On Two Applications of Herschel's Theorem

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Abstract. As a first application of a very old theorem, known as Herschel's theorem, we provide direct elementary proofs of several explicit expressions for some numbers and polynomials that are known in combinatorics. The second application deals with the analytical continuation of the polylogarithmic function of complex argument beyond the circle of convergence.

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HERSCHEL'S THEOREM

Herschel's theorem gives the expression of the expansion of a function of the form $\phi(e^{-t})$ into a Taylor series in ascending powers of t. The proof of Herschel's theorem is straightforward: By Taylor's theorem, we have

$$\begin{aligned}\phi(e^{-t}) &= \phi(1-(1-e^{-t})) = \phi(1)+\phi'(1)\frac{(-1)}{1!}(1-e^{-t})+\phi''(1)\frac{(-1)^2}{2!}(1-e^{-t})^2+ \\ &\quad \cdots+\phi^{(n)}(1)\frac{(-1)^n}{n!}(1-e^{-t})^n+\cdots \\ &= \sum_{n=0}^{\infty}\phi^{(n)}(1)\frac{(-1)^n}{n!}(1-e^{-t})^n. \end{aligned} \tag{1}$$

If we let

$$\phi(e^{-t}) = a_0+a_1t+a_2t^2+\cdots+a_nt^n+\cdots=\sum_{n=0}^{\infty}a_nt^n, \tag{2}$$

then the coefficient a_n will be equal to the sum of the coefficients of t^n in the expansion of all the terms in the right hand side of equation (1). Now, we know that

$$(-1)^k(1-e^{-t})^k=(-1)^k\binom{k}{0}e^{-0.t}+(-1)^{k-1}\binom{k}{1}e^{-t}+\cdots-\binom{k}{k-1}e^{-(k-1)t}+\binom{k}{k}e^{-kt}, \tag{3}$$

and so the coefficient of t^n in the right hand side of equation (3) is equal to

$$\frac{(-1)^n}{n!}\left[(-1)^k\binom{k}{0}0^n+(-1)^{k-1}\binom{k}{1}1^n+\cdots-\binom{k}{k-1}(k-1)^n+\binom{k}{k}k^n\right]. \tag{4}$$

But using the notation of the calculus of finite differences, the last equation can be written as

$$\frac{(-1)^n}{n!}\Delta^k0^n. \tag{5}$$

Combination of the above equations leads to a formula for the coefficient a_n, better known in the old literature as Herschel's theorem [1, chap. 2], [6]:

Theorem 1 (Herschel's Theorem)

$$a_n=\frac{(-1)^n}{n!}\left[\phi(1).0^n+\frac{\phi'(1)}{1!}\Delta0^n+\frac{\phi''(1)}{2!}\Delta^20^n+\cdots+\frac{\phi^{(n)}(1)}{n!}\Delta^n0^n\right]=\frac{(-1)^n}{n!}\sum_{j=0}^{n}\frac{\phi^{(j)}(1)}{j!}\Delta^j0^n. \tag{6}$$

FINDING EXPLICIT FORMULAS USING HERSCHEL'S THEOREM

Several explicit formulas can be deduced from Herschel's Theorem. Let's suppose that we have a series of numbers defined by a generating function for which we want to find an explicit formula. The first step consists in expressing the function as a function of e^{-t}. The second step is to use Theorem 1 to provide the explicit formula using finite differences of 0. We finish the sections with some examples.

Bernoulli Numbers

The generation function of Bernoulli numbers is given by [4, p. 48]:

$$\phi(e^{-t}) = \frac{t}{e^t - 1} = \frac{-\ln(e^{-t})e^{-t}}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}; |t| < 2\pi \tag{7}$$

By setting $X = 1 - e^{-t}$, we have

$$\begin{aligned} \phi(1-X) &= \frac{\ln(1-X)}{X}(1-X) = \frac{1}{X}\left[X + \frac{X^2}{2} + \frac{X^3}{3} + \cdots\right](1-X), \\ &= 1 - \frac{X}{1.2} - \frac{X^2}{2.3} - \cdots - \frac{X^n}{n.(n+1)} - \cdots \end{aligned} \tag{8}$$

where in this case $\frac{\phi^{(j)}(1)}{j!} = -\frac{1}{j(j+1)}$ for $j \geq 1$. Herschel's theorem gives the following well-known explicit formula for Bernoulli numbers:

$$B_n = (-1)^n n! a_n = 1.0^n + (-1)^{n+1}\frac{\Delta 0^n}{1.2} + (-1)^{n+1}\frac{\Delta^2 0^n}{2.3} + \cdots + (-1)^{n+1}\frac{\Delta^n 0^n}{n(n+1)}. \tag{9}$$

