall the definition of parabolic type prehomogeneous spaces (2.2), including their weighted Dynkin diagrams which encode many informations (see Definition 2.2.1 and Remark 2.2.2). General definitions and results about multiplicity free spaces can be found in 2.3. and 2.4. In 2.5, as an example, we describe an important family of irreducible multiplicity free space with a one dimensional quotient, namely the irreducible regular prehomogeneous vector spaces of commutative parabolic type.

Section 3 contains the main result, the classification theorem of the multiplicity free spaces with a one dimensional quotient (Theorem 3.3). The corresponding lists (Tables 2 and 3) take place at the end of the paper.

Section 4 is devoted to the proof of Theorem 3.3. The proof uses case by case examinations from the list by Benson and Ratcliff ([2]) and some tools from the theory of prehomogeneous vector spaces.

Notations: In this paper we will denote by $GL(n)$, $SL(n)$, $SO(n)$ the general linear group, the special linear group, the special orthogonal group of complex matrices of size n respectively. As usual we will denote by $Sp(n)$ the symplectic group of $2n \times 2n$ complex matrices. We will also denote by $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{o}(n)$, $\mathfrak{sp}(n)$ the corresponding Lie algebras. The vector space of $m \times n$ complex matrices will be denoted by $M_{m,n}$, and the space of square $n \times n$ matrices will be denoted by M_n . Finally $Sym(n)$ will denote the $n \times n$ symmetric matrices and $AS(n)$ will denote the skew symmetric matrices. If n is even and if $x \in AS(n)$, then $Pf(x)$ stands for the pfaffian of the matrix x .

2. MULTIPLICITY FREE SPACES. BASIC DEFINITIONS AND PROPERTIES

2.1. Prehomogeneous Vector Spaces.

Let G be a connected algebraic group over \mathbb{C} , and let (G, ρ, V) be a rational representation of G on the (finite dimensional) vector space V . Then the triplet (G, ρ, V) is called a *prehomogeneous vector space* (abbreviated to *PV*) if the action of G on V has a Zariski open

orbit $\Omega \in V$. For the general theory of PV 's, we refer the reader to the book of Kimura [7] or to [20]. The elements in Ω are called *generic*. The PV is said to be *irreducible* if the corresponding representation is irreducible. The *singular set* S of (G, ρ, V) is defined by $S = V \setminus \Omega$. Elements in S are called *singular*. If no confusion can arise we often simply denote the PV by (G, V) . We will also write $g.x$ instead of $\rho(g)x$, for $g \in G$ and $x \in V$. It is easy to see that the condition for a rational representation (G, ρ, V) to be a PV is in fact an infinitesimal condition. More precisely let \mathfrak{g} be the Lie algebra of G and let $d\rho$ be the derived representation of ρ . Then (G, ρ, V) is a PV if and only if there exists $v \in V$ such that the map:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & V \\ X & \longmapsto & d\rho(X)v \end{array}$$

is surjective (we will often write $X.v$ instead of $d\rho(X)v$). Therefore we will call (\mathfrak{g}, V) a PV if the preceding condition is satisfied.

Let (G, V) be a PV . A rational function f on V is called a *relative invariant* of (G, V) if there exists a rational character χ of G such that $f(g.x) = \chi(g)f(x)$ for $g \in G$ and $x \in V$. From the existence of an open orbit it is easy to see that a character χ which is trivial on the isotropy subgroup of an element $x \in \Omega$ determines a unique relative invariant P_χ . Let S_1, S_2, \dots, S_k denote the irreducible components of codimension one of the singular set S . Then there exist irreducible polynomials P_1, P_2, \dots, P_k such that $S_i = \{x \in V \mid P_i(x) = 0\}$. The P_i 's are unique up to nonzero constants. It can be proved that the P_i 's are relative invariants of (G, V) and any nonzero relative invariant f can be written in a unique way $f = cP_1^{n_1}P_2^{n_2} \dots P_k^{n_k}$, where $n_i \in \mathbb{Z}$ and $c \in \mathbb{C}^*$. The polynomials P_1, P_2, \dots, P_k are called the *fundamental relative invariants* of (G, V) . Moreover if the representation (G, V) is irreducible then there exists at most one irreducible polynomial (up to multiplication by a non zero constant) which is relatively invariant.

The prehomogeneous vector space (G, V) is called *regular* if there exists a relative invariant polynomial P whose Hessian $H_P(x)$ is nonzero on Ω . If G is reductive, then (G, V) is regular if and only if the singular set S is a hypersurface, or if and only if the isotropy subgroup of a generic point is reductive. If the PV (G, V) is regular, then the contragredient representation (G, V^*) is again a PV . Regular PV 's are of particular interest, due to the zeta functions that one can associate to their real forms ([21]).

Remark 2.1.1. Let us mention a well known Lemma from the Theory of PV 's, which will be used in section 4. If (G, V) is a PV , and if X_0 is a generic point, then the characters arising as characters of relative invariants are the characters of the quotient group G/H where H is the normal subgroup of G generated by the derived group $[G, G]$ and the generic isotropy subgroup G_{X_0} . This group does not depend on X_0 . For details, see [7], Proposition 2.12. p.28.

2.2. PV 's of parabolic type.

A PV (G, V) is called reductive if the group G is reductive. Among the reductive PV 's there is a family of particular interest, the so-called PV 's of parabolic type. Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} . Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} , and let Σ be the root system of $(\mathfrak{g}, \mathfrak{h})$. We fix once and for all a system of simple roots Ψ for Σ . We denote by Σ^+ (resp. Σ^-) the corresponding set of positive (resp. negative) roots in Σ . Let θ be a subset of Ψ and let us make the standard construction of the parabolic subalgebra $\mathfrak{p}_\theta \subset \mathfrak{g}$ associated to θ . As usual

we denote by $\langle \theta \rangle$ the set of all roots which are linear combinations of elements in θ , and put $\langle \theta \rangle^\pm = \langle \theta \rangle \cap \Sigma^\pm$.

Set

$$\mathfrak{h}_\theta = \theta^\perp = \{X \in \mathfrak{h} \mid \alpha(X) = 0 \ \forall \alpha \in \theta\}, \quad \mathfrak{l}_\theta = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_\theta) = \mathfrak{h} \oplus \sum_{\alpha \in \langle \theta \rangle} \mathfrak{g}^\alpha, \quad \mathfrak{n}_\theta^\pm = \sum_{\alpha \in \Sigma^\pm \setminus \langle \theta \rangle^\pm} \mathfrak{g}^\alpha$$

Then $\mathfrak{p}_\theta = \mathfrak{l}_\theta \oplus \mathfrak{n}_\theta^+$ is the standard parabolic subalgebra associated to θ . There is also a standard \mathbb{Z} -grading of \mathfrak{g} related to these data. Define H_θ to be the unique element of \mathfrak{h}_θ satisfying the linear equations

$$\alpha(H_\theta) = 0 \quad \forall \alpha \in \theta \quad \text{and} \quad \alpha(H_\theta) = 2 \quad \forall \alpha \in \Psi \setminus \theta.$$

The above mentioned grading is just the grading obtained from the eigenspace decomposition of $\text{ad } H_\theta$:

$$d_p(\theta) = \{X \in \mathfrak{g} \mid [H_\theta, X] = 2pX\}.$$

Then we obtain easily:

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} d_p(\theta), \quad \mathfrak{l}_\theta = d_0(\theta), \quad \mathfrak{n}_\theta^+ = \sum_{p \geq 1} d_p(\theta), \quad \mathfrak{n}_\theta^- = \sum_{p \leq -1} d_p(\theta).$$

It is known (using a result of Vinberg [22]) that $(\mathfrak{l}_\theta, d_1(\theta))$ is a prehomogeneous vector space. In fact all the spaces $(\mathfrak{l}_\theta, d_p(\theta))$ with $p \neq 0$ are prehomogeneous, but there is no loss of generality if we only consider $(\mathfrak{l}_\theta, d_1(\theta))$. These spaces have been called prehomogeneous vector spaces of parabolic type ([13]). There are in general neither irreducible nor regular. But they are of particular interest, because in the parabolic context, the group (or more precisely its Lie algebra \mathfrak{l}_θ) and the space (here $d_1(\theta)$) of the PV are embedded into a rich structure, namely the simple Lie algebra \mathfrak{g} . For example the derived representation of the PV is just the adjoint representation of \mathfrak{l}_θ on $d_1(\theta)$. Moreover the Lie algebra \mathfrak{g} also contains the dual PV , namely $(\mathfrak{l}_\theta, d_{-1}(\theta))$.

