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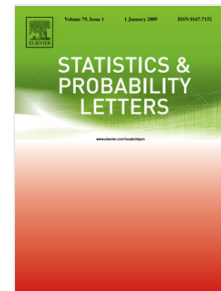
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INARCH(1) Processes: Higher-Order Moments and Jumps

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Abstract

The INARCH(1) model is a simple but practically relevant, two-parameter model for processes of overdispersed counts with an autoregressive serial dependence structure. We derive closed-form expressions for the joint (central) moments and cumulants of the INARCH(1) model up to order 4. These expressions are applied to derive moments of jumps in INARCH(1) processes. We illustrate this kind of application with a real-data example, and outline further potential applications.

Keywords: Count-data time series; cumulants; INARCH(1) model; jumps; moments; overdispersion.

1. Introduction

During the last years, a number of articles considered the analysis and modeling of count-data time series with *overdispersion*, i. e., with a variance greater than the mean, such that the marginal process distribution cannot be described by the popular Poisson model. Typical reasons for such overdispersion are the presence of positive correlation between the monitored events or a variation in the probability of the monitored events, see Weiß (2009a) and references therein, which are commonly observed in real-world applications. Some of these articles discuss ARMA-type models based on an appropriate type of thinning operation, see the survey by Weiß (2008a), but recently, another approach for modeling time series of overdispersed counts became quite popular: the *INGARCH models*, the *integer-valued* counterpart to the usual *generalized autoregressive conditional heteroskedasticity* models. The INGARCH models were introduced by Heinen (2003) and further investigated by Ferland et al. (2006); Fokianos et al. (2009); Fokianos & Fried (2010); Weiß (2009a, 2010); Zhu & Wang (2009, 2010). We consider the particular case of the INGARCH(1, 0) model in the following, and refer to it as the *INARCH(1) model*. This simple two-parameter model with its AR(1)-like serial dependence structure has already proved to be of great practical relevance, with concrete applications being reported for monthly claims counts (Weiß, 2009a), for download counts (Zhu & Wang, 2010), and for monthly strike data (Weiß, 2010).

1.1 Definition (INARCH(1) Model) Let $(X_t)_{\mathbb{Z}}$ be a process with state space $\mathbb{N}_0 = \{0, 1, \dots\}$, let $\beta > 0$ and $0 < \alpha < 1$. The process $(X_t)_{\mathbb{Z}}$ is said to follow an INARCH(1)

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model if X_t , conditioned on X_{t-1}, X_{t-2}, \dots , is Poisson distributed according to $Po(\beta + \alpha \cdot X_{t-1})$.

The INARCH(1) model of Definition 1.1 is a stationary, ergodic Markov chain (Ferland et al., 2006; Zhu & Wang, 2009) with simple Poisson probabilities as the transition probabilities. All moments exist (Ferland et al., 2006), and its autocorrelation function $\rho(k) := \text{Corr}[X_t, X_{t-k}] = \alpha^k$ like in the standard AR(1) case (Weiß, 2009a). The marginal distribution of the INARCH(1) model can be expressed in terms of its cumulants, which can be determined according to the recursive scheme provided by Weiß (2009a, 2010). In particular, using the abbreviation $f_k := \beta / \prod_{i=1}^k (1 - \alpha^i)$, the first four cumulants are given by

$$\kappa_1 = f_1, \quad \kappa_2 = f_2, \quad \kappa_3 = f_3 \cdot (1 + 2\alpha^2), \quad \kappa_4 = f_4 \cdot (1 + 6\alpha^2 + 5\alpha^3 + 6\alpha^5), \quad (1)$$

see Weiß (2009a). As a consequence of formula (1), mean μ , variance σ^2 , skewness and excess of X_t are given by

$$\frac{\beta}{1-\alpha}, \quad \frac{\beta}{(1-\alpha)(1-\alpha^2)}, \quad \frac{1+2\alpha^2}{1+\alpha+\alpha^2} \cdot \sqrt{\frac{1+\alpha}{\beta}} \quad \text{and} \quad \frac{1+6\alpha^2+5\alpha^3+6\alpha^5}{\beta(1+\alpha+\alpha^2)(1+\alpha^2)}, \quad (2)$$

respectively. Since $\sigma^2 > \mu$, the INARCH(1) model allows to describe overdispersion.

In this article, we shall consider joint (central) moments and cumulants of the INARCH(1) model. In Section 2, we derive closed-form expressions for such moments and cumulants up to order 4. These expressions are applied in Section 3 to derive moments of jumps in INARCH(1) processes, and we illustrate this kind of application with a real-data example. Further potential applications are outlined in context of the conclusions in Section 4.

2. Moments and Cumulants of INARCH(1) Processes

The aim of this section is to provide closed-form expressions for the joint (central) moments and cumulants of the INARCH(1) model up to order 4. For this purpose, we introduce the following notations:

$$\begin{aligned} \mu(s_1, \dots, s_{r-1}) &:= E[X_t \cdot X_{t+s_1} \cdots X_{t+s_{r-1}}], \\ \tilde{\mu}(s_1, \dots, s_{r-1}) &:= E[(X_t - \mu)(X_{t+s_1} - \mu) \cdots (X_{t+s_{r-1}} - \mu)], \\ \kappa(s_1, \dots, s_{r-1}) &:= \text{Cum}[X_t, X_{t+s_1}, \dots, X_{t+s_{r-1}}], \end{aligned} \quad (3)$$

with $0 \leq s_1 \leq \dots \leq s_{r-1}$ and $r \in \mathbb{N}$. The case $r = 1$ would correspond to the marginal mean $\mu := E[X_t] = \beta / (1 - \alpha) = f_1$.

2.1 Theorem (Moments of INARCH(1) Process) *Let $(X_t)_{\mathbb{N}}$ be a stationary INARCH(1) process according to Definition 1.1.*

(i) For any $k \geq 0$, we have $\mu(k) = f_2 \cdot (\alpha^k + \beta(1 + \alpha))$.

(ii) For any $l \geq k \geq 0$, we have

$$\mu(k, l) = \frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^l + f_1 f_2 \cdot \alpha^{l-k} - f_3 \cdot (1-\alpha) \cdot \alpha^{l+k+1} + f_1 \cdot \mu(k).$$

(iii) For any $m \geq l \geq k \geq 0$, we have

$$\begin{aligned} \mu(k, l, m) &= \alpha^{m-l} \cdot \left(\alpha^{2l+k+2} \cdot f_4 \cdot (1-\alpha)(2+2\alpha-\alpha^2) \right. \\ &\quad - (\beta+2) \cdot f_3 \cdot \alpha^{2l+1} + \frac{f_2 \cdot \alpha^{2l-k}}{(1-\alpha)(1-\alpha^2)} \cdot (\alpha+2\beta) - \alpha^{2(l-k)+1} \cdot f_3 \cdot \beta \\ &\quad \left. + \frac{\beta+1}{1-\alpha} \cdot \mu(k, l) - f_2 \cdot \mu(k) \cdot (\alpha + \beta(1+\alpha)) \right) + f_1 \cdot \mu(k, l). \end{aligned}$$

The proof of Theorem 2.1 is provided in Appendix A.1. Now, we can apply Theorem 2.1 to derive analogous expressions for the joint central moments and cumulants.

