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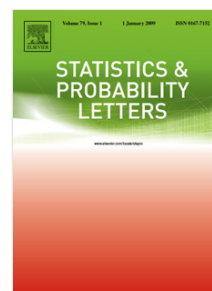
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Consistent estimation of the tail index for dependent data

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Abstract

We consider the estimation of the tail-index for dependent random variables. We establish the consistency of the geometric-type estimator (Brito and Freitas (2003)) for stationary sequences satisfying general mixing conditions and derive a simplified condition, specially adapted for applications.

Key words and phrases: Consistency; mixing conditions; order statistics; parameter estimation; tail indices.

AMS 2000 Classification: 62G05, 62F12, 62G30

1 Introduction

Let $\{X_i\}$ be a sequence of random variables (r.v.'s) with common distribution function (d.f.) G satisfying

$$\bar{G}(x) := 1 - G(x) = P(X_1 > x) = x^{-R}L(x), \quad x > 0, \quad (1)$$

where L is a slowly varying function at infinity and R is a positive constant.

The problem of estimating R or related tail indices has received considerable attention and common applications may be found in a big variety of domains, as for example in economics, applied finance, insurance, business, industry, telecommunications, traffic, geology and sociology. Several estimators for R or $1/R$ have been proposed in the literature

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(see e.g. Dekkers et al. (1989), Bacro and Brito (1995), Csörgő and Viharos (1998) and references therein).

One of the most commonly used estimators for $1/R$ is the Hill estimator (1975), given by

$$\widehat{H}(k_n) = \frac{1}{k_n} \sum_{i=1}^{k_n} \log X_{n-i+1,n} - \log X_{n-k_n,n},$$

where $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$ denote the order statistics (o.s.) of the sample X_1, X_2, \dots, X_n and k_n is a sequence of positive integers satisfying

$$1 \leq k_n < n, \quad \lim_{n \rightarrow \infty} k_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} k_n/n = 0. \quad (2)$$

The asymptotic properties of $\widehat{H}(k_n)$ have been much studied. In the independent context, it is well known that, under some regularity conditions, $\widehat{H}(k_n)$ is strongly consistent (cf. Deheuvels et al. (1988)) with asymptotic normal distribution when properly normalized (cf. Haeusler and Teugels (1985)).

Several extensions of $\widehat{H}(k_n)$ have been proposed, for example, the family of j -moment ratio estimator (cf. Danielsson et al. (1996))

$$\widehat{W}_j = \frac{u_j(X_{n-k_n,n})}{j u_{j-1}(X_{n-k_n,n})},$$

where

$$u_j(s_n) = \frac{1}{k_n} \sum_{i=1}^{k_n} \left(\log \frac{X_{n-i+1,n}}{s_n} \right)^j,$$

for $j = 1, 2, \dots$, and $u_0(s_n) = 0$. For the particular case $j = 1$, \widehat{W}_1 is the Hill estimator.

Other type of scale invariant estimators of the tail index have been introduced, like for example in Politis (2002) and McElroy and Politis (2007).

Based on least squares considerations, Schultze and Steinebach (1996) proposed three estimators, denoted by $\widehat{R}_1(k_n)$, $\widehat{R}_2(k_n)$ and $\widehat{R}_3(k_n)$, for the tail coefficient R , in the i.i.d. case. Brito and Freitas (2003) have introduced a *geometric-type* estimator of R , $\widehat{R}(k_n)$, which is related to the least square estimators $\widehat{R}_1(k_n)$ and $\widehat{R}_3(k_n)$ and defined by

$$\widehat{R}(k_n) = \sqrt{\frac{\sum_{i=1}^{k_n} \log^2(n/i) - \frac{1}{k_n} \left(\sum_{i=1}^{k_n} \log(n/i) \right)^2}{\sum_{i=1}^{k_n} \log^2 X_{n-i+1,n} - \frac{1}{k_n} \left(\sum_{i=1}^{k_n} \log X_{n-i+1,n} \right)^2}}.$$

We recall that these least squares estimators have the interesting property of being universal asymptotically normal under the model (1), property not shared by Hill's estimator. This specific property makes the use of the estimator $\widehat{R}(k_n)$ specially attractive for the case where R is expected to be small. As illustrated by Csörgő and Viharos (1998), the least squares type estimators are, in some situations, more "robust" against deviations of the slowly varying function L from a constant.

