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A Non-Heuristic Reduction Method For Graph Cut Optimization

F. Malgouyres*, N. Lermé †

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Abstract

Graph cuts optimization is now well established for their efficiency but remains limited to the minimization of some Markov Random Fields (MRF) over a small number of variables due to the large memory requirement for storing the graphs. An existing strategy to reduce the graph size consists in testing every node and to create the node satisfying a given local condition. The remaining nodes are typically located in a thin band around the object to segment. However, there does not exist any theoretical guarantee that this strategy permits to construct a global minimizer of the MRF.

In this paper, we propose a local test similar to already existing test for reducing these graphs. A large part of this paper consists in proving that any node satisfying this new test can be safely removed from the non-reduced graph without modifying its max-flow value. The constructed solution is therefore guaranteed to be a global minimizer of the MRF.

Afterwards, we present numerical experiments for segmenting grayscale and color images which confirm this property while globally having memory gains similar to ones obtained with the previous existing local test.

keywords: discrete optimization, graph cuts, segmentation, denoising

AMS classification: 68U10, 94A08, 62M40, 90C27

1 Introduction

The introduction of efficient combinatorial optimization tools based on minimum cuts (min-cut) / maximum flow (max-flow) have deeply modified the landscape of computer vision. Indeed, a wide spectrum of ill-posed problems such as segmentation, restoration or dense field estimation are solved by minimizing

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a functional involving a large number of variables. The latter minimization problems can be solved with a moderate empirical complexity using graph cuts. As a consequence, graph cuts have increased the quality and the quantity of low-level analysis tools.

Although graph cuts stayed behind the scene during one decade, they become more attractive thanks to a fast max-flow algorithm [2] and efficient heuristics for multi-labels problems [3].

In parallel, technological advances have both exploded the amount and the diversity of data to process. Processing and analyzing these data amounts to solve large scale optimization problems. Despite a low running time and good convergence properties, graph cuts sometimes fail to solve such problems due to their memory requirements. This is due to the fact the graphs in which the max-flow is computed usually contains as many vertices as the number of pixels and as many edges as the number of neighboring pixels in the image. This problem has been recently addressed by heuristics (see [12, 13, 14, 8]) and exact methods (see [10, 6, 15]).

To our best knowledge, this problem seems to be first tackled in [12]. The strategy adopted is to compute a graph cut in a graph built from a pre-segmentation. While this approach greatly increases the performance of standard graph cuts, the results depend on the algorithm used for computing the pre-segmentation and better results are obtained when over-segmentation occurs.

Band-based methods have also been proposed in [13, 14, 8]. A low-resolution version of the image is first segmented. Then, the solution is propagated to the finer resolution by only building the graph in a narrow band surrounding the interpolated foreground/background interface at that resolution. Although this strategy clearly improves the performance of standard graph cuts, it is less accurate to segment thin structures like blood vessels or filaments. Notice that this problem is notably reduced in [14] but still present for low-contrasted details. In [8], smaller graphs are obtained by associating an uncertainty measure to each pixel.

Exact methods have been also investigated (see [10, 6, 15]). In [10], binary energy functions are minimized for the shape fitting problem with graph cuts in a narrow band while ensuring the optimality on the solution. One makes a band evolve around the object to delineate by expanding it when the min-cut touches its boundary. This process is iterated until the band no longer evolves. Although the algorithm generally converges in few iterations, an initialization scheme still needs to be designed for pixel classification problems.

In [6], a parallel max-flow algorithm yielding a near-linear speedup with the number of processors is presented. While this method achieves good performance on large scale problems, the algorithm is relatively sensitive to the available amount of physical memory and remains less efficient on small graphs.

In [15], the problem is decomposed into optimizable sub-problems, solved independently and updated according to the results of the adjacent problems [15]. This process is iterated until convergence and optimality is guaranteed by Lagrangian decomposition.

Another band-based method was proposed for reducing graphs in binary

image segmentation [11]. The graph is progressively built by only adding nodes which satisfy a local test. In the manner of previous band-based methods, the graph nodes are typically located in a narrow band surrounding the object edges to segment. This method is able to segment large volumes when standard graph cuts fail while keeping low pixel error. The time for reducing the graph is sometimes compensated by the gain obtained thanks to the non-allocation of the useless nodes and the gain in the computation of the min-cut in a smaller graph. It is empirically observed in [11] that the value of the max-flow in the reduced graph is equal to the value of the max-flow in the original graph. However, there is no formal proof of this statement.

In this paper, we pursue the work of [11] and propose a local test (similar to the test in [11]) to reduce these graphs by discarding a large amount of nodes during the graph construction. While the cost for evaluating this test is slightly higher compared to [11], we give a formal proof that any node satisfying the new local test can be safely removed without modifying the max-flow value, keeping in this way the optimality on the solution.

The rest of this document is organized as follows. We first define some notations about flows and cuts in Section 2, present the new local test for reducing the graph as well as the main theorem of this paper in Section 3. The proof of this theorem is detailed in the next sections. This work is completed with experiments for segmenting large grayscale and color images in Section 7.

2 Notations and preliminaries

We consider a set of pixels $\mathcal{P} \subset \mathbb{Z}^d$, for a positive integer d . We consider two terminal nodes s and t and the set of nodes

$$\mathcal{V} = \mathcal{P} \cup \{s, t\}.$$

We consider a set of directed edges $\mathcal{E} \subset (\mathcal{V} \times \mathcal{V})$ such that $(\mathcal{V}, \mathcal{E})$ is a simple directed graph. We also assume that for every $p \in \mathcal{P}$,

$$(p, s) \notin \mathcal{E} \text{ and } (t, p) \notin \mathcal{E}. \quad (1)$$

We denote the neighbors of any node $p \in \mathcal{V}$ by

$$\sigma_{\mathcal{E}}(p) = \{q \in \mathcal{V}, (p, q) \in \mathcal{E} \text{ or } (q, p) \in \mathcal{E}\}.$$

We denote a walk of positive length $l \in \mathbb{N}$ by $p_0 - p_1 - \dots - p_l$, where $p_i \in \mathcal{V}$, for all $i \in \{0, \dots, l\}$, and $(p_i, p_{i+1}) \in \mathcal{E}$, for all $i \in \{0, \dots, l-1\}$. We also remind that a closed walk is such that $p_0 = p_l$. Moreover, a simple walk is such that $p_i \neq p_j$, for all $(i, j) \in \{0, \dots, l-1\}^2$ satisfying $i \neq j$ and $p_l \neq p_j$, for all $j \in \{1, \dots, l-1\}$. We denote by $W(p, q)$ the set containing all the walks starting at $p \in \mathcal{V}$ and ending at $q \in \mathcal{V}$.

We define the capacities as a mapping $c : (\mathcal{V} \times \mathcal{V}) \rightarrow \mathbb{R}^+$ and denote the capacity of any couple $(p, q) \in (\mathcal{V} \times \mathcal{V})$ by

$$c_{p,q} \geq 0.$$

Although c is defined for any $(p, q) \in (\mathcal{V} \times \mathcal{V})$, we always set

$$c_{p,q} = 0, \text{ when } (p, q) \notin \mathcal{E}, \quad (2)$$

so-that non-null capacities are only defined on existing edges. The purpose for extending the definition capacities to elements of $(\mathcal{V} \times \mathcal{V}) \setminus \mathcal{E}$ in this manner is to simplify notations in many upcoming summations and equations.

We assume, without loss of generality (see [9]), that capacities are such that for every $p \in \mathcal{P}$

$$c_{s,p} \neq 0 \Rightarrow c_{p,t} = 0. \quad (3)$$

We therefore summarize the capacities of the edges linked to the terminal nodes and set for all $p \in \mathcal{P}$

$$c_p = c_{s,p} - c_{p,t}. \quad (4)$$

For any $S \subset \mathcal{P}$, we denote the value of the s - t cut $(S \cup \{s\}, \mathcal{V} \setminus (S \cup \{s\}))$ in \mathcal{G} by $\text{val}_{\mathcal{G}}(S)$. We remind that

$$\text{val}_{\mathcal{G}}(S) = \sum_{\substack{p \in S \cup \{s\} \\ q \notin S \cup \{s\}}} c_{p,q}.$$

Notice that, thanks to (2), we have not clarified that $(p, q) \in \mathcal{E}$ in the above summation.

We use throughout the paper and remind here that, by convention, the empty sum is null:

$$\sum_{n \in \emptyset} a_n = 0 \quad , \text{ whatever } (a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$

We also define flows as any mapping $f : (\mathcal{V} \times \mathcal{V}) \rightarrow \mathbb{R}^+$ satisfying the capacity constraints

$$0 \leq f_{p,q} \leq c_{p,q} \quad , \text{ for all } (p, q) \in (\mathcal{V} \times \mathcal{V}), \quad (5)$$

and the flow conservation

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} f_{q,p} = \sum_{q \in \sigma_{\mathcal{E}}(p)} f_{p,q} \quad , \text{ for all } p \in \mathcal{V}. \quad (6)$$

Again, (2) and (5) guarantee that

$$f_{p,q} = 0, \text{ for any } (p, q) \notin \mathcal{E}. \quad (7)$$

This is the reason why we do not clarify that $(q, p) \in \mathcal{E}$ (resp. $(p, q) \in \mathcal{E}$) in the left (resp. right) hand side sum in (6). As usual, the value of the flow f in \mathcal{G} is defined by

$$\text{val}_{\mathcal{G}}(f) = \sum_{p \in \sigma_{\mathcal{E}}(t)} f_{p,t}. \quad (8)$$

Notice that we use the same notation for the value of a flow and the value of a s - t cut in \mathcal{G} . This abuse of notation will never be ambiguous once in context.