2.2 Corollary (Central Moments and Cumulants) Let $(X_t)_{\mathbb{N}}$ be a stationary INARCH(1) process according to Definition 1.1.

(i) For any $k \geq 0$, we have $\tilde{\mu}(k) = \kappa(k) = f_2 \cdot \alpha^k$.

(ii) For any $l \geq k \geq 0$, we have

$$\tilde{\mu}(k, l) = \kappa(k, l) = f_3 \cdot \alpha^l \cdot (1 + \alpha + \alpha^2 - (1-\alpha) \cdot \alpha^{k+1}).$$

(iii) For any $m \geq l \geq k \geq 0$, we have

$$\begin{aligned} \kappa(k, l, m) &= \alpha^m \cdot f_4 \cdot ((1-\alpha)(2+2\alpha-\alpha^2) \cdot \alpha^{l+k+2} \\ &\quad - (1-\alpha^4) \cdot (2 \cdot \alpha^{l+1} + \alpha^{k+1}) + (1+\alpha+\alpha^2)(1+\alpha^2) \cdot (1+\alpha+\alpha^{l-k+1})), \end{aligned}$$

and

$$\tilde{\mu}(k, l, m) = \kappa(k, l, m) + f_2^2 \cdot (\alpha^{m-l+k} + 2\alpha^{m+l-k}).$$

The proof of Corollary 2.2 is provided in Appendix A.2.

3. Jumps in INARCH(1) Processes

An interesting feature of a count-data process are jumps $J_t := X_t - X_{t-1}$. Such jumps have already been investigated for the case of a Poisson INAR(1) process (Weiß, 2008b, 2009b) and a binomial AR(1) process (Weiß, 2009c). The extent of jumps in a count-data process reflects the extent of serial dependence: If the given count-data process exhibits an only small degree of serial dependence, then quite large jumps can be observed and vice versa. Therefore, the monitoring of jumps offers the potential of application to estimate model parameters related to the serial dependence structure, see, e. g., Weiß (2008b), or to construct control charts to detect changes in the serial dependence structure, as proposed by Weiß (2009b,c).

In view of these potential applications, we shall now investigate stochastic properties of jumps $J_t := X_t - X_{t-1}$ in an INARCH(1) process. The distribution of J_t might be described in terms of its cumulants. As shown in Appendix A.3, the following relation between the cumulant generating function (cgf) of J_t and that of X_t holds for an INARCH(1) process:

$$\kappa_J(z) = \beta \cdot (e^z - 1) + \kappa_X(\alpha \cdot (e^z - 1) - z). \quad (4)$$

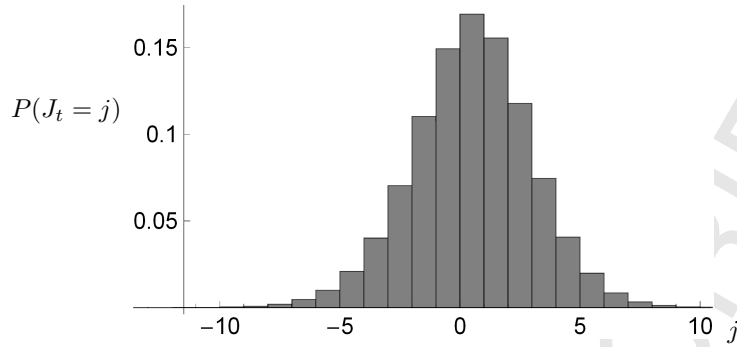


Figure 1: Distribution of J_t for INARCH(1) process with $\beta = 2$ and $\alpha = 0.6$.

So taking derivatives on both sides of this equation, the cumulants of J_t can be expressed in terms of the cumulants of X_t , also see formula (1). Alternatively, and we shall follow this approach in the following, one can apply the results of the previous Section 2 to derive expressions for moments of J_t directly.

3.1 Proposition (Moments of Jumps in INARCH(1) Processes) *Let $(X_t)_{\mathbb{N}}$ be a stationary INARCH(1) process according to Definition 1.1. Then*

$$\begin{aligned} E[J_t] &= 0, & E[J_t^2] &= 2f_2 \cdot (1 - \alpha) = V[J_t], & E[J_t^3] &= -3f_3\alpha^2(1 - \alpha)^2, \\ E[J_t^4] &= 2f_4 \cdot (1 - \alpha)^2(1 + \alpha + 4\alpha^2 + 4\alpha^3 + 2\alpha^4 + 8\alpha^5 - 2\alpha^6) + 12f_2^2 \cdot (1 - \alpha)^2, \\ V[J_t^2] &= 2f_4 \cdot (1 - \alpha)^2(1 + \alpha + 4\alpha^2 + 4\alpha^3 + 2\alpha^4 + 8\alpha^5 - 2\alpha^6) + 8f_2^2 \cdot (1 - \alpha)^2. \end{aligned}$$

The proof is provided in Appendix A.4. It is quite interesting to note that in contrast to the case of the Poisson INAR(1) or binomial AR(1) process (Weiß, 2008b, 2009c), the distribution of jumps in an INARCH(1) process is *not symmetric* but negatively skewed. The effect of the dependence parameter α on the absolute extent of jumps becomes clear from $E[J_t^2] = 2\beta/(1 - \alpha^2)$.

Concerning the autocorrelation structure of the process of jumps, we can simply adapt the result of Theorem 3 in Weiß (2009c), since it is determined only by the autocorrelation function of $(X_t)_{\mathbb{N}}$. So

$$\rho_J(k) = -\frac{1 - \alpha}{2} \cdot \alpha^{k-1}, \quad \rho_{\text{part},J}(k) = -\frac{1 - \alpha}{2 + (k - 1) \cdot (1 - \alpha)} \quad \text{for } k \geq 1, \quad (5)$$

which is always negative.

Let us conclude with a real-data example to see if and how the properties of Proposition 3.1 about theoretical moments can also be identified empirically in a given time series.

3.2 Example (Jumps in Strikes Data) We continue the real-data example from Section 5 of Weiß (2010), where a time series of monthly strike counts (January 1994 to December 2002) of total length $T = 108$ was discussed and shown to be modeled very well by an INARCH(1)

model with parameter values of about $\beta = 2$ and $\alpha = 0.6$. For such an INARCH(1) model, the jumps J_t should have a 2nd order moment (=variance) of $E[J_t^2] = 6.25$, a 3rd order moment of $E[J_t^3] \approx -1.72$ and thus a skewness of ≈ -0.11 . The moderate degree of negative skewness also becomes obvious from the estimated distribution shown in Figure 1, which was obtained from a simulated INARCH(1) time series of length 10^6 .