There is an easy criterion to decide whether or not an irreducible PV of parabolic type is regular and in fact most of the reduced irreducible reductive regular PV 's from Sato-Kimura list are of parabolic type (for details we refer to [14],[15] and [16]).

As these PV 's are in one to one correspondence with the subsets $\theta \subset \Psi$, we will describe them by the mean of the following weighted Dynkin diagram:

Definition 2.2.1. *The diagram of the PV $(\mathfrak{l}_\theta, d_1(\theta))$ is the Dynkin diagram of $(\mathfrak{g}, \mathfrak{h})$ (or Σ) where the vertices corresponding to the simple roots of $\Psi \setminus \theta$ are circled (see an example below).*

This very simple classification by means of diagrams contains nevertheless some immediate and interesting information concerning the PV $(\mathfrak{l}_\theta, d_1(\theta))$ (for all these facts, see [13], [14], [15] or [16]):

Remark 2.2.2.

- The Dynkin diagram of $\mathfrak{l}'_\theta = [\mathfrak{l}_\theta, \mathfrak{l}_\theta]$ (i.e. the semi-simple part of the Lie algebra of the group) is the Dynkin diagram of \mathfrak{g} where we have removed the circled vertices and the edges connected to these vertices.
- In fact as a Lie algebra $\mathfrak{l}_\theta = \mathfrak{l}'_\theta \oplus \mathfrak{h}_\theta$ and $\dim \mathfrak{h}_\theta =$ the number of circled vertices.
- The number of irreducible components of the representation $(\mathfrak{l}_\theta, d_1(\theta))$ is also equal to the number of circled roots, and hence the parabolic PV $(\mathfrak{l}_\theta, d_1(\theta))$ is irreducible if and only if \mathfrak{p}_θ is maximal. More precisely, if α is a (simple) circled root, then any nonzero root vector

$X_\alpha \in \mathfrak{g}^\alpha$ generates an irreducible \mathfrak{l}_θ -module V_α , and $d_1(\theta) = \bigoplus_{\alpha \in \Psi \setminus \theta} V_\alpha$ is the decomposition of $d_1(\theta)$ into irreducibles.

The decomposition of the representation $(\mathfrak{l}_\theta, d_1(\theta))$ into irreducibles can also be described by using the eigenspace decomposition with respect to $\text{ad}(\mathfrak{h}_\theta)$, as we will explain now. For each $\alpha \in \mathfrak{h}^*$, let $\bar{\alpha}$ be the restriction of α to \mathfrak{h}_θ and define

$$\mathfrak{g}^{\bar{\alpha}} = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}_\theta, [H, X] = \bar{\alpha}(H)X\}.$$

Then $\mathfrak{g}^{\bar{0}} = \mathfrak{l}_\theta$ and for $\alpha \in \Psi \setminus \theta$, we have $V_\alpha = \mathfrak{g}^{\bar{\alpha}}$. Hence we can write $d_1(\theta) = \bigoplus_{\alpha \in \Psi \setminus \theta} \mathfrak{g}^{\bar{\alpha}}$.

Moreover one can notice (always for $\alpha \in \Psi \setminus \theta$) that $V_\alpha = \mathfrak{g}^{\bar{\alpha}} = \sum_{\beta \in \sigma_1^\alpha} \mathfrak{g}^\beta$, where σ_1^α is the set of roots which belong to $\alpha + \text{span}(\theta)$.

d) One can also directly read the highest weight of V_α from the diagram. The highest weight of V_α relatively to the negative Borel sub-algebra $\mathfrak{b}_\theta^- = \mathfrak{h} \oplus \sum_{\alpha \in (\theta)^-} \mathfrak{g}^\alpha$ is $\bar{\alpha} = \alpha|_{\mathfrak{h}(\theta)}$. Let ω_β ($\beta \in \theta$) be the fundamental weights of \mathfrak{l}'_θ (*i.e.* the dual basis of $(H_\beta)_{\beta \in \theta}$). For each circled root α (*i.e.* for each $\alpha \in \Psi \setminus \theta$), let $J_\alpha = \{(\beta_i)\}$ be the set of roots in θ ($=$ non-circled) which are connected to α in the diagram. From elementary diagram considerations we know that J_α may be empty and that there are always no more than 3 roots in J_α .

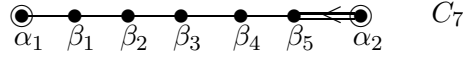
If $J_\alpha = \emptyset$, then V_α is the trivial one dimensional representation of \mathfrak{l}_θ .

If $J_\alpha \neq \emptyset$, then the highest weight $\bar{\alpha}$ of V_α is given by $\bar{\alpha} = \sum_{i \in J_\alpha} c_i \omega_{\beta_i}$ where $c_i = \alpha(H_{\beta_i})$ and where $\alpha(H_{\beta_i})$ can be computed as follows:

$$(R) \begin{cases} \text{if } \|\alpha\| \leq \|\beta_i\|, \text{ then } \alpha(H_{\beta_i}) = -1 ; \\ \text{if } \|\alpha\| > \|\beta_i\| \text{ and if } \alpha \text{ and } \beta_i \text{ are connected by } j \text{ arrows } (1 \leq j \leq 3), \\ \text{then } \alpha(H_{\beta_i}) = -j . \end{cases}$$

Let us illustrate this with an example.

Example 2.2.3. Consider the following diagram:



This diagram is the diagram of a *PV* of parabolic type inside $\mathfrak{g} \simeq \mathfrak{sp}(7) \simeq C_7$. The Lie algebra \mathfrak{l}_θ is isomorphic to $A_5 \oplus \mathfrak{h}_\theta \simeq \mathfrak{sl}(6) \oplus \mathfrak{h}_\theta$ where $\dim \mathfrak{h}_\theta =$ number of circled roots $= 2$. There are two irreducible components V_{α_1} and V_{α_2} , and the highest weight of (A_5, V_{α_1}) (resp. (A_5, V_{α_2})) relatively to the Borel subalgebra \mathfrak{b}_θ^- is ω_1 (resp. $2\omega_5$), where ω_i ($i=1, \dots, 5$) are the fundamental weights of A_5 corresponding respectively to β_1, \dots, β_5 .

2.3. Multiplicity free spaces.

For the results concerning multiplicity free spaces we refer the reader to the survey by Benson and Ratcliff ([1]) or to [9]. Let (G, V) be a finite dimensional rational representation of a connected reductive algebraic group G . Let $\mathbb{C}[V]$ be the algebra of polynomials on V . Then G acts on $\mathbb{C}[V]$ by

$$g \cdot \varphi(x) = \varphi(g^{-1}x) \quad (g \in G, \varphi \in \mathbb{C}[V]).$$

As the space $\mathbb{C}[V]^n$ of homogeneous polynomials of degree n is stable under this action, the representation $(G, \mathbb{C}[V])$ is completely reducible. Let $D(V)$ be the algebra of differential operators with polynomial coefficients. The group G acts also on $D(V)$ by

$$(g \cdot D)(\varphi) = g \cdot (D(g^{-1} \cdot \varphi)) \quad (g \in G, D \in D(V), \varphi \in \mathbb{C}[V]).$$

Definition 2.3.1. *Let G be a connected reductive algebraic group, and let V be the space of a finite dimensional (complex) rational representation of G . The representation (G, V) is said to be multiplicity free if each irreducible representation of G occurs at most once in the representation $(G, \mathbb{C}[V])$.*

From now on "multiplicity-free" will be abbreviated to "MF".

Let us give some results concerning MF spaces (see [1], [5], [9]):

Theorem 2.3.2.

