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Yves Guivarc'H, Emile Le Page

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Spectral gap properties and asymptotics of stationary measures for affine random walks.

Y. Guivarc'h, E. Le Page

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Abstract

Let $V = \mathbb{R}^d$ be the Euclidean d -dimensional space, μ (resp λ) a probability measure on the linear (resp affine) group $G = GL(V)$ (resp $H = Aff(V)$) and assume that μ is the projection of λ on G . We study asymptotic properties of the convolutions $\mu^n * \delta_v$ (resp $\lambda^n * \delta_v$) if $v \in V$, i.e asymptotics of the random walk on V defined by μ (resp λ), if the subsemigroup $T \subset G$ (resp $\Sigma \subset H$) generated by the support of μ (resp λ) is "large". We show spectral gap properties for the convolution operator defined by μ on spaces of homogeneous functions of degree $s \geq 0$ on V , which satisfy Hölder type conditions. As a consequence of our analysis we get precise asymptotics for the potential kernel $\sum_0^\infty \mu^k * \delta_v$, which imply its asymptotic homogeneity. Under natural conditions the H -space V is a λ -boundary ; then we use the above results and radial Fourier Analysis on $V \setminus \{0\}$ to show that the unique λ -stationary measure ρ on V is "homogeneous at infinity" with respect to dilations $v \rightarrow tv(t > 0)$, with a tail measure depending essentially of μ and Σ . Our proofs are based on the simplicity of the dominant Lyapunov exponent for certain products of Markov-dependant random matrices, on the use of a renewal theorem for Markov walks, and on the dynamical properties of a conditional λ -boundary dual to V .

I Introduction, statement of results

We consider the d -dimensional Euclidean space $V = \mathbb{R}^d$, endowed with the natural scalar product $(x, y) \rightarrow \langle x, y \rangle$, the associated norm $x \rightarrow |x|$, the linear group $G = GL(V)$, and the affine group $H = Aff(V)$. Let λ be a probability measure on H with projection μ on G , such that $supp\lambda$ has no fixed point in V . Under natural conditions, including negativity of the dominant Lyapunov exponent L_μ corresponding to μ , for any $v \in V$, $\lambda^n * \delta_v$ converges weakly to ρ ; the probability measure ρ is the unique probability which solves the equation $\lambda * \rho = \rho$, and $supp\rho$ is unbounded. A non trivial property of ρ is the existence of $\alpha > 0$ such that $\int |x|^s d\rho(x) < \infty$ for $s < \alpha$ and $\int |x|^s d\rho(x) = \infty$ for $s \geq \alpha$.

For the asymptotic behaviour of $\lambda^n * \delta_v$ there are four cases of interest :

- a) The "contractive" case where the elements of $supp\mu$ have norms less than 1, ρ exists and is compactly supported.
- b) The "expansive" case where $L_\mu > 0$ and ρ do not exist.
- c) The "critical" case where $L_\mu = 0$ and ρ do not exist.
- d) The "weakly contractive" case where $L_\mu < 0$ and ρ exists with unbounded support.

Heuristically, case d), mentioned above, can be considered as a transition between the cases a), b), which appear to be extreme cases. In this paper we are mainly interested in case d) and in the "shape at infinity" of ρ ; in the corresponding analysis we develop methods and prove results which are of independant interest for products of random matrices. In this context we note the following property of ρ for case d) : there exists a domain $D \subset \mathbb{C}$ which contains the set $\{Rez \in]0, \alpha]\}$ in which for any $v \in V$, the function $z \rightarrow \int \langle x, v \rangle^z d\rho(x)$ is meromorphic and admits, (for most v 's) a unique pole at α which is simple. More generally the behaviour at infinity of certain invariant measures is of interest for various

questions in Probability Theory and Mathematical Physics (see [12], [13], [20], [21], [36]), for the study of dynamical excursions and winding around cusps (see [1], [44], [48]), for the analysis of the H -space (V, ρ) as a λ -boundary and its dynamical consequences (see [2], [16], [17], [31]).

For the study at infinity of ρ , our main geometrical hypothesis here is on the closed sub-semigroup T generated by $\text{supp}\mu$. As in [17] we assume that T satisfies the so-called i-p condition (i-p for irreducibility and proximality) i.e T is strongly irreducible and contains at least one element with a unique simple dominant eigenvalue ; if $d = 1$, we assume furthermore that T is non arithmetic, i.e T is not contained in a subgroup of \mathbb{R}^* of the form $\{\pm a^n; n \in \mathbb{Z}\}$ for some $a > 0$. We observe that for $d > 1$ condition i-p is satisfied by T if and only if it is satisfied by its Zariski closure, hence condition i-p is satisfied if T is "large" (see [23], [45]). On the other hand, the set of probability measures μ on G such that the associated semigroup T satisfies condition i-p is open and dense in the weak topology. Also, if $d > 1$, an essential aperiodicity consequence of condition i-p is the density in \mathbb{R}_+^* of the multiplicative subgroup generated by moduli of dominant eigenvalues of the elements of T (see [26], [29]). If irreducibility is not assumed we can consider the subspace $V^+(T) \subset V$ generated by the dominant eigenvectors of T . Then $V^+(T)$ is T -invariant and, if the restriction T^+ of T to $V^+(T)$ is irreducible then T^+ satisfies condition i-p.

The probability ρ will be analysed in terms of dual objects. We denote by g^* the transposed map of $g \in G$, i.e g^* is the linear map defined by the relation $\langle g^*x, y \rangle = \langle x, gy \rangle$, and μ^* is the push-forward of μ by $g \rightarrow g^*$. For a Radon measure η on $V \setminus \{0\}$ and $t > 0$, we write $t.\eta$ for the push-forward of η by the map $x \rightarrow tx$. For $v \in V \setminus \{0\}$ we denote $H_v^+ = \{x \in V ; \langle x, v \rangle > 1\}$ and if η is a probability measure on $V \setminus \{0\}$ we write $\hat{\eta}(v) = \eta(H_v^+)$. The asymptotic expansion of $\hat{\eta}(tv)$ at $t = o_+$ gives the tail of η in direction v ; also, using the G -equivariant map $\eta \rightarrow \hat{\eta}$ and the affine structures we observe that the equation $\rho = \lambda * \rho$ implies Poisson-type equations on $V \setminus \{0\}$ and $E = (V \setminus \{0\}) \times \mathbb{R}$, for functions associated with $\hat{\rho}$. The space E is \mathbb{R}_+^* -fibered over the space W of affine oriented hyperplanes in V , and has a natural H -homogeneous space structure. The action of $(g, b) \in H$ on $(v, r) \in E$ is given by $(v, r) \rightarrow (g^*v, r + \langle b, v \rangle)$. Our analysis of the equation $\rho = \lambda * \rho$ will rely on the dynamical properties of associated "dual" λ -walks on the H -spaces $E, W, V \setminus \{0\}$ and on the study of Poisson-type equations satisfied by $\hat{\rho}$.

Since the dominant Lyapunov exponent of μ is negative, the linear random walk $S_n(\omega)v$ on $V \setminus \{0\}$ defined by μ converges to zero a.e ; however, since T is unbounded $S_n(\omega)v$ takes arbitrarily large values with a small probability which has a power law decay with degree $\alpha > 0$. As observed in [36], this fluctuation property of $S_n(\omega)v$ is responsible for the "tail homogeneity" of ρ , i.e the existence of $\lim_{t \rightarrow 0_+} t^{-\alpha} \hat{\rho}(tv)$. We consider the more precise

condition of "homogeneity at infinity" of ρ , i.e existence and non triviality of $\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho)$ in the sense of vague convergence. This property is proved here under the above hypothesis and the product structure of the limiting measure is described in terms of twisted μ -stationary measures on the unit sphere \mathbb{S}^{d-1} . This agrees with F. Spitzer's conjecture

to the effect that ρ belongs to the domain of attraction of a stable law if $\alpha < 2$. For another general construction of such measures, in the context of winding of geodesic flow on surfaces see [1]. Furthermore, the property of "homogeneity at infinity" of ρ plays an essential role in extreme value theory (see [19]). The cases where λ is concentrated on non negative matrices or λ has a density on H were studied in [36], tail homogeneity of ρ was proved and these results were extended to the general case in [39], under some restrictive conditions. If $d = 1$, tail homogeneity means homogeneity at infinity ; in this case, more information on the convergence rate of $t^{-\alpha}(t.\rho)$ and on the limiting measure was given in [20], where the more general quasi linear context was also considered. Furthermore, the case where $\text{supp}\lambda$ consists of similarities of V was studied in [8] under general conditions, homogeneity at infinity of ρ in an appropriate sense was proved, and the structure of the limiting measure was described, in terms of T . Here we complete and improve the results of [36], [39], which are simpler in a sense than those of [8] . We observe that the results of [3] on Radon transforms allow to pass from tail homogeneity to homogeneity at infinity if $\alpha \notin \mathbb{N}$. No such general result is valid for $\alpha \in \mathbb{N}$ and the situation is different (see [50]). The arguments for tail homogeneity of ρ depend on a detailed study of asymptotics for the potential kernel $\sum_0^\infty (\mu^*)^k * \delta_v$ and for certain Birkhoff sums along the λ -walk on the H -space W . The homogeneity at infinity of ρ is also closely related to spectral gap properties for operators associated with μ , which are of independant interest. The use of spectral gap properties, meromorphy of related Mellin transforms, and of Wiener-Ikehara theorem allows to give a short proof of the weak convergence of $t^{-\alpha}(t.\rho)$; the non triviality of the limit follows from a lemma of E. Landau, if μ has a density with compact support. Hence we begin by a description of the spectral properties in a general setting, if $d > 1$.

We denote by $|g|$ the norm of $g \in G$ and we write $\gamma(g) = \sup(|g|, |g^{-1}|)$. We write $I_\mu = \{s \geq 0 ; \int |g|^s d\mu(g) < \infty\}$ and we denote $]0, s_\infty[$ the interior of the interval I_μ . We consider the convolution action of μ on continuous functions on $V \setminus \{0\}$ which are homogeneous of degree $s \geq 0$, *i.e* functions f which satisfy : $f(tv) = |t|^s f(v)$ ($t \in \mathbb{R}$). This action reduces to the action of a certain positive operator P^s on $C(\mathbb{P}^{d-1})$, the space of continuous functions on the projective space \mathbb{P}^{d-1} . More precisely, if $f(v) = |v|^s \varphi(\bar{v})$ with $\varphi \in C(\mathbb{P}^{d-1})$, $\bar{v} \in \mathbb{P}^{d-1}$, then $P^s \varphi$ is given by :

$$P^s \varphi(x) = \int |gx|^s \varphi(g.x) d\mu(g),$$

where $x \in \mathbb{P}^{d-1}$, $x \rightarrow g.x$ denotes the projective action of g on x , and $|gx|$ is the norm of any vector gv with $|v| = 1$ and $\bar{v} = x$. Also for $z = s + it \in \mathbb{C}$ we write $P^z \varphi(x) = \int |gx|^z \varphi(g.x) d\mu(g)$. For $\varepsilon > 0$ let $H_\varepsilon(\mathbb{P}^{d-1})$ be the space of ε -Hölder functions on \mathbb{P}^{d-1} , with respect to a certain natural distance. We denote by ℓ^s (resp ℓ) the s -homogeneous (resp Lebesgue) measure on \mathbb{R}_+^* given by $\ell^s(dt) = \frac{dt}{t^{s+1}}$ (resp $\ell(dt) = \frac{dt}{t}$) and we write an s -homogeneous Radon measure η on $\dot{V} = \mathbb{P}^{d-1} \times \mathbb{R}_+^*$ as $\eta = \pi \otimes \ell^s$ where π is a bounded measure on \mathbb{P}^{d-1} . For $s \in I_\mu$ we define $k(s) = \lim_{n \rightarrow \infty} (\int |g|^s d\mu^n(g))^{1/n}$ where μ^n is the n -th convolution power of μ and we observe that $\text{Log}k(s)$ is a convex function on I_μ . A key tool

in our analysis for $d > 1$ is the

Theorem A

Assume $d > 1$ and the subsemigroup $T \subset GL(V)$ generated by $\text{supp}\mu$ satisfies condition i-p. Then, for any $s \in I_\mu$ there exists a unique probability measure ν^s on \mathbb{P}^{d-1} , a unique positive continuous function $e^s \in C(\mathbb{P}^{d-1})$ with $\nu^s(e^s) = 1$ such that $P^s\nu^s = k(s)\nu^s$, $P^se^s = k(s)e^s$. For $s \in I_\mu$, if $\int |g|^s \gamma^\delta(g) d\mu(g) < \infty$ for some $\delta > 0$ and if $\varepsilon > 0$ is sufficiently small, the action of P^s on $H_\varepsilon(\mathbb{P}^{d-1})$ has a spectral gap :

$$P^s = k(s)(\nu^s \otimes e^s + U^s),$$

where $\nu^s \otimes e^s$ is the projection on $\mathbb{C}e^s$ defined by ν^s, e^s and U^s is an operator with spectral radius less than 1 which satisfies $U^s(\nu^s \otimes e^s) = (\nu^s \otimes e^s)U^s = 0$. Furthermore the function $k(s)$ is analytic, strictly convex on $]0, s_\infty[$ and the function $\nu^s \otimes e^s$ from $]0, s_\infty[$ to $\text{End } H_\varepsilon(\mathbb{P}^{d-1})$ is analytic.

The spectral radius of P^z is less than $k(s)$ if $t = \text{Im}z \neq 0$.

We observe that, since condition i-p is open, the last property is robust under perturbation of μ in the weak topology. If $d = 1$, $k(s)$ is the Mellin transform of μ and the above statements are also valid if T is non arithmetic. However, the last property is not robust for $d = 1$. If $s = 0$, P^s reduces to the convolution operator by μ on \mathbb{P}^{d-1} and convergence to the unique μ -stationary measure $\nu^0 = \nu$ was studied in [17] using proximality of the T -action on \mathbb{P}^{d-1} . In this case, spectral gap properties for $s = 0$ were first proved in [39] using the simplicity of the dominant μ -Lyapunov exponent (see [23], [29]). Limit theorems of Probability Theory for the product $S_n = g_n \cdots g_1$ of the random i.i.d matrices g_k , distributed according to μ , are consequences of this result and of radial Fourier Analysis on $V \setminus \{0\}$ used in combination with boundary theory (see [4] [16], [22], [29], [40]). Also, for $s > 0$, the properties described in the theorem were considered in [30] ; they are basic ingredients for the study of precise large deviations of $S_n(\omega)v$ ([41]). The Radon measure $\nu^s \otimes \ell^s$ on \dot{V} satisfies the convolution equation $\mu * (\nu^s \otimes \ell^s) = k(s)\nu^s \otimes \ell^s$ and the support of ν^s is the unique T -minimal subset of \mathbb{P}^{d-1} , the so-called limit set $\Lambda(T)$ of T . The function e^s is an integral transform of the twisted μ^* -eigenmeasure ${}^*\nu^s$. For $s > 0$ and σ a probability measure on \mathbb{P}^{d-1} not concentrated on a proper subspace, $|g|^s$ is comparable to $\int |gx|^s d\sigma(x)$; the uniqueness properties of e^s and ν^s are based on this geometrical fact. The proof of the spectral gap property depends on the simplicity of the dominant Lyapunov exponent for the product of random matrices $S_n = g_n \cdots g_1$ with respect to a natural shift-invariant Markov measure \mathbb{Q}^s on $\Omega = G^{\mathbb{N}}$, which is locally equivalent to the product measure $\mathbb{Q}^0 = \mu^{\otimes \mathbb{N}}$. As in [17] and [6] a martingale construction (here based on ${}^*\nu^s$) plays an essential role in the proof of simplicity and in the comparison of $|S_n(\omega)|$ with $|S_n(\omega)v|$ as in [29].

We observe that spectral gap properties for the family of operators $P^s (s \in \mathbb{R})$ play an important role in various problems, for example in the localisation problem for the Schrödinger operator with random potential on the line (See [4], [24]). In the paper [41], the spectral gap property for $P^s (s > 0)$ is not proved but the large deviations asymptotics

stated there for $|S_n(\omega)v|$ can be justified with the use of Theorem A. If μ has a density with compact support, the above analysis is valid for any $s \in \mathbb{R}$. In general and for $d > 1$, it turns out that the function $k(s)$, as defined above, loses its analyticity at some $s < 0$. In order to develop probabilistic consequences of Theorem A we endow $\Omega = G^{\mathbb{N}}$ with the shift-invariant measure $\mathbb{P} = \mu^{\otimes \mathbb{N}}$ (resp \mathbb{Q}^s). We know that if $\int \text{Log} \gamma(g) d\mu(g)$ (resp $\int |g|^s \text{Log} \gamma(g) d\mu(g)$) is finite the dominant Lyapunov exponent L_μ (resp $L_\mu(s)$) of $S_n = g_n \cdots g_1$ with respect to \mathbb{P} (resp \mathbb{Q}^s) exists and :

$$L_\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log} |S_n(\omega)| d\mathbb{P}(\omega), \quad L_\mu(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log} |S_n(\omega)| d\mathbb{Q}^s(\omega).$$

If $s \in]0, s_\infty[$, $k(s)$ has a derivative $k'(s)$ and $L_\mu(s) = \frac{k'(s)}{k(s)}$. By strict convexity of $\text{Log} k(s)$, if $\lim_{s \rightarrow s_\infty} k(s) \geq 1$ and $s_\infty > 0$, we can define $\alpha > 0$ by $k(\alpha) = 1$. We consider the potential kernel U on \dot{V} defined by $U(v, \cdot) = \sum_0^\infty \mu^k * \delta_v$. Then we have the following multidimensional extensions of the classical renewal theorems of Probability theory (see [15]), which describes the asymptotic homogeneity of $U(v, \cdot)$:

Theorem B

Assume T satisfies condition i-p, $\int \text{Log} \gamma(g) d\mu(g) < \infty$ and $L_\mu > 0$. If $d = 1$ assume furthermore that μ is non-arithmetic. Then, for any $v \in \mathbb{P}^{d-1}$ we have the vague convergence:

$$\lim_{t \rightarrow 0_+} U(tv, \cdot) = \frac{1}{L_\mu} \nu \otimes \ell,$$

where ν is the unique μ -stationary measure on \mathbb{P}^{d-1} .

Theorem B'

Assume T satisfies condition i-p, $\int \text{Log} \gamma(g) d\mu(g) < \infty$, $L_\mu < 0$, $s_\infty > 0$ and there exists $\alpha > 0$ with $k(\alpha) = 1$, $\int |g|^\alpha \text{Log} \gamma(g) d\mu(g) < \infty$. If $d = 1$ assume furthermore that μ is non-arithmetic. Then for any $v \in \mathbb{P}^{d-1}$ we have the vague convergence on \dot{V} :

$$\lim_{t \rightarrow 0_+} t^{-\alpha} U(tv, \cdot) = \frac{e^\alpha(v)}{L_\mu(\alpha)} \nu^\alpha \otimes \ell^\alpha.$$

Up to normalization the Radon measure $\nu^\alpha \otimes \ell^\alpha$ is the unique α -homogeneous measure which satisfies the harmonicity equation $\mu * (\nu^\alpha \otimes \ell^\alpha) = \nu^\alpha \otimes \ell^\alpha$.

Corollary

With the notations of Theorem B', for any $v \in \mathbb{P}^{d-1}$, we have the convergence :

$$\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}\{\exists n \in \mathbb{N}; |S_n v| > t\} = A e^\alpha(v) > 0.$$

The corollary is a matricial version of the famous Cramer estimate for the probability of ruin in collective risk theory (see [15]). Theorems B, B' are consequences of the arguments used in the proof of Theorem A and of a renewal theorem for a class of Markov walks on \mathbb{R} (see [36]). An essential role is played by the law of large numbers for $\text{Log} |S_n v|$ under $\mathbb{Q}^s (s = 0, \alpha)$; the comparison of $|S_n|$ and $|S_n v|$ follows from the finiteness of the limit of $(\text{Log} |S_n| - \text{Log} |S_n v|)$. For the sake of brevity, we have formulated these theorems in the

context of \dot{V} instead of V . Corresponding statements where \mathbb{P}^{d-1} is replaced by \mathbb{S}^{d-1} are given in section 4. Also the above weak convergence can be extended to a larger class of functions. In [36], renewal theorems as above were obtained for non negative matrices, the extension of these results to the general case was an open problem and a partial solution was given in [39]. Theorems B and B' extend these results to a wider setting. In view of the interpretation of $U(v, \cdot)$ as a mean number of visits, Theorem B is a strong reinforcement of the law of large numbers for $S_n(\omega)v$, hence it can be used in some problems of dynamics for group actions on T -spaces. In this respect we observe that the asymptotic homogeneity of U stated in Theorem B has been of essential use in [31] for the description of the T -minimal subsets of the action of a large subsemigroup $T \subset SL(d, \mathbb{Z})$ of automorphisms of the torus \mathbb{T}^d . Also the convergence in Theorem B' gives the convergence of the μ -Martin kernel on \dot{V} to the point $\nu^\alpha \otimes \ell^\alpha$ of the μ -Martin boundary ; if $\int |g|^\alpha \gamma^\delta(g) d\mu(g) < \infty$ for some $\delta > 0$, $\nu^\alpha \otimes \ell^\alpha$ is a minimal point. On the other hand Theorem B' gives a description of the fluctuations of a linear random walk on V with \mathbb{P} -a.e exponential convergence to zero, under condition i-p and the existence in T of a matrix with spectral radius greater than one. These fluctuation properties are responsible for the homogeneity at infinity of stationary measures for affine random walks on V , that we discuss now.