From the strikes data, we obtain a time series of 107 jumps. The empirical 2nd order moment is computed as ≈ 6.47 , which is close to the theoretical value of 6.25 given above. For the empirical 3rd order moment and skewness, we indeed observe negative values, but deviating more clearly from the theoretical ones given above: -8.64 and -0.53 , respectively. This, however, is no contradiction to the assumption of an underlying INARCH(1) process, but can be explained with the inherent variability of the estimators for these quantities: We simulated 10,000 time series of length 108 from an INARCH(1) process with parameters $\beta = 2$ and $\alpha = 0.6$. Each time, we computed the relevant empirical moments and used this data to analyze their stochastic properties. It turned out that

- the empirical 2nd order moment has mean 6.28, standard error 1.13 and skewness 0.63 such that the observed value of 6.47 is indeed very close to the expected one,
- the empirical 3rd order moment has mean -1.82 , standard error 4.63 and skewness -0.80 such that the observed value of -8.64 deviates by less than 2 standard errors from the expected one,
- the skewness has mean -0.11 , standard error 0.26 and skewness -0.14 such that the observed value of -0.53 deviates by less than 2 standard errors from the expected one.

These results also show that the three estimators are at most moderately biased, but especially the 3rd order statistics have quite large standard errors that do not allow to detect a significant deviation from 0.

The last observation is somehow disappointing, since the property of asymmetric jumps distinguishes the INARCH(1) model, e. g., from the Poisson INAR(1) model and therefore offers the potential of deciding between both models. Such tools that allow to discriminate between different model families are generally of great practical importance, see, e. g., the discussion in Jung et al. (2006). Certainly, the standard errors decrease for an increasing length T of the available time series, but only slowly (approximately with $1/\sqrt{T}$) as shown by the following examples (for $T = 250, 500, 1000, 2500$): standard errors 2.99, 2.10, 1.49, 0.93 of empirical 3rd order moments and standard errors 0.18, 0.13, 0.09, 0.06 of empirical skewness, respectively. These numbers are again obtained from simulating 10,000 time series each, and they show that quite long time series are necessary to be able to detect a significant asymmetry if empirical skewness or 3rd order moments are used as statistics. Therefore, future research should try to find alternative statistics related to the symmetry behaviour of jumps, which might be applied already to short time series of counts.

4. Conclusions and Future Research

The INARCH(1) model is a simple but practically relevant, two-parameter model for processes of overdispersed counts with an AR(1)-like serial dependence structure. We derived closed-form expressions for its joint (central) moments and cumulants up to order 4. These expressions are particularly important for several types of application. As an example, we showed how to use them to derive moments of jumps in INARCH(1)

processes. Such jumps offer themselves a lot of applications like parameter estimation or process monitoring, but in this context, see the discussed real-data example, it is also important to analyze moment properties of empirical quantities based on jumps. So an obvious issue for future research would be to apply the expressions for the joint moments presented in this article to derive (asymptotic) moments of jumps-based statistics. Jumps to the power of 3 might be useful for showing a significant asymmetry of the distribution of jumps (and therefore excluding, e. g., a Poisson INAR(1) model, see above), while squared jumps may be applied for parameter estimation: The squared difference estimator

$$SDE := \frac{1}{2(T-1)} \cdot \sum_{t=2}^T (X_t - X_{t-1})^2 = \frac{1}{2(T-1)} \cdot \sum_{t=2}^T J_t^2$$

is an obviously unbiased estimator of the quantity $\sigma^2 \cdot (1 - \alpha)$ and hence a possible alternative to the first-order sample autocovariance as a (biased) estimator of $\sigma^2 \cdot \alpha$.

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Appendix A. Proofs

Appendix A.1. Proof of Theorem 2.1

Let $(X_t)_{\mathbb{N}}$ be a stationary INARCH(1) process according to Definition 1.1. Using the notations of formula (3), an explicit expression for the case $r = 2$ is easily derived using that $V[X_t] = f_2$

according to formula (1) and that the autocorrelation function equals $\rho(k) = \alpha^k$:

$$\mu(k) = Cov[X_t, X_{t+k}] + E[X_t]^2 = f_2 \cdot \alpha^k + f_1^2 = f_2 \cdot (\alpha^k + \beta(1 + \alpha)).$$

Before we continue with higher-order moments, we need the following conditional moments:

$$\begin{aligned} E[X_t | X_{t-1}, \dots] &= \alpha \cdot X_{t-1} + \beta, \\ E[X_t^2 | X_{t-1}, \dots] &= \alpha^2 \cdot X_{t-1}^2 + \alpha(2\beta + 1) \cdot X_{t-1} + \beta(\beta + 1), \\ E[X_t^3 | X_{t-1}, \dots] &= \alpha^3 \cdot X_{t-1}^3 + 3\alpha^2(\beta + 1) \cdot X_{t-1}^2 \\ &\quad + \alpha(3\beta^2 + 6\beta + 1) \cdot X_{t-1} + \beta(\beta^2 + 3\beta + 1). \end{aligned} \quad (\text{Appendix A.1})$$

These expressions follow from the first three moments of the Poisson distribution, since the conditional distribution of X_t given X_{t-1}, \dots is $Po(\beta + \alpha \cdot X_{t-1})$.

Third-Order Moments

In the following, we shall use the well-known formula $E[X] = E[E[X | Z]]$ for conditional expectations together with formula (Appendix A.1). To derive an explicit expression for $\mu(k, l)$ with $0 \leq k \leq l$, we distinguish between the following three cases:

Case 1: $l > k$. Then we have

$$\begin{aligned} \mu(k, l) &= E[X_t X_{t+k} \cdot E[X_{t+l} | X_{t+l-1}, \dots]] = \alpha \cdot \mu(k, l-1) + \beta \cdot \mu(k) \\ &= \dots = \alpha^{l-k} \cdot \mu(k, k) + \beta \cdot \mu(k) \cdot \sum_{j=0}^{l-k-1} \alpha^j \\ &= \alpha^{l-k} \cdot (\mu(k, k) - f_1 \cdot \mu(k)) + f_1 \cdot \mu(k). \end{aligned}$$