- 1) *A finite dimensional representation (G, V) is MF if and only if (B, V) is a prehomogeneous vector space for any Borel subgroup B of G (and hence each MF space (G, V) is a PV).*
- 2) *A finite dimensional representation (G, V) is MF if and only if the algebra $D(V)^G$ of invariant differential operators with polynomial coefficients is commutative.*
- 3) *If (G, V) is a MF space, then the dual space (G, V^*) is also MF. More generally if $(G, W \oplus V)$ is a representation of G where W and V are G -stable, then $(G, W \oplus V)$ is MF if and only if $(G, W \oplus V^*)$ is MF.*

Proof. (Indications)

Part 1) is due to Vinberg and Kimelfeld ([23]), another proof can be found in [9]. Part 2) is due to Howe and Umeda ([5], Proposition 7.1). The first assertion of Part 3), also noted in [5], is a consequence of the G -equivariant isomorphism $\mathbb{C}^i[V^*] \simeq (\mathbb{C}^i[V])^*$. The second assertion of 3) is Corollary 3.3 in [10]. □

Note that the commutativity of $D(V)^G$ for a MF space is just a consequence of the definition, since we have a simultaneous diagonalization of all the operators in $D(V)^G$.

If (G, V) is a MF space, and if B is a Borel subgroup of G , then, as we have seen (G, V) and (B, V) are prehomogeneous spaces. Let us denote by $\Delta_0, \Delta_1, \dots, \Delta_k, \dots, \Delta_r$ the fundamental relative invariants invariants of the PV (B, V) , indexed in such a way that $\Delta_0, \Delta_1, \dots, \Delta_k$ are the fundamental relative invariants of the PV (G, V) and such that the other invariants are ordered by decreasing degree. It is worthwhile noticing that at least Δ_r is of degree one as the highest weight vectors of the irreducible components of V^* must occur. The nonnegative integer $r + 1$ (= the number of fundamental relative invariants under B) is called the *rank* of the MF space (G, V) .

2.4. Multiplicity free spaces with a one dimensional quotient.

Let us now define the main objects this paper deals with.

Definition 2.4.1. (T. Levasseur [11])

- 1) *A prehomogeneous vector space (G, V) is said to be of rank one¹ if there exists an homogeneous polynomial Δ_0 on V such that $\Delta_0 \notin \mathbb{C}[V]^G$ and such that $\mathbb{C}[V]^{G'} = \mathbb{C}[\Delta_0]$.*
- 2) *A multiplicity free space (G, V) is said to have a one-dimensional quotient if it is a PV of rank one. (This implies that the categorical quotient $V//G'$ has dimension 1.)*

Although the following result is implicit in [11] we provide a proof here.

Proposition 2.4.2.

¹It is worth noticing that if (G, V) is multiplicity free, then its rank as a PV is not at all the same as its rank as a MF space.

If (G, V) is a PV of rank one, then the polynomial Δ_0 is the unique fundamental relative invariant of (G, V) . More precisely a PV (G, V) is of rank one if and only if it has a unique fundamental relative invariant.

Proof. We can write $G = G'C$ where $G' = [G, G]$ is the derived group of G and where $C = Z(G)^\circ \simeq (\mathbb{C}^*)^p$ is the connected component of the center of G . Let $g \in C$. Then $\Delta_0(g^{-1}x)$ is again G' -invariant. As $\mathbb{C}[V]^{G'} = \mathbb{C}[\Delta_0]$ and as $\Delta_0(g^{-1}x)$ has the same degree as Δ_0 we obtain that $\Delta_0(g^{-1}x) = \chi(g)\Delta_0(x)$ with $\chi(g) \in \mathbb{C}^*$. Therefore Δ_0 is a relative invariant. Suppose that Δ_0 is not irreducible. Then $\Delta_0 = P_1 \dots P_m$, where the polynomials P_i are irreducible relative invariants and $\partial^\circ(P_i) < \partial^\circ(\Delta_0)$. We should have $P_i \in \mathbb{C}[\Delta_0]$, which is impossible. Hence Δ_0 is irreducible. If f is another fundamental relative invariant then we would have $f \in \mathbb{C}[\Delta_0]$ which is impossible.

It remains to prove that if a PV (G, V) has a unique fundamental relative invariant Δ_0 then it is of rank one. As Δ_0 is non constant we have of course that $\Delta_0 \notin \mathbb{C}[V]^G$. Let $P \in \mathbb{C}[V]^{G'}$. If $P = P_0 + P_1 + \dots + P_m$ with $\partial^\circ(P_i) = i$, then each P_i is G' -invariant. Therefore we can suppose that P has fixed degree n (i.e. $P \in \mathbb{C}[V]^{G'} \cap \mathbb{C}[V]^n$).

Let

$$\mathbb{C}[V]^n = \bigoplus_{i=0}^p M_i$$

be the decomposition of $\mathbb{C}[V]^n$ into G' -isotypic components. We suppose that M_0 is the isotypic component of the trivial G' -module ($M_0 = \mathbb{C}[V]^{G'} \cap \mathbb{C}[V]^n$). Hence $P \in M_0$. As $G = CG'$, the group G stabilizes each M_i . Therefore we can write

$$M_0 = \bigoplus M_\chi$$

where the G -isotypic components M_χ of M_0 are indexed by characters χ of G and given by

$$M_\chi = \{\varphi \in M_0 \mid \varphi(z^{-1}x) = \chi(z)\varphi(x), \forall z \in C, x \in V\}.$$

Hence $P = \sum P_\chi$, $P_\chi \in M_\chi$, and for $z \in C, g' \in G'$ and $x \in V$ we have $P_\chi(zg'x) = \chi^{-1}(z)P_\chi(g'x) = \chi^{-1}(z)P_\chi(x)$. Therefore each P_χ is a relative invariant. But (G, V) has a unique fundamental relative invariant namely Δ_0 . Hence $P_\chi = c_\chi \Delta_0^j$ ($c_\chi \in \mathbb{C}$). The exponent j does not depend on χ , since all the P_χ 's have the same degree. Therefore all the characters χ are the same, namely $\chi = \lambda_0^j$ where λ_0 is the character of Δ_0 . This implies that $M_0 = M_{\lambda_0^j}$, and that $P = c\Delta_0^j$. Hence $\mathbb{C}[V]^{G'} = \mathbb{C}[\Delta_0]$. □

The following result gives a criterion to decide whether or not a PV has rank one. It will be useful in section 4 for the classification of the MF spaces with a one dimensional quotient.

Proposition 2.4.3.

Let G be a connected algebraic group and let (G, V) be a PV . We suppose that $V = V_1 \oplus V_2$ where V_1 and V_2 are G -stable subspaces, and that (G, V_1) is a rank one PV . Let (x_0, y_0) be a generic element in V , with $x_0 \in V_1$ and $y_0 \in V_2$. Let G_{x_0} be the isotropy subgroup of x_0 . We suppose also that the prehomogeneous vector space (G_{x_0}, V_2) has no nontrivial relative invariant (this property does not depend on the choice of (x_0, y_0)). Then (G, V) is a rank one PV .

Proof. For the fact that (G_{x_0}, V_2) is a PV and that y_0 is generic for this PV we refer to [18]. As (G, V_1) is a rank one PV it has a unique fundamental relative invariant $f(x)$ by Proposition 2.4.2. Define $\Delta_0(x, y) = f(x)$ ($x \in V_1, y \in V_2$). Then, as it is irreducible, $\Delta_0(x, y)$ is a fundamental relative invariant of (G, V) . Let $\varphi(x, y)$ be a fundamental relative invariant of (G, V) and consider the function $y \mapsto \varphi(x_0, y)$. For $g \in G_{x_0}$, we have $\varphi(gx_0, gy) = \varphi(x_0, gy) = \chi_\varphi(g)\varphi(x_0, y)$. Hence $y \mapsto \varphi(x_0, y)$ is a relative invariant of (G_{x_0}, V_2) . But by hypothesis (G_{x_0}, V_2) has no nontrivial relative invariant, hence for all $y \in V_2$, we have $\varphi(x_0, y) = \psi(x_0)$ (constant with respect to y). But as this is true for any generic $x_0 \in V_1$, we obtain that $\varphi(x, y) = \psi(x)$, for all $x \in V_1$ and all $y \in V_2$. In other words φ does only depend on the x variable. As φ is irreducible, so is also ψ . And ψ is then a relative invariant of (G, V_1) , hence $\psi = cf$ ($c \in \mathbb{C}$), or equivalently $\varphi(x, y) = c\Delta_0(x, y)$. Then Proposition 2.4.2 implies that (G, V) is a rank one PV . □

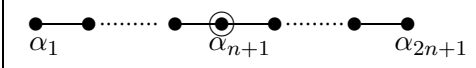
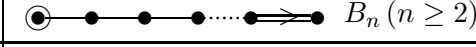
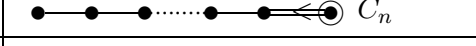
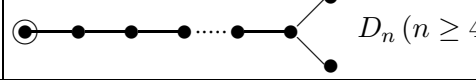
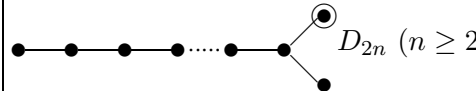
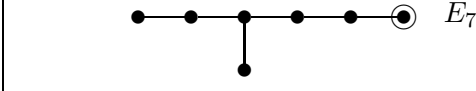
Notation: If (G, V) is a MF space with a one dimensional quotient, we will sometimes say that (G, V) is $QD1$.