Also, an important application in risk theory was first given by Schultze and Steinebach (1996). In particular, these authors establish the consistency of $\widehat{R}_1(k_n)$ and $\widehat{R}_3(k_n)$ for

estimating the adjustment coefficient for a particular subclass of the Sparre Andersen risk process. More recently, Brito and Freitas (2006) have shown that the consistency properties of all these estimators still hold in the Sparre Andersen model under standard conditions.

In this paper we consider the important question of relaxing the independent condition in the problem of estimating the tail coefficient R of the family (1). This question has been studied by several authors, including Resnick and Stărică (1995), Resnick and Stărică (1998) and Hill (2009). In particular, Hsing (1991) investigated the asymptotic behaviour of $\widehat{H}(k_n)$ for stationary sequences in the presence of dependence. The cited works are mainly concerned with the behaviour of the Hill estimator under different dependent structures. Motivated by the nice and specific properties of the least squares type estimators, we focus our study on the consistency of the geometric-type estimator $\widehat{R}(k_n)$ for sequences of identically distributed and eventually dependent r.v.'s. We establish the consistency of $\widehat{R}(k_n)$, under some general assumptions, for strictly stationary sequences satisfying certain mixing conditions (Hsing (1991)). We also give a simplified sufficient condition for consistency, appropriate for applications. This condition is only based on the mixing sequences including, as a particular case, m-dependence.

One well known complex practical problem is the choice of the number of observations included in the estimation of R . In view of the construction of the geometric-type estimator, we may use the heuristic method proposed in Brito and Freitas (2003), based on the minimization of the obtained residuals.

Our study may also be applied to related estimators. In particular, we obtain immediately the consistency of the ratio estimator \widehat{W}_2 .

We present our results in the following section and the corresponding proofs are given in Section 3.

2 Main results

In the sequel, \xrightarrow{P} stands for convergence in probability and we denote by G^{-1} the left continuous inverse of G , i.e., $G^{-1}(s) := \inf\{x: G(x) \geq s\}$.

We investigate the consistency of $\widehat{R}(k_n)$, for stationary sequences under mixing conditions. We recall the strong-mixing condition for a stationary sequence $\{\xi_i\}$:

$$\alpha(l, \{\xi_i\}) := \sup \{ |P[A \cap B] - P[A]P[B]| : A \in \mathcal{F}_1^j\{\xi_i\}, B \in \mathcal{F}_{j+l+1}^\infty\{\xi_i\}, j > 1 \} \rightarrow 0$$

as $l \rightarrow \infty$, where $\mathcal{F}_r^s\{\xi_i\}$ denotes the σ -field $\sigma\{\xi_i : r \leq i \leq s\}$ and $\mathcal{F}_r^\infty\{\xi_i\}$ the σ -field $\sigma\{\xi_i : i \geq r\}$.

Here, we will consider mixing conditions adapted to this context. Namely, the role of l will be played by adequate sequences r_n , as it is usual in Extreme Value Theory, while ξ_i will be appropriate functionals of the r.v.'s X_i .

Before we summarize our results, it is convenient to introduce some notation. For $x \in \mathbb{R}$, x_+ denotes $\max(x, 0)$ and x_- denotes $\max(-x, 0)$. For simplifying the presentation, we write $x_+^2 \equiv (x_+)^2$ and $x_-^2 \equiv (x_-)^2$.

