A stochastic model for elasticity tensors exhibiting uncertainties on material symmetries

Johann Guilleminot, Christian Soize

A stochastic model for elasticity tensors exhibiting uncertainties on material symmetries

Johann Guilleminot*, Christian Soize

Université Paris-Est, Laboratoire Modélisation et Simulation Multi Echelle, MSME UMR8208 CNRS, 5 Bd Descartes, 77454 Marne la Vallée, France

Abstract

This study addresses the stochastic modeling of media whose elasticity tensor exhibits uncertainties on the material symmetry class to which it belongs. More specifically, we focus on the construction of a probabilistic model which allows realizations of random elasticity tensors to be simulated, under the constraint that the mean distance (in a sense to be defined) to a given class of material symmetry is specified. Following the eigensystem characterization of the material symmetries, the proposed approach relies on the probabilistic model derived in [6] which allows the variance of selected eigenvalues of the elasticity tensor to be partially prescribed. A new methodology and parameterization of the model are then defined. The proposed approach is exemplified considering the mean to transverse isotropy. The efficiency of the methodology is demonstrated by computing the mean distance of the random elasticity tensor to this material symmetry class, the distance and projection onto the space of transversely isotropic tensors being defined by considering the Riemannian metric and the Euclidean projection, respectively. It is shown that the methodology allows the above distance to be (partially) reduced as the overall level of statistical fluctuations increases, no matter the initial distance of the mean model used in the simulations. A comparison between this approach and the nonparametric probabilistic approach (with anisotropic fluctuations) proposed in [12] is finally provided.

Key words: Elasticity tensor; Material Symmetry; Maximum Entropy Principle; Probabilistic Model; Uncertainty.

*Corresponding author
Email address: `johann.guilleminot@univ-paris-est.fr` (Johann Guilleminot)

1. Introduction

This study focuses on the probabilistic modeling of random media whose linear behavior is defined with respect to an elasticity tensor exhibiting uncertainties on the material symmetry class to which it belongs. Such a problematic typically arises:

(i) in the experimental identification of material properties at various scales (and especially, for materials whose characterization is performed at a scale close to the characteristic size of the heterogeneities, such as some long-fiber reinforced composites, living tissues and concretes), for which material symmetry properties may be either relaxed or assumed.

(ii) in the computational stochastic modeling of structures, when the randomness induced by fine-scale features may have to be taken into account at a coarse scale (in order to study robust design or prognosis, for instance).

More specifically, we address the modeling of random elasticity tensors whose mean distance (in a sense to be defined) to a given class of material symmetry (for instance, the symmetry class that is usually assumed for the random media under consideration) is specified. Clearly, such a modeling can not be achieved, neither by using a parametric approach (which would imply, by definition, a null distance to the symmetry class, no matter the realization of the random elasticity tensor) nor by considering the nonparametric approach for anisotropic media, introduced in [12] and for which the anisotropic statistical fluctuations make the distance to a given material symmetry class increase with the overall level of fluctuation.
Thus, we propose in this research both the construction of a probabilistic model and the definition of a methodology allowing the numerical simulation (and consequently, the inverse experimental identification) of random elasticity tensors under material symmetry constraints.

2. Definition of distances in the set of elasticity tensors

In the following, we will consider the Kelvin matrix representation $[C] \in \mathbb{M}_6^+(\mathbb{R})$ (where $\mathbb{M}_6^+(\mathbb{R})$ is the set of all the 6×6 symmetric positive-definite real matrices) of the fourth-order tensor $[[C]]$ (with components $[[C]]_{ijk\ell}$) belonging to the set of elasticity tensors (verifying the usual symmetry and positiveness properties). The question of defining the distance between elasticity tensors has received considerable attention, especially within the context of experimental identification (see [3] for an application in geophysics for instance). Indeed, several metrics have been introduced in the literature to quantify the distance between two elasticity tensors, the most widely used metrics being the Euclidean, Log-Euclidean [1] and Riemannian ones [7], denoted by d_E, d_{LE} and d_R respectively, and defined for any elasticity tensors $[[C_1]]$ and $[[C_2]]$ by:

$$d_E([[C_1]],[[C_2]]) = \|[[C_2]] - [[C_1]]\|, \tag{1}$$

$$d_{LE}([[C_1]],[[C_2]]) = \|\log\left([[C_2]]\right) - \log\left([[C_1]]\right)\|, \tag{2}$$

$$d_R([[C_1]],[[C_2]]) = \|\log\left([[C_1]]^{-1/2}[[C_2]][[C_1]]^{-1/2}\right)\|, \tag{3}$$

in which $< [[C]],[[D]] >= [[C]]_{ijk\ell}[[D]]_{ijk\ell}$ and $\|[[C]]\| =< [[C]],[[C]] >^{1/2}$. In particular, the convention retained ensures the preservation of the norm, no matter the representation of the elasticity tensor. Let $\mathcal{C}^{\mathrm{Sym}}$ be a class of elasticity tensors with given symmetries (isotropy, transverse isotropy, orthotropy, etc.). Let $[[C]]$ be a fourth-order elasticity tensor having an arbitrary symmetry, with components $[[C]]_{ijk\ell}$ with respect to a given frame $\mathfrak{R} = (0,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. We then denote by $[[C^{\mathrm{Sym}}]] = \mathcal{P}^{\mathrm{Sym}}\left([[C]]\right)$ the projection of $[[C]]$ onto $\mathcal{C}^{\mathrm{Sym}}$, calculated by using one of the distance d introduced above, such that:

$$[[C^{\mathrm{Sym}}]] = \underset{[[\widetilde{C}]]\in\mathcal{C}^{\mathrm{Sym}}}{\mathrm{Arg\,min}}\; d([[C]],[[\widetilde{C}]]). \tag{4}$$

The matrix representation $[C^{\mathrm{TI}}]$ of the projection $[[C^{\mathrm{TI}}]]$ of $[[C]]$ onto the set of all the elasticity tensors exhibiting transverse isotropy with respect to $\mathbf{e}_3$, defined with respect to the Euclidean distance d_E, can be found in [8] for instance.

3. Probabilistic model derivation

In this work, we consider the eigensystem coordinate-free characterization of the material symmetries [2], according to which a material symmetry class can be defined by both the multiplicities of the eigenvalues and constraints on the related eigenspaces. In this context, it should be pointed out that (**i**) the use of the classical random ensembles from the Random Matrix Theory generally implies all the stochastic eigenvalues to be of multiplicity one, and that (**ii**) the corresponding random eigenspaces cannot be explicitly constrained nor described. It then follows that the mean distance of the random elasticity tensor to a given material symmetry class can be partially controlled by imposing constraints on the variance of a few selected random eigenvalues. The proposed approach thus relies on the probabilistic model for symmetric positive-definite random matrices with prescribed variance on several eigenvalues, derived in [6] and briefly recalled below.

Let $[\mathbf{C}]$ be the $\mathbb{M}_6^+(\mathbb{R})$-valued random matrix representation of the random elasticity tensor under consideration. The construction of the model relies on the use of the Maximum Entropy Principle (see [4] [5] [9]) which consists in maximizing the measure of entropy S, defined as:

$$S = -\int_{\mathbb{M}_6^+(\mathbb{R})} p_{[\mathbf{C}]}([C]) \ln \left(p_{[\mathbf{C}]}([C])\right) \mathrm{d}C, \tag{5}$$

with respect to the probability density function $p_{\mathbf{C}}$, where $[C] \mapsto p_{[\mathbf{C}]}([C])$ is the probability density function from $\mathbb{M}_6^+(\mathbb{R})$ into $\mathbb{R}^+$ defining the probability distribution $P_{[\mathbf{C}]} = p_{[\mathbf{C}]}([C])\,\mathrm{d}C$ of random matrix $[\mathbf{C}]$. The volume measure $\mathrm{d}C$ on $\mathbb{M}_6^S(\mathbb{R})$ is written as $\mathrm{d}C = 2^{15/2}\prod_{1\leq i\leq j\leq 6}\mathrm{d}[C]_{ij}$ [10]. The optimization problem (5) is solved under the following set of constraints:

$$\int_{\mathbb{M}_6^+(\mathbb{R})} p_{[\mathbf{C}]}([C])\,\mathrm{d}C = 1, \tag{6}$$