As for capacities, we summarize the flow passing through the edges linked to the terminal nodes and set for all $p \in \mathcal{P}$

$$f_p = f_{s,p} - f_{p,t}. \quad (9)$$

As is well known (and can easily be shown by induction on the cardinality of S), for any flow f and any $S \subset \mathcal{V}$ the flow entering S is equal to the flow exiting S :

$$\sum_{\substack{p \in S \\ q \notin S}} f_{q,p} = \sum_{\substack{p \in S \\ q \notin S}} f_{p,q}. \quad (10)$$

Considering (3), (5) and (9), we can rewrite (10) and obtain that

$$\text{for any } S \subset \mathcal{P}, \quad \sum_{p \in S} f_p + \sum_{\substack{p \in S \\ q \notin S}} (f_{q,p} - f_{p,q}) = 0. \quad (11)$$

We call max-flow any solution f^* of the linear program

$$\begin{cases} \max_f \text{val}_{\mathcal{G}}(f), \\ \text{under the constraints (5) and (6)}. \end{cases}$$

As is well known (see [5]), the value of the max-flow is equal to the value of the min-cut:

$$\text{val}_{\mathcal{G}}(f^*) = \min_{S \subset \mathcal{P}} \text{val}_{\mathcal{G}}(S).$$

Before going ahead, let us remind the details motivating the study of such a graph: When minimizing a pairwise Markov Random Field of the form

$$E(u) = \beta \sum_{p \in \mathcal{P}} E_p(u_p) + \sum_{(p,q) \in (\mathcal{P} \times \mathcal{P})} E_{p,q}(u_p, u_q), \quad \beta \in \mathbb{R}^+, \quad (12)$$

among $u \in \{0, 1\}^{\mathcal{P}}$ and when the terms $E_{p,q}(\cdot)$ are submodular, [9] proves that (12) can be globally minimized. The proof provides the construction of a graph \mathcal{G} and a constant $K \in \mathbb{R}$ such that for any $S \subset \mathcal{P}$, we have

$$\text{val}_{\mathcal{G}}(S) = E(u^S) + K, \quad (13)$$

where $u^S \in \{0, 1\}^{\mathcal{P}}$ is defined by

$$u_p^S = \begin{cases} 0 & \text{if } p \notin S \\ 1 & \text{if } p \in S \end{cases}, \quad \forall p \in \mathcal{P}. \quad (14)$$

Since (14) makes a one to one correspondence between cuts in \mathcal{G} and $\{0, 1\}^{\mathcal{P}}$, (13) guarantees that a min-cut in \mathcal{G} corresponds to a minimizer of (12). The latter can therefore be efficiently computed with a max-flow algorithm such as described in [2]. The graph constructed in [9] and [2] satisfies all the hypotheses made in this paper.

Throughout the paper, we consider a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, c)$, with \mathcal{V} , \mathcal{E} and c as above. All along this paper, we also denote $B \subset \mathbb{Z}^d$ and assume that B and \mathcal{G} are such that

$$\forall p \in \mathcal{P}, \quad (\sigma_{\mathcal{E}}(p) \cap \mathcal{P}) \subset B_p, \quad (15)$$

where

$$B_p = \{p + q, q \in B\}. \quad (16)$$

In practice, we typically think of B as a ball centered at the origin. In such a case, (15) means that neighbors in the graph \mathcal{G} are close to each other in \mathbb{Z}^d .

3 Main result of the paper

Theorem 1 *Let \mathcal{G} be the graph defined in Section 2, let $B \subset \mathbb{Z}^d$ satisfy (15) and let us assume that $p \in \mathcal{P}$ satisfies*

$$\left\{ \begin{array}{l} \text{either} \quad (\forall q \in B_p, \quad c_q \geq 0 \quad \text{and} \quad c_q \geq \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin B_p}} c_{q,q'}), \\ \text{or} \quad (\forall q \in B_p, \quad c_q \leq 0 \quad \text{and} \quad c_q \leq - \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin B_p}} c_{q',q}). \end{array} \right. \quad (17)$$

Then, there exists a max-flow f in \mathcal{G} such that

$$\forall q \in \sigma_{\mathcal{E}}(p), \quad f_{p,q} = f_{q,p} = 0. \quad (18)$$

As a consequence, removing the node p from the graph \mathcal{G} does not modify its max-flow value.

The proof of this theorem is contained in Section 4, Section 5 and Section 6. For simplicity, we only prove in these sections the theorem when the node p satisfies the first condition of (17).

The statement for the second condition of (17) is easily deduced from the other statement by considering a graph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}', c')$, whose edges are such that

$$\text{for all } (q', q) \in (\mathcal{P} \times \mathcal{P}), \quad (q, q') \in \mathcal{E}' \iff (q', q) \in \mathcal{E}$$

and

$$(s, q) \in \mathcal{E}' \iff (q, t) \in \mathcal{E} \quad \text{and} \quad (q, t) \in \mathcal{E}' \iff (s, q) \in \mathcal{E};$$

and whose capacities are such that

$$c'_{q,q'} = c_{q',q}, \quad \forall (q, q'); \quad \text{and} \quad c'_{s,q} = c_{q,t} \quad \text{and} \quad c'_{q,t} = c_{s,q}, \quad \forall q \in \mathcal{P}.$$

Any pixel $p \in \mathcal{P}$ satisfying the second condition of (17) in \mathcal{G} , satisfies the first condition of (17) in \mathcal{G}' and there exists a max-flow f' in \mathcal{G}' such that

$$\forall q \in \sigma_{\mathcal{E}'}(p), \quad f'_{p,q} = f'_{q,p} = 0.$$

It is not difficult to check that the mapping $f : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ defined by

$$f_{q',q} = f'_{q,q'}, \forall (q, q') \in \mathcal{P} \times \mathcal{P}, \text{ and } f_{q,t} = f'_{s,q} \text{ and } f_{s,q} = f'_{q,t}, \forall q \in \mathcal{P},$$

is a max-flow in \mathcal{G} and satisfies (18).

Algorithmically, the above theorem guarantees that, during the graph construction, we can test every node before it is added to the graph. If the node satisfies (17), it is not useful to the max-flow evaluation and can be removed without alteration of the max-flow value. We can compute the max-flow in the subgraph of \mathcal{G} restricted to $\mathcal{V} \setminus \{p\}$ and extend it using (18) to obtain a max flow f in \mathcal{G} . As a consequence, if we denote by (S, T) the s-t-cut in \mathcal{G} corresponding to this flow f , we trivially obtain that, if p satisfies the first condition of (17) and $c_p > 0$, then $p \in S$. Also, if moreover $q \in \sigma_{\mathcal{E}}(p)$ and $c_{p,q} > 0$, $q \in S$. An analogue reasoning can be done if p satisfies the second condition of (17). Altogether, this permits to deduce the min-cut in the non-reduced graph from the min-cut in the reduced graph.

In practice, if the graph \mathcal{G} is obtained using the construction in [9], we deduce the minimizer $u^* \in \{0, 1\}^{\mathcal{P}}$ of (12), from the minimum cut

$$\left(S \cup \{s\}, ((\mathcal{P} \setminus \{p\}) \setminus S) \cup \{t\} \right),$$

where $S \subset (\mathcal{P} \setminus \{p\})$ has been computed in the graph from which the node p satisfying (17) has been removed, using:

$$u_r^* = \begin{cases} 1 & , \text{ if } r = p \text{ and } \forall q \in B_p, c_q \geq 0 \text{ and } c_q \geq \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin B_p}} c_{q,q'}, \\ 0 & , \text{ if } r = p \text{ and } \forall q \in B_p, c_q \leq 0 \text{ and } c_q \leq - \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin B_p}} c_{q',q}, \\ 1 & , \text{ if } r \neq p \text{ and } r \in S, \\ 0 & , \text{ if } r \neq p \text{ and } r \notin S. \end{cases}$$

Finally, notice that in (17) the condition $c_q \geq 0$ is useless since, for all $q \in B_p$, $c_q \geq \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin B_p}} c_{q,q'} \geq 0$. Similarly, $c_q \leq 0$ is useless. We only write it for the sake of clarity.

4 Avoiding useless flow on closed walks

In this section we remind a known result. We also prove it so-that the paper is self contained.

Proposition 1 *Let \mathcal{G} be the graph defined in Section 2. There exists a max-flow f in \mathcal{G} satisfying*

$$\begin{cases} \text{for any positive } l \text{ and any closed walk } p_0 - p_1 - \dots - p_l \\ \text{of length } l \text{ in } \mathcal{G}, \text{ there exists } i \in \{0, \dots, l-1\} \text{ such that} \\ f_{p_i, p_{i+1}} \leq f_{p_{i+1}, p_i}. \end{cases} \quad (19)$$

Proof. We only prove that there exists a max-flow f in \mathcal{G} satisfying

$$\left\{ \begin{array}{l} \text{for any positive length } l \text{ and any **simple** closed walk } p_0 - p_1 - \dots - p_l \\ \text{of length } l \text{ in } \mathcal{G}, \text{ there exists } i \in \{0, \dots, l-1\} \\ \text{such that } f_{p_i, p_{i+1}} \leq f_{p_{i+1}, p_i}. \end{array} \right. \quad (20)$$

The two statements are indeed equivalent: - The statement of the proposition trivially implies that f satisfies (20); - Conversely, it is possible to extract a simple closed walk from any closed walk $p_0 - p_1 - \dots - p_l$, and the flow f satisfying (20) ends up being such that there exists $i \in \{0, \dots, l\}$ satisfying $f_{p_i, p_{i+1}} \leq f_{p_{i+1}, p_i}$.

Let f be a max-flow in \mathcal{G} . For any positive length $l \in \mathbb{N}$ and any closed simple walk $w = p_0 - p_1 - \dots - p_l$ of length l in \mathcal{G} , we denote by $(P_{f,w})$ the statement:

$$(P_{f,w}) : \forall i \in \{0, \dots, l-1\}, f_{p_i, p_{i+1}} > f_{p_{i+1}, p_i}.$$

We also consider

$$W(f) = \{w, w \text{ is a closed simple walk satisfying } (P_{f,w})\}.$$

Notice first that if

$$\#W(f) = 0, \quad (21)$$

where $\#$ denotes the cardinality of a set, the flow f necessarily satisfies (20) and (19).

We show, in the remaining of the proof, that if f is such that $\#W(f) > 0$, there exist f' such that

$$\#W(f') < \#W(f).$$

Since for any max-flow f the set $W(f)$ is finite, any initial max-flow lead to a max-flow satisfying (21) (and therefore (19)) after a finite number of such recursion.

Let us now assume that f is such that $\#W(f) > 0$. Let us also consider a closed simple walk $w = p_0 - p_1 - \dots - p_l \in W(f)$.

We denote

$$\delta = \min_{i \in \{0, \dots, l-1\}} (f_{p_i, p_{i+1}} - f_{p_{i+1}, p_i}).$$

Since w satisfies $(P_{f,w})$, we have $\delta > 0$.

We define the mapping $f' : (\mathcal{V} \times \mathcal{V}) \rightarrow \mathbb{R}^+$ such that for all $(p, q) \in (\mathcal{V} \times \mathcal{V})$:

$$f'_{p,q} = \begin{cases} f_{p,q} - f_{q,p} - \delta & , \text{ if there is } 0 \leq i < l, \text{ such that } (p, q) = (p_i, p_{i+1}) \\ 0 & , \text{ if there is } 0 \leq i < l, \text{ such that } (p, q) = (p_{i+1}, p_i) \\ f_{p,q} & , \text{ otherwise.} \end{cases} \quad (22)$$

Notice that this definition is not ambiguous since w is a simple walk.