Consider the function b defined by

$$b(t) := G^{-1} \left(1 - \frac{1}{t} \right), \quad t > 1, \quad (3)$$

and define also

$$Y_{ni} := (\log X_i - \log b(n/k_n))_+$$

and

$$I_{ni}(\epsilon, a) := I(\log X_i - \log b(n/ak_n) > \epsilon).$$

Theorem 1 *Assume that G satisfies (1) and k_n is a sequence of positive integers satisfying (2). Suppose that*

$$k_n^{-1} \sum_{i=1}^n (T_{ni} - E(T_{ni})) \xrightarrow{P} 0, \quad (4)$$

for $T_{ni} = Y_{ni}^j$, $j = 1, 2$, and $T_{ni} = I_{ni}(\epsilon, a)$, for every $\epsilon \in \mathbb{R}$ and a in some neighbourhood \mathcal{V}_1 of 1. Then $\widehat{R}(k_n)$ converges to R in probability.

Let r_n be a sequence of positive integers such that $r_n/n \rightarrow 0$ as $n \rightarrow \infty$. Denote by S_{nj} a r.v. measurable with respect to $\mathcal{F}_{(j-1)r_n+1}^{jr_n} \{T_{ni}\}$, $1 \leq j \leq l_n$, where $l_n = \lfloor n/r_n \rfloor$ and T_{ni} is a functional of X_i . Consider the following conditions:

$$k_n^{-1} \sum_{j=1}^{l_n} E(|S_{nj}| I(|S_{nj}| > k_n)) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (5)$$

$$k_n^{-2} \sum_{j=1}^{l_n} E(S_{nj}^2 I(|S_{nj}| \leq k_n)) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (6)$$

We are now in condition to characterize the consistency of $\widehat{R}(k_n)$. Let $S_{nj}(T_{ni}) := \sum_{i=(j-1)r_n+1}^{jr_n} T_{ni}$.

Theorem 2 *Assume that G satisfies (1). Let $\{X_i\}$ be a strictly stationary sequence and k_n a sequence of positive integers satisfying (2). $\widehat{R}(k_n)$ converges to R in probability if the following conditions are verified:*

- (i) *There exists a sequence r_n of positive integers such that $r_n/n \rightarrow 0$, $l_n \alpha(r_n, \{Y_{ni}\}) \rightarrow 0$ as $n \rightarrow \infty$ and the conditions (5) and (6) are satisfied for $S_{nj}(Y_{ni})$.*
- (ii) *For each $\epsilon \in \mathbb{R}$ and a in some neighborhood \mathcal{V}_1 of 1, there exists a sequence r_n of positive integers such that $r_n/n \rightarrow 0$, $l_n \alpha(r_n, \{I_{ni}(\epsilon, a)\}) \rightarrow 0$ as $n \rightarrow \infty$ and the conditions (5) and (6) are satisfied for $S_{nj}(I_{ni}(\epsilon, a))$.*
- (iii) *The conditions (5) and (6) are satisfied for $S_{nj}(Y_{ni}^2)$.*

The consistency of the estimator $\widehat{H}(k_n)$ of $1/R$ was established in Theorem 3.2 of Hsing (1991) under the conditions (i) and (ii) of the theorem above.

The general conditions of Theorem 2 may be much simplified under appropriate assumptions on the dependence structure. In particular, the conditions (5) and (6) are satisfied if the sequence r_n verifies $r_n/k_n \rightarrow 0$. We then obtain a sufficient condition of consistency based only on the mixing sequences, as stated below.

Corollary 1 Assume that G satisfies (1). Let $\{X_i\}$ be a strictly stationary sequence and k_n a sequence of positive integers satisfying (2).

Suppose that for each of the following T_{ni} : $T_{ni} = Y_{ni}$ and $T_{ni} = I_{ni}(\epsilon, a)$, there exists a sequence r_n of positive integers such that $r_n/k_n \rightarrow 0$ and $l_n \alpha(r_n, \{T_{ni}\}) \rightarrow 0$ as $n \rightarrow \infty$. Then, both estimators, $1/\widehat{H}(k_n)$ and $\widehat{R}(k_n)$, converge to R in probability.