$$\mathrm{E}\{[\mathbf{C}]\} = \int_{\mathbb{M}_6^+(\mathbb{R})} [C]\, p_{[\mathbf{C}]}([C])\,\mathrm{d}C = [\underline{C}], \tag{7}$$

$$\int_{\mathbb{M}_6^+(\mathbb{R})} \ln\left(\det([C])\right) p_{[\mathbf{C}]}([C])\,\mathrm{d}C = \beta, \quad |\beta| < +\infty, \tag{8}$$

$$\mathrm{E}\left\{\left(\underline{\boldsymbol{\varphi}}^{i\,\mathrm{T}}\,[\mathbf{C}]\,\underline{\boldsymbol{\varphi}}^{i}\right)^2\right\} = s_i^2 \underline{\lambda}_i^2, \quad i \in \mathcal{I} \subseteq [1,6], \tag{9}$$

where $\mathrm{E}\{\cdot\}$ is the mathematical expectation, $\det([C])$ and $[C]^{\mathrm{T}}$ are the determinant and the transpose of $[C]$, $\left\{\left(\underline{\lambda}_i, \underline{\boldsymbol{\varphi}}^i\right)\right\}_i$ are the eigenvalues and eigenvectors of the mean matrix $[\underline{C}]$. The family $\{s_i\}_{i\in\mathcal{I}}$ is a set of m parameters which are supposed to be either assumed or computed from an experimental inverse identification. The set of constraints defined by Eqs. (6-8) basically corresponds to the one previously considered and largely studied in [10] [11], while the set of constraints (9) allows one to partially prescribe the variances of m ($m \leq 6$) selected random eigenvalues $\{\lambda_i\}_{i=1}^m$

of [**C**] (see [6] for a discussion). Optimization problem (5) can be solved introducing a set of Lagrange multipliers and having recourse to the calculus of variations applied to the resulting Lagragian. It can then be shown that the probability density function $[C] \mapsto p_{[\mathbf{C}]}([C])$ takes the form:

$$\begin{aligned} p_{[\mathbf{C}]}([C]) = \quad & k_1 \left(\det([C])\right)^{\alpha-1} \\ & \times \exp\left(-\operatorname{tr}\left([M_1]^{\mathrm{T}}[C]\right) - \textstyle\sum_{i\in\mathcal{I}} \tau_i' \left(\underline{\varphi}^{i\mathrm{T}}\,[C]\,\underline{\varphi}^i\right)^2\right), \end{aligned} \tag{10}$$

in which k_1 is the normalization constant (depending on the Lagrange multiplier $\mu_0 \in \mathbb{R}$ corresponding to constraint (6)); $[M_1] \in \mathbb{M}_6^S(\mathbb{R})$, $(\alpha - 1) \in \mathbb{R}$ and $\{\tau_i' \in \mathbb{R}\}_{i=1}^6$ are the Lagrange multipliers associated with the constraints (7-9). Further details about the derivation of the model, as well as a strategy for generating realizations of random matrix [**C**], can be found in [6]. Finally, it can be shown that the model can be entirely parameterized by:

- the parameter α, controlling the overall level of statistical fluctuations characterized by $\delta_{\mathbf{C}} = \{\mathrm{E}\{\|[\mathbf{C}] - [\underline{C}]\|_{\mathrm{F}}^2\}/\|[\underline{C}]\|_{\mathrm{F}}^2\}^{1/2}$;
- a set of m parameters τ_i, allowing the variances of the m selected stochastic eigenvalues to be (partially) prescribed.

Here, we propose to characterize the capability of the approach to reduce the (mean) distance to a relevant material symmetry class $\mathcal{C}^{\mathrm{Sym}}$ by studying the mapping $(\alpha, \{\tau_i\}_{i=1}^m) \mapsto \mathrm{E}\{d([\mathbf{C}], [\mathbf{C}^{\mathrm{Sym}}])\}$, where d is any of the distance previously defined. It should be pointed out that in accordance with the philosophy of the Maximum Entropy Principle, fixing values of parameters $(\alpha, \{\tau_i\}_{i=1}^m)$ (in a given admissible space) is strictly equivalent to considering given values of parameters $\{s_i\}_{i\in\mathcal{I}}$. Furthermore, since the Euclidean metric yields a closed-form expression of the projection (which is more suitable for the proposed probabilistic analysis), the use of distance d_E is retained hereafter.