Let λ be a probability measure on the affine group H of V , μ its projection on G , Σ the closed subsemigroup of H generated by $supp\lambda$. As above we assume that the semigroup T generated by $supp\mu$ satisfies condition i-p, and $supp\lambda$ has no fixed point in V . If $d = 1$ we assume that μ is non-arithmetic. We consider the affine stochastic recursion :

$$(R) \quad X_{n+1} = A_{n+1}X_n + B_{n+1},$$

where (A_n, B_n) are λ -distributed i.i.d random variables and $X_n \in V$. From a heuristic point of view, the corresponding affine random walk can be considered as a superposition of an additive random walk on V governed by B_n and a multiplicative random walk on $V \setminus \{0\}$ governed by A_n . Here, as it appears below in Theorem C, which is reminiscent of Theorem B', the non trivial multiplicative part A_n plays a dominant role, while the additive part B_n has a stabilizing effect since ρ is finite and different from δ_0 if B_n is not zero. If $\mathbb{E}(\text{Log}|A_n|) + \mathbb{E}(\text{Log}|B_n|) < \infty$ and the dominant Lyapunov exponent L_μ for the product $A_1 \cdots A_n$ is negative, then $R_n = \sum_0^{n-1} A_1 \cdots A_k B_{k+1}$ converges $\lambda^{\otimes \mathbb{N}} - a.e.$ to R , the law ρ of R is the unique λ -stationary measure on V , (V, ρ) is a λ -boundary and $supp\rho = \Lambda_a(\Sigma)$ is the unique Σ -minimal subset of V . If T contains at least one matrix with spectral radius greater than one then $\Lambda_a(\Sigma)$ is unbounded and if $I_\mu = [0, \infty[$ there exists $\alpha > 0$ with $k(\alpha) = 1$, hence we can inquire about the "shape at infinity" of ρ . According to a conjecture of F. Spitzer the measure ρ should belong to the domain of attraction of a stable law with index α if $\alpha \in [0, 2[$ or a Gaussian law if $\alpha \geq 2$. Here we prove a multidimensional precise form of this conjecture, and more generally the α -homogeneity at infinity of ρ , where we assume that μ satisfies the conditions of Theorem B', λ satisfies moment conditions and $supp\lambda$ has no fixed point in V . The main idea is to express $\hat{\rho}$ as a μ^* -potential of a "small" implicit function on $V \setminus \{0\}$, to use the asymptotics given in Theorem B' and to complete the

argument by a study of the ladder heights for certain Birkhoff sums along the dual λ -walk on W and a Choquet-Deny lemma, for translation-bounded μ -harmonic measures.

In order to state the result we need further notations related to the unit sphere \mathbb{S}^{d-1} of V and to limit sets. We denote by $\tilde{\Lambda}(T)$ the inverse image in \mathbb{S}^{d-1} of the limit set $\Lambda(T) \subset \mathbb{P}^{d-1}$. If T leaves invariant a proper convex cone in V , then $\tilde{\Lambda}(T)$ splits into two T -minimal subsets $\Lambda_+(T)$ and $\Lambda_-(T) = -\Lambda_+(T)$; otherwise $\tilde{\Lambda}(T)$ is T -minimal. We denote by $\tilde{\nu}^\alpha$ the unique symmetric measure on \mathbb{S}^{d-1} with projection ν^α on \mathbb{P}^{d-1} . If $\tilde{\Lambda}(T)$ is not T -minimal we write $\tilde{\nu}^\alpha = \frac{1}{2}(\nu_+^\alpha + \nu_-^\alpha)$ where ν_+^α and ν_-^α are the normalized restrictions of $\tilde{\nu}^\alpha$ to $\Lambda_+(T)$ and $\Lambda_-(T)$ respectively. Then it follows that the α -homogeneous measures $\tilde{\nu}^\alpha \otimes \ell^\alpha$, $\nu_+^\alpha \otimes \ell^\alpha$ and $\nu_-^\alpha \otimes \ell^\alpha$ on $V \setminus \{0\} = \mathbb{S}^{d-1} \times \mathbb{R}_+^*$ are μ -harmonic. If $\tilde{\Lambda}(T)$ is T -minimal, the probability measure $\tilde{\nu}^\alpha$ is uniquely defined by the equation $\mu * (\tilde{\nu}^\alpha \otimes \ell^\alpha) = \tilde{\nu}^\alpha \otimes \ell^\alpha$. If not, any α -homogeneous harmonic measure is a linear combination of $\nu_+^\alpha \otimes \ell^\alpha$ and $\nu_-^\alpha \otimes \ell^\alpha$. We compactify V by adding at infinity the unit sphere \mathbb{S}_∞^{d-1} in the usual way and we denote by $\Lambda^\infty(T)$, $\Lambda_+^\infty(T)$, $\Lambda_-^\infty(T)$ the subsets of \mathbb{S}_∞^{d-1} corresponding to $\tilde{\Lambda}(T)$, $\Lambda_+(T)$, $\Lambda_-(T)$ respectively. If μ has a continuous density, positive at $e \in G$, then $T = G$, $\Lambda(T) = \mathbb{P}^{d-1}$, $\tilde{\Lambda}(T) = \mathbb{S}^{d-1}$ is T -minimal. The "shape at infinity" of ρ depends of the relative positions of the sets $\Lambda^\infty(T)$, $\Lambda_+^\infty(T)$, $\Lambda_-^\infty(T)$ with respect to the closure $\overline{\Lambda_a(\Sigma)}$ of $\Lambda_a(\Sigma)$ in $V \cup \mathbb{S}_\infty^{d-1}$ as described below.

Theorem C

With the above notations we assume that T satisfies condition i-p, $\text{supp}\lambda$ has no fixed point in V , $s_\infty > 0$, $L_\mu < 0$ and $\alpha \in]0, s_\infty[$ satisfies $k(\alpha) = 1$. If $d = 1$ we assume also μ is non-arithmetic. Then, if $\mathbb{E}(|B|^{\alpha+\delta}) < \infty$ and $\mathbb{E}(|A|^{\alpha\gamma^\delta}(A)) < \infty$ for some $\delta > 0$, the unique λ -stationary measure ρ on V satisfies the following vague convergence on $V \setminus \{0\}$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho) = C\sigma^\alpha \otimes \ell^\alpha,$$

where $C > 0$, σ^α is a probability measure on $\tilde{\Lambda}(T)$ and $\sigma^\alpha \otimes \ell^\alpha$ is a μ -harmonic Radon measure supported on $\mathbb{R}^*\tilde{\Lambda}(T)$. If T has no proper convex invariant cone in V , we have $\sigma^\alpha = \tilde{\nu}^\alpha$. In the opposite case, then $C\sigma^\alpha = C_+\nu_+^\alpha + C_-\nu_-^\alpha$, with $C_+ = 0$ (resp $C_- = 0$) if and only if $\overline{\Lambda_a(\Sigma)} \cap \Lambda_+^\infty(T) = \emptyset$ (resp $\overline{\Lambda_a(\Sigma)} \cap \Lambda_-^\infty(T) = \emptyset$). The above convergence is also valid on any Borel function f such that the set of discontinuities of f is $\sigma^\alpha \otimes \ell^\alpha$ -negligible and such that for some $\varepsilon > 0$ the function $|v|^{-\alpha}|\text{Log}|v||^{1+\varepsilon}|f(v)|$ is bounded.

For $d = 1$, the theorem says that $C_+ > 0$ if and only if Σ do not preserve any half-line of the form $] - \infty, a]$. Theorem C was stated in [27]. This statement gives the homogeneity at infinity of ρ and the measure $C\sigma^\alpha \otimes \ell^\alpha$ can be interpreted as the "tail measure" of ρ . In the context of extreme value theory for the process X_n , the convergence in the theorem implies that ρ has "multivariate regular variation" and this property plays an essential role in the theory (see [19]). If the moment condition on $|A|$ is replaced by $\mathbb{E}(|A|^\alpha \text{Log}\gamma(A)) < \infty$, then convergence remains valid on the sets H_v^+ . Since the expression of $\hat{\rho}$ as a μ^* -potential depends on signed and implicit quantities, an important point is the discussion of positivity for C, C_+ and C_- . For $d = 1$, positivity of $C = C_+ + C_-$ was proved in [20] using Levy's

symmetrisation argument, positivity of C_+ and C_- was tackled in [27] by a complex analytic method introduced in [12]. Here our main tool is a detailed analysis of a conditional λ -walk on the H -space $E = (V \setminus \{0\}) \times \mathbb{R}$ considered as a \mathbb{R}_+^* -fibered space over the space W of affine oriented hyperplanes. The space $E = (V \setminus \{0\}) \times \mathbb{R}$ has a right H -homogeneous space structure given by $h(v, r) = (g^*v, r + \langle v, b \rangle)$ where $h = (g, b) \in H = G \ltimes V$ and this formula for $h(v, r)$ defines a right linear representation of H in the vector space $V \times \mathbb{R}$ which preserves the subset $(V \setminus \{0\}) \times \mathbb{R}$; also the factor space of E by the dilation group \mathbb{R}_+^* is $\mathbb{S}^{d-1} \times \mathbb{R} = W$. Hence the corresponding linear λ -random walk on E defines a \mathbb{R}_+^* -valued cocycle of \mathbb{Z} through the H -action on W and the associated λ -walk; also the map $\eta \rightarrow \hat{\eta}$ commutes with the G -actions on $V \setminus \{0\}$ and W . Then, the idea is to study Poisson-type equations on E satisfied by $\hat{\rho}$, and to express C_+ in terms of the ladder process of $\langle R_n, u \rangle_+$ ($u \in \mathbb{S}^{d-1}$) under a conditional measure. The process $\langle R_n, u \rangle_+$ is closely related to the above \mathbb{R}_+^* -valued \mathbb{Z} -cocycle over the λ -walk on the space W and W is a 2-covering of a conditional λ -boundary. Moment estimations of the above conditioned λ -stationary measure on W allows to show, using Kac's recurrence theorem and a suitable stationary measure, that the ladder index of $\langle R_n, u \rangle_+$ has finite expectation, hence to establish positivity of C_+ if $u \in \Lambda_+(T^*)$. If λ has compact support and μ has a density on G , an analytic proof of the asymptotic behaviour of $\hat{\rho}(v)$, based on meromorphy of the Mellin transform of the projection of ρ on the half-line \mathbb{R}_+^*u is given in the Appendix. This approach gives a new expression of C_+ in terms of the corresponding residue at the pole α ; the non vanishing of this residue follows from the fact that $\langle R, u \rangle_+$ is unbounded for $u \in \Lambda_+(T^*)$ and this gives the positivity of C_+ . Under the hypothesis of Theorem C, using Theorem A, we see that the above Mellin transform is meromorphic in an open set D which contains the set $\{0 < \operatorname{Re} z \leq \alpha\}$ and has a unique simple pole at α , hence we can use Wiener-Ikehara theorem instead of the renewal theorem B' , in order to show the tail-homogeneity of ρ . For the homogeneity properties of ρ itself we observe that Theorem B' gives the convergence of $t^{-\alpha}(t.\rho)$ towards $C(\sigma^\alpha \otimes \ell^\alpha)$ on the sets H_v^+ . Then, if $\alpha \notin \mathbb{N}$, the results of [3] and [50] on Radon transforms of positive measures allow to deduce the vague convergence of $t^{-\alpha}(t.\rho)$ ($t \rightarrow 0_+$) towards $C(\sigma^\alpha \otimes \ell^\alpha)$. If $\alpha \in \mathbb{N}$, as general counterexamples show, the argument breaks down. Here we use the μ -harmonicity and boundedness properties of the cluster values of $t^{-\alpha}(t.\rho)$ ($t \rightarrow 0_+$), we apply radial Fourier Analysis on $V \setminus \{0\}$ and the spectral gap property stated in Theorem A, in order to prove a strong form of Choquet-Deny lemma for μ -harmonic measures on $V \setminus \{0\}$.

A natural question is the speed of convergence in Theorem C. For $d = 1$, see [20], if λ has a density. For $d > 1$ and under condition i-p, this question is connected with the possible uniform spectral gap for the operator P^z of Theorem A, $z = s + it$ and s fixed.

We observe that theorem C gives a natural construction for a large class of probability measures in the domain of attraction of an α -stable law. Using also spectral gap and weak dependance properties of the process X_n , Theorem C allows to prove convergence to α -stable laws for normalized Birkhoff sums along the affine λ -walk on V (see [18]). If $d > 1$ and in contrast to [9], this convergence is robust under perturbations of λ in the

weak topology. These convergences to stable laws are connected with the study of random walk in a random medium on the line or the strip (see [21]) if $\alpha < 2$. On the other hand the study of the extremal value behaviour of the process X_n can be fully developed on the basis of Theorem C and on the above weak dependance properties of X_n ; in particular the asymptotics of the extremes of X_n are given by Fréchet-type laws with index α ([32]). As observed in [44] for excursions around the cusps of the modular surface, the famous Sullivan's logarithm law is a simple consequence of Fréchet's law for the continuous fractions expansion of a real number uniformly distributed in $[0, 1]$. Here also a logarithm law is valid for X_n . The arguments developed in the proof of homogeneity at infinity for ρ can also be used in the study of some quasi-linear equations which occur in Mathematical Statistics (see [20]), Fractal analysis (see [42]) or in Statistical Mechanics (see [10], [12], [13]). For example the description of the shape at infinity of the fixed points of the multidimensional version of the "smoothing transformation" considered in [13] depends on such arguments (see [10]). In an econometrical context, the stochastic recursion (R) can be interpreted as a mechanism which, in the long run, produces debt or wealth accumulation; this mechanism "explains" the remarkable power law asymptotic shape of wealth distribution empirically discovered by the economist V. Pareto ([43]).

For information on the role of spectral gap properties in limit theorems for Probability theory and Ergodic theory we refer to [1], [9], [11], [18], [22], [28], [29], [40]. For information on products of random matrices we refer to [4], [16], [25]. Theorem A (resp B, B' and C) is proved in sections 2,3 (resp 4 and 5). Some auxiliary tools are developed in the Appendix. We thank Ch.M. Goldie, I. Melbourne and D. Petritis for useful informations on stochastic recursions.

II Ergodic properties of transfer operators on projective spaces

1) Notations, preliminary results

Let $V = \mathbb{R}^d$ be the Euclidean space endowed with the scalar product $\langle x, y \rangle = \sum_1^d x_i y_i$

and the norm $|x| = \left(\sum_1^d |x_i|^2 \right)^{1/2}$, \dot{V} the factor space of V by the finite group $\{\pm Id\}$. We

denote by \mathbb{P}^{d-1} (resp \mathbb{S}^{d-1}), the projective space (resp unit sphere) of V and by \bar{v} (resp \tilde{v}) the projection of $v \in V$ on \mathbb{P}^{d-1} (resp \mathbb{S}^{d-1}). The linear group $G = GL(V)$ acts on V, \dot{V} by $(g, v) \rightarrow gv$. If $v \in V \setminus \{0\}$, we write $g.v = \frac{gv}{|gv|}$ and we observe that G acts on \mathbb{S}^{d-1} by $(g, x) \rightarrow g.x$. We will also write the action of $g \in G$ on $x \in \mathbb{P}^{d-1}$ by $g.x$; we observe that $|gx|$ is well defined and equal to $|g\tilde{x}|$ if $x \in \mathbb{P}^{d-1}$ and $\tilde{x} \in \mathbb{S}^{d-1}$ has projection $x \in \mathbb{P}^{d-1}$. Also, if $x, y \in \mathbb{P}^{d-1}$, $|\langle x, y \rangle|$ is well defined and equal to $|\langle \tilde{x}, \tilde{y} \rangle|$ where $\tilde{x}, \tilde{y} \in \mathbb{S}^{d-1}$ have projections x, y . Corresponding notations will be taken when convenient. For a subset $A \subset \mathbb{S}^{d-1}$ the convex envelope $Co(A)$ of A is defined as the intersection with \mathbb{S}^{d-1} of the closed convex cone generated by A in V . We denote by $O(V)$ the orthogonal

group of V and by m the $O(V)$ -invariant measure on \mathbb{P}^{d-1} . A positive measure η on \mathbb{P}^{d-1} will be said to be proper if $\eta(U) = 0$ for every proper projective subspace $U \neq \mathbb{P}^{d-1}$.

Let P be a positive kernel on a Polish space E and let e be a positive function on E which satisfies $Pe = ke$ for some $k > 0$. Then we can define a Markov kernel Q_e on E by $Q_e\varphi = \frac{1}{ke}P(\varphi e)$. This procedure will be used frequently here ; in particular if $e = e^s$ depends of a parameter s we will write $Q_e = Q_s$. For a Polish G -space E we denote by $M^1(E)$ the space of probability measures on E . If $\nu \in M^1(E)$, and P is as above, ν will be said to be P -stationary if $P\nu = \nu$, i.e for any Borel function $\varphi : \nu(P\varphi) = \nu(\varphi)$. We will write $C(E)$ (resp $C_b(E)$) for the space of continuous (resp bounded continuous) functions on E . If E is a locally compact G -space, $\mu \in M^1(G)$, and ρ is a Radon measure on E , we recall that the convolution $\mu * \rho$ is defined as a Radon measure by $\mu * \rho = \int \delta_{gx}d\mu(g)d\rho(x)$, where δ_y is the Dirac measure at $y \in E$. A μ -stationary measure on E will be a probability measure $\rho \in M^1(E)$ such that $\mu * \rho = \rho$. In particular, if $E = V$ or \dot{V} and $\mu \in M^1(G)$ we will consider the Markov kernel P on V (resp \dot{P} on \dot{V}) defined by $P(v, \cdot) = \mu * \delta_v$, (resp $\dot{P}(v, \cdot) = \mu * \delta_v$). On \mathbb{P}^{d-1} (resp \mathbb{S}^{d-1}) we will write $\tilde{P}(x, \cdot) = \mu * \delta_x$ (resp $\tilde{P}(x, \cdot) = \mu * \delta_x$). If $u \in \text{End}V$, we denote $u^* \in \text{End}V$ its adjoint map, i.e $\langle u^*x, y \rangle = \langle x, uy \rangle$ if $x, y \in V$. If $\mu \in M^1(G)$ we will write μ^* for its push forward by the map $g \rightarrow g^*$ and we define the kernel $*P$ on V by $*P(v, \cdot) = \mu^* * \delta_v$. For $s \geq 0$ we denote ℓ^s (resp h^s) the s -homogeneous measure (resp function) on $\mathbb{R}_+^* = \{t \in \mathbb{R} ; t > 0\}$ given by $\ell^s(dt) = \frac{dt}{t^{s+1}}$ (resp $h^s(t) = t^s$). For $s = 0$ we write $\ell(dt) = \frac{dt}{t}$.

Using the polar decomposition $V \setminus \{0\} = \mathbb{S}^{d-1} \times \mathbb{R}_+^*$, every s -homogeneous measure η (resp function ψ) on $V \setminus \{0\}$ can be written as :

$$\eta = \pi \otimes \ell^s \quad (\text{resp } \psi = \varphi \otimes h^s)$$

where π (resp φ) is a measure (resp function) on \mathbb{S}^{d-1} . Similar decompositions are valid on $\dot{V} = \mathbb{P}^{d-1} \times \mathbb{R}_+^*$. If $g \in G$, and $\eta = \pi \otimes \ell^s$ (resp $\psi = \varphi \otimes h^s$) the directional component of $g\eta$ (resp $\psi \circ g$) is given by :

$$\rho^s(g)(\eta) = \int |gx|^s \delta_{g.x} d\eta(x) \quad (\text{resp } \rho_s(g)(\psi)(x) = |gx|^s \psi(g.x))$$

The representations ρ^s and ρ_s extend to measures on G by the formulas :

$$\rho^s(\mu)(\eta) = \int |gx|^s \delta_{g.x} d\mu(g) d\pi(x) \quad , \quad \rho_s(\mu)(\psi)(x) = \int |gx|^s \psi(g.x) d\mu(g)$$

We will write, for $\varphi \in C(\mathbb{P}^{d-1})$ (resp $\psi \in C(\mathbb{S}^{d-1})$) :

$$P^s\varphi = \rho_s(\mu)(\varphi), (\text{resp } \tilde{P}^s\psi = \rho_s(\mu)(\psi), \quad *P^s\varphi = \rho_s(\mu^*)(\varphi) \quad , \quad (\text{resp } *\tilde{P}^s\psi = \rho_s(\mu^*)(\psi)).$$

We endow \mathbb{S}^{d-1} (resp \mathbb{P}^{d-1}) with the distance $\tilde{\delta}$ (resp δ) defined by $\tilde{\delta}(x, y) = |x - y|$ (resp $\delta(\bar{x}, \bar{y}) = \text{Inf}\{|x - y|; |x| = |y| = 1\}$).

For $\varepsilon > 0$, $\varphi \in C(\mathbb{P}^{d-1})$ (resp $\psi \in C(\mathbb{S}^{d-1})$), we denote :

$$[\varphi]_\varepsilon = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\delta^\varepsilon(x, y)} \quad (\text{resp } [\psi]_\varepsilon = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{\tilde{\delta}^\varepsilon(x, y)}),$$

$$|\varphi| = \sup\{|\varphi(x)|; x \in \mathbb{P}^{d-1}\} \quad (\text{resp } |\psi| = \sup\{|\psi(x)|; x \in \mathbb{S}^{d-1}\}),$$

and we write $H_\varepsilon(\mathbb{P}^{d-1}) = \{\varphi \in C(\mathbb{P}^{d-1}); [\varphi]_\varepsilon < \infty\}$,

(resp $H_\varepsilon(\mathbb{S}^{d-1}) = \{\psi \in C(\mathbb{S}^{d-1}); [\psi]_\varepsilon < \infty\}$).

