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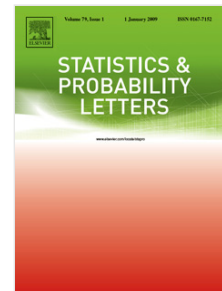
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# Edgeworth expansions for the product of two complex random matrices each with IID components

by

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**Abstract:** Let  $\mathbf{A}$  and  $\mathbf{B}$  be independent  $n \times n$  complex matrices with elements i.i.d. as  $X_1 + iX_2$  and  $Y_1 + iY_2$ , respectively, where  $i = (-1)^{1/2}$ ,  $X_1$  and  $X_2$  are independent and identically distributed,  $Y_1$  and  $Y_2$  are independent and identically distributed. We obtain Edgeworth expansions for the distribution of any element of  $\mathbf{AB}$  (and so for trace  $\mathbf{AB}$ ), in Cartesian coordinates. If  $X_1$  or  $Y_1$  is symmetric about zero, the expansions are given to  $O(n^{-4})$ . A specific example applicable to reverberation systems is the case when  $X_1$  and  $Y_1$  are both Gaussian.

**Keywords:** Complex random matrices; Edgeworth expansions; Gaussian.

## 1 Introduction and summary

Let  $\mathbf{A}$ ,  $\mathbf{B}$  be independent  $n \times n$  complex matrices with elements independent and identically distributed (i.i.d.) as

$$A_{jk} \stackrel{\mathcal{L}}{=} X_1 + iX_2, \quad B_{jk} \stackrel{\mathcal{L}}{=} Y_1 + iY_2,$$

where  $i = (-1)^{1/2}$ ,  $X_1$  and  $X_2$  are i.i.d. as  $X$ ,  $Y_1$  and  $Y_2$  are i.i.d. as  $Y$ , and  $X$ ,  $Y$  are real random variables with finite moments. The need to know the distribution of the product,  $\mathbf{AB}$ , arises in many applied areas. An early problem is in the study of reverberation systems: in systems with  $n$  feedback speakers  $\mathbf{A}$  is the transmission function of the room with the speakers, and  $\mathbf{B}$  is that of the feedback system. Schroeder (1964, page 1720) notes that  $\mathbf{A}$  (that is  $X$ ) is Gaussian distributed by a Central Limit Theorem argument. Such products also arise with respect to the capacity of mobile multiple-antenna communication links (Marzetta and Hochwald, 1999; Abhayapala *et al.*, 2003; Godavarti *et al.*, 2003), dynamics of nearly periodic disordered multi-span beams (Bouzit and Pierre, 2000), and wave component analysis of energy flow in complex structures (Wester and Mace, 2005a, 2005b).

The aim of this note is to obtain the distribution of  $\mathbf{AB}$  and its trace. The distribution of trace  $\mathbf{AB}$  is just that of  $(\mathbf{AB})_{11}$  with the parameter  $n$  replaced by  $n^2$ .

The results are organized as follows. In Section 2, we obtain the moments and cumulants of  $\mathbf{AB}$  when  $n = 1$ . In Section 3, we obtain Edgeworth expansions for the distribution of an arbitrary element of  $\mathbf{AB}$  in Cartesian form. We make specific references to the mentioned example on reverberation systems: see Example 3.1 for specific results, where both  $\mathbf{A}$  and  $\mathbf{B}$  are assumed to be Gaussian.

Elements  $(\mathbf{AB})_{jk}$  and  $(\mathbf{AB})_{lm}$  are independent if and only if  $j \neq l$  and  $k \neq m$ . Expansions for their joint distribution could be obtained for the case when they are dependent by a straightforward extension of the method given here. The same is true for the joint distribution of any fixed number of elements of  $\mathbf{AB}$ .

## 2 Moments and cumulants

An arbitrary element of  $\mathbf{AB}$  may be written as

$$\sum_{j=1}^n (X_{1j} + iX_{2j})(Y_{1j} + iY_{2j}) = S_{nU} + iS_{nV}, \quad (2.1)$$

where

$$S_{nU} = \sum_{j=1}^n U_j, \quad S_{nV} = \sum_{j=1}^n V_j, \quad U_j = X_{1j}Y_{1j} - X_{2j}Y_{2j}, \quad V_j = X_{2j}Y_{1j} + X_{1j}Y_{2j},$$