Case 2: $l = k > 0$. Then we have

$$\begin{aligned} \mu(k, k) &= E[X_t \cdot E[X_{t+k}^2 | X_{t+k-1}, \dots]] \\ &= \alpha^2 \cdot \mu(k-1, k-1) + \alpha(2\beta + 1) \cdot \mu(k-1) + \beta(\beta + 1) \cdot \mu \\ &= \alpha^2 \cdot \mu(k-1, k-1) + (2\beta + 1) \cdot f_2 \cdot \alpha^k + \alpha(2\beta + 1) \cdot f_1^2 + (\beta + 1) \cdot (1 - \alpha) \cdot f_1^2 \\ &= \alpha^2 \cdot \mu(k-1, k-1) + (2\beta + 1) \cdot f_2 \cdot \alpha^k + (1 - \alpha^2) \cdot f_1 \cdot \mu(0) \\ &= \dots = \alpha^{2k} \cdot \mu(0, 0) + (2\beta + 1) \cdot f_2 \cdot \sum_{j=0}^{k-1} \alpha^{k+j} + (1 - \alpha^2) \cdot f_1 \cdot \mu(0) \cdot \sum_{j=0}^{k-1} \alpha^{2j} \\ &= \alpha^{2k} \cdot (\mu(0, 0) - \frac{2\beta+1}{1-\alpha} \cdot f_2 - f_1 \cdot \mu(0)) + \frac{2\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^k + f_1 \cdot \mu(0). \end{aligned}$$

Case 3: $l = k = 0$. Then we have according to formula [1,5] in Douglas (1980) and formula (1) that

$$\begin{aligned} \mu(0, 0) &= E[X_t^3] = \kappa_3 + 3 \cdot \kappa_2 \cdot \mu + \mu^3 = f_3 \cdot (1 + 2\alpha^2) + 3 \cdot f_2 \cdot f_1 + f_1^3 \\ &= \left(\frac{f_2}{1-\alpha} - f_3 \cdot \alpha(1-\alpha) \right) + f_2 \cdot \frac{3\beta}{1-\alpha} + f_2 \cdot \frac{\beta^2(1+\alpha)}{1-\alpha} \\ &= \frac{f_2}{1-\alpha} \cdot (1 + 2\beta) + \frac{f_2}{1-\alpha} \cdot \beta(1 + \beta(1 + \alpha)) - f_3 \cdot \alpha(1-\alpha) = \frac{1+2\beta}{1-\alpha} \cdot f_2 + f_1 \cdot \mu(0) - f_3 \cdot \alpha(1-\alpha). \end{aligned}$$

So the above formula for $\mu(k, k)$ simplifies to

$$\mu(k, k) = -f_3 \cdot \alpha(1-\alpha) \cdot \alpha^{2k} + \frac{2\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^k + f_1 \cdot \mu(0),$$

which also holds for $k = 0$, and it follows that

$$\begin{aligned} \mu(k, l) &= \alpha^{l-k} \cdot (\mu(k, k) - f_1 \cdot \mu(k)) + f_1 \cdot \mu(k) \\ &= \alpha^{l-k} \cdot \left(-f_3 \cdot \alpha(1-\alpha) \cdot \alpha^{2k} + \left(\frac{2\beta+1}{1-\alpha} - f_1 \right) \cdot f_2 \cdot \alpha^k + f_1 \cdot (\mu(0) - f_2 \cdot \beta(1 + \alpha)) \right) + f_1 \cdot \mu(k) \\ &= \alpha^{l-k} \cdot \left(-f_3 \cdot \alpha(1-\alpha) \cdot \alpha^{2k} + \frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^k + f_1 \cdot f_2 \right) + f_1 \cdot \mu(k) \end{aligned}$$

holds for any $0 \leq k \leq l$.

Fourth-Order Moments

To derive an explicit expression for $\mu(k, l, m)$ with $0 \leq k \leq l \leq m$, we proceed in a similar way as above and distinguish between the following four cases:

Case 1: $m > l$. Then we have as above that

$$\mu(k, l, m) = \dots = \alpha^{m-l} \cdot (\mu(k, l, l) - f_1 \cdot \overline{\mu(k, l)}) + f_1 \cdot \mu(k, l).$$

Case 2: $m = l > k$. Then we have

$$\begin{aligned} \mu(k, l, l) &= E[X_t X_{t+k} \cdot E[X_{t+l}^2 \mid X_{t+l-1}, \dots]] \\ &= \alpha^2 \cdot \mu(k, l-1, l-1) + \alpha(2\beta+1) \cdot \mu(k, l-1) + \beta(\beta+1) \cdot \mu(k) \\ &= \alpha^2 \cdot \mu(k, l-1, l-1) + \beta(\beta+1) \cdot \mu(k) + \alpha(2\beta+1) \cdot f_1 \cdot \mu(k) \\ &\quad + \alpha^l \cdot (2\beta+1) \cdot (-f_3 \cdot \alpha(1-\alpha) \cdot \alpha^k + \frac{\beta+1}{1-\alpha} \cdot f_2 + f_1 \cdot f_2 \cdot \alpha^{-k}) \\ &= \alpha^2 \cdot \mu(k, l-1, l-1) + f_1 \cdot (1 + \beta(1 + \alpha)) \cdot \mu(k) \\ &\quad + \alpha^l \cdot (2\beta+1) \cdot (-f_3 \cdot \alpha(1-\alpha) \cdot \alpha^k + \frac{\beta+1}{1-\alpha} \cdot f_2 + f_1 \cdot f_2 \cdot \alpha^{-k}) \\ &= \dots = \alpha^{2(l-k)} \cdot \mu(k, k, k) + f_1 \cdot (1 + \beta(1 + \alpha)) \cdot \mu(k) \cdot \sum_{j=0}^{l-k-1} \alpha^{2j} \\ &\quad + (2\beta+1) \cdot (-f_3 \cdot \alpha(1-\alpha) \cdot \alpha^k + \frac{\beta+1}{1-\alpha} \cdot f_2 + f_1 \cdot f_2 \cdot \alpha^{-k}) \cdot \sum_{j=0}^{l-k-1} \alpha^{l+j} \\ &= \alpha^{2(l-k)} \cdot \mu(k, k, k) + f_2 \cdot (1 + \beta(1 + \alpha)) \cdot \mu(k) \cdot (1 - \alpha^{2(l-k)}) \\ &\quad + \frac{2\beta+1}{1-\alpha} \cdot (-f_3 \cdot \alpha(1-\alpha) \cdot \alpha^k + \frac{\beta+1}{1-\alpha} \cdot f_2 + f_1 \cdot f_2 \cdot \alpha^{-k}) \cdot (\alpha^l - \alpha^{2l-k}) \\ &= \alpha^{2(l-k)} \cdot \mu(k, k, k) - f_2^2 \cdot (1 + \beta(1 + \alpha)) \cdot \alpha^{2l-k} - \frac{2\beta+1}{1-\alpha} \cdot \frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^{2l-k} \\ &\quad - f_2^2 \cdot (1 + \beta(1 + \alpha)) \cdot \beta(1 + \alpha) \cdot \alpha^{2(l-k)} - \frac{2\beta+1}{1-\alpha} \cdot f_1 \cdot f_2 \cdot \alpha^{2(l-k)} \\ &\quad + \frac{2\beta+1}{1-\alpha} \cdot f_3 \cdot \alpha(1-\alpha) \cdot \alpha^{2l} + f_2 \cdot (1 + \beta(1 + \alpha)) \cdot \mu(k) \\ &\quad + \frac{2\beta+1}{1-\alpha} \cdot (-f_3 \cdot \alpha(1-\alpha) \cdot \alpha^k + \frac{\beta+1}{1-\alpha} \cdot f_2 + f_1 \cdot f_2 \cdot \alpha^{-k}) \cdot \alpha^l \\ &= \alpha^{2(l-k)} \cdot \mu(k, k, k) - \frac{f_2 \cdot \alpha^{2l-k}}{(1-\alpha)(1-\alpha^2)} \cdot (\beta + \beta^2(1 + \alpha) + (1 + \alpha)(2\beta + 1)(\beta + 1)) \\ &\quad - f_2^2 \cdot (1 + \alpha) \cdot \alpha^{2(l-k)} \cdot (\beta + \beta^2(1 + \alpha) + 2\beta + 1) \\ &\quad + (2\beta + 1) \cdot f_3 \cdot \alpha^{2l+1} + \mu(0) \cdot \mu(k) + \frac{2\beta+1}{1-\alpha} \cdot (\mu(k, l) - f_1 \cdot \mu(k)) \\ &= \alpha^{2(l-k)} \cdot \mu(k, k, k) - \frac{f_2 \cdot \alpha^{2l-k}}{(1-\alpha)(1-\alpha^2)} \cdot (1 + \alpha + (4 + 3\alpha)\beta + 3(1 + \alpha)\beta^2) + (2\beta + 1) \cdot f_3 \cdot \alpha^{2l+1} \\ &\quad - \alpha^{2(l-k)} \cdot (f_1 \cdot \mu(0, 0) + f_3 \cdot \alpha\beta) + \frac{2\beta+1}{1-\alpha} \cdot \mu(k, l) - f_2 \cdot \mu(k) \cdot (\alpha + \beta(1 + \alpha)). \end{aligned}$$