2.5. An example: the regular commutative PV 's of parabolic type.

Among the PV 's of parabolic type there is a family, the so-called regular commutative PV 's of parabolic type, which are MF spaces with a one dimensional quotient. We will give here a brief description of these objects. Notations and conventions are the same as in section 2.2. The PV 's of parabolic type we are going to describe are irreducible. Therefore there is only one circled root which we denote by α_0 (and then $\theta = \Psi \setminus \{\alpha_0\}$). In this section we will impose the extra condition that the coefficient of α_0 in the highest root is 1. This implies that $d_p(\theta) = \{0\}$ for $p \neq 0, 1, -1$. Hence $\mathfrak{p}_\theta = \mathfrak{l}_\theta \oplus d_1(\theta)$, and the nilradical $d_1(\theta)$ of \mathfrak{p}_θ is a commutative subalgebra. Therefore the spaces $(\mathfrak{l}_\theta, d_1(\theta))$ are called commutative PV 's of parabolic type. It is known that these PV 's are all MF spaces ([12]). By Proposition 2.4.2 those which have a one dimensional quotient are exactly those which have a non trivial relative invariant. From [12] these are also exactly those which are regular, and the list is given in Table 1 below. As they are irreducible, and as the center of \mathfrak{l}_θ is one-dimensional,

these spaces are automatically indecomposable and saturated (see Definition 3.1 below).

Table 1
Regular PV's of commutative parabolic type

	\mathfrak{g}	\mathfrak{l}_θ	$d_1(\theta)$
A_{2n+1}		$\mathfrak{sl}(n+1) \times \mathfrak{gl}(n+1)$	M_n
B_n		$\mathfrak{so}(2n-1) \times \mathbb{C}$	\mathbb{C}^{2n-1}
C_n		$\mathfrak{gl}(n)$	$Sym(n)$
D_n^1		$\mathfrak{so}(2n-2) \times \mathbb{C}$	\mathbb{C}^{2n-2}
D_{2n}^2		$\mathfrak{gl}(2n)$	$AS(2n)$
E_7		$E_6 \times \mathbb{C}$	\mathbb{C}^{27}

3. THE CLASSIFICATION

Let us now explain the classification of MF spaces with a one dimensional quotient. We begin to describe briefly the classification of all MF spaces. Kac ([6]) determined all the cases where the representation (G, V) is irreducible. Brion ([3]) did the case where $G' = [G, G]$ is (almost) simple. Finally Benson-Ratchliff and Leahy did the rest, independently ([1], [2],[10],[9]).

Definition 3.1. (see [9])

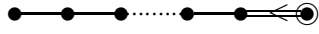
- 1) Two representations (G_1, ρ_1, V_1) and (G_2, ρ_2, V_2) are called geometrically equivalent if there is an isomorphism $\Phi : V_1 \mapsto V_2$ such that $\Phi(\rho_1(G_1))\Phi^{-1} = \rho_2(G_2)$.
- 2) A representation (G, V) is called decomposable if it is geometrically equivalent to a representation of the form $(G_1 \times G_2, V_1 \oplus V_2)$, where V_1 and V_2 are non-zero. It is called indecomposable if it is not decomposable.
- 3) A representation (G, V) is called saturated if the dimension of the center of $\rho(G)$ is equal to the number of irreducible summands of V .

Remark 3.2.

The notion of geometric equivalence is quite natural, once one has remarked that the notion of MF space depends only on $\rho(G)$. It is worthwhile pointing out that any representation is geometrically equivalent to its dual representation (see Theorem 2.3.2). Finally note that any representation can be made saturated by adding a torus.

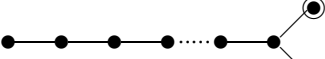
Theorem 3.3. The complete list, up to geometric equivalence, of indecomposable saturated MF spaces with a one dimensional quotient is given by Table 2 (irreducibles) and Table 3 (non irreducibles) at the end of the paper.

4.1.4. $S^2(SL(n)) \times \mathbb{C}^*(n \geq 2)$.

Up to geometric equivalence this is the representation of $GL(n)$ on $Sym(n)$ given by $g.X = gX^tg$ ($g \in GL(n), X \in Sym(n)$). This PV is of commutative parabolic type corresponding to the diagram  C_n .

The unique fundamental relative invariant is the determinant, therefore it is $QD1$. It corresponds to case (2) in Table 2.

4.1.5. $\Lambda^2(SL(n)) \times \mathbb{C}^*(n \geq 4)$.

Up to geometric equivalence this is the representation of $GL(n)$ on $AS(n)$ given by $g.X = gX^tg$ ($g \in GL(n), X \in AS(n)$). This PV is of commutative parabolic type corresponding to the diagram  D_n .

It is well known that there is no relative invariant if n is odd, and that for n even the unique fundamental relative invariant is the pfaffian. Therefore it is $QD1$ if and only if $n = 2p$. It corresponds to case (3) in Table 2.

4.1.6. $SL(n) \otimes SL(m)^* \times \mathbb{C}^*(n, m \geq 2)$.

By Remark 3.2, this representation is geometrically equivalent to case $SL(n) \otimes SL(m) \times \mathbb{C}^*(n, m \geq 2)$ which is considered in Table 1 of Benson-Ratchiff ([2]). This is the representation of $SL(n) \times SL(m)$ on the space $M_{n,m}$ given by $(g_1, g_2).X = g_1Xg_2^{-1}$, $g_1 \in SL(n), g_2 \in SL(m), X \in M_{n,m}$. This is a commutative parabolic PV corresponding to the diagram



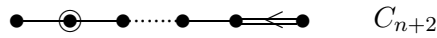
If $n \neq m$, then there is no fundamental relative invariant. If $n = m$, then the unique fundamental relative invariant is the determinant. Hence it is $QD1$ if and only if $n = m$. It corresponds to case (4) in Table 2.

4.1.7. $SL(2) \otimes Sp(n) \times \mathbb{C}^*(n \geq 2)$.

Up to geometric equivalence we can consider that this is the representation of $G = GL(2) \times Sp(n)$ acting on $V = M_{2n,2}$ by

$$(g_1, g_2).X = g_2X^tg_1, g_1 \in GL(2), g_2 \in Sp(n), X \in M_{2n,2}$$

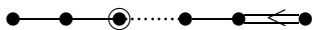
This is a regular irreducible PV of parabolic type (not commutative), corresponding to the diagram



(see [20], [14], [16]). Hence it is $QD1$. According to the computations in [20] (Proposition 17 p.100-101), the fundamental relative invariant is $f(X) = Pf({}^tXJX)$, where $X \in M_{2n,2}(\mathbb{C})$, where $J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$, and where $Pf(\cdot)$ is the pfaffian of a 2×2 skew symmetric matrix. It is case (6) in Table 2.

4.1.8. $SL(3) \otimes Sp(n) \times \mathbb{C}^*(n \geq 2)$.

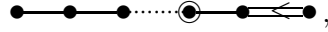
This is a non regular irreducible PV of parabolic type corresponding to the diagram



of type C_{n+3} , it is known ([14] or [16]) that it has no non-trivial relative invariant, hence it is not $QD1$.

4.1.9. $SL(n) \otimes Sp(2) \times \mathbb{C}^*$ ($n \geq 4$).