$\{X_{kj}\}$  are i.i.d. as  $X$ ,  $\{Y_{kj}\}$  are i.i.d. as  $Y$  and  $\{X_{kj}\}$  is independent of  $\{Y_{kj}\}$ . So,  $\left\{\begin{pmatrix} U_j \\ V_j \end{pmatrix}\right\}$  are i.i.d. as  $\begin{pmatrix} U_1 \\ V_1 \end{pmatrix}$ . We assume that  $m_{rX} = EX^r$  and  $m_{rY} = EY^r$  are finite for  $r = 1, 2, 3, \dots$ . Then  $m_{rs} = EU_1^r V_1^s$  is finite for  $r, s = 1, 2, 3, \dots$  and may be expressed in terms of  $\{m_{rX}, m_{sY}\}$ :

$$\begin{aligned} m_{10} &= EU_1 = 0, & m_{01} &= EV_1 = 2m_{1X}m_{1Y}, \\ m_{20} &= EU_1^2 = 2m_{2X}m_{2Y} - 2m_{1X}^2m_{1Y}^2, \\ m_{02} &= EV_1^2 = 2m_{2X}m_{2Y} + 2m_{1X}^2m_{1Y}^2, \\ m_{11} &= EU_1V_1 = 0. \end{aligned}$$

Swapping  $(X_1, Y_1)$  with  $(X_2, Y_2)$  we see that

$$(U_1, V_1) \stackrel{\mathcal{L}}{=} (-U_1, V_1)$$

so that  $m_{jk} = \kappa_{jk} = 0$  for  $j$  odd, where  $\kappa_{jk}$  is the corresponding joint cumulant. Also,  $E \exp(sU_1) = E \exp(sXY)E \exp(-sXY)$ , so

$$m_{r0} = \sum_{j=0}^r \binom{r}{j} (-1)^j m_{jX} m_{jY} m_{r-j,X} m_{r-j,Y},$$

and  $E \exp(sV_1) = \{E \exp(sXY)\}^2$ , so

$$m_{0r} = \sum_{j=0}^r \binom{r}{j} m_{jX} m_{jY} m_{r-j,X} m_{r-j,Y},$$

and the cumulants of  $U_1$  and  $V_1$  are given by

$$\kappa_{r0} = \kappa_r(U_1) = \{1 + (-1)^r\} \kappa_r(XY), \quad \kappa_{0r} = \kappa_r(V_1) = 2\kappa_r(XY).$$

To obtain  $\kappa_r(XY)$  one uses the usual formula in terms of the moments, in this case  $E(XY)^r = m_{rX}m_{rY} = t_r$  say. This can be written as

$$\kappa_r(XY) = \sum_{j=1}^r (-1)^{j-1} (j-1)! B_{rj}(\mathbf{t}),$$

where  $B_{rj}(\mathbf{t})$  is tabled on page 307 of Comtet (1974) up to  $r = 12$ , and is called the partial ordinary Bell polynomial for the sequence  $\mathbf{t} = (t_1, t_2, \dots)$ : see also Withers and Nadarajah (2009).

The Edgeworth expansions up to  $O(n^{-1})$  and  $O(n^{-5})$  require the cross-cumulants of  $(U_1, V_1)$  up to order three and eight, respectively. These are given in Appendix A for the general case up to order three and for the case of  $X$  or  $Y$  symmetric about zero up to order eight.

However, simpler methods for obtaining these are available in special cases using

$$E \exp(uU_1 + vV_1) = EM_Y(uX_1 + vX_2) M_Y(vX_1 - uX_2), \quad (2.2)$$

where  $M_Y(s) = E \exp(sY)$ .

**Example 2.1** Suppose  $Y \sim \mathcal{N}(0, 1)$ . Then  $M_Y(s) = \exp(s^2/2)$ , so the right hand side of (2.2) is equal to  $E \exp(Q/2)$  for  $Q = 2T(X_1^2 + X_2^2)$  and  $2T = u^2 + v^2$ . So,

$$E \exp(Q/2) = \sum_{r=0}^{\infty} M_r T^r / r!,$$

where

$$M_r = E (X_1^2 + X_2^2)^r = \sum_{j=0}^r \binom{r}{j} m_{2j} m_{2r-2j}.$$

Expanding gives

$$m_{2j,2k} = 2^{-r} M_r (2j)!(j!)^{-1} (2k)!(k!)^{-1}, \quad (2.3)$$

where  $r = j + k$ , and other moments being zero. So,

$$(U_1, V_1) \stackrel{L}{=} (V_1, U_1)$$

even if  $X$  is not symmetric. Also the log of the right hand side of (2.2) is equal to

$$2 \log E \exp(TX^2) = 2 \sum_{r=1}^{\infty} \kappa_r(X^2) T^r / r!.$$

Expanding gives

$$\kappa_{2j,2k} = \kappa_r(X^2) 2^{1-r} (r!)^{-1} \binom{r}{k} (2j)!(2k)!,$$

where  $r = j + k$ , and other cumulants being zero. Since  $(U_1, V_1)$  is not altered by swapping  $(X_1, X_2)$  with  $(Y_1, Y_2)$ , this also cover the case when  $X \sim \mathcal{N}(0, 1)$  and  $Y$  is arbitrary.