Also, since f is a flow in \mathcal{G} , we clearly have for all $(p, q) \in (\mathcal{V} \times \mathcal{V})$

$$0 \leq f'_{p,q} \leq f_{p,q} \leq c_{p,q}.$$

In order to prove the flow conservation, we consider $p \in \mathcal{V}$. Let us first assume that $p \neq p_i$, for all $i \in \{0, \dots, l\}$. Then (22) guarantees that $f'_{p,q} = f_{p,q}$ and $f'_{q,p} = f_{q,p}$ for all $q \in \sigma_{\mathcal{E}}(p)$. We therefore trivially get

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} f'_{q,p} = \sum_{q \in \sigma_{\mathcal{E}}(p)} f_{p,q}.$$

Let us now assume that there exists $i \in \{0, \dots, l-1\}$ such that $p = p_i$. We denote (if $i = 0$) $p_{-1} = p_{l-1}$ and we have

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} (f'_{q,p} - f'_{p,q}) = \sum_{\substack{q \in \sigma_{\mathcal{E}}(p) \\ q \neq p_{i+1} \\ q \neq p_{i-1}}} (f'_{q,p} - f'_{p,q}) + (f'_{p_{i+1},p_i} - f'_{p_i,p_{i+1}}) + (f'_{p_{i-1},p_i} - f'_{p_i,p_{i-1}})$$

Using (22), we obtain for each term

$$\sum_{\substack{q \in \sigma_{\mathcal{E}}(p) \\ q \neq p_{i+1} \\ q \neq p_{i-1}}} (f'_{q,p} - f'_{p,q}) = \sum_{\substack{q \in \sigma_{\mathcal{E}}(p) \\ q \neq p_{i+1} \\ q \neq p_{i-1}}} (f_{q,p} - f_{p,q}), \quad (23)$$

$$(f'_{p_{i+1},p_i} - f'_{p_i,p_{i+1}}) = 0 - (f_{p_i,p_{i+1}} - f_{p_{i+1},p_i} - \delta), \quad (24)$$

and

$$(f'_{p_{i-1},p_i} - f'_{p_i,p_{i-1}}) = (f_{p_{i-1},p_i} - f_{p_i,p_{i-1}} - \delta) - 0. \quad (25)$$

Summing (23), (24), (25) and simplifying, we finally get

$$\begin{aligned} \sum_{q \in \sigma_{\mathcal{E}}(p)} (f'_{q,p} - f'_{p,q}) &= \sum_{\substack{q \in \sigma_{\mathcal{E}}(p) \\ q \neq p_{i+1} \\ q \neq p_{i-1}}} (f_{q,p} - f_{p,q}) + (f_{p_{i+1},p_i} - f_{p_i,p_{i+1}}) + (f_{p_{i-1},p_i} - f_{p_i,p_{i-1}}) \\ &= \sum_{q \in \sigma_{\mathcal{E}}(p)} (f_{q,p} - f_{p,q}) \\ &= 0 \end{aligned}$$

As a conclusion, f' is a flow. It is of course a max-flow. Indeed, (1) and (7) guarantee that $f_{t,p} = 0$, for all $p \in \mathcal{P}$. Since w satisfies $(P_{f,w})$, this ensures that

$$t \neq p_i, \forall i \in \{0, \dots, l\}.$$

Using (8) and (22), we finally get

$$\text{val}_{\mathcal{G}}(f') = \text{val}_{\mathcal{G}}(f).$$

We still need to show that

$$\#W(f') < \#W(f).$$

With that in mind, we consider $w' = p'_0 - p'_1 - \dots - p'_{l'}$. We are going to prove that we necessarily have $w' \in W(f)$.

We know that for any $j \in \{0, \dots, l' - 1\}$

$$0 < (f'_{p'_j, p'_{j+1}} - f'_{p'_{j+1}, p'_j}).$$

Together with (22), this guarantees that for any $j \in \{0, \dots, l' - 1\}$

$$(p'_j, p'_{j+1}) \neq (p_{i+1}, p_i), \text{ for all } i \in \{0, \dots, l - 1\}.$$

Therefore, using (22), we either have

$$f'_{p'_j, p'_{j+1}} = f_{p'_j, p'_{j+1}} \quad \text{and} \quad f'_{p'_{j+1}, p'_j} = f_{p'_{j+1}, p'_j};$$

or

$$(p'_j, p'_{j+1}) = (p_i, p_{i+1}), \text{ for some } i \in \{0, \dots, l - 1\}.$$

In the latter case, we have

$$\begin{aligned} 0 < f'_{p'_j, p'_{j+1}} - f'_{p'_{j+1}, p'_j} &= (f_{p'_j, p'_{j+1}} - f_{p'_{j+1}, p'_j} - \delta) - 0 \\ &\leq f_{p'_j, p'_{j+1}} - f_{p'_{j+1}, p'_j}. \end{aligned}$$

As a conclusion, in both cases, we have

$$0 < (f'_{p'_j, p'_{j+1}} - f'_{p'_{j+1}, p'_j}) \leq (f_{p'_j, p'_{j+1}} - f_{p'_{j+1}, p'_j}).$$

This means that $w' \in W(f)$ and finally

$$W(f') \subset W(f).$$

In order to show that this inclusion is strict, we denote

$$i_0 \in \operatorname{argmin}_{i \in \{0, \dots, l-1\}} (f_{p_i, p_{i+1}} - f_{p_{i+1}, p_i}).$$

Using (22), we trivially obtain that

$$f'_{p_{i_0}, p_{i_0+1}} = f_{p_{i_0+1}, p_{i_0}} = 0,$$

and therefore $w \notin W(f')$. This concludes the proof. \square

5 Avoiding useless traversing flow

Throughout this section, we consider a graph \mathcal{G} as constructed in Section 2 and a max-flow f in \mathcal{G} satisfying (19). We also consider $p \in \mathcal{P}$ such that

$$\forall q \in B_p, f_q \geq 0,$$

where B_p is defined in (16).

The purpose of this section is to establish a sufficient condition so-that f can be modified in such a way that

$$f_{p,q} \geq f_{q,p}, \text{ for all } q \in \sigma_{\mathcal{E}}(p).$$

In words, the node p sends more flow to its neighbors than it can receive from them.

In order to do so, our strategy consists in modifying the max-flow f in \mathcal{G} in such a way that it satisfies this property. The modification of f is done by constructing a graph \mathcal{G}' and combining f with a max flow f' in \mathcal{G}' . Heuristically, the construction of \mathcal{G}' permits to redirect (and avoid) the flow traversing p in the grid $\mathcal{E} \cap (\mathcal{P} \times \mathcal{P})$.

In order to construct \mathcal{G}' , we first consider

$$\begin{aligned} \Sigma_i(p) = \{q \in \mathcal{P}, \exists p_0 - \dots - p_l \in W(q, p) \text{ such that} \\ \forall i \in \{0, \dots, l-1\}, f_{p_i, p_{i+1}} > f_{p_{i+1}, p_i}\}, \end{aligned}$$

and

$$\begin{aligned} \Sigma_o(p) = \{q \in B_p, \exists p_0 - \dots - p_l \in W(p, q) \text{ such that} \\ \forall i \in \{0, \dots, l-1\}, f_{p_i, p_{i+1}} > f_{p_{i+1}, p_i}\}, \end{aligned}$$

where we remind that $W(q, p)$ (resp. $W(p, q)$) contains all the walks starting at q (resp. p) and ending at p (resp. q). Examples of sets Σ^+ and Σ^- are drawn on Figure 1.

Let us first notice that, since f satisfies (19),

$$\forall q \in (\Sigma_i(p) \cap \sigma_{\mathcal{E}}(p)), \quad f_{q,p} \geq f_{p,q} \tag{26}$$

and

$$\forall q \in (\Sigma_o(p) \cap \sigma_{\mathcal{E}}(p)), \quad f_{p,q} \geq f_{q,p}. \tag{27}$$

Similarly, since f satisfies (19), we have

$$\Sigma_i(p) \cap \Sigma_o(p) = \emptyset. \tag{28}$$

Moreover,

$$p \notin \Sigma_i(p) \text{ and } p \notin \Sigma_o(p).$$

For simplicity, we denote

$$\Sigma^- = \Sigma_i(p) \text{ and } \Sigma^+ = \Sigma_o(p) \cup \{p\}.$$

Also, since f satisfies (19), we have

$$\forall q \in \Sigma^-, \forall q' \in \Sigma^+, \quad f_{q,q'} \geq f_{q',q}. \tag{29}$$

Otherwise, we could easily build a closed walk contradicting (19).

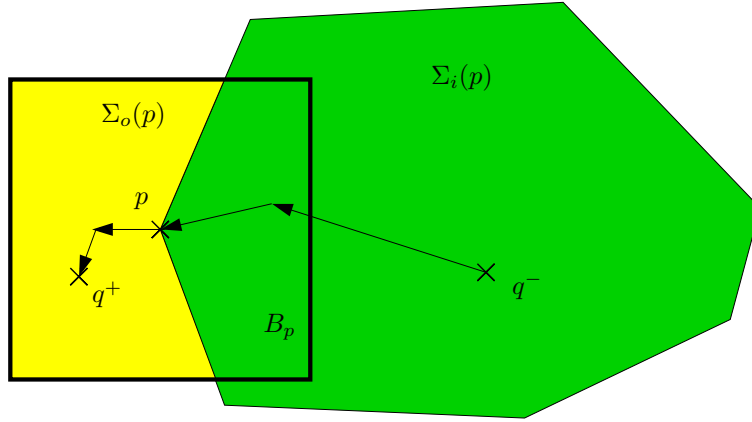


Figure 1: Example of configuration for $\Sigma_o(p)$ (in yellow), $\Sigma_i(p)$ (in green), B_p (the square). The arrows represent the direction of the flow on two walks in $W(q^-, p)$ and $W(p, q^+)$. We know that some flow goes from $q^- \in \Sigma_i(p)$ to $q^+ \in \Sigma_o(p)$ and traverses p . This is the flow we are redirecting in this section. A (too) simplistic solution for avoiding the traversing flow would be to increase f_{s, q^+} , decrease f_{s, q^-} in order to avoid the flow on the walk described by the arrows.