The set of positive integers will be denoted by \mathbb{N} .

Definition 2.1

If $s \in [0, \infty[$ we denote :

$$k(s) = k_\mu(s) = \lim_{n \rightarrow \infty} \left(\int |g|^s d\mu^n(g) \right)^{1/n}, \quad I_\mu = \{s \geq 0; k_\mu(s) < +\infty\}.$$

We observe that the above limit exists, since by subadditivity of $g \rightarrow \text{Log}|g|$, the quantity $u_n(s) = \int |g|^s d\mu^n(g)$ satisfies $u_{m+n}(s) \leq u_m(s)u_n(s)$. Also $k_\mu(s) = \text{Inf}_{n \in \mathbb{N}} (u_n(s))^{1/n}$, which

implies $I_\mu = \{s \geq 0; \int |g|^s d\mu(g) < +\infty\}$. Furthermore, Hölder inequality implies that I_μ is an interval of the form $[0, s_\infty[$ or $[0, s_\infty]$, and $\text{Log}k_\mu(s)$ is convex on I_μ . Also $k_{\mu^*} = k_\mu$ since $|g| = |g^*|$. If μ and μ' commute and $c \in [0, 1]$, $\mu'' = \mu + (1-c)\mu'$ then $k_{\mu''}(s) = ck_\mu(s) + (1-c)k_{\mu'}(s)$, if $s \in I_\mu \cap I_{\mu'}$.

Definition 2.2

- a) An element $g \in \text{End}V$ is said to be proximal if g has a unique and simple eigenvalue $\lambda_g \in \mathbb{R}$ such that $|\lambda_g| = \lim_{n \rightarrow \infty} |g^n|^{1/n}$.
- b) A semigroup $T \subset G$ is said to be strongly irreducible if no finite union of proper subspaces is T -invariant.

Proximality of g means that we can write $V = \mathbb{R}v_g \oplus V_g^<$ with $gv_g = \lambda_g v_g$, $gV_g^< \subset V_g^<$ and the restriction of g to $V_g^<$ has spectral radius less than $|\lambda_g|$. In this case $\lim_{n \rightarrow +\infty} g^n \cdot \bar{x} = \bar{v}_g$ if $x \notin V_g^<$ and we say that λ_g is the dominant eigenvalue of g . If $E \subset G$ we denote by E^{prox} the set of proximal elements of E . The closed subsemigroup (resp group) generated by E will be denoted $[E]$ (resp $\langle E \rangle$). In particular we will consider below the case $E = \text{supp}\mu$ where $\text{supp}\mu$ is the support of $\mu \in M^1(G)$.

Definition 2.3

A semigroup $T \subset G$ is said to satisfy condition i-p if T is strongly irreducible and $T^{\text{prox}} \neq \phi$.

As shown in ([23], [45]) this property is satisfied if it is satisfied by $Zc(T)$, the Zariski closure of T . It can be proved that condition i-p is valid if and only if the connected component of the closed subgroup $Zc(T)$ is locally the product of a similarity group and a semi-simple real Lie group without compact factor which acts proximally and irreducibly on \mathbb{P}^{d-1} . In this sense T is "large". For example, if T is a countable subgroup of G which satisfies condition i-p then T contains a free subgroup with two generators.

We recall that in \mathbb{C}^n , the Zariski closure of $E \subset \mathbb{C}^n$ is the set of zeros of the set of polynomials which vanish on E . The group $G = GL(V)$ can be considered as a Zariski-closed subset of \mathbb{R}^{d^2+1} . If T is a semigroup, then $Zc(T)$ is a closed subgroup of G with a finite number of connected components. If $d = 1$, condition i-p is always satisfied. Hence, when using condition i-p, $d > 1$ will be understood.

Remark

The above definitions will be used below in the analysis of laws of large numbers and renewal theorems. A corresponding analysis has been developed in [36] for the case of

non-negative matrices. We observe that proximality of an element in G as defined above is closely related to the Perron-Frobenius property for a positive matrice. If T is not irreducible, we can consider the subspace $V^+(T)$ generated by the dominant eigenvectors of the elements of T ; then $V^+(T)$ is T -invariant (see [22] p 120) and, if the restriction T^+ of T to $V^+(T)$ is irreducible, then T^+ satisfies condition i-p.

Definition 2.4

Assume T is a subsemigroup of G which satisfies condition i-p. Then the closure of the set $\{\bar{v}_g; g \in T^{prox}\}$ will be called the limit set of T and will be denoted $\Lambda(T)$.

With these definitions we have the

Proposition 2.5

Assume $T \subset G$ is a subsemigroup which satisfies i-p and $S \subset T$ generates T . Then $T\Lambda(T) = \Lambda(T)$ and $\Lambda(T)$ is the unique T -minimal subset of \mathbb{P}^{d-1} . If $\mu \in M^1(G)$ is such that $T = [supp\mu]$ satisfies i-p, there exists a unique μ -stationary measure ν on \mathbb{P}^{d-1} . Also $supp\nu = \Lambda(T)$ and ν is proper. Furthermore, if $d > 1$, the subgroup of \mathbb{R}_+^* generated by the set $\{|\lambda_g|; g \in T^{prox}\}$ is dense in \mathbb{R}_+^* . In particular, if $\varphi \in C(\Lambda(T))$ satisfies for some $t \in \mathbb{R}$, $|e^{it}| = 1 : \varphi(g.x) |gx|^{it} = e^{it}\varphi(x)$ for any $g \in S$, $x \in \Lambda(T)$ then $t = 0$, $e^{it} = 1$, $\varphi =$ constant.

Remark

The first part of the above statement is essentially due to H. Furstenberg ([17], Propositions 4.8, 7.4) ; see also ([22] p 120). The second part is proved in ([29], Proposition 3). For another proof and extensions of this property see [26]. This property plays an essential role in the renewal theorems of section 4 as well as in section 5 for $d > 1$. In the context of non negative matrices a modified form is also valid (see [10]) ; in ([36], Theorem A) under weaker conditions on T , its conclusion is assumed as an hypothesis. If $d = 1$, we will need to assume it, i.e we will assume that T is non-arithmetic ; if $T = [supp\mu]$ satisfies this condition, we say that μ is non-arithmetic.

2) Uniqueness of eigenfunctions and eigenmeasures on \mathbb{P}^{d-1}

Here we consider $s \in I_\mu$ and the operator P^s (resp ${}^*P^s$) on \mathbb{P}^{d-1} defined by :

$$P^s\varphi(x) = \int |gx|^s\varphi(g.x)d\mu(g) \quad (\text{resp } {}^*P^s\varphi(x) = \int |gx|^s\varphi(g.x)d\mu^*(g)).$$

For $z = s + it \in \mathbb{C}$ we will also write $P^z\varphi(x) = \int |gx|^z\varphi(g.x)d\mu(g)$

Below we study existence and uniqueness for eigenfunctions or eigenmeasures of P^s . We show equicontinuity properties of the normalized iterates of P^s and \tilde{P}^s .

Theorem 2.6

Assume $\mu \in M^1(G)$ is such that the semigroup $[supp\mu]$ satisfies (i-p) and let $s \in I_\mu$. If $d = 1$ we assume that μ is non arithmetic. Then the equation $P^s\varphi = k(s)\varphi$ has a unique continuous solution $\varphi = e^s$, up to normalization. The function e^s is positive and \bar{s} -Hölder with $\bar{s} = Inf(1, s)$.

Furthermore there exists a unique $\nu^s \in M^1(\mathbb{P}^{d-1})$ such that $P^s \nu^s$ is proportional to ν^s . One has $P^s \nu^s = k(s) \nu^s$ and $\text{supp} \nu^s = \Lambda([\text{supp} \mu])$. If ${}^* \nu^s \in M^1(\mathbb{P}^{d-1})$ satisfies ${}^* P^s ({}^* \nu^s) = k(s) {}^* \nu^s$, and e^s is normalized by $\nu^s(e^s) = 1$, one has $p(s) e^s(x) = \int |\langle x, y \rangle|^s d{}^* \nu^s(y)$ where $p(s) = \int |\langle x, y \rangle|^s d\nu^s(x) d{}^* \nu^s(y)$.

The map $s \rightarrow \nu^s$ (resp $s \rightarrow e^s$) is continuous in the weak topology (resp uniform topology) and the function $s \rightarrow \text{Log} k(s)$ is strictly convex.

The Markov operator Q^s on \mathbb{P}^{d-1} defined by $Q^s \varphi = \frac{1}{k(s) e^s} P^s(\varphi e^s)$ has a unique stationary measure π^s given by $\pi^s = e^s \nu^s$ and we have for any $\varphi \in C(\mathbb{P}^{d-1})$ the uniform convergence of $(Q^s)^n \varphi$ towards $\pi^s(\varphi)$. If Q^z is defined by $Q^z \varphi = \frac{1}{k(s) e^s} P^z(e^s \varphi)$, the equation $Q^z \varphi = e^{i\theta} \varphi$ with $\varphi \in C(\mathbb{P}^{d-1})$ $\varphi \neq 0$ implies $e^{i\theta} = 1$, $t = 0$, $\varphi = \text{constant}$.

Remarks

- a) If $s = 0$, then $e^s = 1$, and $\nu^s = \nu$ is the unique μ -stationary measure [17]. The fact that ν is proper is of essential use in [40], [4] and [29], for the study of limit theorems.
- b) In section 3 we will also, as in [17], construct a suitable measure-valued martingale which allows to prove that ν^s is proper (see Theorem 3.2) if $s \in I_\mu$. We note that analyticity of $k(s)$ is proved in Corollary 3.20 below. Continuity of the derivative of k will be essential in sections 3,4 and is proved in Theorem 3.10.

The proof of the theorem depends of a proposition and the following lemmas improving corresponding results for positive matrices in [36].

Lemma 2.7

Assume $\sigma \in M^1(\mathbb{P}^{d-1})$ is not supported by a hyperplane. Then, there exists a constant $c_s(\sigma) > 0$ such that, for any u in $\text{End} V$:

$$\int |ux|^s d\sigma(x) \geq c_s(\sigma) |u|^s$$

Proof

Clearly it suffices to show the above inequality if $|u| = 1$. The fonction $u \rightarrow \int |ux|^s d\sigma(x)$ is continuous on $\text{End} V$, hence its attains its infimum $c_s(\sigma)$ on the compact subset of $\text{End} V$ defined by $|u| = 1$. If $c_s(\sigma) = 0$, then for some $u \in \text{End} V$ with $|u| = 1$, we have $\int |ux|^s d\sigma(x) = 0$ hence, $ux = 0$, $\sigma - a.e.$ In other words, $\text{supp} \sigma \subset \text{Ker}(u)$, which contradicts the hypothesis on σ . Hence $c_s(\sigma) > 0$. \square

Lemma 2.8

If $s \in I_\mu$ there exists $\sigma \in M^1(\mathbb{P}^{d-1})$ such that $P^s \sigma = k \sigma$ for some $k > 0$. For any such σ , we have $k = k(s)$ and σ is not supported on a hyperplane.

Furthermore for every $n \in \mathbb{N}$:

$$\int |g|^s d\mu^n(g) \geq k^n(s) \geq c_s(\sigma) \int |g|^s d\mu^n(g)$$

Proof

We consider the non-linear operator \widehat{P}^s on $M^1(\mathbb{P}^{d-1})$ defined by $\widehat{P}^s\sigma = \frac{P^s\sigma}{(P^s\sigma)(1)}$.

Since $\int |g|^s d\mu(g) < +\infty$, this operator is continuous in the weak topology. Since $M^1(\mathbb{P}^{d-1})$ is compact and convex, Schauder-Tychonov theorem implies the existence of $k > 0$ and $\sigma \in M^1(\mathbb{P}^{d-1})$ with $P^s\sigma = k\sigma$, hence $k = (P^s\sigma)(1)$.

For such a σ , the equation

$$k\sigma(\varphi) = \int \varphi(g.x)|gx|^s d\mu(g)d\sigma(x)$$

implies that if $x \in \text{supp}\sigma$, then $g.x \in \text{supp}\sigma$ $\mu - a.e.$

Then for any $g \in \text{supp}\mu : g.\text{supp}\sigma \subset \text{supp}\sigma$. In particular the projective subspace H generated by $\text{supp}\sigma$ satisfies $[\text{supp}\mu].H = H$. Since $[\text{supp}\mu]$ satisfies i-p, we have $H = \mathbb{P}^{d-1}$.

Then Lemma 2.7 gives :

$$\int |gx|^s d\sigma(x) \geq c_s(\sigma)|g|^s.$$

The relation $(P^s)^n\sigma = k^n\sigma$ implies $k^n = \int |gx|^s d\mu^n(g)d\sigma(x)$, hence using Lemma 2.7 : $c_s(\sigma) \int |g|^s d\mu^n(g) \leq k^n \leq \int |g|^s d\mu^n(g)$. It follows $k = \lim_{n \rightarrow +\infty} \left(\int |g|^s d\mu^n(g) \right)^{1/n} = k(s)$. \square

Assume $e \in C(\mathbb{P}^{d-1})$ is positive and satisfies $P^s e = k(s)e$. Then we can define the Markov kernel Q_e^s by $Q_e^s\varphi(x) = \frac{1}{k(s)} \int \varphi(g.x) \frac{e(g.x)}{e(x)} |gx|^s d\mu(g)$. In view of the cocycle property of $\theta_e^s(x, g) = |gx|^s \frac{e(g.x)}{e(x)}$ we can calculate the iterate $(Q_e^s)^n$ by the formula :

$$(Q_e^s)^n\varphi(x) = \int \varphi(g.x) q_{e,n}^s(x, g) d\mu^n(g)$$

with $q_{e,n}^s(x, g) = \frac{1}{k^n(s)} \frac{e(g.x)}{e(x)} |gx|^s$, and $\int q_{e,n}^s(x, g) d\mu^n(g) = 1$.

Lemma 2.9

Assume e is as above, $f \in C(\mathbb{P}^{d-1})$ is real valued and satisfies $Q_e^s f \leq f$. Then, on $\Lambda([\text{supp}\mu])$, f is constant and equal to its infimum on \mathbb{P}^{d-1} .

Proof

Let $M^- = \{x \in \mathbb{P}^{d-1} ; f(x) = \text{Inf}\{f(y); y \in \mathbb{P}^{d-1}\}$. The relation $f(x) \geq \int q_e^s(x, g) f(g.x) d\mu(g)$ implies that if $x \in M^-$ then $g.x \in M^-$, $\mu - a.e.$ Hence $[\text{supp}\mu].M^- \subset M^-$. Since $\Lambda([\text{supp}\mu])$ is the unique $[\text{supp}\mu]$ -minimal subset of \mathbb{P}^{d-1} , we get $\Lambda([\text{supp}\mu]) \subset M^-$, i.e :

$$f(x) = \text{Inf}\{f(y); y \in \mathbb{P}^{d-1}\}, \text{ if } x \in \Lambda([\text{supp}\mu]) \quad \square$$

The following shows the existence of $e \in C(\mathbb{P}^{d-1})$ with $P^s e = k(s)e$, using Lemma 2.8 applied to μ^* .

Lemma 2.10

Assume $\sigma \in M^1(\mathbb{P}^{d-1})$ and $k > 0$ satisfy $*P^s\sigma = k\sigma$. Then the function $\widehat{\sigma}^s$ on \mathbb{P}^{d-1} defined by

$$\widehat{\sigma}^s(x) = \int |\langle x, y \rangle|^s d\sigma(y)$$

satisfies $P^s \widehat{\sigma}^s = k \widehat{\sigma}^s$. Furthermore $\widehat{\sigma}^s$ is positive and Hölder of order $\bar{s} = \text{Inf}(1, s)$.

Proof

We have $|gx|^s \widehat{\sigma}^s(g.x) = \int |\langle x, g^*.y \rangle|^s |g^*.y|^s d\sigma(y)$ and $*P^s \sigma = k\sigma$, hence :

$$P^s \widehat{\sigma}^s(x) = k \int |\langle x, z \rangle|^s d\sigma(z) = k \widehat{\sigma}^s(x).$$

If $\widehat{\sigma}^s(x) = 0$ for some x , then $|\langle x, y \rangle| = 0$ $\sigma - a.e$, hence :

$$\text{supp}\sigma \subset \{y \in \mathbb{P}^{d-1}; \langle x, y \rangle = 0\}.$$

This contradicts Lemma 2.8, since $[\text{supp}\mu]$ satisfies i-p. Hence $\widehat{\sigma}^s$ is positive.

In order to show the Hölder property of $\widehat{\sigma}^s$, we use the inequality $|a^s - b^s| \leq \widehat{s}|a - b|^{\bar{s}}$ where $a, b \in [0, 1]$ $\widehat{s} = \text{sup}(s, 1)$, $\bar{s} = \text{Inf}(s, 1)$. Then

$$|\langle x, y \rangle|^s - |\langle x', y \rangle|^s \leq \widehat{s}|x - x'|^{\bar{s}}|y|^{\bar{s}}, |\widehat{\sigma}^s(x) - \widehat{\sigma}^s(x')| \leq \widehat{s}\delta^{\bar{s}}(x, x'). \quad \square$$

Lemma 2.11

Let e be a positive and \bar{s} -Hölder function on \mathbb{P}^{d-1} with $\bar{s} = \text{Inf}(s, 1)$, $s > 0$. Then for any $(x, y) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$, $g \in G$ and a constant $b_s > 0$:

$$||gx|^s - |gy|^s| \leq (s+1)|g|^s \widetilde{\delta}^{\bar{s}}(x, y), \widetilde{\delta}(g.x, g.y) \leq 2 \frac{|g|}{|gx|} \widetilde{\delta}(x, y), |\theta_e^s(x, g) - \theta_e^s(y, g)| \leq b_s |g|^s \widetilde{\delta}^{\bar{s}}(x, y).$$

Proof

We use the inequality $|a^s - b^s| \leq |a - b|^s$ if $a, b \geq 0$, $s \leq 1$.

We get : $||gx|^s - |gy|^s| \leq ||gx| - |gy||^s \leq |g(x - y)|^s \leq |g|^s |x - y|^s$.

Hence $||gx|^s - |gy|^s| \leq |g|^s \widetilde{\delta}^{\bar{s}}(x, y)$.

If $s > 1$, we use $\frac{1}{s}|a^s - b^s| \leq \text{sup}(a, b)^{s-1}|a - b|$ if $a, b \geq 0$. We get :

$$||gx|^s - |gy|^s| \leq s|g|^{s-1}||gx| - |gy|| \leq s|g|^{s-1}|g(x - y)| \leq s|g|^s |x - y|.$$

Hence we get the first inequality. Furthermore :

$$\widetilde{\delta}(g.x, g.y) = \left| \frac{gx}{|gx|} - \frac{gy}{|gy|} \right| \leq \frac{|g(x - y)|}{|gx|} + |gy| \left| \frac{1}{|gx|} - \frac{1}{|gy|} \right| \leq 2 \frac{|g||x - y|}{|gx|}$$

Hence $\widetilde{\delta}(g.x, g.y) \leq 2 \frac{|g|}{|gx|} \widetilde{\delta}(x, y)$.

We write :

$$|\theta_e^s(x, g) - \theta_e^s(y, g)| \leq ||gx|^s - |gy|^s| \frac{e(g.x)}{e(x)} + |gy|^s \frac{|e(g.x) - e(g.y)|}{e(x)} + |gy|^s e(g.y) \left| \frac{1}{e(x)} - \frac{1}{e(y)} \right|$$

In view of the first inequality the first term satisfies the required bound. The last term also satisfies it, since $\frac{1}{e(x)}$ is \bar{s} -Hölder and $|gy|^s \leq |g|^s$. For the second term we write :

$$|e(g.x) - e(g.y)| \leq [e]_{\bar{s}} \widetilde{\delta}^{\bar{s}}(g.x, g.y) \leq 2^{\bar{s}} [e]_{\bar{s}} \left(\frac{|g|}{|gy|} \right)^{\bar{s}} \widetilde{\delta}^{\bar{s}}(x, y).$$

Hence this term is bounded by :

$$2^{\bar{s}} [e]_{\bar{s}} \widetilde{\delta}^{\bar{s}}(x, y) |gy|^{s-\bar{s}} |g|^{\bar{s}} \leq 2^{\bar{s}} [e]_{\bar{s}} |g|^s \widetilde{\delta}^{\bar{s}}(x, y).$$

The inequalities imply the lemma. \square

The following is well known (see for example [47] Theorem 6)

Lemma 2.12

Let X be a compact metric space, Q a Markov operator on X , which preserves $C(X)$. Assume that all the Q -invariant continuous functions are constant and for any $\varphi \in C(X)$, the sequence $Q^n \varphi$ is equicontinuous. Then Q has a unique stationary measure. Furthermore if the equation $Q\psi = e^{i\theta}\psi$, $\psi \in C(X)$ implies $e^{i\theta} = 1$, then for any $\varphi \in C(X)$ $Q^n \varphi$ converges.

Proposition 2.13

Let $\mu \in M^1(G)$ and assume that the semigroup $[supp\mu]$ satisfies i-p.

Let $s \in I_\mu$, $s > 0$ $\varepsilon \in]0, \bar{s}]$ with $\bar{s} = \text{Inf}(1, s)$, $e \in C(\mathbb{P}^{d-1})$ is positive, \bar{s} -Hölder with $P^s e = k(s)e$.