**Example 2.2** Suppose both  $X$  and  $Y$  are  $\mathcal{N}(0, 1)$ . Then  $X_1^2 + X_2^2$  is  $\chi_2^2$ ,  $E \exp(t\chi_2^2) = (1 - 2t)^{-1}$ , so  $M_r = 2^r r!$  and  $\kappa_r(X^2)2^{1-r}(r!)^{-1} = r^{-1}$ , so (2.3) and (2.4) simplify to

$$m_{2j,2k} = r\kappa_{2j,2k} = \binom{r}{j}(2j)!(2k)!, \quad (2.4)$$

where  $r = j + k$ . So,  $m_{2r,0} = (2r)!$ ,  $\kappa_{2r,0} = 2(2r - 1)!$ ,  $\kappa_{20} = 2$ ,  $\kappa_{40} = 12$ ,  $\kappa_{60} = 240$ ,  $\kappa_{80} = 10080$ ,  $\kappa_{22} = 4$ ,  $\kappa_{42} = 48$ ,  $\kappa_{62} = 1440$ ,  $\kappa_{44} = 864$ , and so on.

**Example 2.3** Similarly, one may show that if  $X \sim \mathcal{N}(\mu_X, v_X)$  and  $Y \sim \mathcal{N}(\mu_Y, v_Y)$  then the c.g.f. of  $(U, V)$  is

$$K_{U,V}(u, v) = -\log(1 - 2Tv_Xv_Y) + (1 - 2Tv_Xv_Y)^{-1}(\tau + 2vm_Xm_Y) - \tau,$$

where  $2T = u^2 + v^2$  and  $\tau = m_X^2/v_X + m_Y^2/v_Y$ . So,  $\kappa_{2i,2j} = \lambda_{ij}(r^{-1} + \tau)$  for  $r = i + j \geq 1$ , and  $\kappa_{2i,2j+1} = (2j+1)\lambda_{ij}(2m_Xm_Y)$  for  $r = i + j \geq 1$ , where  $\lambda_{ij} = r!(2i)!(2j)!(v_Xv_Y)^r / \{i!(r-i)!\}$  for  $r = i + j$ .

**Example 2.4** Suppose that  $X \sim \text{Poisson}(\lambda)$ . Then one may show that the joint non-central moments of  $(U, V)$  are

$$m_{rs} = E \sum_{k=0}^{r+s} \lambda^k B_{r,s,k}(\mathbf{A}),$$

where  $\mathbf{A} = \{A_{rs}\}$ ,  $A_{rs} = Y_1^r Y_2^s + Y_1^s (-Y_2)^r$ , and  $B_{r,s,k}(\mathbf{A})$  is the partial exponential bivariate Bell polynomial. So, one obtains  $\kappa_{r0} = 0$  for  $r$  odd,

$$\begin{aligned} \kappa_{20} &= 2\lambda m_{2Y} + 2\lambda^2 \kappa_{2Y}, \\ \kappa_{40} &= 2\lambda m_{4Y} + 2\lambda^2 (7m_{4Y} + m_{2Y}^2 - 4m_{1Y}m_{3Y}) \\ &\quad + 4\lambda^3 (3m_{4Y} + m_{2Y}^2 - 6m_{3Y}m_{1Y} + 2m_{2Y}m_{1Y}^2) \\ &\quad + 2\lambda^4 (m_{4Y} + m_{2Y}^2 - 4m_{3Y}m_{1Y} + 2m_{2Y}m_{1Y}^2 - 2m_{1Y}^4), \\ \kappa_{01} &= 2\lambda m_{1Y}, \\ \kappa_{02} &= \kappa_{20}, \\ \kappa_{11} &= 0, \\ \kappa_{03} &= 2\lambda m_{3Y} + 6\lambda^2 (m_{3Y} - m_{2Y}m_{1Y}) + 2\lambda^3 \mu_{3Y}, \\ \kappa_{21} &= 2\lambda m_{2Y}m_{1Y} + 2\lambda^2 (m_{3Y} - m_{2Y}m_{1Y}) + 2\lambda^3 \mu_{3Y}, \end{aligned}$$

and so on.

### 3 Cartesian Edgeworth expansions

As noted in (2.1), an arbitrary element of  $\mathbf{AB}$  can be written as

$$S_{nU} + iS_{nV} = \sum_{j=1}^n (U_j + iV_j),$$

where  $(\frac{U_j}{V_j})$  are i.i.d. with cumulants  $\{\kappa_{rs}\}$  given by Section 2. In particular,  $\kappa_{10} = 0$ ,  $\kappa_{01} = 2m_{1X}m_{1Y}$ ,  $\kappa_{20} = \kappa_{02} = 2(m_{2X}m_{2Y} - m_{1X}^2m_{1Y}^2)$ , and  $\kappa_{11} = 0$ . So, as  $n \rightarrow \infty$ ,