We also denote

$$\mathcal{P}' = \Sigma^- \cup \Sigma^+ \quad , \quad \mathcal{V}' = \mathcal{P}' \cup \{s, t\} \quad (30)$$

and construct the graph

$$\mathcal{G}' = (\mathcal{V}', \mathcal{E}', c') ,$$

where \mathcal{E}' and c' are defined below. We set

$$\mathcal{E}' = \mathcal{E}'_t \cup (\mathcal{E}'_n \cap \mathcal{E}^T) , \quad (31)$$

where $\mathcal{E}^T = \{(q, q'), \text{ such that } (q', q) \in \mathcal{E}\}$ and with

$$\mathcal{E}'_t = \{(q, t), \text{ with } q \in \Sigma^- \text{ such that } f_q \geq 0\} \cup (\{s\} \times \Sigma^+) \quad (32)$$

and

$$\mathcal{E}'_n = (\Sigma^+ \times \Sigma^+) \cup ((\Sigma^- \cup \{p\}) \times (\Sigma^- \cup \{p\})) . \quad (33)$$

The capacities c' are defined by

$$c'_{q,t} = f_q \quad , \quad \text{for } q \in \Sigma^- \text{ such that } f_q \geq 0, \quad (34)$$

$$c'_{s,q} = c_q - f_q \quad , \quad \text{for } q \in \Sigma^+, \quad (35)$$

and

$$c'_{q,q'} = \begin{cases} f_{q',q} - f_{q,q'} & , \text{ if } f_{q',q} > f_{q,q'} \\ 0 & , \text{ otherwise} \end{cases} \quad , \quad \text{for } (q, q') \in (\mathcal{E}'_n \cap \mathcal{E}^T). \quad (36)$$

Notice that there exist some nodes in Σ^- which are linked to no terminals. As in Section 2, we artificially extend all the capacities c' and set

$$c'_{q,q'} = 0, \text{ for all } (q, q') \in ((\mathcal{V}' \times \mathcal{V}') \setminus \mathcal{E}').$$

Notice that, in the graph \mathcal{G}' all the flow sent by s goes in Σ^+ and all the flow arriving at t comes from Σ^- . Moreover, all the edges between Σ^+ and Σ^- contain p .

Also, for any $S \subset \mathcal{P}'$, we denote the value of the s - t cut $(S \cup \{s\}, (\mathcal{P}' \setminus S) \cup \{t\})$ in \mathcal{G}' by

$$\text{val}_{\mathcal{G}'}(S) = \sum_{\substack{q \in (S \cup \{s\}) \\ q' \notin (S \cup \{s\})}} c'_{q,q'}.$$

Using (34), (35) and (36), we find

$$\text{val}_{\mathcal{G}'}(S) = E_1 + E_2 + E_3,$$

where we write

$$E_1 = \sum_{q \in (\Sigma^+ \setminus S)} c'_{s,q}, \quad E_2 = \sum_{q \in (\Sigma^- \cap S)} c'_{q,t} \quad \text{and} \quad E_3 = \sum_{\substack{q \in S \\ q' \in (\mathcal{P}' \setminus S)}} c'_{q,q'}. \quad (37)$$

In particular, using (28) and (30), we have

$$\text{val}_{\mathcal{G}'}(\Sigma^+) = \sum_{\substack{q \in \Sigma^+ \\ q' \in \Sigma^-}} c'_{q,q'},$$

which, using (33), (28) becomes

$$\text{val}_{\mathcal{G}'}(\Sigma^+) = \sum_{q \in \Sigma^-} c'_{p,q}.$$

Finally, we obtain using (36) and (26)

$$\text{val}_{\mathcal{G}'}(\Sigma^+) = \sum_{q \in \Sigma^-} (f_{q,p} - f_{p,q}). \quad (38)$$

The following proposition has the most technical proof of the paper but will later provide a condition implying that Σ^+ is a min-cut in \mathcal{G}' .

Proposition 2 *Let \mathcal{G}' be the graph constructed in Section 5. For any $S \subset \mathcal{P}'$,*

$$\text{val}_{\mathcal{G}'}(S) \geq \text{val}_{\mathcal{G}'}(\Sigma^+) + \sum_{q \in \Sigma^+ \setminus (S \cup \{p\})} \left[c_q + \sum_{q' \in \mathcal{P}' \setminus \Sigma^+} (f_{q',q} - f_{q,q'}) \right]. \quad (39)$$

Proof. Let us first decompose E_3 according to

$$E_3 = E'_1 + E'_2 + E'_3 + E'_4,$$

with

$$\begin{aligned} E'_1 &= \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^+ \setminus S)}} c'_{q,q'} &, E'_2 &= \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^- \setminus S)}} c'_{q,q'} \\ E'_3 &= \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^+ \setminus S)}} c'_{q,q'} &, E'_4 &= \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^- \setminus S)}} c'_{q,q'} \end{aligned}$$

We rewrite, using (36),

$$E'_1 = \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^+ \setminus S) \\ f_{q',q} > f_{q,q'}}} (f_{q',q} - f_{q,q'}), E'_2 = \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^- \setminus S) \\ (q,q') \in \mathcal{E}', f_{q',q} > f_{q,q'}}} (f_{q',q} - f_{q,q'}) \quad (40)$$

$$E'_3 = \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^+ \setminus S) \\ (q,q') \in \mathcal{E}', f_{q',q} > f_{q,q'}}} c'_{q,q'}, E'_4 = \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^- \setminus S) \\ f_{q',q} > f_{q,q'}}} (f_{q',q} - f_{q,q'}) \quad (41)$$

Using (33) and (28), then (36) and (26), we immediately find that

$$E'_2 = \begin{cases} \sum_{q \in (\Sigma^- \setminus S)} (f_{q,p} - f_{p,q}) & , \text{ if } p \in S \\ 0 & , \text{ otherwise,} \end{cases} \quad \text{and} \quad E'_3 = 0. \quad (42)$$

Moreover, since the total amount of flow entering and exiting $(S \cap \Sigma^-)$ are equal, we have (see (11))

$$\sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q \geq 0}} f_q + \sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q < 0}} f_q + \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \notin (S \cap \Sigma^-)}} (f_{q',q} - f_{q,q'}) = 0$$

Moreover, if we decompose the last term and reorganize the equation we obtain

$$\sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q \geq 0}} f_q + \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^- \setminus S)}} (f_{q',q} - f_{q,q'}) = - \sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q < 0}} f_q - \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in \Sigma^+}} (f_{q',q} - f_{q,q'})$$

Together with the definition of E_2 in (37), (34) and the definition of E'_4 in (41)

this leads to

$$\begin{aligned}
E_2 + E'_4 &\geq \sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q \geq 0}} f_q + \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^- \setminus S)}} (f_{q',q} - f_{q,q'}) \\
&= - \sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q < 0}} f_q - \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in \Sigma^+}} (f_{q',q} - f_{q,q'}) \\
&\geq \sum_{q \in (S \cap \Sigma^-)} (f_{q,p} - f_{p,q}) + \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^+ \setminus \{p\})}} (f_{q,q'} - f_{q',q}).
\end{aligned}$$

Then, using (29), we immediately obtain

$$E_2 + E'_4 \geq \sum_{q \in (S \cap \Sigma^-)} (f_{q,p} - f_{p,q}).$$

Together with (42) and (38), this leads to the following intermediate result:

$$E_2 + E'_2 + E'_3 + E'_4 \geq \begin{cases} \text{val}_{\mathcal{G}'}(\Sigma^+) & , \text{ if } p \in S \\ \sum_{q \in (S \cap \Sigma^-)} (f_{q,p} - f_{p,q}) & , \text{ otherwise.} \end{cases} \quad (43)$$

In order to finish the proof, let us first notice that using the definition of E_1 in (37), (35) and the definition of E'_1 in (40)

$$E_1 + E'_1 \geq \sum_{q \in (\Sigma^+ \setminus S)} (c_q - f_q) + \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^+ \setminus S)}} (f_{q',q} - f_{q,q'}) \quad (44)$$

Expressing that the total amount of flow entering and exiting $(\Sigma^+ \setminus S)$ are equal, we have (see (11))

$$\sum_{q \in (\Sigma^+ \setminus S)} f_q + \sum_{\substack{q \in (\Sigma^+ \setminus S) \\ q' \in (\Sigma^+ \cap S)}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q \in (\Sigma^+ \setminus S) \\ q' \in \mathcal{P} \setminus \Sigma^+}} (f_{q',q} - f_{q,q'}) = 0.$$

Together with (44), this guarantees that

$$\begin{aligned}
E_1 + E'_1 &\geq \sum_{q \in (\Sigma^+ \setminus S)} c_q + \sum_{\substack{q \in (\Sigma^+ \setminus S) \\ q' \in \mathcal{P} \setminus \Sigma^+}} (f_{q',q} - f_{q,q'}) \\
&\geq \sum_{q \in (\Sigma^+ \setminus S)} \left[c_q + \sum_{q' \in \mathcal{P} \setminus \Sigma^+} (f_{q',q} - f_{q,q'}) \right] \quad (45)
\end{aligned}$$

When $p \in S$, by combining the latter result with (43), we immediately get (39). If $p \notin S$, (45) can be rewritten using (38)

$$E_1 + E'_1 \geq \sum_{q \in (\Sigma^+ \setminus (S \cup \{p\}))} \left[c_q + \sum_{q' \in \mathcal{P} \setminus \Sigma^+} (f_{q',q} - f_{q,q'}) \right] + c_p + \text{val}_{\mathcal{G}'}(\Sigma^+).$$

Since $c_p \geq 0$, and (43) and (26) guarantee that $E_2 + E'_2 + E'_3 + E'_4 \geq 0$, this ensures that (39) holds even when $p \notin S$ and concludes the proof. \square

All along the remaining of this Section, we consider a max-flow f' in \mathcal{G}' . Notice also that \mathcal{G}' satisfies analogues of (1) and (3). Therefore, as in Section 2, we denote

$$f'_q = f'_{s,q} - f'_{q,t},$$

for all $q \in \mathcal{P}'$. We also artificially extend the flow f' and set

$$f'_q = 0, \text{ for all } q \notin \mathcal{P}'$$

and

$$f'_{q,q'} = 0, \text{ for all } (q, q') \in ((\mathcal{V}' \times \mathcal{V}') \setminus \mathcal{E}').$$

We are now going to combine f and f' in order to build a mapping $f'' : \mathcal{E} \rightarrow \mathbb{R}$ which will turn out to be a max-flow in \mathcal{G} such that

$$f''_{p,q} \geq f''_{q,p} = 0 \quad , \forall q \in \sigma_{\mathcal{E}}(p).$$

Let us begin with the definition of f'' . We distinguish in the definition the different possible configuration for edges of \mathcal{E} .