This again is an irreducible PV of parabolic type corresponding to the diagram



of type C_{n+2} . It is known ([14] or [16]) that this PV has a non-trivial relative invariant if and only if $n = 4$. Hence this space is $QD1$ if and only if $n = 4$, and then it is regular. In this case the group $SL(4) \times Sp(2)$ acts on M_4 by $(g_1, g_2).X = g_1 X g_2^{-1}$, and the fundamental relative invariant is the determinant. It is case (7) in Table 2.

4.1.10. $Spin(7) \times \mathbb{C}^*$.

This space is known ([14] or [16]) to be an irreducible regular PV of parabolic type inside F_4 corresponding to the diagram $\bullet \bullet \bullet \rightleftarrows \bullet \bullet \bullet \bullet$. Here the space has dimension 8 and the action is obtained by embedding $Spin(7)$ into $SO(8)$. The fundamental relative invariant is the nondegenerate quadratic form which defines $SO(8)$. It is case (8) in Table 2.

4.1.11. $Spin(9) \times \mathbb{C}^*$.

According to [20], p. 146, number (19) of Table, this is an irreducible regular PV whose fundamental relative invariant is a quadratic form, hence it is $QD1$. According to the diagrammatical rules in Remark 2.2.2 d) it is not of parabolic type. But it has nevertheless an interesting connection with PV 's of parabolic type, see [19], Theorem 5.1. p. 377. It is case (9) in Table 2.

4.1.12. $Spin(10) \times \mathbb{C}^*$.

This is a PV of parabolic type inside E_6 (corresponding to the diagram $\bullet \bullet \bullet \bullet \bullet \bullet \bullet$).

According to the table in ([14] or [16]) it has no non-trivial relative invariant, hence it is not $QD1$.

4.1.13. $G_2 \times \mathbb{C}^*$.

According to [20], p. 146, number (25) of Table, this is an irreducible regular PV whose fundamental relative invariant is a quadratic form, hence it is $QD1$. According to the diagrammatical rules in Remark 2.2.2 d) it is not of parabolic type. But it has nevertheless an interesting connection with PV 's of parabolic type, see [19], Theorem 6.1. p. 381. It is case (10) in Table 2.

4.1.14. $E_6 \times \mathbb{C}^*$.

This space is known ([14] or [16]) to be an irreducible regular PV of parabolic type inside E_7 corresponding to the diagram $\bullet \bullet \bullet \bullet \bullet \bullet \bullet$.

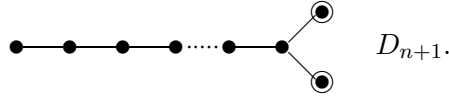
It is the 27-dimensional representation of E_6 . Its fundamental relative invariant has degree 3, it is known as the Freudenthal cubic. This is the case (5) in Table 2.

4.2. Non-irreducible MF spaces.

Here we examine the cases arising in Table II of [2]. We keep the same notations for the representations as before. In addition we adopt also the following notation from [2]. If (G_1, V_1) and (G_2, V_2) are representations of two semi-simple groups G_1 and G_2 which share a common simple factor H , then the notation $G_1 \oplus_H G_2$ denotes the image of the representation on $V_1 \oplus V_2$ where the common factor H acts diagonally. For example $SL(n) \oplus_{SL(n)} SL(n)$ denotes the direct sum representation $(SL(n), \mathbb{C}^n \oplus \mathbb{C}^n)$, and $Spin(8) \oplus_{Spin(8)} SO(8)$ denotes the action of $Spin(8)$ on $\mathbb{C}^8 \oplus \mathbb{C}^8$ via the Spin representation on the first \mathbb{C}^8 factor and via the natural representation of $SO(8)$ on the second factor.

4.2.1. $(SL(n) \oplus_{SL(n)} SL(n)) \times \mathbb{C}^{*2} (n \geq 2)$.

This space is a parabolic PV corresponding to the diagram



Lets us show that this space is $QD1$ if and only if $n = 2$. The representation is given by

$$(g, \lambda, \mu).X = gX \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \text{ where } g \in SL(n), X \in M_{n,2}, \lambda, \mu \in \mathbb{C}^*. \text{ Set } X_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}. \text{ A}$$

simple computation shows that the isotropy subgroup G_{X_0} of X_0 is the subgroup of $SL(n) \times$

$$\mathbb{C}^{*2} \text{ consisting of triplets of the form } \left(\left[\begin{array}{cc|c} \lambda^{-1} & 0 & \beta \\ 0 & \mu^{-1} & \delta \\ \hline 0 & 0 & \delta \end{array} \right], \lambda, \mu \right). \text{ Then, an easy calculation}$$

shows that $\dim G - \dim G_{X_0} = 2n$, and hence the point X_0 is generic. Moreover the preceding computation of G_{X_0} shows that, if $n > 2$, the subgroup generated by the derived group $SL(n) \times \{1\} \times \{1\}$ and G_{X_0} is the full group $SL(n) \times \mathbb{C}^{*2}$. According to Remark 2.1.1 this proves that if $n \neq 2$ there exists no non-trivial relative invariant and hence the space is not $QD1$. If $n = 2$, the space is the space M_2 and then the determinant is the only fundamental relative invariant. This is a particular case of (1) in Table 3.

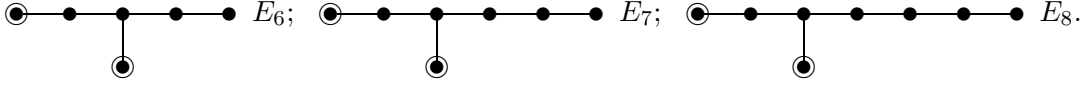
4.2.2. $(SL(n)^* \oplus_{SL(n)} SL(n)) \times \mathbb{C}^{*2} (n \geq 2)$.

This is the representation of $SL(n) \times \mathbb{C}^{*2}$ on $M_{1,2} \times M_{2,1}$ given by $(g, \lambda, \mu)(u, v) = (\lambda u g^{-1}, \mu g v)$ where $\lambda, \mu \in \mathbb{C}^*$, $g \in SL(n)$, $u \in M_{1,2}$ and $v \in M_{2,1}$. This a parabolic PV corresponding

to the diagram It is easily seen that the quadratic form $f(u, v) = uv$ is the unique fundamental relative invariant and as the generic isotropy subgroup is reductive, this PV is regular. This is a particular case of a family of so-called Q -irreducible PV 's of parabolic type. We refer the reader interested into details to Lemma 4.8 in [18]. It is case (1) in Table 3.

4.2.3. $(SL(n) \oplus_{SL(n)} \Lambda^2(SL(n))) \times \mathbb{C}^{*2} (n \geq 4)$.

The representation is given by $(g, \lambda, \mu).(u, x) = (\lambda g u, \mu g x^t g)$ where $\lambda, \mu \in \mathbb{C}^*$, $g \in SL(n)$, $u \in \mathbb{C}^n, x \in AS(n)$. This PV is not of parabolic type except for the cases where $n = 5, 6, 7$ which correspond respectively to the following diagrams:



• Suppose first that $n = 2p$ is even. It is well known that the restriction of the representation to $AS(n)$ is a regular PV of parabolic type, and that its unique fundamental relative invariant is the pfaffian. Moreover the generic isotropy subgroup of this component at the point $J = \begin{bmatrix} 0 & Id_p \\ -Id_p & 0 \end{bmatrix}$ is the subgroup $Sp(p) \times \mathbb{C}^*$. The restriction of the "natural" representation of $SL(2p) \times \mathbb{C}^*$ on \mathbb{C}^{2p} to $Sp(p) \times \mathbb{C}^*$ is well known to have no non-trivial relative invariant. Hence, by Proposition 2.4.3 this space is $QD1$. As the fundamental relative invariant does not depend on all variables, it is not regular. It is case (2)(a) in Table 3.