A simple illustration of the application of this result is given in the following example.

Example Consider the finite moving averages processes

$$X_j = \sum_{i=0}^m c_i Z_{i+j}, \quad j = 1, 2, \dots,$$

where $c_i \in \mathbb{R}^+$, $i = 0, \dots, m$, and Z_k are i.i.d. positive r.v.'s with $P(Z_k > z) = z^{-R}l(z)$, l being a slowly varying function at ∞ and $R \in \mathbb{R}^+$. Then $\{X_i\}$ is a strictly stationary sequence of r.v.'s with common d.f. G satisfying (1). Moreover, for this process, we have that $\alpha(m, \{X_i\}) = 0$ and so $\alpha(m, \{T_{ni}\}) = 0$, for $T_{ni} = (\log X_i - \log b(n/k_n))_+$ and $T_{ni} = I(\log X_i - \log b(n/k_n) > \epsilon)$.

Thus, by Corollary 1, $\widehat{R}(k_n)$ is a consistent estimator of R , for any sequence of positive integers satisfying (2).

Actually, Corollary 1 implies the consistency of $\widehat{R}(k_n)$ (and $1/\widehat{H}(k_n)$) for any m -dependent sequence.

Corollary 2 Assume G satisfies (1). Let $\{X_i\}$ be a strictly stationary m -dependent sequence and k_n a sequence of positive integers satisfying (2). Then, $\widehat{R}(k_n)$ converges to R in probability.

We will consider now the ratio estimator

$$\widehat{W}_2 = \frac{\sum_{i=1}^{k_n} (\log(X_{n-i+1,n}/X_{n-k_n,n}))^2}{2 \sum_{i=1}^{k_n} \log(X_{n-i+1,n}/X_{n-k_n,n})}.$$

We recall that, in the i.i.d. case, and with respect to AMSE criterion, \widehat{W}_1 and \widehat{W}_2 are the only two elements of interest in the class \widehat{W}_j (cf. Danielsson and de Vries (1997)). Noting that, under consistency, $\widehat{W}_2 - 1/\widehat{R}(k_n) = o_p(1)$, we obtain immediately the following result.

Proposition 1 The results of Theorem 2 and Corollaries 1 and 2 still hold when $\widehat{R}(k_n)$ is replaced by $1/\widehat{W}_2$.

3 Proofs

We begin by observing that, since \overline{G} is regularly varying function at ∞ with index $-R$, the function b defined in (3) is regular varying at ∞ with index $1/R$. Moreover,

$$\overline{G}(b(t)) \sim t^{-1} \quad \text{as } t \rightarrow \infty.$$

Note that, as referred by Hsing (1991), the k th moment of $(\log X_1 - \log b(n/k_n))_+$ is such that

$$E(\log X_1 - \log b(n/k_n))_+^k \sim \frac{k_n}{n} \frac{k!}{R^k} \quad \text{as } n \rightarrow \infty. \quad (7)$$

Now, define the random quantities

$$\tilde{H}(k_n) := \frac{1}{k_n} \sum_{i=1}^{k_n} (\log X_{n-i+1,n} - \log b(n/k_n))$$

and

$$H^+(k_n) := \frac{1}{k_n} \sum_{i=1}^n (\log X_i - \log b(n/k_n))_+.$$

We shall also use the following results.

Theorem A (Hsing (1991), Theorem 2.2)

Suppose G satisfies (1) and let k_n be a sequence of positive integers such that (2) holds. Suppose that

$$k_n^{-1} \sum_{i=1}^n (T_{ni} - E(T_{ni})) \xrightarrow{P} 0$$

for $T_{ni} = Y_{ni}$ and $T_{ni} = I_{ni}(\epsilon, a)$ for every $\epsilon \in \mathbb{R}$ and a in some neighbourhood \mathcal{V}_1 of 1. Then $\hat{H}(k_n)$, $\tilde{H}(k_n)$ and $H^+(k_n)$ converge to $1/R$ in probability.