$$f''_{q,q'} = \begin{cases} f_{q,q'} - f_{q',q} & , \text{ if } f_{q,q'} \geq f_{q',q} \\ 0 & , \text{ otherwise,} \end{cases} \quad , \text{ for } (q, q') \notin \mathcal{E}' \quad (46)$$

$$\begin{cases} f''_{s,q} = 0 & \text{and } f''_{q,t} = -f_q & , \text{ for } q \in \mathcal{P}' \text{ such that } f_q < 0 \\ f''_{s,q} = f_q + f'_q & \text{and } f''_{q,t} = 0 & , \text{ for } q \in \mathcal{P}' \text{ such that } f_q \geq 0 \end{cases} \quad (47)$$

$$f''_{q',q} = \begin{cases} f_{q',q} - f_{q,q'} - f'_{q,q'} & , \text{ if } f_{q',q} > f_{q,q'} \\ 0 & , \text{ otherwise} \end{cases} \quad , \text{ for } (q', q) \in \mathcal{P}'^2 \cap \mathcal{E}^T. \quad (48)$$

The equations (46), (47) and (48) permit to define $f''_{q,q'}$ for all $(q, q') \in \mathcal{E}$ and (again) we extend f'' outside \mathcal{E} and set

$$f''_{q,q'} = 0, \text{ for all } (q, q') \in ((\mathcal{V} \times \mathcal{V}) \setminus \mathcal{E}).$$

We also denote

$$f''_q = f''_{s,q} - f''_{q,t} \quad , \forall q \in \mathcal{P}.$$

Notice that, since $f'_q = 0$ for all $q \notin \mathcal{P}'$ as well as for $q \in \mathcal{P}'$ such that $f_q < 0$ (see (34) and (35)), we always have, according to (46) and (47),

$$f''_q = f_q + f'_q \quad , \forall q \in \mathcal{P}. \quad (49)$$

Proposition 3 *The mapping $f'' : (\mathcal{V} \times \mathcal{V}) \rightarrow \mathbb{R}$ is a max-flow in \mathcal{G} .*

Proof. Let us first show that f'' satisfies the capacity constraints. Let $(q', q) \in \mathcal{E}$. We distinguish below the different configuration for (q', q) .

- If $(q', q) \notin \mathcal{E}'$ and using (46) we either have

$$0 \leq f''_{q',q} = f_{q',q} - f_{q,q'} \leq c_{q',q},$$

or

$$0 \leq f''_{q',q} = 0 \leq c_{q',q}.$$

- If $q' \in \Sigma^-$ and $q = s$ or t :
 - If moreover $f_{q'} < 0$, then using (47), $0 \leq f''_{s,q'} = 0 \leq c_{s,q'}$ and $0 \leq f''_{q',t} = f_{q',t} \leq c_{q',t}$.
 - If $f_{q'} \geq 0$, then using (47) and (34), we find that $0 \leq f''_{s,q'} = f_{s,q'} - f'_{q',t} \leq c_{s,q'}$ and $0 \leq f''_{q',t} = 0 \leq c_{q',t}$.
- If $q' \in \Sigma^+$ and $q = s$ or t : since $q' \in B_p$, we necessarily have $f_{q'} \geq 0$, then using (47) and (35), we have $0 \leq f''_{s,q'} = f_{s,q'} + f'_{s,q'} \leq c_{s,q'}$ and $0 \leq f''_{q',t} = 0 \leq c_{q',t}$.
- If $(q', q) \in (\mathcal{P}' \times \mathcal{P}')$:
 - If moreover $f_{q',q} \leq f_{q,q'}$, then (48) guarantees $0 \leq f''_{q',q} = 0 \leq c_{q',q}$.
 - If $f_{q',q} > f_{q,q'}$, using (36), we have

$$0 \leq f'_{q,q'} \leq c'_{q,q'} = f_{q',q} - f_{q,q'},$$

and finally (48) guarantees that

$$0 \leq f''_{q',q} = f_{q',q} - f_{q,q'} - f'_{q,q'} \leq c_{q',q}.$$

Let us now prove the flow conservation. Let $q \in \mathcal{P}$. We distinguish below the different possible position for q .

- If $q \notin \mathcal{P}'$, then for any $q' \in \sigma_{\mathcal{E}}(q)$ the definition of \mathcal{E}' given in (31), (32) and (33) guarantees that both (q, q') and $(q', q) \notin \mathcal{E}'$. Using (46), we obtain $f''_{q,q'} - f''_{q',q} = f_{q,q'} - f_{q',q}$, for all $q' \in \sigma_{\mathcal{E}}(q)$, and therefore

$$\sum_{q' \in \sigma_{\mathcal{E}}(q)} (f''_{q',q} - f''_{q,q'}) = \sum_{q' \in \sigma_{\mathcal{E}}(q)} (f_{q',q} - f_{q,q'}) = 0.$$

- If $q \in \mathcal{P}'$, the flow conservation constraint given by (11) for f and f' at q can be decomposed to provide

$$\begin{aligned} f_q + \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}}(q) \\ q' \notin \sigma_{\mathcal{E}'}(q)}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}'}(q) \\ f_{q',q} > f_{q,q'}}} (f_{q',q} - f_{q,q'}) \\ + \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}'}(q) \\ f_{q',q} \leq f_{q,q'}}} (f_{q',q} - f_{q,q'}) = 0 \end{aligned}$$

and

$$f'_q + \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}'}(q) \\ f'_{q',q} > f_{q,q'}}} (0 - f'_{q,q'}) + \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}'}(q) \\ f'_{q',q} \leq f_{q,q'}}} (f'_{q',q} - 0) = 0.$$

Summing these equalities and using (49), (46) and (48), we obtain

$$\begin{aligned} f''_q + \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}}(q) \\ q' \notin \sigma_{\mathcal{E}'}(q)}} (f''_{q',q} - f''_{q,q'}) + \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}'}(q) \\ f'_{q',q} > f_{q,q'}}} (f''_{q',q} - f''_{q,q'}) \\ + \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}'}(q) \\ f'_{q',q} \leq f_{q,q'}}} (f''_{q',q} - f''_{q,q'}) = 0. \end{aligned}$$

The latter corresponds to flow conservation constraint (11) at the node q for f'' .

Altogether, we now know that f'' is a flow. We still need to show that it is a max-flow. The latter property is in fact trivially obtained since (47) and (46) guarantee that $f''_{q,t} = f_{q,t}$, for all $q \in \mathcal{P}$. Therefore, the value of f'' is equal to the value of f . Since f is a max-flow, this value is maximal and f'' is a max-flow. \square

Proposition 4 *If Σ^+ is a minimum s - t cut in the graph \mathcal{G}' defined in Section 5, then the max-flow f'' is such that*

$$\forall q \in \mathcal{P} \cap \sigma_{\mathcal{E}}(p), \quad f''_{q,p} = 0.$$

As a consequence,

$$\forall q \in \mathcal{P} \cap \sigma_{\mathcal{E}}(p), \quad f''_{p,q} \geq f_{q,p}.$$

Proof. Since f' is a max-flow in \mathcal{G}' and Σ^+ is a min s - t cut in \mathcal{G}' , Ford-Fulkerson theorem guarantees that they have the same value. We therefore have

$$\begin{aligned} \text{val}_{\mathcal{G}'}(f') = \text{val}_{\mathcal{G}'}(\Sigma^+) &= \sum_{\substack{q' \in \Sigma^+ \\ q \notin \Sigma^+ \\ (q',q) \in \mathcal{E}'}} c'_{q',q} \\ &= \sum_{q \in \Sigma^-} c'_{p,q} \end{aligned} \quad (50)$$

Moreover, since f' is a flow, the total amount of flow entering and exiting Σ^+ are equal. Therefore, we have (see (11))

$$\sum_{q \in \Sigma^+} f'_q + \sum_{\substack{q' \in \Sigma^+ \\ q \notin \Sigma^+ \\ q \in \sigma_{\mathcal{E}'}(q')}} (f'_{q,q'} - f'_{q',q}) = 0.$$

Together with (8), (32) and (33), this guarantees that

$$\text{val}_{\mathcal{G}'}(f') = \sum_{\substack{q \in \Sigma^- \\ f_q \geq 0}} f'_q = \sum_{q \in \Sigma^-} (f'_{p,q} - f'_{q,p}),$$

Since the amounts of flow entering and exiting Σ^- are equal. Combined with (50), this provides

$$\sum_{q \in \Sigma^-} c'_{p,q} = \sum_{q \in \Sigma^-} f'_{p,q} - \sum_{q \in \Sigma^-} f'_{q,p}. \quad (51)$$

As a consequence,

$$\sum_{q \in \Sigma^-} f'_{q,p} = \sum_{q \in \Sigma^-} (f'_{p,q} - c'_{p,q}) \leq 0.$$

However, since for all $q \in \Sigma^-$, $f'_{q,p} \geq 0$, we finally obtain that

$$\forall q \in \Sigma^-, f'_{q,p} = 0.$$

Using (51) again, (26) and (36), this provides

$$\forall q \in \Sigma^-, f'_{p,q} = c'_{p,q} = f_{q,p} - f_{p,q}.$$

Therefore, using (26) and (48),

$$\forall q \in \Sigma^-, f''_{q,p} = 0. \quad (52)$$

Moreover, using (27) and (48), we also have

$$\forall q \in (\Sigma^+ \cap \sigma_{\mathcal{E}}(p)), f''_{q,p} = 0. \quad (53)$$

Combining this result with (52), we obtain

$$\forall q \in (\Sigma^+ \cup \Sigma^-) \cap \sigma_{\mathcal{E}}(p), f''_{q,p} = 0. \quad (54)$$

Now, if $q \in \sigma_{\mathcal{E}}(p) \setminus (\Sigma^+ \cup \Sigma^-)$, the definitions of Σ^+ and Σ^- imply that necessarily $f_{p,q} = f_{q,p}$. The definition of \mathcal{E}' also guarantees that (p, q) and $(q, p) \notin \mathcal{E}'$. Together, with (46), we finally obtain that

$$\forall q \in \sigma_{\mathcal{E}}(p) \setminus (\Sigma^+ \cup \Sigma^-), f''_{p,q} = f''_{q,p} = 0.$$

Together with (54), this concludes the proof. \square

Proposition 5 *Let \mathcal{G} be the graph defined in Section 2, let B satisfy (15) and let us assume that $p \in \mathcal{P}$ is such that*

$$\forall q \in B_p, c_q \geq 0 \quad \text{and} \quad c_q \geq \sum_{\substack{q' \in \mathcal{P} \cap \sigma_{\mathcal{E}}(q) \\ q' \in \mathcal{P} \setminus B_p}} c_{q,q'}, \quad (55)$$

then, there exists a max-flow f in \mathcal{G} such that

$$\forall q \in \sigma_{\mathcal{E}}(p), f_{p,q} \geq f_{q,p} = 0. \quad (56)$$

Proof. This is a straightforward consequence of Proposition 3, Proposition 2 and Proposition 4.