Euler Polynomials

Euler polynomials of degree n in x are denoted by $E_n(x)$ and are defined by the generating function

$$\phi(e^{-t}) = \frac{2e^{tx}}{e^t + 1} = \frac{2e^{-t(1-x)}}{1 + e^{-t}} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}; |t| < \pi \tag{10}$$

Again by setting $X = 1 - e^{-t}$, we rewrite the generating function as

$$\phi(1-X) = \frac{(1-X)^{1-x}}{1 - \frac{X}{2}}. \tag{11}$$

Since $|X| < 1$, the generalized binomial theorem gives

$$(1-X)^{1-x} = \sum_{k=0}^{\infty}(-1)^k \binom{1-x}{k} X^k = 1 + (x-1)X + \frac{(x-1)x}{2!}X^2 + \frac{(x-1)x(x+1)}{3!} + \cdots \tag{12}$$

$$\frac{1}{1 - \frac{X}{2}} = 1 + X + \frac{X^2}{2^2} + \frac{X^3}{3^2} + \cdots, \tag{13}$$

The product of the two series provides

$$\phi(1-X) = \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{k=0}^{n}\frac{(-1)^k}{2^k}\binom{1-x}{k}X^n. \tag{14}$$

In this case, $\frac{\phi^{(j)}(1)}{j!} = \frac{(-1)^j}{2^j}\sum_{k=0}^{j}\frac{(-1)^k}{2^k}\binom{1-x}{k}$, and therefore, Herschel's theorem gives the desired explicit formula for Euler polynomials:

$$E_n(x) = (-1)^n n! a_n = (-1)^n \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{k=0}^{j} \frac{(-1)^k}{2^k} \binom{1-x}{k} \Delta^j 0^n. \tag{15}$$

The reader can compare with the formula obtained in [9]. Clearly, one can also obtain an explicit formula for Euler polynomials of higher order.

Eulerian Numbers and Polynomials

It is known that the classical Eulerian polynomials $A_n(\lambda)$, $0 < \lambda < 1$ have the exponential generating function [4, p.51], [2]

$$1 + \sum_{n=1}^{\infty} \frac{A_n(\lambda)}{\lambda} \frac{t^n}{n!} = \frac{1-\lambda}{e^{t(\lambda-1)} - \lambda}, \tag{16}$$

By replacing t by $\frac{t}{\lambda-1}$ in (16), we obtain the following function, [4, chap. 6, p.244]:

$$1 + \sum_{n=1}^{\infty} \frac{A_n(\lambda)}{\lambda(\lambda-1)^n} \frac{t^n}{n!} = \frac{1-\lambda}{e^t - \lambda}, \tag{17}$$

Carlitz [3] denoted the numbers $\frac{A_n(\lambda)}{\lambda(\lambda-1)^n}$ by $H_n(\lambda)$.

A theorem of Frobenius [5] states that the Eulerian polynomials are given by

$$\begin{aligned} A_n(\lambda) &= \lambda \sum_{j=1}^{n} j! S(n,j) (\lambda-1)^{n-j} && (18) \\ &= \lambda \sum_{j=1}^{n} (\lambda-1)^{n-j} \Delta^j 0^n, && (19) \end{aligned}$$

where $S(n,j)$ are the Stirling numbers of the second kind. Note that Stirling numbers of the second kind are defined by

$$S(n,j) = \frac{1}{j!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^n, \tag{20}$$

which can be easily written as a function of k'th forward difference of 0^n:

$$S(n,j) = \frac{1}{j!} \Delta^j 0^n. \tag{21}$$

We now give a new formula for the Eulerian polynomials using Herschel's theorem. The formula complements Frobenius formula. We start by setting $X = 1 - e^{-t}$ and rewriting the generating function (17) as

$$\phi(e^{-t}) = \frac{1-\lambda}{e^t - \lambda} = \frac{(1-\lambda)e^{-t}}{1 - \lambda e^{-t}} = \sum_{n=0}^{\infty} H_n(\lambda) \frac{t^n}{n!}; |t| < \log\lambda + 2\pi. \tag{22}$$

In terms of the variable X, the generating function becomes

$$\phi(1-X) = \frac{1-X}{1 - \frac{\lambda}{\lambda-1}X} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(\lambda-1)^n} X^n. \tag{23}$$

By Herschel's theorem, we obtain $H_0(\lambda) = 1$ and for $n \geq 1$

$$H_n(\lambda) = (-1)^n \sum_{j=1}^{n} (-1)^j \frac{\lambda^{j-1}}{(\lambda-1)^j} \Delta^j 0^n. \tag{24}$$

Thus, $A_0(\lambda) = 1$ and for $n \geq 1$

$$A_n(\lambda) = \sum_{j=1}^{n} \lambda^j (1-\lambda)^{n-j} \Delta^j 0^n. \tag{25}$$

Note that equation (25) reminds of Bernstein polynomials with coefficients as functions of differences of zero.