Theorem B (Hsing (1991), Theorem 3.1)

Let $\{X_i\}$ be a strictly stationary sequence and k_n a sequence of positive integers satisfying (2). Suppose r_n is a sequence of positive integers such that $r_n/n \rightarrow 0$ as $n \rightarrow \infty$. Assume that, as $n \rightarrow \infty$, $l_n \alpha(r_n, \{T_{ni}\}) \rightarrow 0$ and conditions (5) and (6) hold. Then,

$$k_n^{-1} \sum_{j=1}^{l_n} (S_{nj} - E(S_{nj})) \xrightarrow{P} 0.$$

Lemma A (Brito and Freitas (2003), Lemma 1)

Let k_n be a sequence of positive integers such that $1 \leq k_n \leq n$, and consider the sequence $i_n(k_n) := \frac{1}{k_n} \sum_{i=1}^{k_n} \log^2(n/i) - \left(\frac{1}{k_n} \sum_{i=1}^{k_n} \log(n/i) \right)^2$. Then

$$i_n(k_n) = 1 + O\left(\frac{\log^2 k_n}{k_n}\right).$$

We turn now to the consistency of the estimator $\hat{R}(k_n)$. First note that $\hat{R}(k_n)$ can be written in the form

$$\widehat{R}(k_n) = \sqrt{\frac{i_n(k_n)}{\frac{1}{k_n} \sum_{i=1}^{k_n} \log^2 X_{n-i+1,n} - \frac{1}{k_n^2} \left(\sum_{i=1}^{k_n} \log X_{n-i+1,n} \right)^2}}.$$

By Lemma A, we have that $i_n(k_n) \rightarrow 1$ as $n \rightarrow \infty$, and so $\widehat{R}(k_n)$ converges to R in probability if and only if $\widehat{N}(k_n)$ converges to $1/R^2$ in probability, where

$$\begin{aligned} \widehat{N}(k_n) &:= \frac{1}{k_n} \sum_{i=1}^{k_n} \log^2 X_{n-i+1,n} - \frac{1}{k_n^2} \left(\sum_{i=1}^{k_n} \log X_{n-i+1,n} \right)^2 \\ &= \frac{1}{k_n} \sum_{i=1}^{k_n} (\log X_{n-i+1,n} - \log b(n/k_n))^2 - \frac{1}{k_n^2} \left(\sum_{i=1}^{k_n} (\log X_{n-i+1,n} - \log b(n/k_n)) \right)^2. \end{aligned}$$

Proof of Theorem 1. Define

$$N^+(k_n) := \frac{1}{k_n} \sum_{i=1}^n (\log X_i - \log b(n/k_n))_+^2 - \frac{1}{k_n^2} \left(\sum_{i=1}^n (\log X_i - \log b(n/k_n))_+ \right)^2.$$

We start by showing that $N^+(k_n) \xrightarrow{P} 1/R^2$. By Theorem A, $\widetilde{H}(k_n)$ and $H^+(k_n)$ converge in probability to $1/R$ and, consequently, $\widetilde{H}^2(k_n)$ and $H^{+2}(k_n)$ converge in probability to $1/R^2$.

Moreover, (7) implies that

$$\frac{n}{k_n} E(\log X_1 - \log b(n/k_n))_+^2 \rightarrow 2/R^2$$

as $n \rightarrow \infty$. Using condition (4) for Y_{ni}^2 , we conclude that

$$\frac{1}{k_n} \sum_{i=1}^n (\log X_i - \log b(n/k_n))_+^2 \xrightarrow{P} 2/R^2.$$

Thus,

$$N^+(k_n) = \frac{1}{k_n} \sum_{i=1}^n (\log X_i - \log b(n/k_n))_+^2 - H^{+2}(k_n) \xrightarrow{P} 1/R^2.$$