$$\mathbf{Y}_n = (n\kappa_{20})^{-1/2} \begin{pmatrix} S_{nU} & \\ S_{nV} & -n\kappa_{01} \end{pmatrix} \xrightarrow{\mathcal{L}} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix},$$

where  $N_1$  and  $N_2$  are i.i.d.  $\mathcal{N}(0, 1)$  random variables. Also by Bhattacharya and Rao (1976), if  $X$  and  $Y$  are nonlattice, then so is  $(U_1, V_1)$ , so  $\mathbf{Y}_n$  has the Edgeworth expansions

$$P_{\mathbf{Y}_n}(\mathbf{y}) = P(\mathbf{Y}_n \leq \mathbf{y}) = \sum_{j=0}^{\infty} n^{-j/2} P_j(\mathbf{y}),$$

$$p_{\mathbf{Y}_n}(\mathbf{y}) = \partial^2 P_{\mathbf{Y}_n}(\mathbf{y}) / \partial y_1 \partial y_2 = \sum_{j=0}^{\infty} n^{-j/2} p_j(\mathbf{y}),$$

where

$$P_j(\mathbf{y}) = \tilde{P}_j(-\partial/\partial\mathbf{y}) \Phi(y_1) \Phi(y_2),$$

$$p_j(\mathbf{y}) = \partial^2 P_j(\mathbf{y}) / \partial y_1 \partial y_2 = \tilde{P}_j(-\partial/\partial\mathbf{y}) \phi(y_1) \phi(y_2),$$

where  $\Phi(\cdot)$  and  $\phi(\cdot)$  are the distribution and density of a standard normal, and  $\tilde{P}_j(\mathbf{t})$  are certain polynomials in  $\mathbf{t} = (t_1, t_2)$ . In particular,

$$\tilde{P}_0(\mathbf{t}) = 1, \tilde{P}_1 = \nabla_1, \tilde{P}_2 = \nabla_2 + \nabla_1^2/2, \tilde{P}_3 = \nabla_3 + \nabla_2\nabla_1 + \nabla_1^3/6, \dots,$$

where

$$\nabla_j(\mathbf{t}) = \sum_{r+s=j+2} t_1^r t_2^s K_{rs}, \quad K_{rs} = \kappa_{20}^{-(r+s)/2} \kappa_{rs} / (r!s!).$$

So,

$$\tilde{P}_1(\mathbf{t}) = \sum_{r+s=3} t_1^r t_2^s K_{rs} = t_1^3 K_{30} + t_1^2 t_2 K_{21} + t_1 t_2^2 K_{12} + t_2^3 K_{03},$$

and

$$p_1(\mathbf{y}) = (H_{31}K_{30} + H_{21}H_{12}K_{21} + H_{12}H_{22}K_{12} + H_{32}K_{03}) \phi(Y_1) \phi(Y_2),$$

where  $H_{jk} = H_j(y_k)$  and  $H_j(x) = \phi(x)^{-1}(-\partial/\partial x)^j \phi(x)$ , the  $j$ th Hermite polynomial, expressible as  $H_j(x) = \text{Re } E(x + i\mathcal{N}(0, 1))^j$ . Similarly,

$$P_1(\mathbf{y}) = -(H_{21}\phi_1\Phi_2K_{30} + H_{22}\Phi_1\phi_2K_{03} + H_{11}\phi_1\phi_2K_{21} + H_{12}\phi_1\phi_2K_{12}),$$

where  $\phi_i = \phi(y_i)$  and  $\Phi_i = \Phi(y_i)$ . Higher order  $p_j(\mathbf{y})$  and  $P_j(\mathbf{y})$  can be written down similarly.

Now suppose that  $X$  is symmetric about zero. Then by Appendix A,  $\kappa_{rs} = 0$  unless both  $r$  and  $s$  are even, so  $\tilde{P}_j = \nabla_j = 0$  for  $j$  odd,  $\tilde{P}_2 = \nabla_2$ ,  $\tilde{P}_4 = \nabla_4 + \nabla_2^2/2$ ,  $\tilde{P}_6 = \nabla_6 + \nabla_4\nabla_2 + \nabla_2^3/6$ , and so on. Also  $\kappa_{rs} = \kappa_{sr}$ , so

$$\tilde{P}_2(\mathbf{t}) = \nabla_2(\mathbf{t}) = \sum_{i=1}^2 t_i^4 K_{40} + t_1^2 t_2^2 K_{22},$$

$$\nabla_4(\mathbf{t}) = \sum_{i=1}^2 (t_i^6 K_{60} + t_i^4 t_j^2 K_{42}),$$

$$\nabla_6(\mathbf{t}) = \sum_{i=1}^2 (t_i^8 K_{80} + t_i^6 t_j^2 K_{62}) + t_1^4 t_2^4 K_{44},$$