Then there exists $a_s \geq 0$ such that for any $n \in \mathbb{N}$ and any ε -Hölder function φ on \mathbb{P}^{d-1} :

$$[(Q_e^s)^n \varphi]_\varepsilon \leq a_s |\varphi| + \rho_{n,s}(\varepsilon) [\varphi]_\varepsilon$$

where $\rho_{n,s}(\varepsilon) = \text{Sup}_{x,y} \int q_{e,n}^s(x,g) \frac{\delta^\varepsilon(g.x, g.y)}{\delta^\varepsilon(x,y)} d\mu^n(g)$ is bounded independantly of n .

In particular for any $\psi \in C(\mathbb{P}^{d-1})$ the sequence $(Q_e^s)^n \psi$ is equicontinuous.

Furthermore any continuous Q_e^s -invariant function is constant and Q_e^s has a unique stationary measure π_e^s . If $s = 0$ and $\varphi \in C(\mathbb{P}^{d-1})$ we have the uniform convergence : $\lim_{n \rightarrow \infty} \bar{P}^n \varphi = \nu(\varphi)$.

Proof

The definition of Q_e^s gives for any ε -Hölder function φ :

$$|(Q_e^s)^n \varphi(x) - (Q_e^s)^n \varphi(y)| \leq |\varphi| \int |q_{e,n}^s(x,g) - q_{e,n}^s(y,g)| d\mu^n(g) + [\varphi]_\varepsilon \int q_{e,n}^s(y,g) \delta^\varepsilon(g.x, g.y) d\mu^n(g).$$

Lemma 2.11 shows that the first integral is dominated by $\frac{b_s}{k^n(s)} \delta^{\bar{s}}(x,y) \int |g|^s d\mu^n(g)$ and Lemma 2.8 gives $k^n(s) \geq c_s \int |g|^s d\mu^n(g)$. Hence : $[(Q_e^s)^n \varphi]_\varepsilon \leq a_s |\varphi| + \rho_{n,s}(\varepsilon) [\varphi]_\varepsilon$ with $a_s = b_s/c_s$.

Lemma 2.4 allows to bound $\rho_{n,s}(\varepsilon)$:

$$\delta^\varepsilon(g.x, g.y) \leq 2^\varepsilon \frac{|g|^\varepsilon}{|gx|^\varepsilon} \delta^\varepsilon(x,y), \quad \rho_{n,s}(\varepsilon) \leq \frac{2^\varepsilon}{k^n(s)} \sup_x \int \frac{e(g.x)}{e(x)} |gx|^{s-\varepsilon} |g|^\varepsilon d\mu^n(g).$$

We denote $c = \sup_{g,x} \frac{e(g.x)}{e(x)} < \infty$ hence using $s \geq \varepsilon$, $|gx|^{s-\varepsilon} \leq |g|^{s-\varepsilon}$ we get :

$$\frac{e(g.x)}{e(x)} |gx|^{s-\varepsilon} |g|^\varepsilon \leq c |g|^s, \quad \rho_{n,s}(\varepsilon) \leq \frac{2^\varepsilon}{k^n(s)} \int |g|^s d\mu^n(g) \leq c \frac{2^\varepsilon}{c_s}.$$

Assume that $\varphi \in C(\mathbb{P}^{d-1})$ satisfies $Q_e^s \varphi = \varphi$ and denote :

$$M^+ = \{x \in \mathbb{P}^{d-1}; \varphi(x) = \sup_{y \in \mathbb{P}^{d-1}} \varphi(y)\}, \quad M^- = \{x \in \mathbb{P}^{d-1}; \varphi(x) = \text{Inf}_{y \in \mathbb{P}^{d-1}} \varphi(y)\}.$$

Then, as in the proof of Lemma 2.9, $\text{supp}\mu.M^+ \subset M^+$, $\text{supp}\mu.M^- \subset M^-$, hence by minimality of $\Lambda([\text{supp}\mu])$ we have $\Lambda([\text{supp}\mu]) \subset M^+ \cap M^-$.

It follows $M^+ \cap M^- \neq \emptyset$, $\varphi = \text{cte}$.

If $\psi \in C(\mathbb{P}^{d-1})$ is ε -Hölder the above inequality gives for any $x, y \in \mathbb{P}^{d-1}$:

$$|(Q_e^s)^n \varphi(x) - (Q_e^s)^n \varphi(y)| \leq (a_s |\varphi| + \rho_{n,s}(\varepsilon)) [\varphi]_\varepsilon \delta^\varepsilon(x, y).$$

Since $\rho_{n,s}(\varepsilon)$ is bounded this shows that the sequence $(Q_e^s)^n \psi$ is equicontinuous. By density this remains valid for any $\psi \in C(\mathbb{P}^{d-1})$.

Hence we can apply Lemma 2.12 to $Q = Q_e^s$: there is a unique Q_e^s -stationary measure. If $s = 0$, we have $\varepsilon = 0$, hence the above inequality do not show the equicontinuity of $\overline{P}^n \varphi$.

In this case the equicontinuity follows from Theorem 3.2 in the next section : we have for

$\varepsilon > 0 \lim_{n \rightarrow \infty} \int \delta^\varepsilon(g.x, g.y) d\mu^n(g) = 0$, which implies for $\varphi \in H_\varepsilon(\mathbb{P}^{d-1})$:

$$\begin{aligned} & (|\overline{P}^n \varphi(x) - \overline{P}^n \varphi(y)| \leq [\varphi]_\varepsilon \int \delta^\varepsilon(g.x, g.y) d\mu^n(g). \\ |\overline{P}^n \varphi(x) - \nu(\varphi)| & \leq [\varphi]_\varepsilon \int \delta^\varepsilon(g.x, g.y) d\nu(y) d\mu^n(g), \lim_{n \rightarrow \infty} |\overline{P}^n \varphi(x) - \nu(\varphi)| = 0. \quad \square \end{aligned}$$

Remarks

a) If for some $\delta > 0 \int |g|^\delta \gamma^\delta(g) d\mu(g) < \infty$ then for any $s \in [0, s_\infty[\lim_{n \rightarrow \infty} \rho_{n,s}(\varepsilon) = 0$, hence $\rho_{n,s}(\varepsilon) < 1$ for some $n = n_0$. Hence $(Q_e^s)^{n_0}$ satisfies a so called Doeblin-Fortet inequality (see [27]).

b) Let \tilde{Q}_e^s be the Markov kernel on \mathbb{S}^{d-1} defined by $\tilde{Q}_e^s \varphi = \frac{1}{k(s)e} \tilde{P}^s(\varphi e)$ where e still denotes the function on \mathbb{S}^{d-1} corresponding to $e \in C(\mathbb{P}^{d-1})$. Then the inequality and its proof remain valid for \tilde{Q}_e^s instead of Q_e^s . In particular for any $\psi \in C(\mathbb{S}^{d-1})$ the sequence $(\tilde{Q}_e^s)^n \psi$ is equicontinuous. This fact will be used in the next paragraph

Proof of the theorem

As in the proof of Lemma 2.8, we consider the non linear operator $*\hat{P}^s$ on $M^1(\mathbb{P}^{d-1})$ defined by $*\hat{P}^s \sigma = \frac{*P^s \sigma}{(*P^s \sigma)(1)}$. The same argument gives the existence of k and $\sigma \in M^1(\mathbb{P}^{d-1})$ such that $*P^s \sigma = k\sigma$, with $k = (*P^s \sigma)(1) > 0$. We consider only the case $d > 1$.

Since $[\text{supp}\mu^*] = [\text{supp}\mu]^*$ satisfies i-p, Lemma 2.8 applied to μ^* gives $k = k(s)$ and σ is not supported by a hyperplane. Then Lemma 2.10 implies that $\hat{\sigma}^s(x) = \int | \langle x, y \rangle |^s d\sigma(y)$ satisfies $P^s \hat{\sigma}^s = k(s) \hat{\sigma}^s$ and is positive, Hölder continuous of order $\bar{s} = \text{Inf}(1, s)$. Hence we can apply Proposition 2.13 with $e = \hat{\sigma}^s$; then we get existence and uniqueness of e^s with $P^s e^s = k(s) e^s$, $\nu^s(e^s) = 1$ and e^s satisfies $p(s) e^s(x) = \int | \langle x, y \rangle |^s d * \sigma(y)$. Also $Q^s = Q_e^s$ has a unique stationary measure π^s . The uniqueness of $\nu^s \in M^1(\mathbb{P}^{d-1})$ with $P^s \nu^s = k(s) \nu^s$ follows. Also $\sigma = * \nu^s$ by the same proof.

Lemma 2.8 implies that if some $\eta \in M^1(\mathbb{P}^{d-1})$ satisfies $P^s \eta = k\eta$, then $k = k(s)$. Since $\text{supp}\nu^s$ is $[\text{supp}\mu]$ -invariant and $\Lambda([\text{supp}\mu])$ is minimal we get $\text{supp}\nu^s \supset \Lambda([\text{supp}\mu])$. We can again use Schauder-Tychonoff theorem in order to construct $\sigma' \in M^1(\mathbb{P}^{d-1})$ with $\text{supp}\sigma' \subset \Lambda([\text{supp}\mu])$, $P^s \sigma' = k\sigma'$. Since $\sigma' = \nu^s$, we get finally $\text{supp}\nu^s = \Lambda([\text{supp}\mu])$.

In order to show the continuity of $s \rightarrow \nu^s$, $s \rightarrow e^s$ we observe that, from above, ν^s is uniquely defined by : $P^s \nu^s = k(s) \nu^s$, $\nu^s \in M^1(\mathbb{P}^{d-1})$. Also, by convexity, $k(s)$ is continuous. On the other hand, the uniform continuity of $(x, s) \rightarrow |gx|^s$ and the fact that $|gx|^s \leq |g|^s$ is bounded by the μ -integrable function $\sup(|g|^{s_1}, |g|^{s_2})$ on $[s_1, s_2] \subset I_\mu$ implies the uniform continuity of $P^s \varphi$ if φ is fixed. Then we consider a sequence $s_n \in I_\mu$, $s_0 \in I_\mu$ with $\lim_{s_n \rightarrow s_0} \nu^{s_n} = \eta \in M^1(\mathbb{P}^{d-1})$. We have

$$P^{s_n} \nu^{s_n}(\varphi) = \nu^{s_n}(P^{s_n} \varphi), \quad \lim_{s_n \rightarrow s_0} P^{s_n} \nu^{s_n}(\varphi) = \lim_{s_n \rightarrow s_0} k(s_n) \nu^{s_n}(\varphi) = k(s_0) \eta(\varphi).$$

Then the uniform continuity in (s, x) of $P^s \varphi(x)$ implies $P^{s_0} \eta = k(s_0) \eta$. The uniqueness of ν^{s_0} implies $\nu^{s_0} = \eta$, and the arbitrariness of s_n gives the continuity of $s \rightarrow \nu^s$ at s_0 . The same property is true for the operator $*P^s$ and the measure $*\nu^s$ defined by $*P^s (*\nu^s) = k(s) (*\nu^s)$, $*\nu^s \in M^1(\mathbb{P}^{d-1})$.

Lemma 2.10, implies $p(s)e^s(x) = \int | \langle x, y \rangle |^s d*\nu^s(y)$, and since the set of functions $x \rightarrow | \langle x, y \rangle |^s$ ($y \in \mathbb{P}^{d-1}$, $s \in I_\mu$) is locally equicontinuous : $\lim_{s \rightarrow s_0} |e^s - e^{s_0}| = 0$.

In order to show the strict convexity of $\text{Log}k(s)$ we take $s, t \in I_\mu$, $p \in [0, 1]$ and we observe that from Hölder inequality, $P^{ps+(1-p)t}[(e^s)^p(e^t)^{1-p}] \leq k^p(s)k^{1-p}(t)(e^s)^p(e^t)^{1-p}$.

We denote $f = (e^s)^p(e^t)^{1-p}$ and assume $k(ps+(1-p)t) = k^p(s)k^{1-p}(t)$ for some $s \neq t$.

Then Lemma 2.9 can be used with $e = e^{ps+(1-p)t}$ and $Q_e^{pq+(1-p)t} \varphi = \frac{1}{k(ps+(1-p)t)} P^{ps+(1-p)t}(\varphi e)$.

It gives on $\Lambda([supp\mu])$: $f = ce^{ps+(1-p)t}$ for some constant $c > 0$.

Hence, on $\Lambda([supp\mu])$ we have :

$$P^{ps+(1-p)t}[(e^s)^p(e^t)^{1-p}] = k^p(s)k^{1-p}(t)(e^s)^p(e^t)^{1-p}.$$

This means that there is equality in the above Hölder inequality. It follows that, for some positive function $c(x)$ and any x in $\Lambda([supp\mu])$, $g \in supp\mu$:

$$|gx|^s \frac{e^s(g.x)}{e^s(x)} = c(x) |gx|^t \frac{e^t(g.x)}{e^t(x)}.$$

Integration with respect to μ gives : $c(x) = \frac{k(s)}{k(t)}$. Since $s \neq t$, we get, for some constant $c > 0$ and $\varphi \in C(\mathbb{P}^{d-1})$ positive, for any (x, g) as above : $|gx| = c \frac{\varphi(g.x)}{\varphi(x)}$. It follows, if $g \in (supp\mu)^n$ and $x \in \Lambda([supp\mu])$, $|gx| = c^n \frac{\varphi(g.x)}{\varphi(x)}$. If $g \in [supp\mu]^{prox}$, we get $|\lambda_g| \in c^{\mathbb{N}}$. This contradicts Proposition 2.5.

In order to show the convergence of $(Q^s)^n \varphi$, since by Proposition 2.13 the family $(Q^s)^n \varphi$ is equicontinuous, it suffices to show in view of Lemma 2.12 that the relation $Q^s \varphi = e^{i\theta} \varphi$ with $\varphi \in C(\mathbb{P}^{d-1})$, $|e^{i\theta}| = 1$ implies $e^{i\theta} = 1$, $\varphi = cte$.

Taking absolute values we get $|\varphi| \leq Q^s |\varphi|$. As in Lemma 2.9, we get that for any x in $\Lambda([supp\mu])$: $|\varphi(x)| = \sup\{|\varphi(y)|; y \in \mathbb{P}^{d-1}\}$.

Hence we can assume $|\varphi(x)| = 1$ on $\Lambda([supp\mu])$. Now we can use the equation

$$e^{i\theta} \varphi(x) = \int q^s(x, g) \varphi(g.x) d\mu(g)$$

where $q^s(x, g) = \frac{1}{k(s)} \frac{e^s(g.x)}{e^s(x)} |gx|^s$, hence $\int q^s(x, g) d\mu(g) = 1$. Strict convexity gives : $e^{i\theta} \varphi(x) = \varphi(g.x)$, for any $x \in \Lambda([supp\mu])$, $g \in supp\mu$.

We know, from Proposition 2.13 that $\bar{P}^n \varphi$ converges uniformly to $\nu(\varphi)$ where ν is the unique \bar{P} -stationary measure on \mathbb{P}^{d-1} . Furthermore, on $\Lambda([supp\mu])$ we have $\bar{P}^n \varphi = e^{in\theta} \varphi$. The above convergence gives $e^{i\theta} = 1$, since $\varphi \neq 0$ on $\Lambda([supp\mu])$. The fact that φ is constant follows from Proposition 2.13.

In order to show the last assertion in case $d > 1$ we write $Q^z \varphi(x)$ as

$$Q^z \varphi(x) = \int |gx|^{it} q^s(x, g) \varphi(g.x) d\mu(g).$$

We observe that the absolute value of the function $Q^z \varphi$ is bounded by the function $Q^s |\varphi|$. Hence, from above, the equation $Q^z \varphi = e^{i\theta} \varphi$ gives $Q^s |\varphi| \geq |\varphi|$, hence $Q^s |\varphi| = |\varphi|$ and $|\varphi| = \text{cte}$. Then the equation $Q^z \varphi = e^{i\theta} \varphi$ gives for any x and $g \in supp\mu$, since $\int q^s(x, g) d\mu(g) = 1$: $|gx|^{it} \frac{\varphi(g.x)}{\varphi(x)} = e^{i\theta} \mu - a.e.$

This contradicts Proposition 2.5 if $t \neq 0$. If $t = 0$ we have $e^{i\theta} = 1$, $\varphi = \text{cte}$ from above. \square

3) Eigenfunctions and eigenmeasures on \mathbb{S}^{d-1}

Here we study the operator \tilde{P}^s on \mathbb{S}^{d-1} defined by $\tilde{P}^s \varphi(x) = \int \varphi(g.x) |gx|^s \mu(dg)$. We show that there are 2 cases, depending of the existence of a $[supp\mu]$ -invariant proper convex cone in V or not. We still denote by e^s the function on \mathbb{S}^{d-1} lifted from $e^s \in C(\mathbb{P}^{d-1})$. We denote \tilde{Q}^s the operator on \mathbb{S}^{d-1} defined by $\tilde{Q}^s \varphi = \frac{1}{k(s)e^s} \tilde{P}^s(\varphi e^s)$.

We already know, using the remark which follows Proposition 2.13, that for $s > 0$ and any given $\varphi \in C(\mathbb{S}^{d-1})$, the sequence $(\tilde{Q}^s)^n \varphi$ is equicontinuous. For any subsemigroup T of G satisfying condition i-p, we denote by $\tilde{\Lambda}(T)$ the inverse image of $\Lambda(T)$ in \mathbb{S}^{d-1} . We begin by considering the dynamics of T on \mathbb{S}^{d-1} . For analogous results in more general situations see [25].

Proposition 2.15

Assume $T \subset G$ is a subsemigroup which satisfies condition i-p. If $d = 1$, we assume that T is non-arithmetic. Then the action of T on \mathbb{S}^{d-1} has one or two minimal sets whose union is $\tilde{\Lambda}(T)$:

Case I There is no T -invariant proper convex cone in V .
Then $\tilde{\Lambda}(T)$ is the unique T -minimal subset of \mathbb{S}^{d-1} .

Case II T preserves a closed proper convex cone $C \subset V$.
Then the action of T on \mathbb{S}^{d-1} has two and only two minimal subsets $\Lambda_+(T)$, $\Lambda_-(T)$ with $\Lambda_-(T) = -\Lambda_+(T)$, $\Lambda_+(T) \subset \mathbb{S}^{d-1} \cap C$. The convex cone generated by $\Lambda_+(T)$ is proper and T -invariant.

The proof depends of the following Lemma.

Lemma 2.16

Let $V_i (1 \leq i \leq r)$ be vector subspaces of V . If condition i-p is valid, then there exists $g \in T^{prox}$ such that the hyperplane $V_g^<$ does not contain any $V_i (1 \leq i \leq r)$.

Proof

The dual semigroup T^* of T satisfies also condition i-p hence we can also consider its limit set $\Lambda(T^*) \subset \mathbb{P}(V^*)$. Let $\bar{v}(g^*)$ be the point of $\mathbb{P}(V^*)$ corresponding to a dominant eigenvector of g^* . Observe that the condition that an hyperplane contains V_i defines a subspace of V^* . If for any $g^* \in (T^*)^{prox}$ the hyperplane $\bar{v}(g^*)$ contains some V_i then by density any $x \in \Lambda(T^*)$ contains some V_i . Then the T^* -invariance of $\Lambda(T^*)$ implies that T^* leaves invariant a finite union of subspaces of $\mathbb{P}(V^*)$, which contradicts condition i-p. \square

Proof of the proposition

Let $x \in \tilde{\Lambda}(T)$ and $S = \overline{T.x}$. We observe that if $y \in \mathbb{S}^{d-1}$, then $\overline{T.y}$ contains x or $-x$, since the projection of $\overline{T.x}$ in \mathbb{P}^{d-1} contains $\Lambda(T)$. Assume first $-x \notin \overline{T.x}$. If $y \in \overline{T.x}$, then $\overline{T.y} \subset S$, hence $x \in \overline{T.y}$. This shows the T -minimality of S . The same argument shows that $-y \notin S$, hence $S \cap -S = \phi$. Since the projection of S in \mathbb{P}^{d-1} is $\Lambda(T)$, we see that the projection of \mathbb{S}^{d-1} on \mathbb{P}^{d-1} gives a T -equivariant homeomorphism of S on $\Lambda(T)$. Since $-x \notin S$, there are two T -minimal sets, S and $-S$. Since for any $y \in \mathbb{S}^{d-1}$, $\overline{T.y}$ contains S or $-S$, these sets are the unique minimal sets.

Assume now $-x \in \overline{T.x}$, hence $S = -S$. Since the projection of S in \mathbb{P}^{d-1} is $\Lambda(T)$, we see that $S = \tilde{\Lambda}(T)$.

Assume now that C is a T -invariant closed proper convex cone, that we can suppose closed. Then $C \cap \mathbb{S}^{d-1}$ is T -invariant and closed, hence $C \cap \mathbb{S}^{d-1} \supset \Lambda_+(T)$ or $\Lambda_-(T)$ in the first situation, ($-x \notin \overline{T.x}$). In the second situation C cannot exist, since $C \cap \mathbb{S}^{d-1}$ would contain $\tilde{\Lambda}(T)$, which is symmetric.