• Suppose now that $n = 2p+1$ is odd. Rather than the group $SL(n) \times \mathbb{C}^{*2}$, we will here consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n) \times \mathbb{C}$ acting on $V = \mathbb{C}^n \oplus AS(n)$ by $(U, \lambda)(v, x) = (\lambda v + Uv, Ux + x^t U)$ where $\lambda \in \mathbb{C}, U \in \mathfrak{gl}(n), v \in \mathbb{C}^n, x \in AS(n)$. Once we identify $\mathfrak{gl}(n)$ with $\mathfrak{sl}(n) \times \mathbb{C}$ this is essentially the derived representation of $(SL(n) \oplus_{SL(n)} \Lambda^2(SL(n)) \times \mathbb{C}^{*2})$. Consider the

point $X_0 = (v_0, x_0) \in \mathbb{C}^n \oplus AS(n)$ where $v_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ and where $x_0 = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$. An easy

computation shows that the isotropy subalgebra \mathfrak{g}_{X_0} is given by

$$\mathfrak{g}_{X_0} = \left\{ \left(\begin{bmatrix} A & 0 \\ 0 & -\lambda \end{bmatrix}, \lambda \right), \lambda \in \mathbb{C}, A \in Sp(p) \right\}.$$

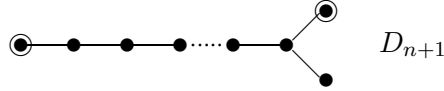
As $\dim \mathfrak{g} - \dim \mathfrak{g}_{X_0} = \dim V$, the point X_0 is generic. The Lie algebra \mathfrak{g}_{X_0} is the Lie algebra of a reductive subgroup. Hence this space is regular. As $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}(n)$, the Lie algebra generated by \mathfrak{g}_{X_0} and $[\mathfrak{g}, \mathfrak{g}]$ is equal to $\mathfrak{sl}(n)$, and hence $\mathfrak{g}/\mathfrak{sl}(n) \simeq \mathbb{C}$. According to Remark 2.1.1, there exists exactly one (up to constants) fundamental relative invariant, and hence this space is $QD1$. Keeping the same notations as above it is easy to see that the polynomial

$$f(v, x) = Pf \left(\begin{bmatrix} x & v \\ -x^t & 0 \end{bmatrix} \right), v \in \mathbb{C}^n, x \in AS(2p+1)$$

is a fundamental relative invariant. These spaces are Q -irreducible in the sense of [18] (see Remark 4.15 in [18]). It is case (2)(b) in Table 3.

4.2.4. $(SL(n)^* \oplus_{SL(n)} \Lambda^2(SL(n)) \times \mathbb{C}^{*2} (n \geq 4))$.

This PV is always of parabolic type. The corresponding diagram is the following:



Up to geometric equivalence we can take here $G = GL(n) \times \mathbb{C}^*$ acting on $V = M_{1,n} \oplus AS(n)$ by $(g, \lambda).(v, x) = (\lambda v U^{-1}, Ux^t U)$.

• Suppose first that $n = 2p$ is even. The restriction of the representation to $AS(n)$ is a regular PV whose fundamental relative invariant is the pfaffian. The partial isotropy of $J = \begin{bmatrix} 0 & Id_p \\ -Id_p & 0 \end{bmatrix} \in AS(n)$ is equal to $Sp(p) \times \mathbb{C}^*$, and it is well known that the action of

$Sp(p) \times \mathbb{C}^*$ on $M_{1,n}$ has non non-trivial relative invariant. Therefore, from Proposition 2.4.3, we obtain that this MF space is $QD1$. As the fundamental relative invariant does not depend on all variables, it is not regular. It is case (3) in Table 3.

• Suppose now that $n = 2p + 1$ is odd. Rather than the group action, we will consider here the infinitesimal action. In other words we consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n) \times \mathbb{C}$ acting on $V = M_{1,2p+1} \oplus AS(2p+1)$ by $(U, \lambda)(v, x) = (\lambda v - vU, Ux + x^t U)$ where $\lambda \in \mathbb{C}, U \in \mathfrak{gl}(n), v \in \mathbb{C}^n, x \in AS(n)$. Consider the element $X_0 = (v_0, x_0) \in V$ which is defined by $v_0 = (1, 0, \dots, 0)$ and $x_0 = \left[\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array} \right] \in AS(2p+1)$ with $J = \left[\begin{array}{c|c} 0 & Id_p \\ \hline -Id_p & 0 \end{array} \right]$. A computation shows that the

isotropy subalgebra \mathfrak{g}_{X_0} is the set of couples of the form $\left(\left[\begin{array}{c|c} A & B \\ \hline 0 & D \end{array} \right], \lambda \right)$,

– where $A = \left[\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & -{}^t\alpha \end{array} \right]$ with $\alpha = \left[\begin{array}{c} \lambda, 0, \dots, 0 \\ A_1 \end{array} \right]$, $A_1 \in M_{p-1,p}$; $\beta = \left[\begin{array}{cc} 0 & 0 \\ 0 & b \end{array} \right]$, $b \in Sym(p-1)$;
 $\gamma \in Sym(p)$.

– where $B = \left[\begin{array}{c} 0 \\ \tilde{B} \end{array} \right]$, $\tilde{B} \in \mathbb{C}^{2p-1}$

– and where $D, \lambda \in \mathbb{C}$.

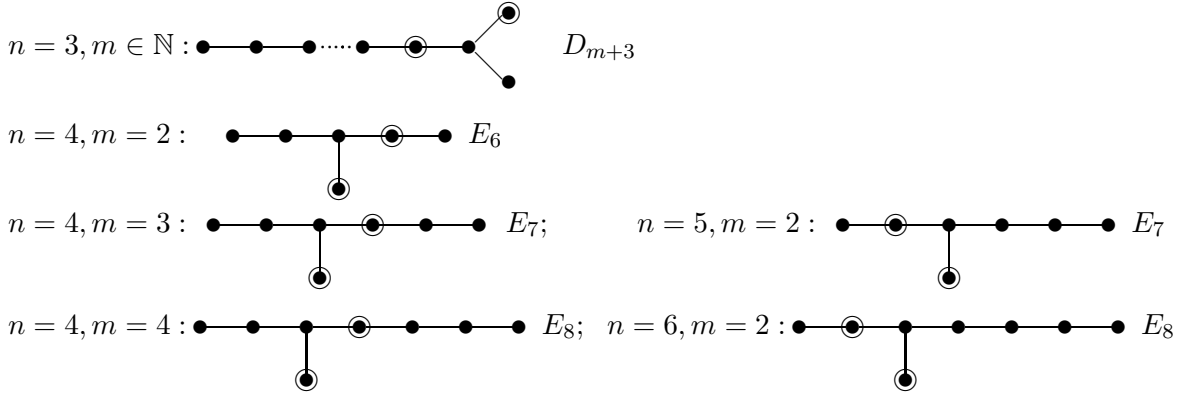
Then one verifies that $\dim \mathfrak{g} - \dim \mathfrak{g}_{X_0} = \dim V$, and hence X_0 is generic. Moreover the Lie subalgebra generated by \mathfrak{g}_{X_0} and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}(n) \times \{0\}$ is equal to \mathfrak{g} . According again to Remark 2.1.1, this shows that there is no non-trivial relative invariant, and therefore this space is not $QD1$.

4.2.5. $(SL(n) \oplus_{SL(n)} (SL(n) \otimes SL(m))) \times \mathbb{C}^{*2}$ ($n, m \geq 2$).

It is convenient here to replace this representation by the representation (G, V) where $G = GL(n) \times GL(m)$ acts on $V = M_{n,1} \oplus M_{n,m}$ by

$$(g_1, g_2)(v, x) = (g_1 v, g_1 x g_2^{-1}), g_1 \in GL(n), g_2 \in GL(m), v \in M_{n,1}, x \in M_{n,m}.$$

Due to Remark 3.2, this representation is geometrically equivalent to the first one. It is not of parabolic type except for the following cases:



• Suppose $n = m$. Then the component $M_{n,n} = M_n$ has a unique fundamental relative invariant, namely the determinant. The point $X_0 = (e_1, Id_n)$, where e_1 is the first vector of the canonical basis of $M_{n,1} \simeq \mathbb{C}^n$, is generic. And the partial isotropy subgroup $G_{(0, Id_n)}$ is the diagonal subgroup $\{(g, g) \in GL(n) \times GL(n)\}$. Therefore the action of $G_{(0, Id_n)}$ on $M_{n,1}$ has no relative invariant. According to Proposition 2.4.3 this space is $QD1$ in the case $m = n$.

As the fundamental relative invariant does not depend on all variables, it is not regular. It is case (4)(a) in Table 3.