Case 3: $m = l = k > 0$. Then we have

$$\begin{aligned}
\mu(k, k, k) &= E[X_t \cdot E[X_{t+k}^3 | X_{t+k-1}, \dots]] \\
&= \alpha^3 \cdot \mu(k-1, k-1, k-1) + 3\alpha^2(\beta+1) \cdot \mu(k-1, k-1) \\
&\quad + \alpha(3\beta^2 + 6\beta + 1) \cdot \mu(k-1) + \beta(\beta^2 + 3\beta + 1) \cdot \mu \\
&= \alpha^3 \cdot \mu(k-1, k-1, k-1) - 3(\beta+1) \cdot f_3 \cdot \alpha(1-\alpha) \cdot \alpha^{2k} \\
&\quad + 3\alpha(\beta+1) \cdot \frac{2\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^k + (3\beta^2 + 6\beta + 1) \cdot f_2 \cdot \alpha^k \\
&\quad + 3\alpha^2(\beta+1) \cdot f_1 \cdot \mu(0) + \alpha(3\beta^2 + 6\beta + 1) \cdot f_2 \cdot \beta(1+\alpha) + \beta(\beta^2 + 3\beta + 1) \cdot f_1 \\
&= \alpha^3 \cdot \mu(k-1, k-1, k-1) - 3(\beta+1) \cdot f_3 \cdot \alpha(1-\alpha) \cdot \alpha^{2k} \\
&\quad + \frac{f_2}{1-\alpha} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) \cdot \alpha^k \\
&\quad + f_1 f_2 \cdot (1+2\alpha^2+3(1+\alpha+\alpha^2)\beta+(1+\alpha)(1+\alpha+\alpha^2)\beta^2) \\
&= \alpha^3 \cdot \mu(k-1, k-1, k-1) - 3(\beta+1) \cdot f_3 \cdot \alpha(1-\alpha) \cdot \alpha^{2k} \\
&\quad + \frac{f_2}{1-\alpha} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) \cdot \alpha^k + f_1 \cdot (1-\alpha^3) \cdot \mu(0, 0) \\
&= \dots = \alpha^{3k} \cdot \mu(0, 0, 0) - 3(\beta+1) \cdot f_3 \cdot \alpha(1-\alpha) \cdot \sum_{j=0}^{k-1} \alpha^{2k+j} \\
&\quad + \frac{f_2}{1-\alpha} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) \cdot \sum_{j=0}^{k-1} \alpha^{k+2j} + f_1 \cdot (1-\alpha^3) \cdot \mu(0, 0) \cdot \sum_{j=0}^{k-1} \alpha^{3j} \\
&= \alpha^{3k} \cdot \mu(0, 0, 0) - 3(\beta+1) \cdot f_3 \cdot \alpha \cdot (\alpha^{2k} - \alpha^{3k}) \\
&\quad + \frac{f_2}{(1-\alpha)(1-\alpha^2)} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) \cdot (\alpha^k - \alpha^{3k}) + f_1 \cdot \mu(0, 0) \cdot (1 - \alpha^{3k}) \\
&= \alpha^{3k} \cdot \mu(0, 0, 0) + 3(\beta+1) \cdot f_3 \cdot \alpha \cdot \alpha^{3k} \\
&\quad - \frac{f_2}{(1-\alpha)(1-\alpha^2)} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) \cdot \alpha^{3k} - f_1 \cdot \mu(0, 0) \cdot \alpha^{3k} \\
&\quad + \frac{f_2 \cdot \alpha^k}{(1-\alpha)(1-\alpha^2)} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) + f_1 \cdot \mu(0, 0) - 3(\beta+1) \cdot f_3 \cdot \alpha^{2k+1} \\
&= \alpha^{3k} \cdot \left(\mu(0, 0, 0) - \frac{f_3}{1-\alpha^2} \cdot (1+3\alpha^2+5\alpha^3+(7+7\alpha+11\alpha^2+8\alpha^3)\beta \right. \\
&\quad \left. + 6(1+\alpha)(1+\alpha+\alpha^2)\beta^2+(1+\alpha)^2(1+\alpha+\alpha^2)\beta^3) \right) \\
&\quad + \frac{f_2 \cdot \alpha^k}{(1-\alpha)(1-\alpha^2)} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) + f_1 \cdot \mu(0, 0) - 3(\beta+1) \cdot f_3 \cdot \alpha^{2k+1} \\
&= \alpha^{3k} \cdot \left(\mu(0, 0, 0) - f_4 \cdot (1+4\alpha^2+5\alpha^3+3\alpha^4+5\alpha^5+(1+\alpha^2)(7+7\alpha+11\alpha^2+8\alpha^3)\beta \right. \\
&\quad \left. + 6(1+\alpha)(1+\alpha^2)(1+\alpha+\alpha^2)\beta^2+(1+\alpha)^2(1+\alpha^2)(1+\alpha+\alpha^2)\beta^3) \right) \\
&\quad + \frac{f_2 \cdot \alpha^k}{(1-\alpha)(1-\alpha^2)} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) + f_1 \cdot \mu(0, 0) - 3(\beta+1) \cdot f_3 \cdot \alpha^{2k+1}.
\end{aligned}$$