Next, we are going to prove that $N^+(k_n) - \widehat{N}(k_n) \xrightarrow{P} 0$. Note we can write

$$N^+(k_n) - \widehat{N}(k_n) = A_n + B_n + \widetilde{H}^2(k_n) - H^{2+}(k_n),$$

where

$$A_n = -\frac{1}{k_n} \sum_{j=1}^{k_n} (\log X_{n-j+1,n} - \log b(n/k_n))_-^2$$

and

$$B_n = \frac{1}{k_n} \sum_{j=k_n+1}^n (\log X_{n-j+1,n} - \log b(n/k_n))_+^2.$$

As $\widetilde{H}^2(k_n) - H^{2+}(k_n) \xrightarrow{P} 0$, it is enough to show that $A_n \xrightarrow{P} 0$ and $B_n \xrightarrow{P} 0$.

For this we begin by recalling that

$$\log X_{n-[ak_n]+1,n} - \log b(n/ak_n) \xrightarrow{P} 0, \quad (8)$$

for all $a \in \mathcal{V}_1$, as was proved by Hsing (1991). For completeness and for future reference we include here a proof of this fact. Let $\epsilon > 0$ and $a \in \mathcal{V}_1$. We can write

$$\begin{aligned} P[\log X_{n-[ak_n]+1,n} - \log b(n/ak_n) > \epsilon] &= P\left[\sum_{i=1}^n I_{ni}(\epsilon, a) \geq [ak_n]\right] \\ &= P\left[k_n^{-1} \sum_{i=1}^n (I_{ni}(\epsilon, a) - E(I_{ni}(\epsilon, a))) \geq k_n^{-1}([ak_n] - nE(I_{ni}(\epsilon, a)))\right] \end{aligned}$$

and

$$\begin{aligned} E(I_{ni}(\epsilon, a)) &= P[\log X_i - \log b(n/ak_n) > \epsilon] \\ &= \bar{G}(e^\epsilon b(n/ak_n)) \\ &\sim a \frac{k_n}{n} e^{-R\epsilon} \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus,

$$k_n^{-1}([ak_n] - nE(I_{ni}(\epsilon, a))) \rightarrow a(1 - e^{-R\epsilon}) > 0.$$

And, as by hypothesis, $k_n^{-1} \sum_{i=1}^n (I_{ni}(\epsilon, a) - E(I_{ni}(\epsilon, a))) \xrightarrow{P} 0$, we deduce that

$$P[\log X_{n-[ak_n]+1,n} - \log b(n/ak_n) > \epsilon] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In the same way, we can see that

$$P[\log X_{n-[ak_n]+1,n} - \log b(n/ak_n) < -\epsilon] \rightarrow 0 \text{ as } n \rightarrow \infty,$$

for every $\epsilon > 0$ and $a \in \mathcal{V}_1$.

By definition of A_n we easily verify that

$$|A_n| \leq (\log X_{n-k_n+1,n} - \log b(n/k_n))_-^2.$$

So, given $\epsilon > 0$,

$$P[|A_n| > \epsilon] \leq P[(\log X_{n-k_n+1,n} - \log b(n/k_n))_-^2 > \epsilon].$$

Using (8) with $a = 1$, we deduce that $A_n \xrightarrow{P} 0$.

Consider now the term B_n . Taking $\delta \in \mathbb{R}^+$ such that $(1 - \delta, 1 + \delta) \subset \mathcal{V}_1$, we may write

$$B_n = C_n + D_n,$$

where

$$C_n = \frac{1}{k_n} \sum_{j=k_n+1}^{[(1+\delta)k_n]} (\log X_{n-j+1,n} - \log b(n/k_n))_+^2,$$

and

$$D_n = \frac{1}{k_n} \sum_{j=[(1+\delta)k_n]+1}^n (\log X_{n-j+1,n} - \log b(n/k_n))_+^2.$$