where

$$\sum^2 f(t_1, t_2) = f(t_1, t_2) + f(t_2, t_1).$$

So,

$$\tilde{P}_4(\mathbf{t}) = \sum^2 (t_1^8 P_{80} + t_1^6 t_2^2 P_{62} + t_1^6 K_{60} + t_1^4 t_2^2 K_{42}) + t_1^4 t_2^4 P_{44},$$

where

$$P_{80} = K_{40}^2/2, \quad P_{62} = K_{40}K_{22}, \quad P_{44} = K_{40}^2 + K_{22}^2/2,$$

and

$$\begin{aligned} \tilde{P}_6(\mathbf{t}) = & \sum^2 (t_1^{12} P_{12,0} + t_1^{10} t_2^2 P_{10,2} + t_1^8 t_2^4 P_{84} + t_1^{10} P_{10,0} + t_1^8 t_2^2 P_{82} + t_1^6 t_2^4 P_{64} + t_1^8 K_{80}) t_1^6 t_2^6 P_{66} \\ & + t_1^4 t_2^4 K_{44}, \end{aligned}$$

where

$$\begin{aligned} P_{12,0} &= K_{40}^3/6, \quad P_{10,2} = K_{40}^2 K_{22}/2, \quad P_{84} = (K_{40}^3 + K_{40}K_{22}^2)/2, \\ P_{10,0} &= K_{60}K_{40}, \quad P_{82} = K_{42}K_{40} + K_{60}K_{22}, \\ P_{64} &= K_{60}K_{40} + K_{42}K_{40} + K_{42}K_{22}, \\ P_{66} &= K_{22}^3/6 + 6K_{40}^2 K_{22}. \end{aligned}$$

So,

$$p_{\mathbf{Y}_n}(\mathbf{y}) = \sum_{j=0}^{\infty} n^{-j} p_{2j}(\mathbf{y}), \quad (3.1)$$

and

$$P(\mathbf{Y}_n \leq \mathbf{y}) = \sum_{j=0}^{\infty} n^{-j} P_{2j}(\mathbf{y}), \quad (3.2)$$

where  $P_0(\mathbf{y}) = \Phi_1 \Phi_2$ ,  $p_0(\mathbf{y}) = \phi_1 \phi_2$ , and, for  $r = 2, 4, \dots$ ,

$$P_r(\mathbf{y}) = \left\{ \sum^2 a_r(\mathbf{y}) + b_r(\mathbf{y}) \right\} \phi_1 \phi_2, \quad (3.3)$$

and

$$P_r(\mathbf{y}) = - \sum^2 A_r(\mathbf{y}) \Phi_1 \Phi_2 + \left\{ B_r(\mathbf{y}) + \sum^2 C_r(\mathbf{y}) \right\} \phi_1 \phi_2, \quad (3.4)$$

where

$$\sum^2 f(\mathbf{y}) = f(y_1, y_2) + f(y_2, y_1).$$



In particular,

$$\begin{aligned}
a_2(\mathbf{y}) &= H_{41}K_{40}, \quad b_2(\mathbf{y}) = H_{21}H_{22}K_{22}, \\
A_2(\mathbf{y}) &= H_{31}K_{40}, \quad B_2(\mathbf{y}) = H_{11}H_{12}K_{22}, \quad C_2(\mathbf{y}) = 0, \\
a_4(\mathbf{y}) &= H_{81}P_{80} + H_{61}K_{60} + H_{61}H_{22}P_{62} + H_{41}H_{22}K_{42}, \\
b_4(\mathbf{y}) &= H_{41}H_{42}P_{44}, \\
A_4(\mathbf{y}) &= H_{71}P_{80} + H_{51}K_{60}, \quad B_4(\mathbf{y}) = H_{31}H_{32}P_{44}, \\
C_4(\mathbf{y}) &= H_{51}H_{12}P_{62} + H_{31}H_{12}K_{42}, \\
a_6(\mathbf{y}) &= H_{12,1}P_{12,0} + H_{10,1}H_{22}P_{10,2} + H_{81}H_{42}P_{84} \\
&\quad + H_{10,1}P_{10,0} + H_{81}H_{22}P_{82} + H_{61}H_{42}P_{64} + H_{81}K_{80}, \\
b_6(\mathbf{y}) &= H_{61}H_{62}P_{66} + H_{41}H_{42}K_{44}, \\
A_6(\mathbf{y}) &= H_{11,1}P_{12,0} + H_{91}P_{10,0} + H_{71}K_{80}, \\
B_6(\mathbf{y}) &= H_{51}H_{52}P_{66} + H_{31}H_{32}K_{44}, \\
C_6(\mathbf{y}) &= H_{91}H_{12}P_{10,2} + H_{71}H_{32}P_{84} + H_{71}H_{12}P_{82} + H_{51}H_{32}P_{64},
\end{aligned}$$

and so on.