• Suppose that $n < m$. A simple calculation shows that the point $X_0 = (e_1, x_0)$ where $x_0 = [Id_n \mid 0]$ is generic and that its isotropy subgroup G_{X_0} is the set of pairs of matrices of the form $(A, \left[\begin{array}{c|c} A & 0 \\ \hline B & C \end{array} \right])$, where $B \in M_{m-n,n}, C \in GL(m-n)$, and where $A = \left[\begin{array}{c|c} 1 & A_1 \\ \hline 0 & A_2 \end{array} \right]$, with $A_1 \in M_{1,n-1}$ and $A_2 \in GL(n-1)$. This implies that the subgroup of $G = GL(n) \times GL(m)$ generated by G_{X_0} and the derived group $SL(n) \times SL(m)$ is G itself. Hence from Remark 2.1.1, we know that there is no non-trivial relative invariant, and therefore it is not $QD1$.

• Suppose that $n > m + 1$. Then the element $X_0 = (e_n, x_0)$ where $x_0 = \left[\begin{array}{c} Id_m \\ 0 \end{array} \right]$ is generic and the isotropy subgroup G_{X_0} is the set of pairs of matrices of the form $(\left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right], A)$ where $A \in GL(m)$, where $B \in M_{m,n-m}$ is of the form $B = [B' \mid 0]$ with $B' \in M_{m,n-m-1}$, and where $C \in GL(n-m)$ is of the form $C = \left[\begin{array}{c|c} C_1 & 0 \\ \hline C_2 & 1 \end{array} \right]$ with $C_1 \in GL(n-m-1)$ and $C_2 \in M_{1,n-m-1}$. Again this implies that the subgroup generated by G_{X_0} and $[G, G]$ is equal to G . Hence by Remark 2.1.1, we obtain that this space has no non-trivial relative invariant, and hence it is not $QD1$.

• Suppose finally that $n = m + 1$. Then the same calculation as before holds. But now as $n - m = 1$, the isotropy subgroup G_{X_0} is the set of pairs of matrices of the form $(\left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right], A)$ where $A \in GL(m)$. The subgroup \tilde{G} generated by G_{X_0} and $[G, G]$ is equal to $\{(g_1, g_2) \in G \mid \det(g_1) = \det(g_2)\}$. This implies that $\dim G/\tilde{G} = 1$ and hence by Remark 2.1.1, we obtain that this space has one fundamental relative invariant, and therefore it is $QD1$. As the generic isotropy subgroup is reductive, it is regular. It is easy to see that $f(v, x) = \det(v; x)$, where $(x; v)$ is the $n \times n$ matrix obtained by putting the column vector v left to the $m \times n$ matrix x , is the fundamental relative invariant. It is case (4)(b) in Table 3.

4.2.6. $(SL(n)^* \oplus_{SL(n)} (SL(n) \otimes SL(m))) \times \mathbb{C}^{*2} (n \geq 3, m \geq 2)$.

It is convenient here to consider the representation (G, V) where $G = GL(n) \times GL(m)$ acts on $V = M_{1,n} \oplus M_{n,m}$ by

$$(g_1, g_2)(v, x) = (vg_1^{-1}, g_1 x g_2^{-1}), g_1 \in GL(n), g_2 \in GL(m), v \in M_{1,n}, x \in M_{n,m}.$$

This representation is geometrically equivalent to the original one. This PV is of parabolic type and corresponds to the diagram:

$$\bullet \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n+1}} \bullet \xrightarrow{\alpha_{n+m}} \bullet \quad A_{n+m}$$

The element (e_1, x_0) where $e_1 = (1, 0, \dots, 0)$ and where $x_0 = Id_n$ if $n = m$, $x_0 = [Id_n \mid 0]$ if $n < m$, and $x_0 = \left[\begin{array}{c} Id_m \\ 0 \end{array} \right]$ if $n > m$, is generic and almost the same calculations as in 4.2.5 show that this MF space is $QD1$ if and only if $n = m$. Moreover by the same argument it is not a regular PV . It is case (5) in Table 3.

Up to geometric equivalence we can take $G = Sp(n) \times GL(2) \times Sp(m) \times \mathbb{C}^*$ acting on $V = V_1 \oplus V_2$ where $V_1 = M_{2n,2}$ and $V_2 = M_{2m,2}$ by

$$(g_1, g_2, g_3, \lambda).(X, Y) = (g_1 X^t g_2, \lambda g_3 Y^t g_2),$$

where $g_1 \in Sp(n), g_2 \in GL(2), g_3 \in Sp(m), \lambda \in \mathbb{C}^*, X \in M_{2n,2}, Y \in M_{2m,2}$.

According to the diagrammatical rules in Remark 2.2.2 d) this space is not of parabolic type.. As each of the representations (G, V_1) and (G, V_2) is of the type seen in 4.1.7. above, they have each their own fundamental relative invariant. Therefore this *MF* space is not *QD1*.

4.2.11. $Sp(n) \oplus_{Sp(n)} Sp(n) \times \mathbb{C}^{*2}, (n \geq 2)$.

Here $G = Sp(n) \times \mathbb{C}^{*2}$ acts on $V = M_{2n,1} \oplus M_{2n,1}$ by

$$(g, \lambda, \mu).(u, v) = (\lambda g u, \mu g v), g \in Sp(n), \lambda, \mu \in \mathbb{C}, u, v \in M_{2n,1}.$$

At the infinitesimal level the Lie algebra $\mathfrak{g} = \mathfrak{sp}(n) \times \mathbb{C}^2$ acts on V by

$$(x, \lambda, \mu)(u, v) = (\lambda u + x u, \mu v + x v), x \in \mathfrak{sp}(n), \lambda, \mu \in \mathbb{C}, u, v \in M_{2n,1}.$$

First of all let us remark that there is at least one fundamental relative invariant, namely

$$f(u, v) = {}^t u J v \text{ where } J = \begin{bmatrix} 0 & Id_n \\ -Id_n & 0 \end{bmatrix}.$$

Consider the element $X_0 = (e_1, e_{n+1}) \in M_{2n,1} \oplus M_{2n,1}$ where e_j is the j -th vector of the canonical base of $M_{2n,1} \simeq \mathbb{C}^{2n}$. An easy calculation shows that the isotropy subalgebra \mathfrak{g}_{X_0} of X_0 is given by:

$$\mathfrak{g}_{X_0} = \left\{ \left(\begin{array}{cc|cc} -\lambda & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & \lambda & 0 \\ 0 & C & 0 & -{}^t A \end{array} \right), \lambda, -\lambda, A \in \mathfrak{gl}(n-1), B, C \in Sym(n-1), \lambda \in \mathbb{C}^* \right\}.$$

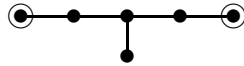
As $\dim \mathfrak{g} - \dim \mathfrak{g}_{X_0} = \dim V$, the point X_0 is generic. As \mathfrak{g}_{X_0} is the Lie algebra of a reductive subgroup, this *PV* is regular. Let $\tilde{\mathfrak{g}}$ the Lie algebra generated by $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sp}(n) \times \{0\} \times \{0\}$ and \mathfrak{g}_{X_0} . We have $\dim(\mathfrak{g}/\tilde{\mathfrak{g}}) = 1$. Then according to Remark 2.1.1, the polynomial $f(u, v) = {}^t u J v$ is the only fundamental relative invariant. Therefore this space is *QD1*. According to Remark 2.2.2 d), it is not of parabolic type. It is case (9) in Table 3.

4.2.12. $Spin(8) \oplus_{Spin(8)} SO(8) \times \mathbb{C}^{*2}$.

Let ρ be one of the Spin representations of $Spin(8)$. Here $G = Spin(8) \times \mathbb{C}^{*2}$ acts on $V = \mathbb{C}^8 \oplus \mathbb{C}^8$ by

$$(g, \lambda, \mu)(u, v) = (\lambda g u, \mu \rho(g) v), g \in Spin(8), \lambda, \mu \in \mathbb{C}^*, u, v \in \mathbb{C}^8.$$

This is a parabolic *PV* in E_6 corresponding to the diagram:



As each of the two summands of this representation has his own fundamental relative invariant (a quadratic form), this space is not *QD1*.

**Tables of indecomposable, saturated, multiplicity free representations
with one dimensional quotient**

Table 2: Irreducible representations
(Notations for representations as in [2], see section 4.1.)