Indeed, if (55) holds, we know that for any max-flow f in \mathcal{G} as in Proposition 1 and any $S \subset \mathcal{P}'$

$$\sum_{q \in \Sigma^+ \setminus (S \cup \{p\})} \left[c_q + \sum_{q' \in \mathcal{P} \setminus \Sigma^+} (f_{q',q} - f_{q,q'}) \right] \geq \sum_{q \in \Sigma^+ \setminus (S \cup \{p\})} \left[c_q + \sum_{q' \in \mathcal{P} \setminus B_p} (f_{q',q} - f_{q,q'}) \right] \geq 0,$$

since for all $q' \in B_p \setminus \Sigma^+$, $f_{q',q} - f_{q,q'} \geq 0$. Therefore, Proposition 2 guarantees that Σ^+ is a min s - t cut in \mathcal{G}' . Then, Proposition 3 guarantees that f'' is a max-flow in \mathcal{G} and Proposition 4 guarantees that f'' satisfies (56). \square

6 A useless nodes

Throughout this section, we consider a graph \mathcal{G} as constructed in Section 2, a set B satisfying (15), a pixel $p \in \mathcal{P}$ satisfying (55) and a max-flow f in \mathcal{G} satisfying (56).

The purpose of this section is to modify f so-that it remains a max-flow in \mathcal{G} and satisfies

$$\forall q \in \sigma_{\mathcal{E}}(p), \quad f_{p,q} = f_{q,p} = 0.$$

The latter obviously implies that the node p is useless when computing the max-flow in \mathcal{G} .

Notice that, since the flow f satisfies (56), the only flow entering p comes from the source s . Therefore, in this section we want to decrease $f_{s,p}$. However, since the flow $f_{s,p}$ entering p contributes to the flow exiting B_p , we need to compensate $f_{s,p}$ by increasing $f_{s,q}$, for $q \in B_p$. Similarly to Section 5, this intuitive (and too simplistic) strategy is strengthened by considering a max-flow f' in an appropriate graph \mathcal{G}' .

Since the method for modifying f is analogous to the one used in Section 5, we chose to use the same notations for the objects playing the same role. Beware not to confuse their definition.

First, we denote

$$\mathcal{P}' = B_p \setminus \{p\}, \quad \Sigma^+ = B_p \setminus \{p\} \quad \text{and} \quad \Sigma^- = \{p\}. \quad (57)$$

In order to modify f , we build a graph $\mathcal{G}' = (\mathcal{P}', \mathcal{E}', c')$ where \mathcal{E}' and c' are defined below. We consider

$$\mathcal{E}' = (\mathcal{E} \cap (\Sigma^+ \times \Sigma^+)) \cup ((\sigma_{\mathcal{E}}(p) \cap \Sigma^+) \times \Sigma^-) \cup (\{s\} \times \Sigma^+) \cup \{(p, t)\}. \quad (58)$$

We define the capacities c' by

$$c'_{q,q'} = c_{q,q'} - f_{q,q'} + f_{q',q}, \forall (q, q') \in (\mathcal{E} \cap (\Sigma^+ \times \Sigma^+)) \quad (59)$$

$$c'_{q,p} = f_{p,q}, \quad \forall q \in (\sigma_{\mathcal{E}}(p) \cap \Sigma^+) \quad (60)$$

$$c'_{s,q} = c_q - f_q, \quad \forall q \in \Sigma^+ \quad (61)$$

$$c'_{p,t} = f_p \quad (62)$$

As usual, in order to simplify the notations, we artificially set

$$c'_{q,q'} = 0, \quad \forall (q, q') \in (\mathcal{P}' \times \mathcal{P}') \setminus \mathcal{E}', \quad (63)$$

and we write

$$c'_q = c'_{s,q} - c'_{q,t}, \quad \forall q \in \mathcal{P}'. \quad (64)$$

Notice first that, for any $S \subset \mathcal{P}'$, the value of the s - t cut $((S \cup \{s\}), (\mathcal{P}' \setminus S) \cup \{t\})$ in \mathcal{G}' depends on whether $p \in S$ or $p \notin S$. If $p \in S$, we have

$$\text{val}_{\mathcal{G}'}(S) = c'_{p,t} + \sum_{q \in (\Sigma^+ \setminus S)} c'_q + \sum_{\substack{q \in S \\ q' \in (\mathcal{P}' \setminus S)}} c'_{q,q'}.$$

Therefore, we trivially have using (59)-(64)

$$\text{val}_{\mathcal{G}'}(S) \geq c'_{p,t} = f_p, \quad \text{if } p \in S. \quad (65)$$

Moreover, for any $S \subset \mathcal{P}'$, the value of the s - t cut $((S \cup \{s\}), (\mathcal{P}' \setminus S) \cup \{t\})$ in \mathcal{G}' is given by

$$\text{val}_{\mathcal{G}'}(S) = \sum_{q \in (\Sigma^+ \setminus S)} c'_q + \sum_{\substack{q \in S \\ q' \in (\mathcal{P}' \setminus S)}} c'_{q,q'}, \quad \text{if } p \notin S. \quad (66)$$

In particular, if $S = \Sigma^+$, we obtain using (60), the conservation of the flow f at p and (56) that

$$\begin{aligned} \text{val}_{\mathcal{G}'}(\Sigma^+) &= \sum_{q \in (\Sigma^+ \cap \sigma_{\mathcal{E}'}(p))} c'_{q,p}, \\ &= \sum_{q \in (\Sigma^+ \cap \sigma_{\mathcal{E}}(p))} f_{p,q}, \\ &= f_p. \end{aligned} \quad (67)$$

The following proposition will later give a sufficient condition for Σ^+ to be a min cut in \mathcal{G}' .

Proposition 6 *Let \mathcal{G}' be the graph constructed in Section 6. For any $S \subset \mathcal{P}'$,*

- if $p \notin S$

$$\text{val}_{\mathcal{G}'}(S) = \text{val}_{\mathcal{G}'}(\Sigma^+) + \sum_{\substack{q \in S \\ q' \in (\Sigma^+ \setminus S)}} c_{q,q'} + \sum_{q \in (\Sigma^+ \setminus S)} \left[c_q + \sum_{q' \in \mathcal{P}' \setminus \mathcal{P}'} (f_{q',q} - f_{q,q'}) \right], \quad (68)$$

- if $p \in S$

$$\text{val}_{G'}(S) \geq \text{val}_{G'}(\Sigma^+). \quad (69)$$

Proof. Notice first that, if $p \in S$, (69) is a straightforward consequence of (65) and (67). Let us assume from now on that $p \notin S$.

Since f is a flow, the total amount of flow entering and exiting $(\mathcal{P}' \setminus S)$ are equal (see (11)) and therefore, using (57)

$$f_p + \sum_{q \in (\Sigma^+ \setminus S)} f_q + \sum_{\substack{q \in (\mathcal{P}' \setminus S) \\ q' \in \mathcal{P}' \setminus (\mathcal{P}' \setminus S)}} (f_{q',q} - f_{q,q'}) = 0.$$

Using (67), (61) and (64), we obtain

$$\text{val}_{G'}(\Sigma^+) + \sum_{q \in (\Sigma^+ \setminus S)} (c_q - c'_q) + \sum_{\substack{q \in (\mathcal{P}' \setminus S) \\ q' \in \mathcal{P}' \setminus (\mathcal{P}' \setminus S)}} (f_{q',q} - f_{q,q'}) = 0.$$

Combined with (66), this becomes

$$\text{val}_{G'}(S) = \text{val}_{G'}(\Sigma^+) + \sum_{q \in (\Sigma^+ \setminus S)} c_q + \sum_{\substack{q \in (\mathcal{P}' \setminus S) \\ q' \in \mathcal{P}' \setminus (\mathcal{P}' \setminus S)}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q \in S \\ q' \in (\mathcal{P}' \setminus S)}} c'_{q,q'}. \quad (70)$$

We now decompose the last term of the above equation using (59), (60) and (56) and write

$$\begin{aligned} \sum_{\substack{q \in S \\ q' \in (\mathcal{P}' \setminus S)}} c'_{q,q'} &= \sum_{\substack{q \in S \\ q' \in (\Sigma^+ \setminus S)}} (c_{q,q'} - f_{q,q'} + f_{q',q}) + \sum_{q \in S} f_{p,q}, \\ &= \sum_{\substack{q \in S \\ q' \in (\Sigma^+ \setminus S)}} c_{q,q'} - \sum_{\substack{q' \in S \\ q \in (\Sigma^+ \setminus S)}} (f_{q',q} - f_{q,q'}) + \sum_{q' \in S} (f_{p,q'} - f_{q',p}), \\ &= \sum_{\substack{q \in S \\ q' \in (\Sigma^+ \setminus S)}} c_{q,q'} - \sum_{\substack{q \in (\mathcal{P}' \setminus S) \\ q' \in S}} (f_{q',q} - f_{q,q'}). \end{aligned}$$

Combining the latter with (70), we finally obtain

$$\text{val}_{G'}(S) = \text{val}_{G'}(\Sigma^+) + \sum_{q \in (\Sigma^+ \setminus S)} c_q + \sum_{\substack{q \in S \\ q' \in (\Sigma^+ \setminus S)}} c_{q,q'} + \sum_{\substack{q \in (\mathcal{P}' \setminus S) \\ q' \in \mathcal{P}' \setminus \mathcal{P}'}} (f_{q',q} - f_{q,q'}).$$

Using (15), we remark that for any $q' \notin \mathcal{P}' \setminus \mathcal{P}'$, we have $q' \notin \sigma_{\mathcal{E}}(p)$ and we can finally deduce that (68) holds for all $S \subset \mathcal{P}'$ such that $p \notin S$. \square

As in Section 5, we will from now on consider a max flow f' in the graph \mathcal{G}' built in the current section. We also artificially extend the flow f' and set

$$f'_{q,q'} = 0, \text{ for all } (q, q') \in ((\mathcal{V}' \times \mathcal{V}') \setminus \mathcal{E}'). \quad (71)$$

Once again, the graph \mathcal{G}' satisfies analogues of (1) and (3), therefore, as usual, we denote for simplicity

$$f'_q = f'_{s,q} - f'_{q,t}, \quad \forall q \in \mathcal{P}'. \quad (72)$$

We are now going to combine f and f' in order to build a mapping $f'' : \mathcal{E} \rightarrow \mathbb{R}$ which will turn out to be a max-flow in \mathcal{G} such that

$$f''_{p,q} = f''_{q,p} = 0, \quad \forall q \in \sigma_{\mathcal{E}}(p).$$

As for \mathcal{G}' and f' , beware that the mapping f'' is different in Section 5 and in the current section.