**Note 3.1** We have  $p_{\mathbf{Y}_n}(\mathbf{y}) = p_{\mathbf{Y}_n}(y_1, -y_2) = p_{\mathbf{Y}_n}(-y_1, y_2)$ , so  $\mathbf{Z}_n = (|Y_{n1}|, |Y_{n2}|)'$  has density

$$p_{\mathbf{Z}_n}(\mathbf{y}) = 4p_{\mathbf{Y}_n}(\mathbf{y})$$

on  $(0, \infty)^2$  and

$$P(\mathbf{Z}_n \leq \mathbf{y}) = 4\{P_{\mathbf{Y}_n}(\mathbf{y}) - P_{\mathbf{Y}_n}(y_1, 0) - P_{\mathbf{Y}_n}(0, y_2) - P_{\mathbf{Y}_n}(\mathbf{0})\}.$$

Also  $P_{\mathbf{Y}_n}(\mathbf{0}) = 1/4$  and  $p_{\mathbf{Z}_n}(\mathbf{y}) = p_{\mathbf{Z}_n}(y_2, y_1)$ .

**Example 3.1** Suppose  $X$  and  $Y$  are  $\mathcal{N}(0, 1)$ . Then by Example 2.2

$$K_{2j,2k} = 2^{-j-k}(j+k)^{-1} \binom{j+k}{j}.$$

In particular,  $K_{40} = 2^{-3}$ ,  $K_{60} = 2^{-3}/3$ ,  $K_{80} = 2^{-6}$ ,  $K_{22} = 2^{-2}$ ,  $K_{42} = 2^{-3}$ ,  $K_{62} = 2^{-3}$ ,  $K_{44} = 3 \cdot 2^{-5}$ ,  $P_{80} = 2^{-7}$ ,  $P_{62} = 2^{-5}$ ,  $P_{44} = 3 \cdot 2^{-6}$ ,  $P_{12,0} = 2^{-10}/3$ ,  $P_{10,2} = 2^{-9}$ ,  $P_{84} = 5 \cdot 2^{-10}$ ,  $P_{10,0} = 2^{-6}/3$ ,  $P_{82} = 5 \cdot 2^{-6}/3$ ,  $P_{64} = 5 \cdot 2^{-5}/3$  and  $P_{66} = 5 \cdot 2^{-6}/3$ . So,

$$\mathbf{Y}_n = (2n)^{-1/2} (S_{nU}, S_{nV})'$$

satisfies (3.1)-(3.4) with

$$\begin{aligned}
a_2(\mathbf{y}) &= 2^{-3}H_{41}, \quad b_2(\mathbf{y}) = 2^{-2}H_{21}H_{22}, \\
A_2(\mathbf{y}) &= 2^{-3}H_{31}, \quad B_2(\mathbf{y}) = 2^{-2}H_{11}H_{12}, \quad C_2(\mathbf{y}) = 0, \\
a_4(\mathbf{y}) &= 2^{-7}H_{81} + 2^{-5}H_{61}(4/3 + H_{22}) + 2^{-3}H_{41}H_{22}, \\
b_4(\mathbf{y}) &= 3 \cdot 2^{-6}H_{41}H_{42}, \\
A_4(\mathbf{y}) &= 2^{-7}H_{71} + 2^{-3}H_{51}/3, \quad B_4(\mathbf{y}) = 3 \cdot 2^{-6}H_{31}H_{32}, \\
C_4(\mathbf{y}) &= 2^{-5}H_{41}H_{12} + 2^{-3}H_{31}H_{12}, \\
a_6(\mathbf{y}) &= 2^{-10}H_{12,1}/3 + 2^{-9}H_{10,1}H_{22} + 5 \cdot 2^{-10}H_{81}H_{42} \\
&\quad + 10^{-6}H_{10,1}/3 + 5 \cdot 2^{-6}H_{81}H_{22}/3 + 5 \cdot 2^{-6}H_{61}H_{42}/3 + 2^{-6}H_{81}, \\
b_6(\mathbf{y}) &= 5 \cdot 2^{-3}H_{61}H_{62}/3 + 3 \cdot 2^{-5}H_{41}H_{42}, \\
A_6(\mathbf{y}) &= 2^{-10}H_{11,1}/3 + 10^{-6}H_{91}/3 + 2^{-6}H_{71}, \\
B_6(\mathbf{y}) &= 5 \cdot 2^{-3}H_{51}H_{52}/3 + 2^{-3}H_{31}H_{32}, \\
C_6(\mathbf{y}) &= 2^{-9}H_{91}H_{12} + 5 \cdot 2^{-10}H_{71}H_{32} + 5 \cdot 2^{-6}H_{71}H_{12}/3 + 5 \cdot 2^{-6}H_{51}H_{32}/3.
\end{aligned}$$