It remains to show that, in the first situation, there exists a T -invariant closed proper convex cone. Let C be the convex cone generated by $\Lambda_+(T)$ and let us show $C \cap -C = \{0\}$. Assume $C \cap -C \neq \{0\}$; then we can find $y_1, \dots, y_p \in C$, $z_1, \dots, z_q \in -C$ and convex combinations $y = \sum_1^p \alpha_i y_i$, $z = \sum_1^q \beta_j z_j$ with $y = z$. The lemma shows that there exists $g \in T^{prox}$ such that $y_i (1 \leq i \leq p)$ and $z_j (1 \leq j \leq q)$ do not belong to $V_g^<$. Hence, with $n \in 2\mathbb{N}$:

$$\lim_{n \rightarrow +\infty} \frac{g^n y}{|g^n|} = \lim_{n \rightarrow \infty} \sum_1^p \alpha_i \frac{g^n y_i}{|g^n y_i|} \frac{|g^n y_i|}{|g^n|} = \left(\sum_1^p \alpha_i u_i \right) v_g$$

where $u_i = \lim_{n \rightarrow \infty} \frac{|g^n y_i|}{|g^n|} > 0$ and $v_g \in \Lambda_+(T)$ is the unique dominant eigenvector of g in $\Lambda_+(T)$. In the same way :

$$\lim_{n \rightarrow \infty} \frac{g^n z}{|g^n|} = - \left(\sum_1^q \beta_j u'_j \right) v_g$$

with $u'_j > 0$. Since $y = z$ we have a contradiction. Hence we have the required dichotomy. The last assertion follows. \square

We denote by $\tilde{\nu}^s$ the symmetric measure on \mathbb{S}^{d-1} with projection ν^s on \mathbb{P}^{d-1} . In case II, we denote by ν_+^s (resp ν_-^s) the normalized restrictions of $\tilde{\nu}^s$ to $\Lambda_+(T)$ (resp $\Lambda_-(T)$)

Theorem 2.17

Let $\mu \in M^1(G)$, $s \in I_\mu$ and assume $T = [supp\mu]$ satisfies i-p. If $d = 1$ we assume that μ is non-arithmetic. Then for any $\varphi \in C(\mathbb{S}^{d-1})$, $x \in \mathbb{S}^{d-1}$, we have the uniform convergence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (\tilde{Q}^s)^n \varphi(x) = \tilde{\pi}^s(x)(\varphi),$$

where, $\tilde{\pi}^s(x) \in M^1(\mathbb{S}^{d-1})$ is supported on $\tilde{\Lambda}(T)$ and is \tilde{Q}^s -stationary.

Furthermore there are 2 cases given by Proposition 2.15.

Case I, \tilde{Q}^s has a unique stationary measure $\tilde{\pi}^s$ with $supp\tilde{\pi}^s = \tilde{\Lambda}(T)$ and $\tilde{\pi}^s(x) = \tilde{\pi}^s$ for any $x \in \mathbb{S}^{d-1}$. The \tilde{Q}^s -invariant functions are constant.

We have $\tilde{\pi}^s = e^s \tilde{\nu}^s$ and $\tilde{P}^s \tilde{\nu}^s = k(s) \tilde{\nu}^s$.

Case II, \tilde{Q}^s has two and only two extremal stationary measures π_+^s, π_-^s . We have $supp\pi_+^s = \Lambda_+(T)$ and π_-^s is symmetric of π_+^s . If $\pi_+^s = e^s \nu_+^s$, then $\tilde{P}^s \nu_+^s = k(s) \nu_+^s$.

Also, there are 2 minimal \tilde{Q}^s -invariant continuous functions p_+^s, p_-^s and we have :

$$\tilde{\pi}^s(x) = p_+^s(x) \pi_+^s + p_-^s(x) \pi_-^s.$$

Furthermore $p_+^s(x)$ is equal to the entrance probability in the set $Co(\Lambda_+(T))$ for the Markov chain defined by \tilde{Q}^s . In particular $p_+^s(x) = 1$ (resp $p_+^s(x) = 0$) if $x \in \Lambda_+(T)$ (resp $\Lambda_-(T)$).

If $*\nu_+^s \in M^1(\Lambda_+(T^*))$ satisfies $*\tilde{P}^s *\nu_+^s = k(s) *\nu_+^s$, we have for $u \in \mathbb{S}^{d-1}$,

$$p(s) e_+^s(u) = \int \langle u, u' \rangle_+^s d *\nu_+^s(u') \text{ with } \langle u, u' \rangle_+ = \sup(0, \langle u, u' \rangle) \text{ and } e_+^s = p_+^s e^s$$

(resp $e_-^s = p_-^s e^s$) satisfies $\tilde{P}^s e_+^s = k(s) e_+^s$ (resp $\tilde{P}^s e_-^s = k(s) e_-^s$).

The space of continuous \tilde{Q}^s -invariant (resp \tilde{P}^s)-eigenfunctions is generated by p_+^s and p_-^s (resp e_+^s and e_-^s).

For $s = 0$ we will need the following lemma, which uses results of section 3.

Lemma 2.18

For $u \in \mathbb{S}^{d-1}$, we denote $\Delta_u^t = \{y \in \mathbb{P}^{d-1} ; |\langle u, y \rangle| < t\}$. Then, for any $\varepsilon, t > 0$, $x, y \in \mathbb{S}^{d-1}$:

$$\limsup_{n \rightarrow \infty} \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g) \leq \frac{2^\varepsilon}{t^\varepsilon} \tilde{\delta}^\varepsilon(x, y) + 2^\varepsilon \nu(\Delta_x^t).$$

In particular, for any $\varphi \in H_\varepsilon(\mathbb{S}^{d-1})$ the sequence $\tilde{P}^n \varphi$ is equicontinuous.

Proof

We write :

$$\int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g) = \int 1_{[1/t, \infty[}(\frac{|S_n|}{|S_n x|}) \tilde{\delta}^\varepsilon(S_n.x, S_n.y) d\mathbb{P}(\omega) + \int 1_{]0, 1/t[}(\frac{|S_n|}{|S_n x|}) \tilde{\delta}^\varepsilon(S_n.x, S_n.y) d\mathbb{P}(\omega).$$

Using Lemma 2.11 we have :

$$\tilde{\delta}^\varepsilon(S_n.x, S_n.y) \leq (2 \frac{|S_n|}{|S_n x|} \tilde{\delta}(x, y))^\varepsilon.$$

On the other hand, using Theorem 3.2, we know that : $\lim_{n \rightarrow \infty} \frac{|S_n|}{|S_n x|} = \frac{1}{|\langle z^*(\omega), x \rangle|}$

where $z^*(\omega) \in \mathbb{P}^{d-1}$ has law ν . Hence :

$$\limsup_{n \rightarrow \infty} \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g) \leq \frac{2^\varepsilon}{t^\varepsilon} \tilde{\delta}^\varepsilon(x, y) + 2^\varepsilon \mathbb{P}\{|\langle z^*(\omega), x \rangle| < t\} = \frac{2^\varepsilon}{t^\varepsilon} \tilde{\delta}^\varepsilon(x, y) + 2^\varepsilon \nu(\Delta_x^t).$$

We have $|\tilde{P}^n \varphi(x) - \tilde{P}^n \varphi(y)| \leq [\varphi]_\varepsilon \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g)$. From Theorem 3.2, we know that ν is proper, hence $\lim_{t \rightarrow 0} \nu(\Delta_x^t) = 0$. Then, for x fixed we use the above estimation of

$\limsup_{n \rightarrow \infty} \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g)$ to choose t sufficiently small in order to get the continuity of

$\limsup_{n \rightarrow \infty} \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g)$. This gives that $|\tilde{P}^n \varphi(x) - \tilde{P}^n \varphi(y)|$ depends continuously of y , hence the equicontinuity of the sequence $\tilde{P}^n \varphi$. \square

Proof of the theorem

As observed in remark *b* after Proposition 2.13, if $s > 0$ for any $\varphi \in C(\mathbb{S}^{d-1})$ the set of functions $\{(\tilde{Q}^s)^n \varphi ; n \in \mathbb{N}\}$ is equicontinuous. In view of Lemma 2.18, this is also valid if $s = 0$. Hence we can use here Lemma 2.12 and the results of [47]. This gives the first convergence. Since $\tilde{\pi}^s(x)$ is \tilde{Q}^s -stationary, its projection on \mathbb{P}^{d-1} is equal to the unique Q^s -stationary measure π^s , hence $\text{supp} \tilde{\pi}^s(x) \subset \tilde{\Lambda}(T)$. On the other hand $\text{supp} \tilde{\pi}^s(x)$ is closed and T -invariant, hence contains a T -minimal set.

In case I, $\tilde{\Lambda}(T)$ is the unique minimal set, hence $\text{supp} \tilde{\pi}^s(x) = \tilde{\Lambda}(T)$. Furthermore, if $\varphi \in C(\mathbb{S}^{d-1})$ is \tilde{Q}^s -invariant, the sets :

$$M^- = \{x ; \varphi(x) = \text{Inf}\{\varphi(y); y \in \mathbb{P}^{d-1}\}\}, \quad M^+ = \{x ; \varphi(x) = \text{sup}\{\varphi(y); y \in \mathbb{P}^{d-1}\}\},$$

are closed and T -invariant, hence contains minimal sets. Since $\tilde{\Lambda}(T)$ is the unique minimal set, $M^+ \cap M^- \supset \tilde{\Lambda}(T) \neq \emptyset$, hence $\varphi = \text{cte}$.

Then, using Proposition 2.13 and Lemma 2.12, we get that there exists a unique stationary measure $\tilde{\pi}^s$. It follows that $\tilde{\pi}^s$ is symmetric with projection π^s on \mathbb{P}^{d-1} and $\tilde{\pi}^s = e^s \tilde{\nu}^s$.

In case II, the restriction to $Co(\Lambda_+(T)) = \Phi$ of the projection on \mathbb{P}^{d-1} is a T -equivariant homeomorphism. If we denote by i_+ its inverse, we get that $i_+(\pi^s)$ is the unique \tilde{Q}^s -stationary measure supported in Φ . Hence $i_+(\pi^s) = \pi_+^s$. Then π_+^s and π_-^s are extremal \tilde{Q}^s -stationary measures.

Since the projection of $\tilde{\pi}^s(x)$ on \mathbb{P}^{d-1} is π^s , we can write :

$$\tilde{\pi}^s(x) = \int (p_+^s(x, y) \delta_y + p_-^s(x, y) \delta_{-y}) d\pi^s(y) = p_+^s(x) \pi_+^s + p_-^s(x) \pi_-^s$$

where $p_+^s(x) = p_+^s(x, \cdot)$ and $p_-^s(x) = p_-^s(x, \cdot)$ are Borel functions of $y \in \tilde{\Lambda}(T)$ such that $p_+^s(x) + p_-^s(x) = 1$. Then $p_+^s(x) \pi_+^s$ is the restriction of $\tilde{\pi}^s(x)$ to $\Lambda_+(T)$, hence is a \tilde{Q}^s -invariant measure. In view of the uniqueness of the stationary measure of \tilde{Q}^s restricted to $\Lambda_+(T)$, we get that $p_+^s(x) \pi_+^s$ is proportional to π_+^s , i.e $p_+^s(x)$ is independant of y $\pi_+^s - a.e$.

Hence, the first assertion of the theorem implies that the only extremal \tilde{Q}^s -stationary measures are π_+^s and π_-^s . The corresponding facts for ν_+^s and ν_-^s follow.

Also the operator defined by $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_0^{n-1} (\tilde{Q}^s)^k$ is the projection on the space of \tilde{Q}^s -invariant functions and is equal to $p_+^s(x)\pi_+^s + p_-^s(x)\pi_-^s$. The continuity and the extremality of the \tilde{Q}^s -invariant functions $p_+^s(x)$ and $p_-^s(x)$ follows. The corresponding facts for e_+^s and e_-^s follow.

If we restrict the convergence of $\frac{1}{n} \sum_0^{n-1} (\tilde{Q}^s)^k(\delta_x)$ to $x \in \Phi$, in view of the fact that the restriction to Φ of the projection on \mathbb{P}^{d-1} is a homeomorphism onto its image, we get : $p_+^s(x) = 1, p_-^s(x) = 0$ if $x \in \Phi$.

Let us denote by τ the entrance time of $S_n(\omega).x$ in $\Phi \cup -\Phi$ and by ${}^a\mathbb{E}_x^s$ the expectation symbol associated with the Markov chain $S_n(\omega).x$ defined by \tilde{Q}^s . Using theorem 2.6 we get ${}^a\mathbb{E}_x^s(1_{\Phi \cup -\Phi}(S_\tau.x)) = 1$. Since $p_+^s(x)$ is a \tilde{Q}^s -invariant function $p_+^s(S_n.x)$ is a martingale, hence $p_+^s(x) = {}^a\mathbb{E}_x^s(p_+^s(S_\tau.x))$. Since $p_+^s(x) = 1$ on Φ and $p_+^s(x) = 0$ on $-\Phi$ we get $p_+^s(x) = {}^a\mathbb{E}_x^s(1_\Phi(S_\tau.x))$, hence the stated interpretation for $p_+^s(x)$.

As in Lemma 2.10, we verify easily that the function $\varphi(u) = p(s) \int \langle u, u' \rangle_+^s d^* \nu_+^s(u')$ on \mathbb{S}^{d-1} satisfies $\tilde{P}^s \varphi = k(s)\varphi$, hence the function $\frac{\varphi}{e^s}$ satisfies $\tilde{Q}^s(\frac{\varphi}{e^s}) = \frac{\varphi}{e^s}$. By duality, cases II for μ and μ^* are the same, hence there are two minimal T^* -invariant subsets $\Lambda_+(T^*)$ and $\Lambda_-(T^*) = -\Lambda_+(T^*)$. On the other hand the set :

$$\hat{\Lambda}_+(T) = \{u \in \mathbb{S}^{d-1} ; \text{for any } u' \text{ in } \Lambda_+(T^*), \langle u, u' \rangle \geq 0\}$$

is non trivial, closed, T -invariant and has non zero interior, hence $\hat{\Lambda}_+(T)$ contains either $\Lambda_+(T)$ or $\Lambda_-(T)$ and has trivial intersection with one of them. We can assume $\hat{\Lambda}_+(T) \supset \Lambda_+(T)$. Then, for $u \in \Lambda_+(T)$ and any $u' \in \Lambda_+(T^*)$, we have $\langle u, u' \rangle_+ = \langle u, u' \rangle$, hence :

$$\varphi(u) = p(s) \int \langle u, u' \rangle_+^s d^* \nu_+^s(u') = e^s(u),$$

i.e $\frac{\varphi}{e^s} = 1$ on $\Lambda_+(T)$. Also we have $\langle u, u' \rangle_+ = 0$ for $u \in \Lambda_-(T), u' \in \Lambda_+(T^*)$. Since $\tilde{Q}(\frac{\varphi}{e^s}) = \frac{\varphi}{e^s} \geq 0$ and $\frac{\varphi}{e^s} = 1$ on $\Lambda_+(T)$, we conclude from above that $p_+^s = \frac{\varphi}{e^s}$, hence we get the last formula and last assertions with $e_-^s(u) = p(s) \int \langle u, x \rangle_+^s d^* \nu_-^s(x) = p_-^s(u)e^s(u)$. \square

From above we know that if $s \geq 0$ and $\varphi \in C(\mathbb{S}^{d-1})$ the sequence $(\tilde{Q}^s)^n \varphi$ is equicontinuous. Lemma 2.12 reduces the discussion of the behaviour of $(\tilde{Q}^s)^n \varphi$ to the existence of eigenvalues z of \tilde{Q}^s with $|z| = 1$. In this direction we have the following.

Corollary 2.19

For $s \in I_\mu$, the equation $\tilde{Q}^s \varphi = e^{i\theta} \varphi$ with $e^{i\theta} \neq 1, \varphi \in C(\mathbb{S}^{d-1})$ has a non trivial solution only in case I. In that case $e^{i\theta} = -1, \varphi$ is antisymmetric, satisfies on $\text{supp} \mu \times \tilde{\Lambda}(T)$ $\varphi(g.x) = -\varphi(x)$ and is uniquely defined up to a coefficient.

Proof

We observe that, since φ satisfies $\tilde{Q}^s \varphi = e^{i\theta} \varphi$, the function φ' defined by $\varphi'(x) = \varphi(-x)$ satisfies also $\tilde{Q}^s \varphi' = e^{i\theta} \varphi'$. Then $\varphi + \varphi'$ is symmetric and defines a function $\bar{\varphi}$ in $C(\mathbb{P}^{d-1})$

with $Q^s \bar{\varphi} = e^{i\theta} \bar{\varphi}$. If $e^{i\theta} \neq 1$, Theorem 2.6 gives $\bar{\varphi} = 0$, i.e φ is antisymmetric. Furthermore, in case II, the restriction of φ to $\tilde{\Lambda}_+(T)$ satisfies the same equation and the projection of $\tilde{\Lambda}_+(T)$ on $\Lambda(T)$ is an equivariant homomorphism. Then Theorem 2.6 gives a contradiction. Hence if $\varphi \in C(\mathbb{S}^{d-1})$ satisfies $\tilde{Q}^s \varphi = e^{i\theta} \varphi$, then we are in case I. Also, passing to absolute values as in the proof of the theorem we get $\tilde{Q}^s |\varphi| = |\varphi|$, $|\varphi| = \text{cte}$. Furthermore by strict convexity we have on $\text{supp}\mu \times \tilde{\Lambda}(T)$, $\varphi(g.x) = e^{i\theta} \varphi(x)$, hence $\varphi^2(g.x) = e^{2i\theta} \varphi^2(x)$. Since φ^2 is symmetric and satisfies $\tilde{Q}^s \varphi^2 = e^{2i\theta} \varphi^2$, we get $e^{2i\theta} = 1$, i.e $e^{i\theta} = -1$; in particular $\varphi(g.x) = -\varphi(x)$ on $\text{supp}\mu \times \tilde{\Lambda}(T)$. If $\varphi' \in C(\mathbb{S}^{d-1})$ satisfies also $\tilde{Q}^s \varphi' = -\varphi'$, we get from above $\tilde{Q}^s \frac{\varphi'}{\varphi} = \frac{\varphi'}{\varphi}$, $\frac{\varphi'}{\varphi}(-x) = \frac{\varphi'}{\varphi}(x)$, hence φ' is proportional to φ . \square

III Laws of large numbers and spectral gaps.

1) Notations

As in section 2, we assume that condition i-p is valid for $[\text{supp}\mu]$. If $d = 1$ we assume that μ is non arithmetic. For $s \in I_\mu$ we consider the functions q^s and q_n^s ($n > 0$) on $\mathbb{P}^{d-1} \times G$, defined by :

$$q^s(x, g) = \frac{1}{k(s)} \frac{e^s(g.x)}{e^s(x)} |gx|^s = q_1^s(x, g), \quad q_n^s(x, g) = \frac{1}{k^n(s)} \frac{e^s(g.x)}{e^s(x)} |gx|^s,$$

hence by definition of e^s : $\int q_n^s(x, g) d\mu^n(g) = 1$.

We denote by \mathbb{P}_2^{d-1} the flag manifold of planes and by $\mathbb{P}_{1,2}^{d-1}$ the manifold of contact elements on \mathbb{P}^{d-1} . Such a plane is defined up to normalisation by a 2-vector $x \wedge y \in \wedge^2 V$ and we can assume $|x \wedge y| = 1$. Also a contact element ξ is defined by its origin $x \in \mathbb{P}^{d-1}$ and a line through x . Hence we can write $\xi = (x, x \wedge y)$ where $|x| = |x \wedge y| = 1$. The following additive cocycles of the actions of G on \mathbb{P}^{d-1} , \mathbb{P}_2^{d-1} , $\mathbb{P}_{1,2}^{d-1}$ will play an essential role :

$$\sigma_1(g, x) = \text{Log}|gx|, \quad \sigma_2(g, x \wedge y) = \text{Log}|g(x \wedge y)|, \quad \sigma(g, \xi) = \text{Log}|g(x \wedge y)| - 2\text{Log}|gx|.$$

In addition to the norm of g we will need to use the quantity $\gamma(g) = \sup(|g|, |g^{-1}|) \geq 1$.

Clearly, for any $x \in V$, with $|x| = 1$, we have $-\text{Log} \gamma(g) \leq \text{Log}|gx| \leq \text{Log} \gamma(g)$.

For a finite sequence $\omega = (g_1, g_2, \dots, g_n)$ we write :

$$S_n(\omega) = g_n \cdots g_1 \in G, \quad q_n^s(x, \omega) = \prod_{k=1}^n q^s(S_{k-1}.x, g_k).$$

We denote by Ω_n the space of finite sequences $\omega = (g_1, g_2, \dots, g_n)$ and we write $\Omega = G^{\mathbb{N}}$.

We observe that $\theta^s(x, g) = |gx|^s \frac{e^s(g.x)}{e^s(x)}$ satisfies the cocycle relation $\theta^s(x, gg') = \theta^s(x, g'.x) \theta^s(g', x)$, hence $q_n^s(x, \omega) = \frac{1}{k^n(s)} \theta^s(x, S_n(\omega))$.

Definition 3.1

We denote $\mathbb{Q}_x^s \in M^1(\Omega)$ the limit of the projective system of probability measures $q_k^s(x, \cdot) \mu^{\otimes k}$ on Ω_k . We write $\mathbb{Q}^s = \int \mathbb{Q}_x^s d\pi^s(x)$ where π^s is the unique Q^s -stationary measure on \mathbb{P}^{d-1} . The corresponding expectation symbol will be written \mathbb{E}_x^s and the shift on Ω will be denoted by θ . We write also $\mathbb{E}^s(\varphi) = \int \mathbb{E}_x^s(\varphi) d\pi^s(x)$.