4. Application

In this application, we consider the following mean model, corresponding to a random perturbation of the elasticity tensor of a carbon-epoxy unidirectional composite and being characterized by a small distance to transverse isotropy:

$$[\underline{C}] = \begin{bmatrix} 10.1036 & 0.5391 & 2.9625 & -0.0040 & 0.0071 & -0.0165 \\ 0.5391 & 10.1061 & 2.9782 & -0.0041 & -0.0070 & -0.0036 \\ 2.9625 & 2.9782 & 182.690 & 0.0197 & 0.0016 & 0.0145 \\ -0.0040 & -0.0041 & 0.0197 & 14.0339 & 0.0068 & 0.0008 \\ 0.0071 & -0.0070 & 0.0016 & 0.0068 & 14.0121 & -0.0103 \\ -0.0165 & -0.0036 & 0.0145 & 0.0008 & -0.0103 & 9.5552 \end{bmatrix}.$$

The mean distance to transverse isotropy is controlled by enforcing a small variance on the random eigenvalues λ_1, λ_2, λ_4 and λ_5, that is to say, by setting all the parameters controlling the variances to the same, large, value τ ($\tau_1 = \tau_2 = \tau_4 = \tau_5 = \tau$, $\tau_3 = \tau_6 = 0$). For a given value of parameter α, the influence of parameter τ can be visualized on Fig. 1, where the probability density functions (estimated using the kernel density estimation method) of the five first stochastic eigenvalues are plotted for $\tau = 1$ (black solid line) and $\tau = 10^4$ (red solid line).

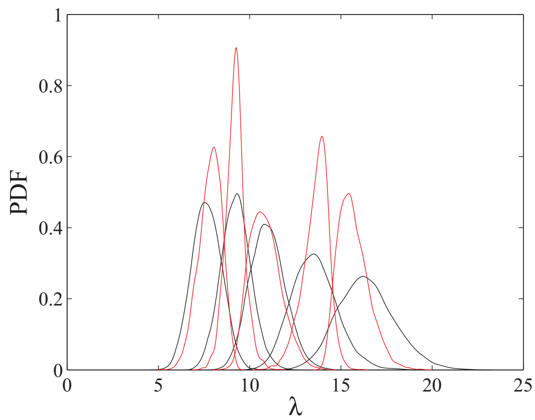

Figure 1: Plot of the probability density functions of the random eigenvalues λ_i, $i = 1, ..., 5$, for $\tau = 1$ (black solid line) and $\tau = 10^4$ (red solid line).

It is clearly seen that setting a large value of τ allows to reduce the variance of the constrained eigenvalues (while the variance of the third random eigenvalue remains unchanged). However, it is worth noting that no matter the value of τ, all the eigenvalues are stochastic, so that (**i**) the mean distance to a given symmetry class can be prescribed in a limited extent and (**ii**) the variances of the constrained eigenvalues tend to constant nonzero values as τ tends to infinity. The capability of the proposed approach to reduce the mean Riemannian distance to the considered symmetry class is illustrated on Fig. 2, where the plot of $\tau \mapsto \mathrm{E}\left\{d_R\left([\mathbf{C}], [\mathbf{C}^{\mathrm{TI}}]\right)\right\}$ is reported in semi-log scale for $\alpha = 60$ (corresponding to $\delta_{\mathbf{C}} = 0.15$) and for τ ranging from 10^{-1} to 10^4. This result is also illustrated for different values of α in Fig. 3.

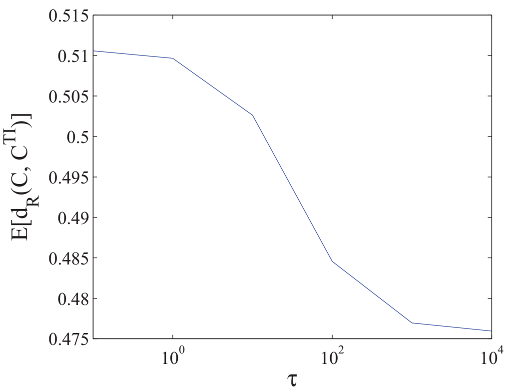

Figure 2: Plot of $\tau \mapsto \mathrm{E}\left\{d_R\left([\mathbf{C}], [\mathbf{C}^{\mathrm{TI}}]\right)\right\}$ for $\alpha = 60$.