## Appendix A

Here, we give expressions for the joint moments  $m_{rs} = EU^rV^s$  and corresponding cumulants  $\kappa_{rs}$  for  $U = X_1Y_1 - X_2Y_2$  and  $V = X_2Y_1 + X_1Y_2$ , where  $X_1, X_2 \stackrel{\mathcal{L}}{=} X$  and  $Y_1, Y_2 \stackrel{\mathcal{L}}{=} Y$  and  $X_1, X_2, Y_1, Y_2$  are independent real random variables with finite moments  $m_{rX} = EX^r$ ,  $m_{rY} = EY^r$  for  $r = 1, 2, \dots$ . In Section 2, we showed that  $m_{rs} = \kappa_{rs} = 0$  for  $r$  odd, and gave expressions for general  $m_{r0}, m_{0r}$ . So,

$$\begin{aligned}
\kappa_{10} &= m_{10} = 0, \quad \kappa_{01} = m_{01} = 2m_{1X}m_{1Y}, \\
\kappa_{20} &= m_{20} = \kappa_{02} = 2(m_{2X}m_{2Y} - m_{1X}^2m_{1Y}^2), \\
\kappa_{11} &= m_{11} = 0, \quad \kappa_{30} = m_{30} = 0, \quad \kappa_{12} = m_{12} = 0.
\end{aligned}$$

Also

$$m_{21} = 2(m_{1X}m_{2X}m_{3Y} - 2m_{1X}m_{2X}m_{1Y}m_{2Y} + m_{3X}m_{1Y}m_{2Y}),$$

so

$$\kappa_{21} = 2(m_{1X}m_{2X}m_{3Y} - 4m_{1X}m_{2X}m_{1Y}m_{2Y} + m_{3X}m_{1Y}m_{2Y} + 2m_{1X}^3m_{1Y}^3).$$

This gives all the cumulants needed for the Edgeworth expansions up to  $O(n^{-1})$ .

We now assume  $Y$  is symmetric about zero, or equivalently,  $X$  is symmetric about zero. We give those cumulants needed for the Edgeworth expansions up to  $O(n^{-5})$ . So,  $(U, V) \stackrel{\mathcal{L}}{=} (X_1Y_1 + X_2Y_2, X_2Y_1 - X_1Y_2) \stackrel{\mathcal{L}}{=} (V, U)$ . So,  $m_{rs} = m_{sr}$ ,  $\kappa_{rs} = \kappa_{sr}$ ,  $m_{rs} = \kappa_{rs} = 0$  unless both  $r$  and  $s$  are even,

$$m_{2r,0} = \sum_{j=0}^r \binom{2r}{2j} m_{2j,X} m_{2j,Y} m_{2r-2j,X} m_{2r-2j,Y}$$

and

$$m_{2r,2s} = \sum_{j=0}^{2r} \binom{2r}{j} \sum_{k=0}^{2s} \binom{2s}{k} m_{j+2s-k,X} m_{2r-j+k,X} m_{j+k,Y} m_{2r-j+2s-k,Y} \times \{I(j, k \text{ even}) - I(j, k \text{ odd})\},$$

where  $I(A) = 1$  or  $0$  for  $A$  true or false. So,

$$m_{20} = 2m_{2X}m_{2Y}, \quad m_{40} = 2(m_{4X}m_{4Y} + 3m_{2X}^2m_{2Y}^2), \quad (\text{A.1})$$

$$m_{60} = 2(m_{6X}m_{6Y} + 15m_{4X}m_{2X}m_{4Y}m_{2Y}), \quad (\text{A.2})$$

$$m_{80} = 2(m_{8X}m_{8Y} + 28m_{6X}m_{2X}m_{6Y}m_{2Y} + 35m_{4X}^2m_{4Y}^2), \quad (\text{A.3})$$

$$m_{22} = 2 \left( \sum_{i=0}^2 m_{2X}^2 m_{4Y} - 2m_{2X}^2 m_{2Y}^2 \right), \quad (\text{A.4})$$

$$m_{42} = 2 \left( \sum_{i=0}^2 m_{4X} m_{2X} m_{6Y} - 2m_{4X} m_{2X} m_{4Y} m_{2Y} \right), \quad (\text{A.5})$$

$$m_{62} = 2 \left( \sum_{i=0}^2 m_{8X} m_{6Y} m_{2Y} - 12m_{6X} m_{2X} m_{6Y} m_{2Y} + 15 \sum_{i=0}^2 m_{6X} m_{2X} m_{4Y} - 20m_{4X}^2 m_{4Y}^2 \right),$$

$$m_{44} = 2 \left\{ \sum_{i=0}^2 m_{4X}^2 (m_{8Y} - 16m_{6Y} m_{2Y}) + 12m_{6X} m_{2X} m_{6Y} m_{2Y} + 18m_{4X}^2 m_{4Y}^2 m_{4Y}^2 \right\},$$

where  $\sum^2 f(X, Y) = f(X, Y) + f(Y, X)$ .