The path space of the Markov chain defined by Q^s is a factor space of ${}^a\Omega = \mathbb{P}^{d-1} \times \Omega$, and the corresponding shift on ${}^a\Omega$ will be written ${}^a\theta : {}^a\theta(x, \omega) = (g_1(\omega).x, \theta\omega)$. Hence $(\mathbb{P}^{d-1} \times \Omega, {}^a\theta)$ is a skew product over (Ω, θ) . The projection on Ω of the Markov measure ${}^aQ_x^s = \delta_x \otimes Q_x^s$ is Q_x^s , hence ${}^aQ^s = \int \delta_x \otimes Q_x^s d\pi^s(x)$ projects on Q^s . The uniqueness of the Q^s -stationary measure π^s implies the ergodicity of the ${}^a\theta$ -invariant measure ${}^aQ^s$, hence Q^s is also θ -invariant and ergodic.

If $s = 0$, the random variables $g_k(\omega)$ are i.i.d with law μ and $Q^0 = \mathbb{P} = \mu^{\otimes \mathbb{N}}$. Here, under condition (i-p), we extend the results of [29] to the case $s \geq 0$, in particular we construct a suitable measure-valued martingale with contraction properties as in [17]. This will allow us to prove strong forms of the law of large numbers for $S_n(\omega)$ and to compare the measures Q_x^s when x varies. Then we can deduce the simplicity of the dominant Lyapunov exponent of $S_n(\omega)$ under the θ -invariant probability Q^s for $s \geq 0$. Spectral gap properties for twisted convolution operators on the projective space and on the unit sphere will follow.

2) A martingale and the equivalence of Q_x^s to Q^s

When convenient we identify $x \in \mathbb{P}^{d-1}$ with one of its representants \tilde{x} in \mathbb{S}^{d-1} . We recall that the Markov kernel ${}^*Q^s$ is defined by ${}^*Q^s\varphi = \frac{1}{k(s){}^*e^s} {}^*P^s(\varphi{}^*e^s)$ where ${}^*P^s\varphi(x) = \int \varphi(g.x)|gx|^s d\mu^*(g)$, ${}^*P^s({}^*e^s) = k(s){}^*e^s$ and ${}^*Q^s$ has a unique stationary measure ${}^*\pi^s$. Furthermore we have ${}^*\pi^s = {}^*e^s{}^*\nu^s$ where ${}^*\nu^s \in M^1(\mathbb{P}^{d-1})$ is the unique solution of ${}^*P^s({}^*\nu^s) = k(s){}^*\nu^s$. We denote by m the unique rotation invariant probability measure on \mathbb{P}^{d-1} .

Theorem 3.2

Let $\Omega' \subset \Omega$ be the (shift-invariant) Borel subset of elements $\omega \in \Omega$ such that $S_n^*(\omega).m$ converges to a Dirac measure $\delta_{z^*}(\omega)$. Then $g_1^*.z^*(\theta\omega) = z^*(\omega)$, $Q^s(\Omega') = 1$, the law of $z^*(\omega)$ under Q^s is ${}^*\pi^s$ and ${}^*\pi^s$ is proper.

In particular if $\omega \in \Omega'$ and $|\langle x, z^*(\omega) \rangle \langle y, z^*(\omega) \rangle| \neq 0$, then :

$$\lim_{n \rightarrow \infty} \delta(S_n(\omega).x, S_n(\omega).y) = 0.$$

If $\omega \in \Omega'$ and $\xi = (x, x \wedge y) \in \mathbb{P}_{1,2}^{d-1}$

$$\lim_{n \rightarrow \infty} \frac{|S_n(\omega).x|}{|S_n(\omega)|} = |\langle z^*(\omega), x \rangle|, \quad \lim_{n \rightarrow \infty} S_n^*.m = \delta_{z^*(\omega)}.$$

In particular, if $\langle z^*(\omega), x \rangle \neq 0$ then $\lim_{n \rightarrow \infty} \sigma(S_n, \xi) = -\infty$

Also for any $x \in \mathbb{P}^{d-1}$ Q_x^s is equivalent to Q^s and $\frac{dQ_x^s}{dQ^s}(\omega) = \left| \frac{\langle z^*(\omega), x \rangle}{\langle z^*(\omega), y \rangle} \right|^s \frac{e^s(y)}{e^s(x)}$.

The proof of Theorem 3.2 is based on the following lemmas, in particular on the study of a measure-valued martingale.

Lemma 3.3

Assume $z^* \in \mathbb{P}^{d-1}$ and $u_n \in G$ is a sequence such that : $\lim_{n \rightarrow \infty} u_n^*.m = \delta_{z^*}$. Then, for any $x, y \in \mathbb{P}^{d-1}$ with $|\langle z^*, x \rangle \langle z^*, y \rangle| \neq 0$, $\lim_{n \rightarrow \infty} \delta(u_n.x, u_n.y) = 0$. If $\xi = (x, x \wedge y) \in \mathbb{P}_{1,2}^{d-1}$ and $\langle z^*, x \rangle \neq 0$:

$$\lim_{n \rightarrow \infty} \frac{|u_n x|}{|u_n|} = |\langle z^*, x \rangle|, \quad \lim_{n \rightarrow \infty} \sigma(u_n, \xi) = -\infty.$$

These convergences are uniform on the compact subsets on which $\langle z^*, x \rangle$ do not vanish.

Proof

We denote by $e_i (1 \leq i \leq d)$ an orthonormal basis of V , by \bar{e}_i the projection of e_i in \mathbb{P}^{d-1} , by \bar{A}^+ the set of diagonal matrices $a = \text{diag}(a^1, a^2, \dots, a^d)$ and $a^1 \geq a^2 \geq \dots \geq a^d > 0$. We write $u_n = k_n a_n k'_n$ with $a_n \in \bar{A}^+$, $k_n, k'_n \in O(d)$. Then, for $x \in \mathbb{P}^{d-1}$:

$$|u_n x|^2 = |a_n k'_n x|^2 = \sum_1^d (a_n^i)^2 |\langle k'_n x, e_i \rangle|^2.$$

Also, $u_n^*.m = (k'_n)^{-1} a_n.m$ converges to $z^* \in \mathbb{P}^{d-1}$, which implies :

$$\lim_{n \rightarrow \infty} a_n.m = \delta_{\bar{e}_1}, \quad \lim_{n \rightarrow \infty} (k'_n)^{-1}.\bar{e}_1 = z^*.$$

In particular, if $i > 1$, we have $a_n^i = o(a_n^1)$ and

$$\lim_{n \rightarrow \infty} |\langle k'_n x, e_1 \rangle| = |\langle z^*, x \rangle| \neq 0.$$

It follows that $|u_n x| \sim a_n^1 |\langle z^*, x \rangle|$. Since $|u_n| = a_n^1$, we get : $\lim_{n \rightarrow \infty} \frac{|u_n x|}{|u_n|} = |\langle z^*, x \rangle|$, as asserted. We get also, if $|\langle y, z^* \rangle| \neq 0$, $|u_n y| \sim a_n^1 |\langle z^*, y \rangle|$. On the exterior product space $\wedge^2 V$ there exists an $O(d)$ -invariant scalar product such that on a decomposable 2-vector $x \wedge y$:

$$|x \wedge y|^2 = |x|^2 |y|^2 - |\langle x, y \rangle|^2.$$

For $x, y \in \mathbb{P}^{d-1}$ and corresponding $\tilde{x}, \tilde{y} \in \mathbb{S}^{d-1}$ we write $|x \wedge y| = |\tilde{x} \wedge \tilde{y}|$. Then on \mathbb{P}^{d-1} , there is an associated distance δ_1 given by $\delta_1(x, y) = |x \wedge y|$ and we have $\frac{1}{2}\delta \leq \delta_1 \leq \delta$. We observe that $\delta_1(u_n.x, u_n.y) = \frac{|u_n x \wedge u_n y|}{|u_n x| |u_n y|}$.

Also $|u_n x \wedge u_n y|^2 = \sum_{i>j} (a_n^i a_n^j)^2 |\langle k_n(\tilde{x} \wedge \tilde{y}), e_i \wedge e_j \rangle|^2 \leq \frac{d(d-1)}{2} (a_n^1 a_n^2 |\tilde{x} \wedge \tilde{y}|)^2$.

It follows

$$\delta_1(u_n.x, u_n.y) \leq \left(\frac{d(d-1)}{2} \right)^{1/2} \frac{a_n^1 a_n^2}{|u_n x| |u_n y|} |\tilde{x} \wedge \tilde{y}|.$$

Since $|u_n x| \sim a_n^1 | \langle z^*, x \rangle |$, $|u_n y| \sim a_n^1 | \langle z^*, y \rangle |$ and $a_n^2 = o(a_n^1)$, $\langle z^*, x \rangle \langle z^*, y \rangle \neq 0$, we get :

$$\lim_{n \rightarrow \infty} \delta_1(u_n \cdot x, u_n \cdot y) = 0.$$

It follows, for any $x, y \in \mathbb{P}^{d-1}$: $\lim_{n \rightarrow \infty} \delta(u_n \cdot x, u_n \cdot y) = 0$.

Also, since $a_n^2 = o(a_n^1)$, and $\langle z^*, x \rangle \neq 0$ we get $\lim_{n \rightarrow \infty} \sigma(u_n, \xi) = -\infty$.

The above calculations imply the uniformities in the convergences. \square

Lemma 3.4

Assume $\nu_n \in M^1(\mathbb{P}^{d-1})$ is a sequence such that ν_n is relatively compact in variation, and each ν_n is proper. Let $u_n \in G$ be a sequence such that $u_n^* \cdot \nu_n$ converges weakly to δ_{z^*} ($z^* \in \mathbb{P}^{d-1}$). Then for any proper $\rho \in M^1(\mathbb{P}^{d-1})$, $u_n^* \cdot \rho$ converges weakly to δ_{z^*} .

Proof

We can assume, in variation, $\lim_{n \rightarrow \infty} \nu_n = \nu_0$ where ν_0 is proper. Also we can assume going to subsequences that u_n^* converges to a quasi-projective map of the form u^* , defined and continuous outside a projective subspace $H \subset \mathbb{P}^{d-1}$. Let $\varphi \in C(\mathbb{P}^{d-1})$ and denote :

$$I_n = (u_n^* \cdot \nu_n)(\varphi) - (u^* \cdot \nu_0)(\varphi) = (\nu_n - \nu_0)(\varphi \circ u_n^*) + \nu_0(\varphi \circ u_n^* - \varphi \circ u^*)$$

The first term is bounded by $|\varphi| \|\nu_n - \nu_0\|$, hence it converges to zero. Since $\nu_0(H) = 0$ and $\varphi \circ u_n^*$ converges to $\varphi \circ u^*$ outside H , we can use dominated convergence for the second term : $\lim_{n \rightarrow \infty} \nu_0(\varphi \circ u_n^* - \varphi \circ u^*) = 0$, hence $\lim_{n \rightarrow \infty} I_n = 0$. Then $u_n^* \cdot \nu_n$ converges to $u^* \cdot \nu_0$ weakly. In particular $u^* \cdot \nu_0 = \delta_{z^*}$, hence $u^* \cdot y = z^*$ $\nu_0 - a.e.$ Since $\nu_0(H) = 0$, we have $u^* \cdot y = z^*$ on $\mathbb{P}^{d-1} \setminus H$.

Since ρ is proper $u^* \cdot \rho = \delta_{z^*}$, hence $\lim_{n \rightarrow \infty} u_n^* \cdot \rho = \delta_{z^*}$. \square

Lemma 3.5

For $x, y \in \mathbb{P}^{d-1}$ the total variation measure of $\mathbb{Q}_x^s - \mathbb{Q}_y^s$ is bounded by $B\delta^{\bar{s}}(x, y)\mathbb{Q}^s$.

Then exists $c(s) > 0$ such that, for any $x \in \mathbb{P}^{d-1}$, $\mathbb{Q}_x^s \leq c(s) \mathbb{Q}^s$

Proof

We write $q_n^s(g) = \int q_n^s(x, g) d\pi^s(x)$ and we observe that for any measurable φ depending on the first n coordinates :

$$\int \varphi(\omega) d\mathbb{Q}^s(\omega) = \int q_n^s(S_n(\omega)) \varphi(\omega) d\mu^{\otimes n}(\omega).$$

Also : $|(\mathbb{Q}_x^s - \mathbb{Q}_y^s)(\varphi)| \leq \int |q_n^s(x, S_n) - q_n^s(y, S_n)| |\varphi(\omega)| d\mu^{\otimes n}$. Using Lemma 2.11 we have for any $g \in G$:

$$|q_n^s(x, g) - q_n^s(y, g)| \leq b_s \frac{|g|^s}{k^n(s)} \delta^{\bar{s}}(x, y).$$

Using Theorem 2.6 for some $b > 0$:

$q_n^s(g) \geq b \frac{|g|^s}{k^n(s)}$, hence $|q_n^s(x, g) - q_n^s(y, g)| \leq \frac{b_s}{b} q_n^s(g) \delta^{\bar{s}}(x, y)$.

It follows :

$$|(\mathbb{Q}_x^s - \mathbb{Q}_y^s)(\varphi)| \leq \frac{b_s}{b} \delta^{\bar{s}}(x, y) \int |\varphi(\omega)| d\mathbb{Q}^s(\omega)$$

hence the first conclusion with $B = \frac{b_s}{b}$.

Integrating with respect to π we get, since $\delta(x, y) \leq \sqrt{2} : \mathbb{Q}_x^s \leq (1 + B(\sqrt{2})^{\bar{s}})\mathbb{Q}^s$ hence the second formula with $c(s) = 1 + B(\sqrt{2})^{\bar{s}}$. \square

Lemma 3.6

We consider the positive kernel ν_x^s from \mathbb{P}^{d-1} to \mathbb{P}^{d-1} given by $\nu_x^s = \frac{|\langle x, \cdot \rangle|^s}{e^s(x)} * \nu^s$. Then:

$\int g^* \cdot \nu_{g \cdot x}^s q^s(x, g) d\mu(g) = \nu_x^s$, $\nu_x^s(1) = \frac{1}{e^s(x)} \int |\langle x, y \rangle|^s d^* \nu^s(y) = p(s) \in]0, 1]$ and $x \rightarrow \nu_x^s$ is continuous in variation.

In particular $S_n^* \cdot \nu_{S_n \cdot x}^s$ is a bounded martingale with respect to \mathbb{Q}_x^s and the natural filtration.

Proof

We consider the s -homogeneous measure λ^s on V defined by $\lambda^s = * \nu^s \otimes \ell^s$. By definition of $* \nu^s$, $\int g^* \lambda^s d\mu(g) = k(s) \lambda^s$. Then the Radon measure λ_v^s defined by $\lambda_v^s = |\langle v, \cdot \rangle|^s \lambda^s$ satisfies $\int g^* \lambda_{g \cdot v}^s d\mu(g) = k(s) \lambda_v^s$. This can be written, by definition of ν_x^s and $q^s(x, g)$:

$$\int g^* \cdot \nu_{g \cdot x}^s q^s(x, g) d\mu(g) = \nu_x^s.$$

The martingale property of $S_n^* \cdot \nu_{S_n \cdot x}^s$ follows.

Furthermore since $p(s) e^s(x)$ is equal to $\int |\langle x, y \rangle|^s d^* \nu^s(y)$, Lemma 2.10 gives :

$$\nu_x^s(1) = \frac{1}{e^s(x)} \int |\langle x, y \rangle|^s d^* \nu^s(y) = p(s) \in]0, 1].$$

The continuity in variation of $x \rightarrow \nu_x^s$ follows from the definition.

Lemma 3.7

Let $\rho \in M^1(\mathbb{P}^{d-1})$, \mathcal{H} be the set of projective subspaces H of minimal dimension such that $\rho(H) > 0$. Then the subset of elements $H \in \mathcal{H}$ such that $\rho(H) = \sup\{\rho(L) ; L \in \mathcal{H}\}$ is finite and non void. Furthermore, there exists $\varepsilon_\rho > 0$ such that for any $H \in \mathcal{H}$:

$$\rho(H) = c_\rho \text{ or } \rho(H) \leq c_\rho - \varepsilon_\rho,$$

where $c_\rho = \sup\{\rho(L) ; L \in \mathcal{H}\}$.

Proof

If $H, H' \in \mathcal{H}$, $H \neq H'$, then $\dim H \cap H' < \dim H$, hence $\rho(H \cap H') = 0$. Then, for any $\beta > 0$, the cardinality of the set of elements $H \in \mathcal{H}$ with $\rho(H) \geq \beta$ is bounded by $\frac{1}{\beta}$. The first assertion follows. Assume the second assertion is false. Then there exists a sequence

$H_n \in \mathcal{H}$ with $\frac{c_\rho}{2} < \rho(H_n) < c_\rho$, $\lim_{n \rightarrow \infty} \rho(H_n) = c_\rho$, and $\rho(H_n) \neq \rho(H_m)$ if $n \neq m$. This contradicts the fact that the cardinality of the sequence H_n is at most $\frac{2}{c_\rho}$.

Lemma 3.8

Assume that the Markovian kernel $x \rightarrow \nu_x \in M^1(\mathbb{P}^{d-1})$ is continuous in variation and satisfies :

$$\nu_x = \int q^s(x, g) g^* \cdot \nu_{g.x} d\mu(g)$$

Let $\mathcal{H}_{p,r}$ the set of finite unions of r distinct subspaces of dimension p and let h be the function $h(x) = \sup\{\nu_x(W) ; W \in \mathcal{H}_{p,r}\}$. Then h is continuous and the set

$$X = \{x ; h(x) = \sup h(y), y \in \mathbb{P}^{d-1}\}$$

is closed and $[\text{supp}\mu]$ -invariant

Proof

If $W \in \mathcal{H}_{p,r}$ is fixed the function $x \rightarrow \nu_x(W)$ is continuous since $|\nu_x(W) - \nu_y(W)| \leq \|\nu_x - \nu_y\|$. This implies $|h(x) - h(y)| \leq \|\nu_x - \nu_y\|$, hence the continuity of h .

We have for any $W \in \mathcal{H}_{p,r}$:

$$\nu_x(W) = \int q^s(x, g) \nu_{g.x}((g^*)^{-1}W) d\mu(g).$$

Hence : $h(x) \leq \int q^s(x, g) h(g.x) d\mu(g)$. Then, as in Lemma 2.9, X is $[\text{supp}\mu]$ -invariant and closed.

Lemma 3.9

Let ν_x be as in Lemma 3.8. Then for any $x \in \mathbb{P}^{d-1}$, ν_x is proper.

Proof

We write $\pi_x = \frac{\nu_x}{\nu_x(1)}$, denote by \mathcal{H}_k be the set of projective subspaces of dimension k and : $\mathcal{H} = \bigcup_{k \geq 0} \mathcal{H}_k$, $d(x) = \text{Inf}\{\dim H ; H \in \mathcal{H}, \pi_x(H) > 0\}$,

$$m(x) = \sup\{\pi_x(H) ; H \in \mathcal{H}, \dim H = d(x)\}, \mathcal{W}(x) = \{H \in \mathcal{H} ; \pi_x(H) = m(x)\}.$$

Lemma 3.7 implies that the set $\mathcal{W}(x)$ has finite cardinality $n(x) > 0$. Also we denote $p = \text{Inf}\{d(x) ; x \in \mathbb{P}^{d-1}\}$, $h_p(x) = \sup\{\pi_x(H) ; H \in \mathcal{H}_p\}$.

Lemma 3.8 shows that $h_p(x)$ reaches its maximum β on a closed $[\text{supp}\mu]$ -invariant subset $X \subset \mathbb{P}^{d-1}$. Hence on $\Lambda([\text{supp}\mu])$ we have $h_p(x) = \beta = m(x)$. It follows $d(x) = p$ on $\Lambda([\text{supp}\mu])$. The relation $n(x)m(x) \leq 1$ implies $n(x) \leq \frac{1}{\beta}$ on $\Lambda([\text{supp}\mu])$.

Let $r = \sup\{n(x) ; x \in \Lambda([\text{supp}\mu])\}$ and denote $h_{p,r}(x) = \sup\{\pi_x(W) ; W \in \mathcal{H}_{p,r}\}$. Then Lemma 3.8 implies : $h_{p,r}(x) = r\beta$ on $\Lambda([\text{supp}\mu])$. Since $m(x) = \beta$, this relation implies $n(x) = r$ on $\Lambda([\text{supp}\mu])$. Let $W(x) = \bigcup\{H ; H \in \mathcal{W}(x)\}$ and let us show the local constancy of the function $W(x)$. Using Lemma 3.7 we get,

$$\beta(x) = \sup\{\pi_x(H) ; H \in \mathcal{H}_p, H \notin \mathcal{W}(x)\} < \beta.$$

Let $x \in \Lambda([\text{supp}\mu])$, $U_x = \{y ; \|\pi_y - \pi_x\| < \beta - \beta(x)\}$ and $H_y \in \mathcal{H}_p$ with $\pi_y(H_y) = \beta$. Then:

$$\beta - \pi_x(H_y) = \pi_y(H_y) - \pi_x(H_y) \leq \|\pi_y - \pi_x\| < \beta - \beta(x).$$

Hence $\pi_x(H_y) > \beta(x)$ and, using Lemma 3.8, we get $H_y \in \mathcal{W}(x)$ for any $y \in U_x$. Since π_x is continuous in variation, U_x is a neighbourhood of x , hence $W(x)$ is locally constant. Since $\Lambda([supp\mu])$ is compact, $W = \bigcup_{x \in \Lambda([supp\mu])} W(x)$ is a finite union of subspaces.