Case 4: $m = l = k = 0$. Then we have according to formula [1,5] in Douglas (1980) and formula (1) that

$$\begin{aligned}
\mu(0, 0, 0) &= E[X_t^4] = \kappa_4 + 4 \cdot \kappa_3 \cdot \mu + 3\kappa_2^2 + 6 \cdot \kappa_2 \cdot \mu^2 + \mu^4 \\
&= f_4 \cdot (1+6\alpha^2+5\alpha^3+6\alpha^5+(1+\alpha^2)(7+7\alpha+11\alpha^2+8\alpha^3)\beta \\
&\quad + 6(1+\alpha)(1+\alpha^2)(1+\alpha+\alpha^2)\beta^2+(1+\alpha)^2(1+\alpha^2)(1+\alpha+\alpha^2)\beta^3).
\end{aligned}$$

So the above formula for $\mu(k, k, k)$ simplifies to

$$\begin{aligned}
\mu(k, k, k) &= \alpha^{3k} \cdot f_4 \cdot \alpha^2(1-\alpha)(2+2\alpha-\alpha^2) - 3(\beta+1) \cdot f_3 \cdot \alpha^{2k+1} \\
&\quad + \frac{f_2 \cdot \alpha^k}{(1-\alpha)(1-\alpha^2)} \cdot (1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) + f_1 \cdot \mu(0, 0).
\end{aligned}$$

Inserting into the formula for $\mu(k, l, l)$, we obtain

$$\begin{aligned}
\mu(k, l, l) &= \alpha^{2(l-k)} \cdot \mu(k, k, k) - \frac{f_2 \cdot \alpha^{2l-k}}{(1-\alpha)(1-\alpha^2)} \cdot (1 + \alpha + (4 + 3\alpha)\beta + 3(1 + \alpha)\beta^2) \\
&\quad - \alpha^{2(l-k)} \cdot (f_1 \cdot \mu(0, 0) + f_3 \cdot \alpha\beta) + (2\beta + 1) \cdot f_3 \cdot \alpha^{2l+1} \\
&\quad + \frac{2\beta+1}{1-\alpha} \cdot \mu(k, l) - f_2 \cdot \mu(k) \cdot (\alpha + \beta(1 + \alpha)) \\
&= \alpha^{2l+k} \cdot f_4 \cdot \alpha^2(1-\alpha)(2+2\alpha-\alpha^2) - 3(\beta+1) \cdot f_3 \cdot \alpha^{2l+1} + (2\beta+1) \cdot f_3 \cdot \alpha^{2l+1} \\
&\quad + \frac{f_2 \cdot \alpha^{2l-k}}{(1-\alpha)(1-\alpha^2)} \cdot \left((1+2\alpha+3(2+\alpha)\beta+3(1+\alpha)\beta^2) - (1+\alpha+(4+3\alpha)\beta+3(1+\alpha)\beta^2) \right) \\
&\quad + \alpha^{2(l-k)} \cdot f_1 \cdot \mu(0, 0) - \alpha^{2(l-k)} \cdot (f_1 \cdot \mu(0, 0) + f_3 \cdot \alpha\beta) \\
&\quad + \frac{2\beta+1}{1-\alpha} \cdot \mu(k, l) - f_2 \cdot \mu(k) \cdot (\alpha + \beta(1 + \alpha)) \\
&= \alpha^{2l+k} \cdot f_4 \cdot \alpha^2(1-\alpha)(2+2\alpha-\alpha^2) - (\beta+2) \cdot f_3 \cdot \alpha^{2l+1} + \frac{f_2 \cdot \alpha^{2l-k}}{(1-\alpha)(1-\alpha^2)} \cdot (\alpha + 2\beta) \\
&\quad - \alpha^{2(l-k)} \cdot f_3 \cdot \alpha\beta + \frac{2\beta+1}{1-\alpha} \cdot \mu(k, l) - f_2 \cdot \mu(k) \cdot (\alpha + \beta(1 + \alpha)).
\end{aligned}$$

So it follows that for all $0 \leq k \leq l \leq m$, we have

$$\begin{aligned}
\mu(k, l, m) &= \alpha^{m-l} \cdot (\mu(k, l, l) - f_1 \cdot \mu(k, l)) + f_1 \cdot \mu(k, l) \\
&= \alpha^{m-l} \cdot \left(\alpha^{2l+k} \cdot f_4 \cdot \alpha^2(1-\alpha)(2+2\alpha-\alpha^2) - (\beta+2) \cdot f_3 \cdot \alpha^{2l+1} + \frac{f_2 \cdot \alpha^{2l-k}}{(1-\alpha)(1-\alpha^2)} \cdot (\alpha + 2\beta) \right. \\
&\quad \left. - \alpha^{2(l-k)} \cdot f_3 \cdot \alpha\beta + \frac{2\beta+1}{1-\alpha} \cdot \mu(k, l) - f_2 \cdot \mu(k) \cdot (\alpha + \beta(1 + \alpha)) \right) + f_1 \cdot \mu(k, l).
\end{aligned}$$

Appendix A.2. Proof of Corollary 2.2

In the following, we shall apply the general relations between joint moments and joint cumulants stated by Bakouch (2010), as well as the notations of formula (3).

Second-Order Central Moments and Cumulants

For $k \geq 0$, we have $\kappa(k) = \mu(k) - \mu^2 = Cov[X_t, X_{t+k}] = f_2 \cdot \alpha^k$, and obviously, also $\tilde{\mu}(k) = Cov[X_t, X_{t+k}]$ coincides with this expression.