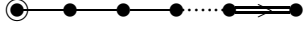
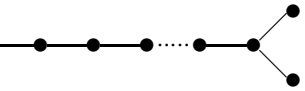
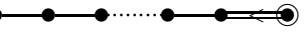
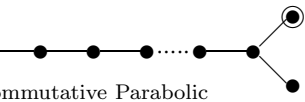
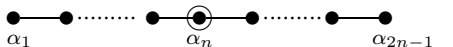
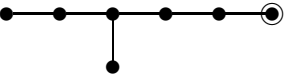


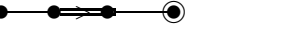

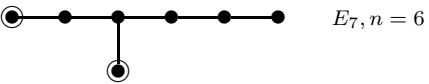
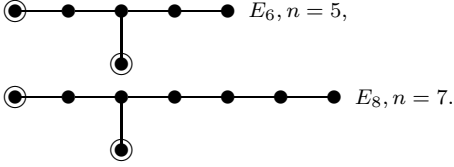
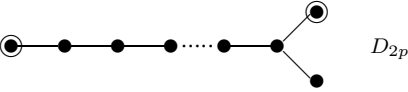
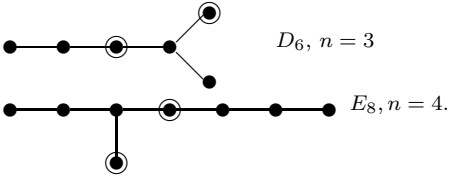
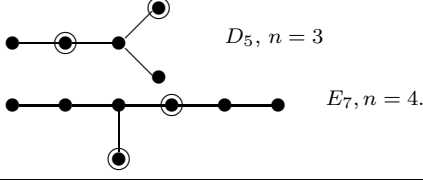
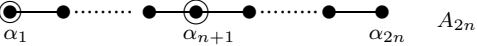

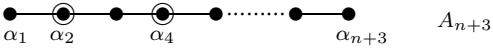
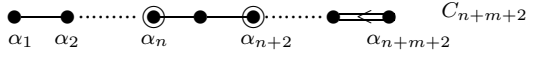
Representation, rank	Weighted Dynkin diagram (if parabolic type)	Regular	Fundamental invariant
(1) $SO(n) \times \mathbb{C}^*$ ($n \geq 3$) ($n \geq 3$), rank=2	$n = 2p + 1$  B_{p+1} $n = 2p$  D_{p+1} Commutative Parabolic (both)	Yes	Non degenerate quadratic form
(2) $S^2(SL(n)) \times \mathbb{C}^*$ ($n \geq 2$), rank=n	 C_n Commutative Parabolic	Yes	Determinant on symmetric matrices
(3) $\Lambda^2(SL(n)) \times \mathbb{C}^*$ ($n \geq 4$) and $n = 2p$ rank=p	 D_{2p} Commutative Parabolic	Yes	pfaffian on skew symmetric matrices
(4) $(SL(n)^* \otimes SL(n)) \times \mathbb{C}^*$ ($n \geq 2$), rank=n	 A_{2p-1} Commutative Parabolic	Yes	Determinant on full matrix space
(5) $E_6 \times \mathbb{C}^*$ (dim=27) rank=3	 E_7 Commutative Parabolic	Yes	Freudenthal cubic
(6) $(SL(2) \otimes Sp(n)) \times \mathbb{C}^*$ ($n \geq 2$), rank=3	 C_{n+2}	Yes	$Pf({}^t X J X)$ $X \in M(2n, 2)$ Pf =pfaffian of 2×2 matrices
(7) $SL(4) \times Sp(2) \times \mathbb{C}^*$ rank=6	 C_6	Yes	Det(X), $X \in M(4)$
(8) $Spin(7) \times \mathbb{C}^*$ rank=2	 F_4	Yes	Non degenerate quadratic form ($Spin(7) \hookrightarrow SO(8)$)
(9) $Spin(9) \times \mathbb{C}^*$ rank=3	Non parabolic	Yes	Non degenerate quadratic form
(10) $G_2 \times \mathbb{C}^*$ (dim = 7) rank=2	Non parabolic	Yes	Non degenerate quadratic form $G_2 \hookrightarrow SO(7)$

Table 3: Non Irreducible representations
(Notations for representations as in [2], see section 4.2.)

Representation	Weighted Dynkin diagram (if parabolic type)	Regular	Fundamental invariant
(1) $(SL(n)^* \oplus_{SL(n)} SL(n)) \times (\mathbb{C}^*)^2$ $n \geq 2$ rank=3	 A_{n+1}	Yes	$f(u, v) = uv$ on $M(1, n) \oplus M(n, 1)$
(2)(a) $(SL(n) \oplus_{SL(n)} \Lambda^2(SL(n))) \times (\mathbb{C}^*)^2$ $(n \geq 4, n = 2p \text{ even})$ rank= $n=2p$	non parabolic except for the case:  $E_{7, n = 6}$	No	pfaffian on skew symmetric matrices (on 2nd component)
(2)(b) $(SL(n) \oplus_{SL(n)} \Lambda^2(SL(n))) \times (\mathbb{C}^*)^2$ $(n \geq 4, n = 2p + 1 \text{ odd})$ rank= $n=2p+1$	non parabolic except for the cases:  $E_{6, n = 5},$ $E_{8, n = 7}.$	Yes	$f(v, x) = Pf\left(\begin{bmatrix} x & v \\ -t_v & 0 \end{bmatrix}\right)$ $v \in \mathbb{C}^n$ $x \in AS(2p + 1)$
(3) $(SL(n)^* \oplus_{SL(n)} \Lambda^2(SL(n))) \times (\mathbb{C}^*)^2$ $(n \geq 4, n = 2p \text{ even})$ rank= n	 D_{2p}	No	$Pf(x)$ (pfaffian) (on 2nd component)
(4)(a) $SL(n) \oplus_{SL(n)} (SL(n) \otimes SL(n)) \times (\mathbb{C}^*)^2, n \geq 2$	non parabolic except for the cases:  $D_{6, n = 3}$ $E_{8, n = 4}.$	No	Determinant (on 2nd component)
(4)(b) $(SL(n) \oplus_{SL(n)} (SL(n) \otimes SL(n - 1))) \times (\mathbb{C}^*)^2, n \geq 3$	non parabolic except for the cases:  $D_{5, n = 3}$ $E_{7, n = 4}.$	Yes	$\det(v; x)$ $v \in M_{n,1}, x \in M_{n,n-1}$
(5) $SL(n)^* \oplus_{SL(n)} (SL(n) \otimes SL(n)) \times (\mathbb{C}^*)^2, n \geq 3$ rank= $2n$	 A_{2n}	No	Determinant (on 2nd component)

Continued next page.

Table 3(continued): Non Irreducible representations
 (Notations for representations as in [2], see section 4.2)

Representation, rank	Weighted Dynkin diagram (if parabolic type)	Regular	Fundamental invariant
(6) $(SL(2) \oplus_{SL(2)} (SL(2) \otimes Sp(n)))$ $(\times \mathbb{C}^*)^2$ $n \geq 2$ rank=3	 C_{n+3}	No	$Pf({}^t X J X)$ $X \in M(2n, 2)$ $Pf = pfaffian$ (on 2nd component)
(7) $(SL(2) \otimes SL(2)) \oplus_{SL(2)} (SL(2) \otimes SL(n))$ $\times (\mathbb{C}^*)^2$ $(n \geq 3)$ rank=5	 A_{n+3}	No	$Det(X), X \in M(2, 2)$ (on 1st component)
(8) $(SL(n) \otimes SL(2)) \oplus_{SL(2)} (SL(2) \otimes Sp(m))$ $\times (\mathbb{C}^*)^2$ $(n \geq 3, m \geq 2)$ rank=6	 C_{n+m+2}	No	$Pf({}^t X J X)$ $X \in M(2m, 2)$ $Pf = pfaffian$ (on 2nd component)
(9) $(Sp(n) \oplus_{Sp(n)} Sp(n)) \times (\mathbb{C}^*)^2, n \geq 2$ rank=4	Non parabolic	Yes	$f(u, v) = {}^t u J v$ on $M(1, 2n) \oplus M(1, 2n)$

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