Genocchi Numbers

Genocchi numbers are defined by the generating function [4, p. 49].

$$\frac{2t}{e^t+1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}. \tag{26}$$

Using the change of variable $X = 1 - e^{-t}$, the generating function becomes

$$\phi(e^{-t}) = \phi(1-X) = -\frac{(1-X)\log(1-X)}{1-\frac{X}{2}} \tag{27}$$

$$\log(1-X)(1-X) = X - \frac{X^2}{1.2} - \frac{X^3}{2.3} - \cdots - \frac{X^n}{(n-1).n} - \cdots, \tag{28}$$

$$\frac{1}{1-\frac{X}{2}} = 1 + \frac{X}{2} + \frac{X^2}{2^2} + \cdots + \frac{X^n}{2^n} - \cdots \tag{29}$$

Multiplying the two power series, the coefficient of X^0 is zero, the coefficient of X is equal to 1 and the coefficient of X^n is given by

$$(-1)^n \frac{\phi^{(n)}(1)}{n!} = \frac{1}{2^{n-1}} \left[1 - \sum_{k=2}^{n} \frac{2^{k-1}}{(k-1)k} \right], n \geq 2. \tag{30}$$

Finally, an application of Herschel's theorem yields the formula for the Genocchi numbers:

$$G_n = (-1)^n n! a_n = (-1)^n \left\{ 1 + \sum_{j=2}^{n} \frac{1}{2^{j-1}} \left[1 - \sum_{k=2}^{n} \frac{2^{k-1}}{(k-1)k} \right] \Delta^j 0^n \right\}. \tag{31}$$

AN ANALYTIC CONTINUATION OF THE POLYLOGARITHM

The polylogarithm $\mathrm{Li}_s(x)$ is defined by the power series

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}. \tag{32}$$

The above definition is valid for all complex values s and all complex values of x such that $|x| < 1$.
The conformal mapping

$$x = 1 - e^{-t}, \tag{33}$$

$$t = -\mathrm{Log}(1-x) \tag{34}$$

maps the part of the x-plane between two half-lines starting from the point $(1,0)$ to a strip parallel to the x-axis in the t plane. Moreover, the function $1-e^{-t}$ is conformal at each point $t \in \mathbb{C}$, since its derivative does not vanish at t. Its restriction to the horizontal strip $\{|\mathrm{Im}(t)| < \pi\}$ is a conformal mapping of the strip onto the cut plane $\mathbb{C} \setminus [1,\infty)$.

The principal branch $\mathrm{Log}(1-x)$ of $\log(1-x)$ is also a conformal mapping of the cut plane $\mathbb{C} \setminus [1,\infty)$ onto the horizontal strip $\{|\mathrm{Im}(t)| < \pi\}$.

It is easy to verify that $\mathrm{Li}_s(x)$ has one finite singularity, namely the point $x = 1$. The point $x = 1$ is mapped to ∞ by the conformal mapping (34). Making the substitution (33) into (32), the series becomes

$$\mathrm{Li}_s(1-e^{-t}) = \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{n^s}. \tag{35}$$

Expanding the right hand side of (35) into a series in powers of t, we get

$$\mathrm{Li}_s(1-e^{-t}) = \sum_{n=0}^{\infty} a_n t^n. \tag{36}$$

where the a_n may be calculated using Herschel's theorem. Indeed, we have here

$$(-1)^n \frac{\phi^{(n)}(1)}{n!} = \frac{1}{n^s}, n \geq 1; \phi(1) = 0. \tag{37}$$

By Herschel's theorem, we thus have

$$a_0 = 0, \quad a_n = \frac{(-1)^n}{n!} \sum_{j=1}^{n} \frac{(-1)^j}{j^s} \Delta^j 0^n, \quad n \geq 1. \tag{38}$$

Now if the function $\mathrm{Li}_s(x)$ is regular in the plane $\mathbb{C}$ minus the semiaxis $\mathrm{Re}(x) > 1$, the series (36) is necessarily convergent in the circle $|t| < 1$. Conversely, if the series (36) is convergent in the circle $|t| < 1$, then the function $\mathrm{Li}_s(x)$ is regular in the cut plane $\mathbb{C} \setminus [1,\infty)$. Therefore, we can assert that $\mathrm{Li}_s(x)$ can be represented at any point of the cut plane $\mathbb{C}$ by the following expansion

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n!} \sum_{j=1}^{n} \frac{(-1)^j}{j^s} \Delta^j 0^n \right) (\mathrm{Log}(1-x))^n. \tag{39}$$

There exists other integral and series relations which provide the analytic continuation of the polylogarithm beyond the circle of convergence $|x| = 1$ of the defining power series. But these relations are valid for all but some exceptional values of s, [8, p. 139-140], [7, 11, 10]. To the author's knowledge, the series (39) is the only series that defines the polylogarithm for all values of $s \in \mathbb{C}$ and all values of $x \in \mathbb{C}$.

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