We are going to show that both terms C_n and D_n converge in probability to 0. Notice that

$$D_n > 0 \Rightarrow \log X_{n-[(1+\delta)k_n],n} - \log b(n/k_n) > 0$$

and, so,

$$P[D_n > 0] \leq P[\log X_{n-[(1+\delta)k_n]+1,n} - \log b(n/k_n) > 0].$$

Now,

$$\begin{aligned} & P[\log X_{n-[(1+\delta)k_n]+1,n} - \log b(n/k_n) > 0] \\ &= P[\log X_{n-[(1+\delta)k_n]+1,n} - \log b(n/(1+\delta)k_n) > \log b(n/k_n) - \log b(n/(1+\delta)k_n)]. \end{aligned}$$

By (8),

$$\log X_{n-[(1+\delta)k_n]+1,n} - \log b(n/(1+\delta)k_n) \xrightarrow{P} 0.$$

And because b is regular varying at ∞ with index $1/R$, we have that

$$\liminf_{n \rightarrow \infty} (\log b(n/k_n) - \log b(n/(1+\delta)k_n)) = \lim_{n \rightarrow \infty} \log \frac{b(n/k_n)}{b(n/(1+\delta)k_n)} = \frac{1}{R} \log(1+\delta) > 0.$$

Thus, $P[D_n > 0] \rightarrow 0$ as $n \rightarrow \infty$.

Regarding the term C_n , we may observe that

$$\begin{aligned} C_n &\leq \frac{1}{k_n} ([(1+\delta)k_n] - k_n) (\log X_{n-k_n,n} - \log b(n/k_n))_+^2 \\ &\leq \delta (\log X_{n-k_n,n} - \log b(n/k_n))_+^2. \end{aligned}$$

So, for $\epsilon > 0$,

$$\begin{aligned} P[|C_n| > \epsilon] &\leq P[(\log X_{n-k_n,n} - \log b(n/k_n))_+^2 > \epsilon \delta^{-1}] \\ &\leq P[(\log X_{n-k_n+1,n} - \log b(n/k_n))_+ > \sqrt{\epsilon \delta^{-1}}]. \end{aligned}$$

Using again (8) with $a = 1$, we conclude that $C_n \xrightarrow{P} 0$. \square

Proof of Theorem 2. We begin by observing that, under (i), $l_n \alpha(r_n, \{Y_{ni}^2\}) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, using conditions (i), (ii) and (iii), it also follows, from Theorem B, that

$$k_n^{-1} \sum_{i=1}^{l_n r_n} (T_{ni} - E(T_{ni})) = k_n^{-1} \sum_{j=1}^{l_n} \left(\sum_{i=(j-1)r_n+1}^{j r_n} (T_{ni} - E(T_{ni})) \right) \xrightarrow{P} 0,$$

for $T_{ni} = Y_{ni}^j$, $j = 1, 2$, and $T_{ni} = I_{ni}(\epsilon, a)$, respectively.

If we show that $E(k_n^{-1} \sum_{i=l_n r_n+1}^n T_{ni}) \rightarrow 0$ as $n \rightarrow \infty$, then, by Markov inequality, we have that

$$k_n^{-1} \sum_{i=l_n r_n+1}^n (T_{ni} - E(T_{ni})) \xrightarrow{P} 0,$$

and, so,

$$\begin{aligned} k_n^{-1} \sum_{i=1}^n (T_{ni} - E(T_{ni})) &= k_n^{-1} \sum_{i=1}^{l_n r_n} (T_{ni} - E(T_{ni})) + k_n^{-1} \sum_{i=l_n r_n+1}^n (T_{ni} - E(T_{ni})) \\ &\xrightarrow{P} 0. \end{aligned}$$

Thus, the result will follow from Theorem 1.