On the other hand, the relations :

$$r\beta = \pi_x(W(x)) = \int q^s(x, g) g^* \cdot \pi_{g.x}(W(x)) d\mu(g) \quad , \quad r\beta \geq (g^* \cdot \pi_{g.x})(W(x))$$

imply that, for any $x \in \Lambda([supp\mu])$, $r\beta = g^* \cdot \pi_{g.x}(W(x)) \mu - a.e.$

By definition of $W(g.x)$, we get : $(g^*)^{-1}W(x) = W(g.x) \mu - a.e.$

Hence, for any $g \in supp\mu$: $(g^*)^{-1}(W(x)) = W(g.x)$.

The relation $(g^*)^{-1}(W) = \bigcup_{x \in \Lambda([supp\mu])} (g^*)^{-1}(W(x)) = \bigcup_{x \in \Lambda([supp\mu])} W(g.x)$ shows that W is $[supp\mu^*]$ -invariant. Then condition i-p implies $W = \mathbb{P}^{d-1}$, $r = 1, p = d - 1$, $d(x) = d - 1, m(x) = 1$ for any $x \in \mathbb{P}^{d-1}$, hence the Lemma. \square

Proof of the theorem

We use the Markov kernel π_x^s defined by $\pi_x^s = \frac{\nu_x^s}{\nu_x^s(1)}$ with ν_x^s given in Lemma 3.6.

Then we have the harmonicity equation :

$$\pi_x^s = \int q^s(x, g) g^* \cdot \pi_{g.x}^s d\mu(g)$$

and the continuity in variation of π_x^s . The above equation implies that the sequence $S_n^*(\omega) \cdot \pi_{S_n(\omega).x}^s$ is a \mathbb{Q}_x^s -martingale with respect to the natural filtration on Ω . Since $\pi_x^s \in M^1(\mathbb{P}^{d-1})$ we can apply the martingale convergence theorem.

Since, by Lemma 3.9, π_x^s is proper and, by definition, $x \rightarrow \pi_x^s$ is continuous in variation, we can use the same method as in ([6], [29]) : because of i-p condition, the martingale $S_n^*(\omega) \cdot \pi_{S_n(\omega).x}^s$ converges $\mathbb{Q}_x^s - a.e$ to a Dirac measure. Then, using Lemma 3.4, $S_n^* \cdot m$ converges $\mathbb{Q}_x^s - a.e$ to a Dirac measure $\delta_{z^*(\omega)}$. Then, from above, for any $x \in \mathbb{P}^{d-1}$:

$$\mathbb{Q}_x^s(\Omega') = 1 \quad , \quad z^*(\mathbb{Q}_x^s) = \pi_x^s.$$

Hence the law of $z^*(\omega)$ under \mathbb{Q}_x^s is π_x^s . It follows, by integration :

$$\mathbb{Q}^s(\Omega') = 1 \quad , \quad z^*(\mathbb{Q}^s) = \int z^*(\mathbb{Q}_x^s) d\pi^s(x) = \int \pi_x^s d\pi^s(x).$$

In view of the formulas for ν_x^s , π_x^s and the relation $\pi^s = \frac{e^s \nu^s}{\nu^s(e^s)}$, we get $z^*(\mathbb{Q}^s) = {}^* \pi^s$. Then Lemma 3.9 and the definition of π_x^s give that ${}^* \pi^s$ is proper. The relations :

$\lim_{n \rightarrow \infty} \delta(S_n(\omega).x, S_n(\omega).y) = 0$, $\lim_{n \rightarrow \infty} \frac{|S_n(\omega)x|}{|S_n(\omega)|} = |\langle z^*(\omega), x \rangle|$, $\lim_{n \rightarrow \infty} \sigma(S_n, \xi) = -\infty$, follow from the geometrical Lemma 3.3, since $S_n^* \cdot m$ converges to $\delta_{z^*(\omega)}$.

Using Lemma 3.5 we know that \mathbb{Q}_x^s is absolutely continuous with respect to \mathbb{Q}^s . We calculate $\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega)$ as follows.

By definition of \mathbb{Q}_x^s and \mathbb{Q}^s :

$$\mathbb{E}_x^s\left(\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega)|g_1, \dots, g_n\right) = \frac{q_n^s(x, S_n(\omega))}{\int q_n^s(y, S_n(\omega))d\pi^s(y)}$$

Furthermore :

$$\frac{q_n^s(x, S_n(\omega))}{q_n^s(y, S_n(\omega))} = \frac{|S_n(\omega)x|^s}{|S_n(\omega)y|^s} \frac{e^s(S_n(\omega), x)}{e^s(S_n(\omega), y)} \frac{e^s(y)}{e^s(x)}.$$

The martingale convergence theorem gives :

$$\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega) = \lim_{n \rightarrow \infty} \frac{q_n^s(x, S_n(\omega))}{\int q_n^s(y, S_n(\omega))d\pi^s(y)}.$$

Using the relation $\lim_{n \rightarrow \infty} \frac{|S_n(\omega)x|}{|S_n(\omega)|} = |\langle z(\omega), x \rangle|$, if $\omega \in \Omega'$, we get :

$$\lim_{n \rightarrow \infty} \frac{q_n^s(x, S_n(\omega))}{q_n^s(y, S_n(\omega))} = \left| \frac{\langle z^*(\omega), x \rangle}{\langle z^*(\omega), y \rangle} \right|^s \frac{e^s(y)}{e^s(x)}.$$

Hence $\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega) = \frac{|\langle z^*(\omega), x \rangle|^s}{e^s(x)} \left[\int \frac{|\langle z^*(\omega), y \rangle|^s}{e^s(y)} d\pi^s(y) \right]^{-1}$. Since, from above π^s is proper and the \mathbb{Q}^s -law of $z^*(\omega)$ is π^s , we have for any $x \in \mathbb{P}^{d-1}$: $|\langle z^*(\omega), x \rangle| > 0$, $\mathbb{Q}^s - a.e.$

Hence $\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega) > 0$ $\mathbb{Q}^s - a.e.$, i.e \mathbb{Q}_x^s is equivalent to \mathbb{Q}^s ; Also, using the formulas above :

$$\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}_y^s}(\omega) = \left| \frac{\langle z^*(\omega), x \rangle}{\langle z^*(\omega), y \rangle} \right|^s \frac{e^s(y)}{e^s(x)} \quad \square$$

3) The law of large numbers for $\text{Log}|S_n(\omega)x|$ with respect to \mathbb{Q}^s

Here, by derivatives of a function φ at the boundaries of an interval $[a, b]$ we will mean finite half derivatives i.e we write :

$$\varphi'(a) = \varphi'(a_+) \in \mathbb{R} \quad , \quad \varphi'(b) = \varphi'(b_-) \in \mathbb{R}$$

Theorem 3.10

Let $\mu \in M^1(G)$, $s \in I_\mu$. Assume $[\text{supp}\mu]$ satisfies i-p, and $\text{Log}\gamma(g)$ is μ -integrable. Assume also that $|g|^s \text{Log}\gamma(g)$ is μ -integrable and write

$$L_\mu(s) = \int \text{Log}|gx|q^s(x, g)d\pi^s(x) d\mu(g).$$

Then, for any $x \in \mathbb{P}^{d-1}$, we have $\mathbb{Q}^s - a.e$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)x| = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)| = L_\mu(s).$$

This convergence is valid in $\mathbb{L}^1(\mathbb{Q}^s)$ and in $\mathbb{L}^1(\mathbb{Q}_x^s)$ for any $x \in \mathbb{P}^{d-1}$. Furthermore $k(t)$ has a continuous derivative on $[0, s]$ and if $t \in [0, s]$, $x \in \mathbb{P}^{d-1}$:

$$L_\mu(t) = \frac{k'(t)}{k(t)} = \lim_{n \rightarrow \infty} \frac{1}{nk^n(t)} \int |gx|^t \text{Log}|gx| d\mu^n(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \frac{\int |g|^t \text{Log}|g| d\mu^n(g)}{\int |g|^t d\mu^n(g)}.$$

In particular if $\alpha > 0$ satisfies $k(\alpha) = 1$, then $k'(\alpha) > 0$.

Proof

We consider the function $f(x, \omega)$ on ${}^a\Omega$ defined by $f(x, \omega) = \text{Log}|g_1(\omega)x|$. If $|x| = 1$, we have $-\text{Log}|g^{-1}| \leq \text{Log}|gx| \leq \text{Log}|g|$, hence $f(x, \omega)$ is ${}^a\mathbb{Q}^s$ -integrable.

Moreover $\int f(x, \omega) d^a\mathbb{Q}(x, \omega) = \int q^s(x, g) \text{Log}|gx| d\nu^s(x) d\mu(g) = L_\mu(s)$, and

$$\sum_0^{n-1} (f \circ {}^a\theta^k)(x, \omega) = \text{Log}|S_n(\omega)x|.$$

As mentioned above ${}^a\mathbb{Q}^s$ is ${}^a\theta$ -ergodic, hence we get using Birkhoff's theorem :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)x| = L_\mu(s), \quad {}^a\mathbb{Q}^s - a.e.$$

On the other hand, we can apply the subadditive ergodic theorem to the sequence $\text{Log}|S_n(\omega)|$ and to the ergodic system $(\Omega, \theta, \mathbb{Q}^s)$.

This gives that there exists $L(s) \in \mathbb{R}$ such that, $\mathbb{Q}^s - a.e$ and in $\mathbb{L}^1(\mathbb{Q}^s)$, the sequence $\frac{1}{n} \text{Log}|S_n(\omega)|$ converges to $L(s)$. We know, using Theorem 3.2 that, for fixed x and $\mathbb{Q}^s - a.e$,

$$\lim_{n \rightarrow \infty} \frac{|S_n(\omega)x|}{|S_n(\omega)|} = | \langle z^*(\omega), x \rangle |,$$

and furthermore the law of $z^*(\omega)$ under \mathbb{Q}^s is proper. Hence, for fixed x we have :

$$| \langle z^*(\omega), x \rangle | > 0, \quad \mathbb{Q}^s - a.e.$$

Then for fixed $x \in \mathbb{P}^{d-1}$ and $\mathbb{Q}^s - a.e$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)x| = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)| = L_\mu(s).$$

Using Lemma 3.6, since $\mathbb{Q}_x^s \leq c(s) \mathbb{Q}^s$ this convergence is also valid $\mathbb{Q}_x^s - a.e$. Hence by definition of ${}^a\mathbb{Q}^s$, we have $L(s) = L_\mu(s)$. The first assertion follows.

In order to get the \mathbb{L}^1 -convergences, we observe that Fatou's Lemma gives :

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)x| d\mathbb{Q}^s(\omega) \geq L_\mu(s).$$

On the other hand, the subadditive ergodic theorem gives :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)| d\mathbb{Q}^s(\omega) = L(s) = L_\mu(s).$$

Since $|S_n(\omega)x| \leq |S_n(\omega)|$ if $|x| = 1$, these two relations imply, for every $x \in \mathbb{P}^{d-1}$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)x| d\mathbb{Q}^s(\omega) = L_\mu(s).$$

Now we write :

$$\frac{1}{n}|Log|S_n(\omega)x| - L(s)| \leq \frac{1}{n}(Log|S_n(\omega)| - Log|S_n(\omega)x|) + \frac{1}{n}|Log|S_n(\omega)| - L(s)|.$$

From the above calculation, the integral of the first term converge to zero. The subadditive ergodic theorem implies the same for the second term.

Hence $\lim_{n \rightarrow +\infty} \int |Log|S_n(\omega)x| - L(s)|d\mathbb{Q}^s(\omega) = 0$. Since $\mathbb{Q}_x^s \leq c(s)\mathbb{Q}^s$, this convergence is also valid in $\mathbb{L}^1(\mathbb{Q}_x^s)$ for any fixed x . This gives the second assertion, in particular :

$$L_\mu(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \int Log|S_n(\omega)x|d\mathbb{Q}_x^s(\omega) = \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}_x^s(Log|S_n(\omega)x|).$$

The above limit can be expressed as follows.

Let φ be a continuous function on \mathbb{P}^{d-1} . Then Theorem 2.6 implies :

$$\lim_{n \rightarrow \infty} \mathbb{E}_x^s(\varphi(S_n(\omega).x)) = \lim_{n \rightarrow \infty} (Q^s)^n \varphi(x) = \pi^s(\varphi),$$

uniformly in $x \in \mathbb{P}^{d-1}$.

Hence $L_\mu(s)\pi_s(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x^s(\varphi(S_n(\omega).x)Log|S_n(\omega)x|)$.

In particular with $\varphi = \frac{1}{e^s}$ and any x :

$$L_\mu(s) = \lim_{n \rightarrow \infty} \frac{1}{nk^n(s)} \int |gx|^s Log|gx|d\mu^n(g).$$

We denote $v_n(s) = \int |gx|^s d\mu^n(g)$ and we observe that $v'_n(s) = \int |gx|^s Log|gx| d\mu^n(g)$.

Using Theorem 2.6, we get : $\lim_{n \rightarrow \infty} \frac{v_n(s)}{k^n(s)} = \pi^s \left(\frac{1}{e^s} \right) = 1$.

Then the above formula for $L_\mu(s)$ reduces to $L_\mu(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \frac{v'_n(s)}{v_n(s)}$.

On the other-hand : $\frac{1}{n} Logv_n(s) = \frac{1}{n} \int_0^s \frac{v'_n(t)}{v_n(t)} dt$, $\lim_{n \rightarrow \infty} \frac{1}{n} Logv_n(s) = Logk(s)$.

The convexity of $Logv_n$ on $[0, s]$ gives :

$$\frac{v'_n(0)}{v_n(0)} \leq \frac{v'_n(t)}{v_n(t)} \leq \frac{v'_n(s)}{v_n(s)}.$$

Then the convergence of the sequences $\frac{1}{n} \frac{v'_n(0)}{v_n(0)}$ and $\frac{1}{n} \frac{v'_n(s)}{v_n(s)}$ implies that the sequence $\frac{1}{n} \frac{v'_n(t)}{v_n(t)}$ is uniformly bounded on $[0, s]$. On the other hand, Hölder inequality implies the μ -integrability of $|g|^t Log|g|$ if $t \in [0, s]$, and the bound :

$$\int |g|^t |Log|g||d\mu(g) \leq \left(\int |Log|g||d\mu(g) \right)^{\frac{s-t}{s}} \left(\int |g|^s |Log|g||d\mu(g) \right)^{t/s}.$$

Hence, as above, we have the convergence of $\frac{1}{n} \frac{v'_n(t)}{v_n(t)}$, to $L(t)$. Then dominated convergence gives the convergence of $\frac{1}{n} Logv_n(s)$ to $Logk(s) = \int_0^s L(t)dt$.

We have $L(t) = \int \text{Log}|gx|q^t(x, g)d\mu(g)d\pi^t(x)$, and the continuity of q^s, π^s given by Theorem 2.6. Then the bound of $\int |g|^t|\text{Log}|g||d\mu(g)$ given above imply the continuity of $L(t)$ on $[0, s]$. The integral expression of $\text{Log}k(s)$ in terms of $L(t)$ implies that k has a derivative and $\frac{k'(t)}{k(t)} = L(t)$ if $t \in [0, s]$. This gives the first part of the last relation in the theorem. In order to get the rest, we consider $u_n(t) = \int |g|^t d\mu^n(g)$ and write for $t \in [0, s]$:

$$\frac{u'_n(t)}{u_n(t)} = \frac{\int |g|^t \text{Log}|g| d\mu^n(g)}{\int |g|^t d\mu^n(g)}$$

The convergence of $\frac{1}{n}\text{Log}u_n(t)$ to $\text{Log}k(t)$ and the convexity of the functions $\text{Log}u_n(t), \text{Log}k(t)$ give for $t \in [0, s]$:

$$\frac{k'(t_-)}{k(t)} \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \frac{u'_n(t)}{u_n(t)} \leq \limsup_{n \rightarrow \infty} \frac{u'_n(t)}{u_n(t)} \leq \frac{k'(t_+)}{k(t)}.$$

Since if $t \in]0, s[$, we have $k'(t_-) = k'(t_+) = k'(t)$, we get $\lim_{n \rightarrow \infty} \frac{1}{n} \frac{u'_n(t)}{u_n(t)} = \frac{k'(t)}{k(t)}$ if $t < s$.

Furthermore, by continuity we have $\lim_{n \rightarrow \infty} \frac{1}{n} \frac{u'_n(s)}{u_n(s)} = \frac{k'(s_-)}{k(s)}$. Now the rest of the formula follows from the expression of $\frac{u'_n(t)}{u_n(t)}$ given above. The relation $L_\mu(\alpha) > 0$ follows from the formula $L_\mu(t) = \frac{k'(t)}{k(t)}$ and the strict convexity of $\text{Log} k(t)$. \square

4) Lyapunov exponents and spectral gaps.

We begin with a more general situation than above. As special cases, it contains the Markov chains on \mathbb{P}^{d-1} considered in section 2. In particular simplicity of the dominant Lyapunov exponent, given by Theorem 3.17 below, will be a simple consequence of their special properties and of the general Proposition 3.11 below. For $s = 0$, this result was shown in [29] under condition i-p. For the use of the Zariski closure as a tool to show condition i-p see [23], [45]. We give corresponding notations.

Let X be a compact metric space, $\mathcal{C}(X, X)$ the semigroup of continuous maps of X into itself which is endowed with by the topology of uniform convergence. We denote by $g.x$ the action of $g \in \mathcal{C}(X, X)$ on $x \in X$ and we consider a closed subsemigroup Σ of $\mathcal{C}(X, X)$. Let μ be a probability measure on Σ and let $q(x, g)$ be a continuous non negative function on $X \times \text{supp}\mu$ such that $\int q(x, g)d\mu(g) = 1$. We will denote by $(X, q \otimes \mu, \Sigma)$ this set of datas and we will say that $(X, q \otimes \mu, \Sigma)$ is a Markov system on (X, Σ) . We write $\Omega = \Sigma^{\mathbb{N}}$, we denote by Ω_n the set of finite sequences of length n on Σ and for $\omega = (g_1, g_2, \dots, g_n)$ in $\Omega_n, x \in X$, we write $q_n(x, \omega) = \prod_1^n q(S_{k-1}.x, g_k)$ where $S_n = g_n \cdots g_1, S_0 = Id$.

We define a probability measure \mathbb{Q}_x^n on Ω_n by $\mathbb{Q}_x^n = q_n(x, \cdot)\mu^{\otimes n}$ and we denote by \mathbb{Q}_x the probability measure on Ω which is the projective limit of this system. If ν is a probability measure on X we set $\mathbb{Q}_\nu = \int \mathbb{Q}_x d\nu(x)$. We will consider the extended shift ${}^a\theta$ on ${}^a\Omega = X \times \Omega$

which is defined by : ${}^a\theta(x, \omega) = (g_1.x, \theta\omega)$, and also the Markov chain on X with kernel Q defined by $Q\varphi(x) = \int \varphi(g.x)q(x, g)d\mu(g)$. Clearly, when endowed with the corresponding shift, the space of paths of this Markov chain is a factor system of $(X \times \Omega, {}^a\theta)$. If π is a Q -stationary measure on X , the measure \mathbb{Q}_π on Ω is shift-invariant and ${}^a\mathbb{Q}_\pi = \int \delta_x \otimes \mathbb{Q}_x d\pi(x)$ is ${}^a\theta$ -invariant. In this situation we will say that $(X, q \otimes \mu, \Sigma, \pi)$ is a stationary Markov system. If π is Q -ergodic, then ${}^a\mathbb{Q}_\pi$ is ${}^a\theta$ -ergodic and \mathbb{Q}_π is θ -ergodic. We will denote by $\mathbb{E}_x, \mathbb{E}_\pi$ the corresponding expectations symbols.

In particular we will consider below Markov systems of the form $(X, q \otimes \mu, T)$ where $\Sigma = T \subset GL(d, \mathbb{R})$ ($d > 1$), and also flag-extensions of them. We can extend the action of $g \in T$ to $X \times \mathbb{P}^{d-1}$ by $g(x, v) = (g.x, g.v)$ and define a new Markov chain with kernel Q_1 by $Q_1\varphi(x, v) = \int \varphi(g.x, g.v)q(x, g)d\mu(g)$. Given a Q -stationary probability measure π , we will denote by C_1 the compact convex set of probability measures on $X \times \mathbb{P}^{d-1}$ which have projection π on X . The same considerations apply if \mathbb{P}^{d-1} is replaced by \mathbb{P}_2^{d-1} , the manifold of 2-planes or $\mathbb{P}_{1,2}^{d-1}$, the manifold of contact elements in \mathbb{P}^{d-1} . Then we define similarly the kernels $Q_2, Q_{1,2}$ and the convex sets $C_2, C_{1,2}$.

Since $S_n = g_n \cdots g_1$ and the g_k are \mathbb{Q}_π -stationary random variables where \mathbb{Q}_π is θ -invariant and ergodic, the Lyapunov exponents of the product S_n are well defined as soon as :

$$\int \text{Log}|g_1(\omega)|d\mathbb{Q}_\pi(\omega) \text{ and } \int \text{Log}|g_1^{-1}(\omega)|d\mathbb{Q}_\pi(\omega)$$

are finite (see [46]). In particular the two largest ones γ_1 and γ_2 are given by :

$$\gamma_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)|d\mathbb{Q}_\pi(\omega), \quad \gamma_1 + \gamma_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|\wedge^2 S_n(\omega)|d\mathbb{Q}_\pi(\omega)$$

In order to study the values of γ_1, γ_2 we need to consider the above Markov operators $Q_1, Q_2, Q_{1,2}$ and the convex sets $C_1, C_2, C_{1,2}$ of corresponding stationary measures. We denote $q(g) = |q(\cdot, g)|$ and we assume $\int \text{Log}q(g)d\mu(g) < \infty$. For $\eta_1 \in C_1$, we will write $I_1(\eta_1) = \int \sigma_1(g, v)d\eta_1(x, v)d\mu(g)$. Similarly with $\eta_2 \in C_2, \eta_{1,2} \in C_{1,2}$ we define $I_2(\eta_2), I_{1,2}(\eta_{1,2})$. The following result will be a basic tool in this subsection.