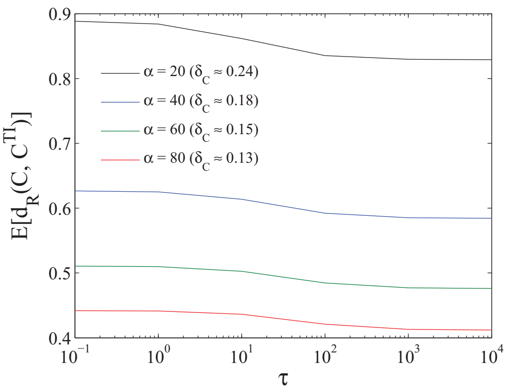

Figure 3: Plot of $\tau \mapsto \mathrm{E}\left\{d_R\left([\mathbf{C}], [\mathbf{C}^{\mathrm{TI}}]\right)\right\}$ for several values of parameter α.

Finally, the proposed approach and the nonparametric probabilistic model for anisotropic media are compared. It is shown that the methodology presented allows the reduction of the mean distance to transverse isotropy, no matter the mean model or the level of statistical fluctuations $\delta_{\mathbf{C}}$ used in the simulations.

5. Conclusion

In this paper, we address the stochastic modeling for elasticity tensors with uncertain material symmetries. The approach, based on the eigensystem characterization of the symmetry classes, allows the mean distance of the elasticity tensor to a given symmetry class to be partially controlled. Making use of a probabilistic model recently derived in the literature, we introduce and exemplify the methodology for the case of a prescribed distance to transverse isotropy, typically corresponding to unidirectional fibre-reinforced composites. We also provide a comparison between the proposed approach and the nonparametric probabilistic model for anisotropic media. It is worth noticing that beyond its capability to represent different classes of symmetries, the probabilistic model exhibits more parameters than any other stochastic model previously developed within a nonparametric framework and may thus be more suitable for the fundamental issue of inverse experimental identification under material symmetry uncertainties. It can also be used as a prior stochastic model for the development of computational approaches, where the underlying randomness arising from fine scale features may have to be taken into account at a coarse scale, for instance.

References

[1] V. Arsigny, P. Fillard, X. Pennec, and N. Ayache. Log-Euclidean metrics for fast and simple calculus on diffusion tensors. *Magnetic Resonance in Medicine*, 56:411–421, 2006.

[2] A. Bóna, I. Bucataru, and M. A. Slawinski. Coordinate-free characterization of the symmetry classes of elasticity tensors. *Journal of Elasticity*, 87:109–132, 2007.

[3] J.T. Browaeys and S. Chevrot. Decomposition of the elastic tensor and geophysical applications. *Geophys. J. Int.*, 159:667–678, 2004.

[4] E. T. Jaynes. Information theory and statistical mechanics. *Physical Review*, 106(4):620–630, 1957.

[5] E. T. Jaynes. Information theory and statistical mechanics. *Physical Review*, 108(2):171–190, 1957.

[6] M. P. Mignolet and C. Soize. Nonparametric stochastic modeling of linear systems with prescribed variance of several natural frequencies. *Probabilistic Engineering Mechanics*, 23:267–278, 2008.

[7] M. Moakher. On the averaging of symmetric positivedefinite tensors. *Journal of Elasticity*, 82:273–296, 2006.

[8] M. Moakher and A. N. Norris. The closest elastic tensor of arbitrary symmetry to an elasticity tensor of lower symmetry. *Journal of Elasticity*, 85:215–263, 2006.

[9] C. E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423/623–659, 1948.

[10] C. Soize. A nonparametric model of random uncertainties on reduced matrix model in structural dynamics. *Probabilistic Engineering Mechanics*, 15(3):277–294, 2000.

[11] C. Soize. Maximum entropy approach for modeling random uncertainties in transient elastodynamics. *Journal of the Acoutical Society of America*, 109(5):1979–1996, 2001.

[12] C. Soize. Tensor-valued random fields for meso-scale stochastic model of anisotropic elastic microstructure and probabilistic analysis of representative volume element size. *Probabilistic Engineering Mechanics*, 23:307–323, 2008.