Let us begin with the definition of f'' . We distinguish below the different possible configurations for the elements of \mathcal{E} .

$$f''_q = f_q \quad \forall q \notin \mathcal{P}', \quad (73)$$

$$f''_{q,q'} = f_{q,q'} \quad \forall (q, q') \in \mathcal{E}, \text{ with } q \notin \mathcal{P}' \text{ or } q' \notin \mathcal{P}' \quad (74)$$

$$f''_q = f_q + f'_q \quad \forall q \in \mathcal{P}', \quad (75)$$

and for $(q, q') \in (\mathcal{E} \cap (\Sigma^+)^2)$

$$f''_{q,q'} = \begin{cases} (f_{q,q'} + f'_{q,q'}) - (f'_{q',q} + f'_{q',q}) & \text{if } f_{q,q'} + f'_{q,q'} \geq f'_{q',q} + f'_{q',q} \\ 0 & \text{if } f_{q,q'} + f'_{q,q'} < f'_{q',q} + f'_{q',q} \end{cases} \quad (76)$$

and

$$f''_{p,q} = f_{p,q} - f'_{q,p} \quad \forall q \in (\mathcal{P}' \cap \sigma_{\mathcal{E}}(p)) \quad (77)$$

$$f''_{q,p} = 0 \quad \forall q \in (\mathcal{P}' \cap \sigma_{\mathcal{E}}(p)) \quad (78)$$

We also define

$$f''_{s,q} = \max(f''_q, 0) \quad \text{and} \quad f''_{q,t} = \max(-f''_q, 0), \quad \forall q \in \mathcal{P}. \quad (79)$$

Notice that the equation (73)-(79) permit to define $f''_{q,q'}$ for all $(q, q') \in \mathcal{E}$. Once again, we extend f'' outside \mathcal{E} and set

$$f''_{q,q'} = 0, \text{ for all } (q, q') \in ((\mathcal{V} \times \mathcal{V}) \setminus \mathcal{E}).$$

The following proposition holds.

Proposition 7 *The mapping $f'' : (\mathcal{V} \times \mathcal{V}) \rightarrow \mathbb{R}$ is max-flow in \mathcal{G} .*

Proof. Notice first that, if f'' is a flow in \mathcal{G} it is necessarily a max flow since, according to (55), $(\sigma_{\mathcal{E}}(t) \cap \mathcal{P}') = \emptyset$ and therefore, using (73), we always have $f''_{q,t} = f_{q,t}$, for all $q \in \sigma_{\mathcal{E}}(t)$. Therefore, we have $\text{val}_{\mathcal{G}}(f'') = \text{val}_{\mathcal{G}}(f)$ and, if f'' is a flow in \mathcal{G} , f'' is necessarily a max-flow in \mathcal{G} .

In order to show that f'' is a flow we first show that it satisfies the capacity constraints. Let $(q, q') \in \mathcal{E}$. We distinguish below the different possible configurations for the elements of (q, q') .

- If $q = s$ and $q' \notin B_p$ or if $q \notin B_p$ and $q' = t$, using (73) and (79), we know that

$$0 \leq f''_{q,q'} = f_{q,q'} \leq c_{q,q'} \quad \text{and} \quad 0 \leq f''_{q',q} = f_{q',q} \leq c_{q',q}.$$

- If $q \notin B_p$ or $q' \notin B_p$, using (74), we obtain again

$$0 \leq f''_{q,q'} = f_{q,q'} \leq c_{q,q'}.$$

- If $q = s$ and $q' \in \Sigma^+$, using (75) and (61), we get

$$0 \leq f''_{q,q'} = f_{s,q'} + f'_{s,q'} \leq c_{q,q'}.$$

- If $q = s$ and $q' = p$, using (75) and (62), we get

$$0 \leq f''_{q,q'} = f_{s,p} - f'_{p,t} \leq c_{q,q'}.$$

- If $(q, q') \in (\Sigma^+)^2$ and $f_{q,q'} + f'_{q,q'} \geq f_{q',q} + f'_{q',q}$, using (76) and (59), we obtain

$$0 \leq f''_{q,q'} = f_{q,q'} + f'_{q,q'} - f_{q',q} - f'_{q',q} \leq c_{q,q'} - f'_{q',q} \leq c_{q,q'}.$$

- If $(q, q') \in (\Sigma^+)^2$ and $f_{q,q'} + f'_{q,q'} < f_{q',q} + f'_{q',q}$, using (76), we trivially have

$$0 \leq f''_{q,q'} = 0 \leq c_{q,q'}.$$

- If $q = p$ and $q' \in (B_p \cap \sigma_{\mathcal{E}}(p))$, using (77) and (60), we get

$$0 \leq f''_{q,q'} = f_{p,q'} - f'_{q',p} \leq c_{q,q'}.$$

- If $q \in (B_p \cap \sigma_{\mathcal{E}}(p))$ and $q' = p$, then (78) trivially guarantees that

$$0 \leq f''_{q,q'} = 0 \leq c_{q,q'}.$$

In order to show the flow conservation constraints, we consider, from now on, $q \in \mathcal{P}$. We distinguish below the different possible position for q .

- If $q \in \mathcal{P} \setminus \mathcal{P}'$, we have, using (73) and (74), we have $f''_{q,q'} = f_{q,q'}$ and $f''_{q',q} = f_{q',q}$, for all $q' \in \sigma_{\mathcal{E}}(q)$. Therefore,

$$\sum_{q' \in \sigma_{\mathcal{E}}(q)} f''_{q',q} = \sum_{q' \in \sigma_{\mathcal{E}}(q)} f_{q',q} = \sum_{q' \in \sigma_{\mathcal{E}}(q)} f_{q,q'} = \sum_{q' \in \sigma_{\mathcal{E}}(q)} f''_{q,q'}.$$

- If $q \in \Sigma^+$, expressing that the two flows f and f' are conserved at q , we obtain using (7) and (56)

$$f_q + \sum_{\substack{q' \in (\mathcal{P} \cap \sigma_{\mathcal{E}}(q)) \\ q' \in \mathcal{P} \setminus \mathcal{P}'}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q' \in (\mathcal{P} \cap \sigma_{\mathcal{E}}(q)) \\ q' \in \Sigma^+}} (f_{q',q} - f_{q,q'}) + f_{p,q} = 0$$

and

$$f'_q + \sum_{\substack{q' \in (\mathcal{P} \cap \sigma_{\mathcal{E}}(q)) \\ q' \in \Sigma^+}} (f'_{q',q} - f'_{q,q'}) - f'_{q,p} = 0.$$

Summing those inequalities and using (74)-(78), we obtain

$$f''_q + \sum_{\substack{q' \in (\mathcal{P} \cap \sigma_{\mathcal{E}}(q)) \\ q' \in \mathcal{P} \setminus \mathcal{P}'}} (f''_{q',q} - f''_{q,q'}) + \sum_{\substack{q' \in (\mathcal{P} \cap \sigma_{\mathcal{E}}(q)) \\ q' \in \Sigma^+}} (f''_{q',q} - f''_{q,q'}) + (f''_{p,q} - f''_{q,p}) = 0.$$

The latter expresses that f'' is conserved at the node q .

- If $q = p$, then using (75), (77) and (78) as well as (15) and (62), we obtain

$$\sum_{q' \in \sigma_{\mathcal{E}}(p)} (f''_{q',p} - f''_{p,q'}) = f_{s,p} - f'_{p,t} - \sum_{q' \in (\mathcal{P}' \cap \sigma_{\mathcal{E}}(p))} (f_{p,q'} - f'_{q',p}).$$

Using that $f_{p,t} = 0$ (see (55), (4) and (3)), $f'_{s,p} = 0$ (see (58) and (63)), $f_{q',p} = 0$ (see (56)) and $f'_{p,q'} = 0$ (see (58) and (63)), we obtain

$$\begin{aligned} \sum_{q' \in \sigma_{\mathcal{E}}(p)} (f''_{q',p} - f''_{p,q'}) &= (f_{s,p} - f_{p,t}) + (f'_{s,p} - f'_{p,t}) \\ &\quad - \sum_{q' \in (\mathcal{P}' \cap \sigma_{\mathcal{E}}(p))} [(f_{p,q'} - f_{q',p}) + (f'_{p,q'} - f'_{q',p})]. \end{aligned} \quad (80)$$

Simplifying, we finally obtain

$$\begin{aligned} \sum_{q' \in \sigma_{\mathcal{E}}(p)} (f''_{q',p} - f''_{p,q'}) &= \sum_{q' \in \sigma_{\mathcal{E}}(p)} (f_{q',p} - f_{p,q'}) + \sum_{q' \in \sigma_{\mathcal{E}}(p)} (f'_{q',p} - f'_{p,q'}), \\ &= 0, \end{aligned}$$

since the two flows f and f' are conserved at p .

This concludes the proof. \square

Proposition 8 *If Σ^+ is a minimum s - t cut in the graph \mathcal{G}' defined in Section 6, then the max-flow f'' is such that*

$$\forall q \in \sigma_{\mathcal{E}}(p), \quad f''_{q,p} = f''_{p,q} = 0.$$

As a consequence, removing the node p from the graph \mathcal{G} does not modify its maximal flow value.

Proof. If Σ^+ is a minimum s - t cut in the graph \mathcal{G}' defined in Section 6, then Ford-Fulkerson theorem, (67) and (58) guarantee that

$$f_p = \text{val}_{\mathcal{G}'}(\Sigma^+) = \text{val}_{\mathcal{G}'}(f') = f'_{p,t}.$$

Using (75), (72) and (71) this yields

$$f''_p = f_p - f'_{p,t} = 0,$$

which, using (79), provides

$$f''_{s,p} = f''_{p,t} = 0.$$

Together with (78), this guarantees that

$$\text{for all } q \in \sigma_{\mathcal{E}}(p), \quad f''_{q,p} = 0. \quad (81)$$

Expressing the flow conservation constraint at p for f'' , we deduce from (81) that

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} f''_{p,q} = \sum_{q \in \sigma_{\mathcal{E}}(p)} f''_{q,p} = 0,$$

which guarantees that

$$\text{for all } q \in \sigma_{\mathcal{E}}(p), \quad f''_{p,q} = 0,$$

since $f''_{p,q} \geq 0$, for all $q \in \sigma_{\mathcal{E}}(p)$.

Together with (81), this concludes the proof. \square

We can now conclude with the following proposition.

Proposition 9 *Let \mathcal{G} be the graph defined in Section 2, let B satisfy (15) and let us assume that $p \in \mathcal{P}$ satisfies (55). Then, there exists a max-flow f in \mathcal{G} such that*

$$\forall q \in \sigma_{\mathcal{E}}(p), \quad f_{p,q} = f_{q,p} = 0. \quad (82)$$

As a consequence, removing the node p from the graph \mathcal{G} does not modify its maximal flow value.