Third-Order Central Moments and Cumulants

For $l \geq k \geq 0$, we have

$$\begin{aligned}
\kappa(k, l) &= \mu(k, l) - \mu \cdot \mu(k) - \mu \cdot (\mu(l-k) + \mu(l) - 2\mu^2) \\
&= \frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^l + f_1 f_2 \cdot \alpha^{l-k} - f_3 \cdot (1-\alpha) \cdot \alpha^{l+k+1} - f_1 \cdot (f_2 \cdot \alpha^{l-k} + f_2 \cdot \alpha^l) \\
&= \frac{f_2}{1-\alpha} \cdot \alpha^l - f_3 \cdot (1-\alpha) \cdot \alpha^{l+k+1} = f_3 \cdot \alpha^l \cdot (1 + \alpha + \alpha^2 - (1-\alpha) \cdot \alpha^{k+1}).
\end{aligned}$$

Furthermore, it follows that simply

$$\begin{aligned}
\tilde{\mu}(k, l) &= E[(X_t - \mu)(X_{t+k} - \mu)(X_{t+l} - \mu)] \\
&= E[(X_t X_{t+k} - \mu \cdot X_t - \mu \cdot X_{t+k} + \mu^2) \cdot X_{t+l}] - \mu \cdot \tilde{\mu}(k) \\
&= \mu(k, l) - \mu \cdot \mu(l) - \mu \cdot \mu(l-k) + \mu^3 - \mu \cdot \mu(k) + \mu^3 = \kappa(k, l).
\end{aligned}$$

Fourth-Order Central Moments and Cumulants

For $m \geq l \geq k \geq 0$, we have

$$\begin{aligned}
\kappa(k, l, m) &= \mu(k, l, m) - \mu \cdot \mu(k, l) \\
&\quad - \mu \cdot (\mu(l - k, m - k) - \mu \cdot \mu(l - k) + \mu(l, m) - \mu \cdot \mu(l) + \mu(k, m) - \mu \cdot \mu(k)) \\
&\quad - (\mu(k) - \mu^2) \cdot (\mu(m - l) - \mu^2) - (\mu(l) - \mu^2) \cdot (\mu(m - k) - \mu^2) \\
&\quad - (\mu(l - k) - \mu^2) \cdot (\mu(m) - \mu^2) + \mu^2 \cdot (\mu(m) + \mu(m - k) + \mu(m - l) - 3\mu^2) \\
&= \alpha^{m-l} \cdot \left(\alpha^{2l+k} \cdot f_4 \cdot \alpha^2(1-\alpha)(2+2\alpha-\alpha^2) - (\beta+2) \cdot f_3 \cdot \alpha^{2l+1} \right. \\
&\quad \left. + \frac{f_2 \cdot \alpha^{2l-k}}{(1-\alpha)(1-\alpha^2)} \cdot (\alpha+2\beta) - \alpha^{2(l-k)} \cdot f_3 \cdot \alpha\beta + \frac{\beta+1}{1-\alpha} \cdot \mu(k, l) - f_2 \cdot \mu(k) \cdot (\alpha+\beta(1+\alpha)) \right) \\
&\quad - f_1 \cdot \left(\frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^{m-k} + f_1 f_2 \cdot \alpha^{m-l} - f_3 \cdot (1-\alpha) \cdot \alpha^{m+l-2k+1} + \frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^m \right. \\
&\quad \left. - f_3 \cdot (1-\alpha) \cdot \alpha^{m+l+1} + f_1 f_2 \cdot \alpha^{m-l} + \frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^m + f_1 f_2 \cdot \alpha^{m-k} \right. \\
&\quad \left. - f_3 \cdot (1-\alpha) \cdot \alpha^{m+k+1} \right) - f_2^2 \cdot (\alpha^{m-l+k} + 2\alpha^{m+l-k}) + f_1^2 f_2 \cdot (\alpha^m + \alpha^{m-k} + \alpha^{m-l}) \\
&= \alpha^{m+l+k+2} \cdot f_4 \cdot (1-\alpha)(2+2\alpha-\alpha^2) - (\beta+2) \cdot f_3 \cdot \alpha^{m+l+1} + \frac{f_2 \cdot \alpha^{m+l-k}}{(1-\alpha)(1-\alpha^2)} \cdot (\alpha+2\beta) \\
&\quad - \alpha^{m+l-2k+1} \cdot f_3 \cdot \beta + \frac{\beta+1}{1-\alpha} \cdot \alpha^{m-l} \cdot (\mu(k, l) - f_1 \cdot \mu(k)) + \alpha^{m-l} \cdot f_2 \cdot (\mu(k) - f_1^2) \\
&\quad - f_1 \cdot \left(\frac{2\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^{m-k} + 2f_1 f_2 \cdot \alpha^{m-l} - f_3 \cdot (1-\alpha) \cdot \alpha^{m+l-2k+1} \right. \\
&\quad \left. + 2 \frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^m - f_3 \cdot (1-\alpha) \cdot \alpha^{m+l+1} - f_3 \cdot (1-\alpha) \cdot \alpha^{m+k+1} \right) \\
&\quad - f_2^2 \cdot (\alpha^{m-l+k} + 2\alpha^{m+l-k}) + f_1^2 f_2 \cdot (\alpha^m + \alpha^{m-k} + 2\alpha^{m-l}) \\
&= \alpha^{m+l+k+2} \cdot f_4 \cdot (1-\alpha)(2+2\alpha-\alpha^2) - 2 \cdot f_3 \cdot \alpha^{m+l+1} + \frac{f_2 \cdot \alpha^{m+l-k+1}}{(1-\alpha)(1-\alpha^2)} \\
&\quad + \frac{\beta+1}{1-\alpha} \cdot \alpha^{m-l} \cdot \left(\frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^l + f_1 f_2 \cdot \alpha^{l-k} - f_3 \cdot (1-\alpha) \cdot \alpha^{l+k+1} \right) + f_2^2 \cdot \alpha^{m-l+k} \\
&\quad - f_1 \cdot \left(\frac{\beta+1}{1-\alpha} \cdot f_2 \cdot \alpha^{m-k} + \frac{\beta+2}{1-\alpha} \cdot f_2 \cdot \alpha^m - f_3 \cdot (1-\alpha) \cdot \alpha^{m+k+1} \right) - f_2^2 \cdot \alpha^{m-l+k} \\
&= \alpha^{m+l+k+2} \cdot f_4 \cdot (1-\alpha)(2+2\alpha-\alpha^2) - 2 \cdot f_3 \cdot \alpha^{m+l+1} + \frac{f_2 \cdot \alpha^{m+l-k+1}}{(1-\alpha)(1-\alpha^2)} \\
&\quad + \frac{f_2}{(1-\alpha)^2} \cdot \alpha^m - f_3 \cdot \alpha^{m+k+1} \\
&= \alpha^m \cdot f_4 \cdot ((1-\alpha)(2+2\alpha-\alpha^2) \cdot \alpha^{l+k+2} - (1-\alpha^4) \cdot (2 \cdot \alpha^{l+1} + \alpha^{k+1}) \\
&\quad + (1+\alpha+\alpha^2)(1+\alpha^2) \cdot (1+\alpha+\alpha^{l-k+1})).
\end{aligned}$$