The condition that  $Y \stackrel{\mathcal{L}}{=} -Y$  can be weakened. For example, (A.1) and (A.4) hold if  $EY = 0$ , (A.2), (A.3) and (A.5) hold and  $\kappa_{21} = 0$  if  $EY = EY^3 = 0$ , and so on. So, if  $EY = EY^3 = 0$  (or  $EX = EX^3 = 0$ ) then the error in the Central Limit Theorem approximation is  $O(n^{-1})$  not just  $O(n^{-1/2})$ .

So,

$$\begin{aligned}
\kappa_{20} &= 2m_{2X}m_{2Y}, \\
\kappa_{40} &= 2(m_{4X}m_{4Y} - 3m_{2X}^2m_{2Y}^2), \\
\kappa_{60} &= 2(m_{6X}m_{6Y} - 15m_{4X}m_{2X}m_{4Y}m_{2Y} + 30m_{2X}^3m_{2Y}^3), \\
\kappa_{80} &= 2m_{8X}m_{8Y} \\
&\quad + 14(-4m_{6X}m_{2X}m_{6Y}m_{2Y} - 5m_{4X}^2m_{4Y}^2 + 60m_{4X}m_{2X}m_{4Y}m_{2Y} - 90m_{2X}^4m_{2Y}^4), \\
\kappa_{22} &= \mu_{22} - \mu_{20}\mu_{02} - 2\mu_{11}^2 = 2\left(\sum^2 m_{2X}^2m_{4Y} - 4m_{2X}^2m_{2Y}^2\right), \\
\kappa_{42} &= \mu_{42} - 6\mu_{20}\mu_{22} - 8\mu_{11}\mu_{31} - \mu_{02}\mu_{40} + 6\mu_{20}^2\mu_{02} + 24\mu_{11}^2\mu_{20} \\
&= 2\left\{\sum^2 m_{4X}m_{2X}(m_{6Y} - 12m_{2Y}^3) - 4m_{4X}m_{2X}m_{4Y}m_{2Y} + 42m_{2X}^3m_{2Y}^3\right\}, \\
\kappa_{62} &= \mu_{62} - \mu_{02}\mu_{60} - 12\mu_{11}\mu_{51} - 15\mu_{20}\mu_{42} - 15\mu_{40}\mu_{22} \\
&\quad - 20\mu_{31}^2 + 90\mu_{22}\mu_{20}^2 + 30\mu_{40}(\mu_{20}\mu_{02} + 2\mu_{11}^2) \\
&\quad + 240\mu_{31}\mu_{11}\mu_{20} - 270\mu_{02}\mu_{20}^3 - 360\mu_{11}^2\mu_{20}^2 \\
&= 2\left\{\sum^2 m_{8X}m_{6Y}m_{2Y} - 14m_{6X}m_{2X}m_{6Y}m_{2Y} + 15\sum^2 m_{6X}m_{2X}(m_{4Y}^2 - 2m_{4Y}m_{2Y}^2) \right. \\
&\quad - 20m_{4X}^2m_{4Y}^2 + 210m_{4X}m_{2X}^2m_{4Y}m_{2Y}^2 - 30\sum^2 m_{4X}^2m_{4Y}m_{2Y}^2 \\
&\quad \left. + 270\sum^2 m_{4X}m_{2X}^2m_{2Y}^4 - 2340m_{2X}^4m_{2Y}^4\right\}, \\
\kappa_{44} &= \mu_{44} - 6\mu_{20}\mu_{24} - 6\mu_{02}\mu_{42} - 16\mu_{11}\mu_{33} - \mu_{40}\mu_{04} - 16\mu_{31}\mu_{13} \\
&\quad - 18\mu_{22}^2 + 6\mu_{40}\mu_{02}^2 + 6\mu_{04}\mu_{20}^2 + 96\mu_{11}(\mu_{31}\mu_{02} + \mu_{13}\mu_{20}) \\
&\quad + 72\mu_{22}(\mu_{20}\mu_{02} + 2\mu_{11}^2) - 18(\mu_{20}^2\mu_{02}^2 + 12\mu_{20}\mu_{02}\mu_{11}^2 + 8\mu_{11}^4) \\
&= 2\left\{\sum^2 m_{4X}^2(m_{8Y} - 16m_{6Y}m_{2Y}) + 12m_{6X}m_{2X}m_{6Y}m_{2Y} \right. \\
&\quad + 16m_{4X}^2m_{4Y}^2 - 24\sum^2 m_{6X}m_{2X}m_{4Y}m_{2Y}^2 + 12m_{4X}m_{2X}^2m_{4Y}m_{2Y}^2 \\
&\quad \left. - 36\sum^2 m_{4X}^2m_{2Y}^4 + 432\sum^2 m_{4X}m_{2X}^2m_{2Y}^4 - 738m_{2X}^4m_{2Y}^4\right\}.
\end{aligned}$$

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