Proof. This is a straightforward consequence of Proposition 5, Proposition 6, Proposition 7 and Proposition 8.

Indeed, if (55) holds, we know that there is max-flow f in \mathcal{G} satisfying (56). Therefore, using the notations of Section 6, we know that for any $S \subset \mathcal{P}'$ such that $p \notin S$

$$\sum_{q \in \Sigma^+ \setminus S} \left[c_q + \sum_{q' \in \mathcal{P} \setminus \mathcal{P}'} (f_{q',q} - f_{q,q'}) \right] \geq 0.$$

Therefore, for \mathcal{G}' as defined in Section 6, Proposition 6 guarantees that for any $S \subset \mathcal{P}'$

$$\text{val}_{\mathcal{G}'}(S) \geq \text{val}_{\mathcal{G}'}(\Sigma^+),$$

and therefore Σ^+ is a min s - t cut in \mathcal{G}' . Then, Proposition 7 guarantees that f'' is a max-flow in \mathcal{G} and Proposition 8 guarantees that f'' satisfies (82). \square

7 Numerical experiments

In this section, we evaluate the performance of the test (17) against standard graph cuts (SGC) in terms of memory and segmentation accuracy for reducing graphs involved in binary image segmentation. For doing so, we compare the relative reduced graph sizes obtained from the tests (17) and the test described in [11] which is defined by

$$\rho = \frac{\#\mathcal{V}^*}{\#\mathcal{V}} \times 100, \quad (83)$$

where \mathcal{V}^* denote the set of useful nodes actually present in the reduced graph. Lower values of (83) therefore means lower memory consumption. In the experiments, B corresponds to a square window of size 3 centered at the origin. Moreover, we use the interactive image segmentation model described in [1] in connectivity 1¹. Let us now briefly remind this model. Consider an image $I : \mathcal{P} \rightarrow [0, 1]^c$ as a function, mapping each pixel $p \in \mathcal{P}$ to a vector $I_p \in [0, 1]^c$. For any pixel $p \in \mathcal{P}$, the data term $E_p(\cdot)$ is defined as

$$\begin{cases} E_p(1) = -\log \mathbb{P}(I_p \mid p \in \mathcal{O}), \\ E_p(0) = -\log \mathbb{P}(I_p \mid p \in \mathcal{B}). \end{cases} \quad (84)$$

where \mathcal{O} and \mathcal{B} denote respectively the object and background seeds interactively given by the user. In (84), the probability density function of color distribution for the object and the background seeds are estimated using normalized histograms with a number of bins respectively equal to 256 and 50 for grayscale and color images. For any pixels pair $(p, q) \in (\mathcal{P} \times \mathcal{P})$, the smoothness term $E_{p,q}(\cdot)$ corresponds to a contrast-sensitive Ising model

$$E_{p,q}(u_p, u_q) = \begin{cases} 0 & \text{if } u_p = u_q, \\ \frac{1}{\|p-q\|_2} \exp\left(-\frac{\|I_p - I_q\|_2^2}{2\sigma^2}\right) & \text{otherwise,} \end{cases} \quad (85)$$

where $\|\cdot\|_2$ denotes the Euclidean norm (either in \mathbb{R}^d or \mathbb{R}^c) and σ is a parameter generally linked to the contrast in the image. As an illustration, when the intensities I_p and I_q in (85) appear in the same range, we have $\|I_p - I_q\|_2 < \sigma$. This implies a large cost in the exponential and therefore discourages any cut between the nodes p and q . The opposite situation is also valid when the nodes p and q are located on both sides of a contour. The following experiments are performed on an Athlon Dual Core 6000+ 3GHz with 2Gb RAM using the max-flow algorithm of [2]. Times include the graph construction, the max-flow

¹This term corresponds to 8 neighbors in 2D images and 26 neighbors in 2D+t and 3D images.

computation as well as the construction of the solution and are averaged over 10 runs.

Let us now describe the experimental procedure for segmenting images segmented in [11]. For each image, the seeds and the model parameters are manually optimized for getting the best segmentation. Using these seeds and parameters, a reference segmentation is computed with SGC. Afterwards, a second segmentation is computed with the test (17) using the same seeds and parameters. The differences between both segmentations are then estimated using the Hausdorff distance. We also measure the difference (denoted by $\Delta\rho^*$) between the relative reduced graph size (denoted by ρ^*) obtained with the test (17) and the test in [11]. In words, the test studied in [11] is more efficient (from a memory point of view) than the test (17) when $\Delta\rho^* > 0$ and conversely. The obtained results for these experiments are summarized in Table 1 and illustrated in Figure 2.

As the test [11], the test (17) globally outperforms SGC in terms of memory while keeping a Hausdorff distance null ². SGC fail to segment some of the 2D+t and 3D volumes while the test (17) permits to segment them while keeping a low memory usage. In a similar manner to the test [11], one can also observe in Figure 2 that reduced graphs have a larger size when β is low (denotes a strong regularization of the model). Indeed, since t-links (edges linking a node $p \in \mathcal{P}$ to s or t) capacities are multiplied by β , the test (17) becomes harder to satisfy when β diminishes and conversely. This situation is typical of noisy images since a lot of nodes inside B are connected to opposite terminals (see image "zen-garden-c" in Figure 2). An ideal situation therefore consists as in [11] of large area of nodes linked to the same terminal and separated by rough borders (see images "book", "fluo-cell-c" and "ct-thorax" in Figure 2).

In Table 1, one also observe that the test (17) is globally less efficient than the test [11] in terms of memory usage with an average negative $\Delta\rho^*$ over all images. This least performance is strenghtened when the amount of regularization is large. Indeed, in such a situation, the test [11] can be relaxed when varying the window radius r unlike the test (17). However, when the amount of regularization is of moderate level, the memory gains are almost the same. We have also measured no differences between the segmentation obtained with the test (17) and the test [11]. This clearly demonstrates that the test [11] is an heuristic achieving very good results with respect to the test (17). The exactness of the test (17) is however at the expense of a larger computational cost compared to the test [11] since the worst-case complexity now depends both on the size of B and the graph connectivity.

Finally, we want to highlight that the test (17) for reducing the graphs is not limited to a particular energy model or more generally, to image segmentation. Such an approach can be for instance adapted to solve Chan-Vese model [4] in image segmentation or Total Variation-based models [7] in image restoration.

²We omit this information in the experiments since this value is always the same.

Volume name	Size	SGC	Test (17)	ρ^* (%)	$\Delta\rho^*$ (%)
zen-garden-c	481 × 321	22.90 Mb	23.39 Mb	90.24	+0.51
red-flowers-c	481 × 321	22.90 Mb	10.40 Mb	46.26	-22.96
book	3012 × 2048	917.26 Mb	78.95 Mb	7.91	+0.27
cells-z	512 × 512	38.91 Mb	23.39 Mb	61.65	-12.74
interview-man-2c	426 × 240 × 180	7.55 Gb (*)	228.44 Mb	3.21	0.0
plane-take-off-c	492 × 276 × 180	10.03 Gb (*)	532.00 Mb	6.09	+0.11
fluo-cell-c	478 × 396 × 121	9.39 Gb (*)	514.00 Mb	5.88	0.0
ct-thorax	245 × 245 × 151	3.71 Gb (*)	771.00 Mb	17.30	0.0
cells	230 × 230 × 57	1.23 Gb	771.00 Mb	59.38	-8.0
brain	181 × 217 × 181	2.91 Gb (*)	771.00 Mb	24.22	+0.16

Table 1: The memory required by SGC and graph reduced with the test (17) are compared when segmenting 2D (top rows), 2D+t (middle rows) and 3D (bottom rows) images. Color images names are suffixed by "c". (*) means that SGC fail to segment the image due to a too large memory usage. The memory consumption is obtained by multiplying the number of nodes and edges by the memory size of the structure representing them.

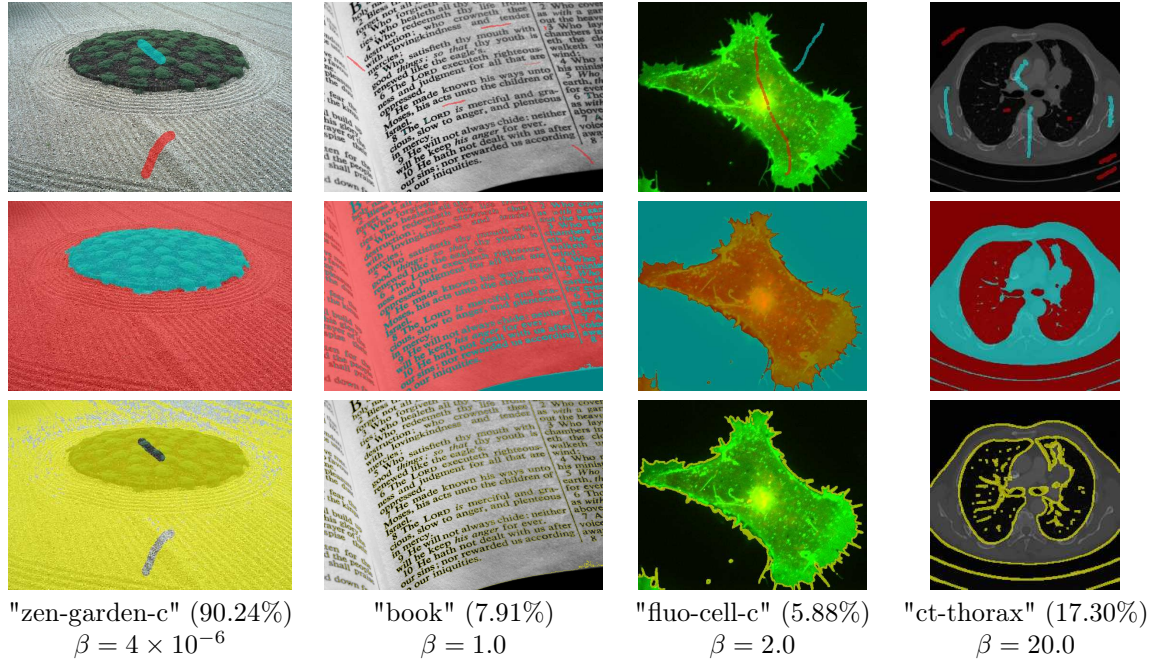


Figure 2: Seeds (top row), segmentations (middle row) and reduced graphs (bottom row) for a subset of images of Table 1. Reduced graphs are superimposed in yellow to the original image. Relative reduced graph sizes as well as β values are indicated below each image.

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