For the corresponding central moment, we have

$$\begin{aligned}
\tilde{\mu}(k, l, m) &= E[(X_t - \mu)(X_{t+k} - \mu)(X_{t+l} - \mu)(X_{t+m} - \mu)] \\
&= E[(X_t X_{t+k} X_{t+l} - \mu \cdot X_t X_{t+l} - \mu \cdot X_{t+k} X_{t+l} + \mu^2 \cdot X_{t+l}) \cdot X_{t+m}] \\
&\quad - E[\mu \cdot X_t X_{t+k} - \mu^2 \cdot X_t - \mu^2 \cdot X_{t+k} + \mu^3] \cdot X_{t+m}] - \mu \cdot \tilde{\mu}(k, l) \\
&= \mu(k, l, m) - \mu \cdot (\mu(l - k, m - k) + \mu(l, m) + \mu(k, l) + \mu(k, m)) \\
&\quad + \mu^2 \cdot (\mu(k) + \mu(l) + \mu(m) + \mu(l - k) + \mu(m - k) + \mu(m - l)) - 3\mu^4 \\
&= \kappa(k, l, m) + (\mu(k) - \mu^2) \cdot (\mu(m - l) - \mu^2) + (\mu(l) - \mu^2) \cdot (\mu(m - k) - \mu^2) \\
&\quad + (\mu(m) - \mu^2) \cdot (\mu(l - k) - \mu^2) \\
&= \kappa(k, l, m) + f_2^2 \cdot \alpha^{m-l+k} + 2 \cdot f_2^2 \cdot \alpha^{m+l-k}.
\end{aligned}$$

Appendix A.3. Proof of Formula (4)

A Poisson random variable $Y \sim Po(\lambda)$ has the moment generating function (mgf) $\mu_Y(z) = \exp(\lambda(e^z - 1))$. Using that X_t , conditioned on X_{t-1}, \dots , is Poisson distributed according to

$Po(\beta + \alpha \cdot X_{t-1})$, see Definition 1.1, it follows that the mgf of J_t is given by

$$\begin{aligned}\mu_J(z) &= E[\exp(z(X_t - X_{t-1}))] = E[\exp(-zX_{t-1}) \cdot E[\exp(zX_t) \mid X_{t-1}, \dots]] \\ &= E[\exp(-zX_{t-1}) \cdot \exp((\beta + \alpha \cdot X_{t-1})(e^z - 1))] \\ &= \exp(\beta(e^z - 1)) \cdot E[\exp(-zX_{t-1}) \cdot \exp(\alpha(e^z - 1) \cdot X_{t-1})] \\ &= \exp(\beta(e^z - 1)) \cdot \mu_X(\alpha(e^z - 1) - z).\end{aligned}$$

Taking the logarithm on both sides, we obtain the relation of formula (4).

Appendix A.4. Proof of Proposition 3.1

Since $J_t = (X_t - \mu) - (X_{t-1} - \mu)$, we can express moments of J_t in terms of the central moments given in Corollary 2.2. For this purpose, we need the following results:

$$\begin{aligned}\tilde{\mu}(0, 1) &= f_3 \cdot \alpha(1 + 2\alpha^2), & \tilde{\mu}(1, 1) &= f_3 \cdot \alpha(1 + \alpha + \alpha^3), \\ \tilde{\mu}(0, 0, 0) &= f_4 \cdot (1 + 6\alpha^2 + 5\alpha^3 + 6\alpha^5) + 3f_2^2, & \tilde{\mu}(0, 0, 1) &= \alpha \cdot \tilde{\mu}(0, 0, 0), \\ \tilde{\mu}(0, 1, 1) &= f_4 \cdot \alpha(1 + \alpha + 2\alpha^2 + 6\alpha^3 + 4\alpha^4 + 4\alpha^6) + f_2^2 \cdot (1 + 2\alpha^2), \\ \tilde{\mu}(1, 1, 1) &= f_4 \cdot \alpha(1 + 3\alpha + \alpha^2 + 5\alpha^3 + 5\alpha^4 + 2\alpha^5 + \alpha^7) + 3f_2^2 \cdot \alpha.\end{aligned}$$

Now it follows that

$$\begin{aligned}E[J_t^2] &= E[(X_t - \mu)^2 + (X_{t-1} - \mu)^2 - 2(X_{t-1} - \mu)(X_t - \mu)] = 2(\tilde{\mu}(0) - \tilde{\mu}(1)) = 2f_2 \cdot (1 - \alpha); \\ E[J_t^3] &= \tilde{\mu}(0, 0) - 3\tilde{\mu}(1, 1) + 3\tilde{\mu}(0, 1) - \tilde{\mu}(0, 0) \\ &= 3f_3 \cdot \alpha((1 + 2\alpha^2) - (1 + \alpha + \alpha^3)) = -3f_3 \cdot \alpha^2(1 - \alpha)^2; \\ E[J_t^4] &= 2\tilde{\mu}(0, 0, 0) - 4\tilde{\mu}(1, 1, 1) + 6\tilde{\mu}(0, 1, 1) - 4\tilde{\mu}(0, 0, 1) \\ &= 2(1 - 2\alpha) \cdot \tilde{\mu}(0, 0, 0) - 4\tilde{\mu}(1, 1, 1) + 6\tilde{\mu}(0, 1, 1) \\ &= 2(1 - 2\alpha) \cdot f_4 \cdot (1 + 6\alpha^2 + 5\alpha^3 + 6\alpha^5) - 4f_4 \cdot \alpha(1 + 3\alpha + \alpha^2 + 5\alpha^3 + 5\alpha^4 + 2\alpha^5 + \alpha^7) \\ &\quad + 6f_4 \cdot \alpha(1 + \alpha + 2\alpha^2 + 6\alpha^3 + 4\alpha^4 + 4\alpha^6) + 2(1 - 2\alpha) \cdot 3f_2^2 - 12f_2^2 \cdot \alpha + 6f_2^2 \cdot (1 + 2\alpha^2) \\ &= 2f_4 \cdot (1 - 2\alpha + 6\alpha^2 - 7\alpha^3 - 10\alpha^4 + 6\alpha^5 - 12\alpha^6) - 4f_4 \cdot (\alpha + 3\alpha^2 + \alpha^3 + 5\alpha^4 + 5\alpha^5 + 2\alpha^6 + \alpha^8) \\ &\quad + 6f_4 \cdot (\alpha + \alpha^2 + 2\alpha^3 + 6\alpha^4 + 4\alpha^5 + 4\alpha^7) + 6f_2^2 \cdot (1 - 2\alpha - 2\alpha + 1 + 2\alpha^2) \\ &= 2f_4 \cdot (1 - \alpha + 3\alpha^2 - 3\alpha^3 - 2\alpha^4 + 8\alpha^5 - 16\alpha^6 + 12\alpha^7 - 2\alpha^8) + 12f_2^2 \cdot (1 - \alpha)^2 \\ &= 2f_4 \cdot (1 - \alpha)^2(1 + \alpha + 4\alpha^2 + 4\alpha^3 + 2\alpha^4 + 8\alpha^5 - 2\alpha^6) + 12f_2^2 \cdot (1 - \alpha)^2.\end{aligned}$$