We start by showing that $E(k_n^{-1} \sum_{i=l_n r_n+1}^n Y_{ni}) \rightarrow 0$ as $n \rightarrow \infty$.

$$E\left(k_n^{-1} \sum_{i=l_n r_n+1}^n Y_{ni}\right) = k_n^{-1}(n - l_n r_n)E(Y_{ni}).$$

By (7), $E(Y_{ni}) \sim \frac{k_n}{n} R^{-1}$ as $n \rightarrow \infty$, and, consequently,

$$E\left(k_n^{-1} \sum_{i=l_n r_n+1}^n Y_{ni}\right) \sim R^{-1}(1 - (l_n r_n)/n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Using (7) with $k = 2$ we can prove in an analogous way that

$$E\left(k_n^{-1} \sum_{i=l_n r_n+1}^n Y_{ni}^2\right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Finally, consider the sequence $(k_n^{-1} \sum_{i=l_n r_n+1}^n I_{ni}(\epsilon, a))$. In the proof of Theorem 1, we have already seen that

$$E(I_{ni}(\epsilon, a)) \sim a \frac{k_n}{n} e^{-R\epsilon} \text{ as } n \rightarrow \infty.$$

Thus, as $n \rightarrow \infty$, we have that

$$E\left(k_n^{-1} \sum_{i=l_n r_n+1}^n I_{ni}(\epsilon, a)\right) = k_n^{-1}(n - l_n r_n)E(I_{ni}(\epsilon, a)) \sim a e^{-R\epsilon}(1 - (l_n r_n)/n),$$

concluding that $E(k_n^{-1} \sum_{i=l_n r_n+1}^n I_{ni}(\epsilon, a)) \rightarrow 0$, which completes the proof. \square

Proof of Corollary 1. We will show that for a sequence r_n of positive integers such that $r_n/k_n \rightarrow 0$, the conditions (5) and (6) hold for $S_{n,j}(T_{ni})$, for $T_{ni} = Y_{ni}$, $T_{ni} = Y_{ni}^2$ and $T_{ni} = I_{ni}(\epsilon, a)$. Then the consistency of $1/\hat{H}(k_n)$ and $\hat{R}(k_n)$ will follow from Theorem 3.2 of Hsing (1991) and Theorem 2, respectively.

Beginning by considering condition (5), note that

$$\begin{aligned} k_n^{-1} \sum_{j=1}^{l_n} E(|S_{nj}| I(|S_{nj}| > k_n)) &= \frac{l_n}{k_n} E(S_{n1} I(S_{n1} > k_n)) \\ &\leq \frac{l_n}{k_n^2} E(S_{n1}^2). \end{aligned}$$

Concerning condition (6),

$$\begin{aligned} k_n^{-2} \sum_{j=1}^{l_n} E(S_{nj}^2 I(|S_{nj}| \leq k_n)) &= \frac{l_n}{k_n^2} E(S_{n1}^2 I(S_{n1} \leq k_n)) \\ &\leq \frac{l_n}{k_n^2} E(S_{n1}^2). \end{aligned}$$

Now, note that

$$\frac{l_n}{k_n^2} E(S_{n1}^2(T_{ni})) = \frac{l_n}{k_n^2} E\left(\left(\sum_{i=1}^{r_n} T_{ni}\right)^2\right) \leq \frac{l_n}{k_n^2} r_n^2 E(T_{n1}^2).$$

To conclude the proof of the theorem, it is then enough to show that $\frac{l_n}{k_n^2} r_n^2 E(T_{n1}^2) \rightarrow 0$ as $n \rightarrow \infty$.

Considering $T_{n1} = I_{n1}$, we have that

$$E(T_{n1}^2) = E(I_{n1}) = O\left(\frac{k_n}{n}\right).$$

On the other hand, for $T_{n1} = Y_{n1}$ and $T_{n1} = Y_{n1}^2$, we obtain, by (7),

$$E(T_{n1}^2) = O\left(\frac{k_n}{n}\right).$$

Thus,

$$\begin{aligned} \frac{l_n}{k_n^2} E(S_{n1}^2(T_{ni})) &\leq \frac{[n/r_n]}{k_n^2} r_n^2 O\left(\frac{k_n}{n}\right) \\ &= O\left(\frac{r_n}{k_n}\right), \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$, once that $r_n/k_n \rightarrow 0$ as $n \rightarrow \infty$. \square

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