Proposition 3.11

With the above notations, let T be a closed subsemigroup of $GL(d, \mathbb{R})$, $(X, q \otimes \mu, T, \pi)$ a stationary and uniquely ergodic Markov system. Assume that $S_n^*.m$ converges $\mathbb{Q}_\pi - a.e$ to a Dirac measure $\delta_{z^*(\omega)}$ such that for any $v \in \mathbb{P}^{d-1}$, $\langle v, z^*(\omega) \rangle \neq 0$ $\mathbb{Q}_\pi - a.e$. Assume that $\int \text{Log}q(g)d\mu(g)$ is finite. Then we have $\gamma_2 - \gamma_1 = \sup\{I_{1,2}(\eta); \eta \in C_{1,2}\} < 0$, and the sequence $\frac{1}{n} \sup_{x, v, v'} \mathbb{E}_x(\text{Log} \frac{\delta(S_n(\omega).v, S_n(\omega).v')}{\delta(v, v')})$ converges to $\gamma_2 - \gamma_1 < 0$.

The proof uses the same arguments as in [6], [30]. The condition $\lim_{n \rightarrow \infty} S_n^*.m = \delta_{z^*(\omega)}$ is satisfied in the examples of subsection 3.2 (see Theorem 3.2).

Lemma 3.12

Let m_p be the natural $SO(d)$ -invariant probability measure on the submanifold of p -decomposable unit multivectors $x = v_1 \wedge v_2 \wedge \cdots \wedge v_p$. Then there exists $c > 0$ such

that for any $u \in \text{End}V : 0 < \text{Log}|\wedge^p u| - \int \text{Log}|(\wedge^p u)(x)|dm_p(x) \leq c$.

Proof

We write u in polar form $u = kak'$ with $k, k' \in SO(d), a = \text{diag}(a_1, a_2 \cdots, a_d)$ and $a_1 \geq a_2 \geq \cdots \geq a_d > 0$. We write also $x = k''\varepsilon^p$ with $k'' \in SO(d), \varepsilon^p = e_1 \wedge e_2 \wedge \cdots \wedge e_p$, hence :

$$\int \text{Log}|\wedge^p ux|dm_p(x) = \int \text{Log}|\wedge^p ak'k''\varepsilon^p|d\tilde{m}(k'') = \int \text{Log}|\wedge^p ak\varepsilon^p|d\tilde{m}(k),$$

where \tilde{m} is the normalized Haar measure on $SO(d)$. Furthermore :

$$\begin{aligned} |\wedge^p ak\varepsilon^p| &\geq | \langle \wedge^p ak\varepsilon^p, \varepsilon^p \rangle | = | \wedge^p a | \langle k\varepsilon^p, \varepsilon^p \rangle |, \\ \int \text{Log}|\wedge^p ux|dm_p(x) &\geq \text{Log}|\wedge^p u| + \int \text{Log}|\langle k\varepsilon^p, \varepsilon^p \rangle|d\tilde{m}(k) \\ &= \text{Log}|\wedge^p u| + \int \text{Log}|\langle x, \varepsilon^p \rangle|dm_p(x). \end{aligned}$$

Hence it suffices to show the finiteness of the integral in the right hand side. But the set of unit decomposable p -vectors is an algebraic submanifold of the unit sphere of $\wedge^p V$ and m_p is its natural Riemannian measure. Since the map $x \rightarrow \langle x, \varepsilon^p \rangle^2$ is polynomial, there exists $q \in \mathbb{N}, c > 0$ such that : $ct^q \leq m_p\{x; \langle x, \varepsilon^p \rangle^2 \leq t\} \leq 1$.

The push forward σ of m_p by this map is an absolutely continuous probability measure on $[0,1]$ which satisfies $\sigma(0, t) \geq ct^{q/2}$. Then :

$$\int \text{Log}|\langle x, \varepsilon^p \rangle|dm_p(x) = \int_0^1 \text{Log} t d\sigma(t) > -\infty,$$

since the integral $\int_0^1 t^{q/2-1} dt$ is finite for $q > 0$. \square

We recall from [38] the following :

Lemma 3.13

Let (E, θ, ν) be a dynamical system where ν is a θ -invariant probability measure, f a ν -integrable function. If $\sum_1^n f \circ \theta^k$ converges $\nu - a.e$ to $-\infty$, then one has $\nu(f) < 0$.

Lemma 3.14

Let E be a compact metric space, P a Markov kernel on E which preserves the space of continuous functions on E , $C(P)$ the compact convex set of P -stationary measures. Then, for every continuous function f on E , the sequence $\sup_{x \in E} \frac{1}{n} \sum_1^n P^k f(x)$ converges to $\sup\{\eta(f); \eta \in C(P)\}$. In particular, if for all $\eta, \eta' \in C(P)$ we have $\eta(f) = \eta'(f)$, then we have the uniform convergence : $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_1^n P^k f(x) = \eta(f)$.

Proof

Let $J \subset \mathbb{R}$ be the set of cluster values of the sequences $\frac{1}{n} \sum_0^{n-1} (P^k f)(x_n)$ with $x_n \in E$. We will show that the convex envelope of J is equal to $\{\eta(f) ; \eta \in C(P)\}$. If the sequence $\frac{1}{n_k} \sum_0^{n_k} (P^i f)(x_{n_k})$ converges to $c \in \mathbb{R}$, we can consider the sequence of probability measures

$\eta_k = \frac{1}{n_k} \sum_0^{n_k-1} P^i \delta_{x_{n_k}}$ and extract a convergent subsequence with limit $\eta \in C(P)$. Then, since f is continuous :

$$\eta(f) = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_0^{n_k-1} (P^i f)(x_{n_k}) = c.$$

Conversely, if $\eta \in C(P)$ is ergodic, Birkhoff's theorem applied to the sequence $\frac{1}{n} \sum_0^{n-1} (P^i f)(x)$ gives $\eta - a.e$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_0^{n-1} (P^i f)(x) = \eta(f),$$

hence there exists $x \in E$ such that $\eta(f)$ is the limit of $\frac{1}{n} \sum_0^{n-1} (P^i f)(x)$. If η is not ergodic, η is a barycenter of ergodic measures, hence $\eta(f)$ belongs to the convex envelope of J . In view of the above, this shows the claim. Since J is closed, the convex envelope of J is a closed interval $[a, b]$, hence $b = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in E} \sum_0^{n-1} (P^i f)(x) = \sup_{\eta \in C(P)} \eta(f)$. \square

Lemma 3.15

We have the formulas

$$\gamma_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x,v} \int \text{Log}|S_n(\omega)v| d\mathbb{Q}_x(\omega) = \sup_{\eta \in C_1} I_1(\eta),$$

$$\gamma_1 + \gamma_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x,v,v'} \int \text{Log}|S_n(\omega)v \wedge S_n(\omega)v'| d\mathbb{Q}_x(\omega) = \sup_{\eta \in C_2} I_2(\eta).$$

Proof

We consider the Markov chain on $X \times \mathbb{P}^{d-1}$ with kernel Q_1 defined by :

$$Q_1 \varphi(x, v) = \int \varphi(g.x, g.v) q(x, g) d\mu(g),$$

and the function $\psi(x, v) = \int \sigma_1(g, v) q(x, g) d\mu(g)$.

We observe that : $\int \sigma_1(S_n(\omega), v) d\mathbb{Q}_x(\omega) = \sum_0^{n-1} Q_1^k \psi(x, v)$,

and ψ is continuous since $\int \text{Log} \gamma(g) q(g) d\mu(g) < \infty$. Also, since π is the unique Q -stationary measure, any Q_1 -stationary measure has projection π on X . Then, using Lemma 3.14 :

$$\sup_{\eta \in C_1} I_1(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x,v} \int \sigma_1(S_n(\omega), v) d\mathbb{Q}_x(\omega),$$

which gives the second part of the first formula. In order to show the first part we consider $\eta \in C_1$ which is Q_1 -ergodic, the extended shift $\tilde{\theta}$ on $X \times \mathbb{P}^{d-1} \times \Omega$ and the function $f(x, v, \omega) = \sigma_1(g_1(\omega), v)$. Then :

$$\tilde{\theta}(x, v, \omega) = (g_1.x, g_1.v, \theta\omega) \text{ and } \sigma_1(S_n(\omega), v) = \sum_0^{n-1} f \circ \tilde{\theta}^k(x, v, \omega).$$

Also, $f(x, v, \omega)$ is $\hat{\mathbb{Q}}_\eta$ -integrable where $\hat{\mathbb{Q}}_\eta = \int \delta_{(x,v)} \otimes \mathbb{Q}_x d\eta(x, v)$.

Using the subadditive ergodic theorem :

$$I_1(\eta) = \frac{1}{n} \int \sigma_1(S_n(\omega), v) d\mathbb{Q}_x(\omega) d\eta(x, v) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)| d\mathbb{Q}_\pi(\omega) = \gamma_1.$$

We show now that for some $\eta \in C_1$ we have $\gamma_1 = I_1(\eta)$.

Using Lemma 3.12, we know that :

$$0 \leq \text{Log}|S_n(\omega)| - \int \text{Log}|S_n(\omega)v|dm(v) \leq c,$$

hence, integrating with respect to \mathbb{Q}_π :

$$0 \leq \int dm(v) \int (\text{Log}|S_n(\omega)| - \text{Log}|S_n(\omega)v|)d\mathbb{Q}_\pi(\omega) \leq c.$$

Then the sequence of non negative functions $h_n(v)$ on \mathbb{P}^{d-1} given by :

$$h_n(v) = \frac{1}{n} \int (\text{Log}|S_n(\omega)| - \text{Log}|S_n(\omega)v|)d\mathbb{Q}_\pi(\omega)$$

satisfies $0 \leq h_n(v) \leq \frac{c}{n}$ with c given by Lemma 3.11, $\lim_{n \rightarrow \infty} \int h_n(v)dm(v) = 0$. It follows that we can find a subsequence h_{n_j} such that $h_{n_j}(v)$ converges to zero $m - a.e$, hence :

$$\gamma_1 = \lim_{j \rightarrow \infty} \frac{1}{n_j} \int \sigma_1(S_{n_j}(\omega), v)d\mathbb{Q}_\pi(\omega) \quad m - a.e,$$

in particular the convergence is valid for some $v \in \mathbb{P}^{d-1}$. The sequence of probability measures $\frac{1}{n_j} \sum_1^{n_j} \mathbb{Q}_1^k(\pi \otimes \delta_v)$ has a weakly convergent subsequence η_j with a \mathbb{Q}_1 -invariant limit η . Furthermore, the function ψ considered above is continuous, hence :

$$\eta(\psi) = \lim_{j \rightarrow \infty} \frac{1}{n_j} \int \sigma_1(S_{n_j}(\omega)v)d\mathbb{Q}_\pi(\omega) = \gamma_1, \quad \gamma_1 = I_1(\eta) = \sup_{\eta_1 \in C_1} I_1(\eta_1).$$

The same argument is valid for $\text{Log}|S_n(\omega)(v \wedge v')|$ with m replaced by m_2 , hence the second formula. \square

Lemma 3.16

For any $\eta \in C_1$, we have $\gamma_1 = I_1(\eta)$

Proof

As in the proof of Lemma 3.15 we have $I_1(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \sigma_1(S_n(\omega), v) \quad \mathbb{Q}_\eta - a.e$, hence the existence of $v \in \mathbb{P}^{d-1}$ such that $\mathbb{Q}_\pi - a.e$:

$$I_1(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)v|$$

Then, using Theorem 3.2 and Lemma 3.15 we get, $\mathbb{Q}_\pi - a.e$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} \frac{|S_n(\omega)v|}{|S_n(\omega)|} = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} |\langle z^*(\omega), v \rangle| = 0,$$

since $\langle z^*(\omega), v \rangle \neq 0$, $\mathbb{Q}_\pi - a.e$.

Hence $I_1(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)|d\mathbb{Q}_\pi(\omega) = \gamma_1 \quad \square$

Proof of the Proposition

We have $\gamma_2 - \gamma_1 = (\gamma_1 + \gamma_2) - 2\gamma_1$, $\gamma_1 = I_1(\eta_1)$ for any $\eta_1 \in C_1$ and $\gamma_1 + \gamma_2 = \sup_{\eta_2 \in C_2} I_2(\eta_2)$.

Using the theorem of Markov-Kakutani, for the inverse image of $\eta_2 \in C_2$ in $C_{1,2}$ we know

that any $\eta_2 \in C_2$ is the projection of some $\eta_{1,2} \in C_{1,2}$, hence $\gamma_1 + \gamma_2 = \sup_{\eta_{1,2} \in C_{1,2}} I_2(\eta_{1,2})$. If

η'_1 is the projection of $\eta_{1,2}$ on \mathbb{P}^{d-1} , we have $I_{1,2}(\eta_{1,2}) = I_2(\eta_{1,2}) - 2I_1(\eta'_1)$ and from Lemma 3.16, $\gamma_1 = I_1(\eta'_1)$. It follows $\gamma_2 - \gamma_1 = \sup_{\eta_{1,2} \in C_{1,2}} I_{1,2}(\eta_{1,2})$.

Since $I_{1,2}(\eta_{1,2})$ depends continuously of $\eta_{1,2}$ and $C_{1,2}$ is compact, in order to show that $\gamma_2 - \gamma_1$ is negative it suffices to prove $I_{1,2}(\eta) < 0$, for any $\eta \in C_{1,2}$. We consider the extended shift $\tilde{\theta}$ on $X \times \mathbb{P}_{1,2}^{d-1} \times \Omega$ defined by $\tilde{\theta}(x, \xi, \omega) = (g_1.x, g_1.\xi, \theta\omega)$, the function $f(\xi, \omega) = \sigma(g_1, \xi)$ and the $\tilde{\theta}$ -invariant measure $\tilde{\mathbb{Q}}_\eta = \int \delta_{(x, \xi)} \otimes \mathbb{Q}_x d\eta(x, \xi)$. Since $S_n^*.m$ converges $\mathbb{Q}_\pi - a.e$ to $\delta_{z^*(\omega)}$, Lemma 3.4 implies $\lim_{n \rightarrow \infty} \sigma(S_n(\omega), \xi) = -\infty, \mathbb{Q}_\pi - a.e$ if the origin v of ξ satisfies $\langle v, z^*(\omega) \rangle \neq 0$. By hypothesis, this condition is satisfied for any ξ and $\mathbb{Q}_\pi - a.e$, hence we have $\lim_{n \rightarrow \infty} \sum_1^n f \circ \theta^k = -\infty \mathbb{Q}_\pi - a.e$ for any ξ . It follows that this convergence is valid $\tilde{\mathbb{Q}}_\eta - a.e$, hence Lemma 3.13 gives : $\eta(f) = I_{1,2}(\eta) < 0$.

We consider $I_n = \frac{1}{n} \mathbb{E}_x \left(\text{Log} \frac{\delta(S_n(\omega).v, S_n(\omega).v')}{\delta(v, v')} \right)$. With $|v| = |v'| = 1$, $\delta_1(v, v') = |v \wedge v'|$:

$$\text{Log} \frac{\delta_1(S_n(\omega).v, S_n(\omega).v')}{\delta_1(v, v')} = \text{Log} \frac{|S_n(\omega)v \wedge S_n(\omega)v'|}{|v \wedge v'|} - \text{Log}|S_n(\omega)v| - \text{Log}|S_n(\omega)v'|.$$

By Lemma 3.15, we have :

$$\gamma_1 + \gamma_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x, v, v'} \mathbb{E}_x \left(\text{Log} \frac{|(S_n(\omega)v \wedge S_n(\omega)v')|}{|v \wedge v'|} \right).$$

Also, by Lemmas 3.14 and 3.16, we have the convergence of $\sup_{x, v} \frac{1}{n} \mathbb{E}_x(\text{Log}|S_n(\omega)v|)$ and

$\inf_{x, v} \frac{1}{n} \mathbb{E}_x(\text{Log}|S_n(\omega)v|)$ to $I_1(\eta) = \gamma_1$. The uniform convergence of $\frac{1}{n} \mathbb{E}_x(\text{Log}|S_n(\omega)v|)$ to γ_1

follows. Then the equivalence of δ_1, δ implies that $\sup_{x, v, v'} \frac{1}{n} I_n$ converges to $\gamma_2 - \gamma_1$. \square

With the notations of paragraph 3 above we have the following corollaries, for products of random matrices.

Theorem 3.17

Assume $d > 1$, the closed subsemigroup $T \subset GL(d, \mathbb{R})$ satisfies condition i-p, $s \in I_\mu$ and $\int |g|^s \text{Log} \gamma(g) d\mu(g)$ is finite. Then the dominant Lyapunov exponent of $S_n(\omega)$ is simple and :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x, v, v'} \mathbb{E}_x^s \left(\text{Log} \frac{\delta(S_n(\omega).v, S_n(\omega).v')}{\delta(v, v')} \right) = L_{\mu, 2}(s) - L_{\mu, 1}(s) < 0$$

where $L_{\mu, 1}(s), L_{\mu, 2}(s)$ are the two highest Lyapunov exponents of $S_n(\omega)$ with respect to \mathbb{Q}^s .

In particular : $\lim_{n \rightarrow \infty} \frac{1}{n} \sup_{v, v'} \mathbb{E}^s \left(\text{Log} \frac{\delta(S_n(\omega)v, S_n(\omega)v')}{\delta(v, v')} \right) \leq L_{\mu, 2}(s) - L_{\mu, 1}(s) < 0$.

Proof

In view of Theorems 3.2, 2.6, the conditions of Proposition 3.11 are satisfied by :
 $X = \mathbb{P}^{d-1}, q \otimes \mu = q^s \otimes \mu, \pi = \pi^s$. On the other hand we have $\mathbb{Q}^s = \int \mathbb{Q}_x^s d\pi(x)$, hence the second formula. \square

We will use Theorem 3.17 to establish certain functional inequalities for the operators Q^s, \tilde{Q}^s on $\mathbb{P}^{d-1}, \mathbb{S}^{d-1}$ defined in section 2 and acting on $H_\varepsilon(\mathbb{P}^{d-1})$ or $H_\varepsilon(\mathbb{S}^{d-1})$. Using [34], spectral gap properties will follow. We will say that Q satisfies a "Doebelin-Fortet inequality" on $H_\varepsilon(X)$, where X is a compact metric space if we have for any $\varphi \in H_\varepsilon(X)$: $[Q^{n_0}\varphi]_\varepsilon \leq \rho[\varphi]_\varepsilon + D|\varphi|$ for some $n_0 \in \mathbb{N}$ where $\rho < 1, D \geq 0$.

Corollary 3.18

For ε sufficiently small and $0 \leq s < s_\infty$, if $\int |g|^s \gamma^\delta(g) d\mu(g) < \infty$ for some $\delta > 0$,

$$\lim_{n \rightarrow \infty} (\sup_{x,y} \mathbb{E}^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)} \right))^{1/n} = \rho(\varepsilon) < 1.$$

If $k'(s) > 0$ then $\lim_{n \rightarrow \infty} (\sup_x \mathbb{E}^s \left(\frac{1}{|S_n x|^\varepsilon} \right))^{1/n} < 1$.

Proof

The proof of the first formula is based on the theorem and is given below. The proof of the second formula follows from a similar argument (see also [39], Theorem 1, for $s = 0$).

We denote $\alpha_n(x, y, \omega) = \text{Log} \frac{\delta(S_n(\omega) \cdot x, S_n(\omega) \cdot y)}{\delta(x, y)}$ and we observe that :

$$e^{\varepsilon \alpha_n} \leq 1 + \varepsilon \alpha_n + \varepsilon^2 \alpha_n^2 e^{|\alpha_n|}, \quad |\alpha_n| \leq 2 \text{Log} \gamma(S_n)$$

Since $t^2 e^{|t|} \leq e^{3|t|}$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$:

$$\alpha_n^2 e^{\varepsilon |\alpha_n|} \leq \frac{1}{\varepsilon_0^2} e^{3\varepsilon_0 |\alpha_n|} \leq \frac{1}{\varepsilon_0^2} (\gamma(S_n))^{\varepsilon_0}.$$

We observe that $I_n = \frac{1}{\varepsilon_0^2} \mathbb{E}^s(\gamma^{6\varepsilon_0}(S_n))$ is finite for $s < s_\infty$ and ε_0 sufficiently small (see the proof of Corollary 3.20. below).

It follows :

$$\begin{aligned} \mathbb{E}^s(e^{\varepsilon \alpha_n(x, y, \omega)}) &\leq 1 + \varepsilon \mathbb{E}^s(\alpha_n(x, y, \omega)) + \varepsilon^2 I_n, \\ \sup_{x,y} \mathbb{E}^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)} \right) &\leq 1 + \varepsilon \sup_{x,y} \mathbb{E}^s \left(\text{Log} \frac{\delta(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)} \right) + \varepsilon^2 I_n. \end{aligned}$$

Also the quantity $\rho_n(\varepsilon) = \sup_{x,y} \mathbb{E}^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)} \right)$ satisfies $\rho_{m+n}(\varepsilon) \leq \rho_m(\varepsilon) \rho_n(\varepsilon)$, hence

we have $\rho(\varepsilon) = \lim_{n \rightarrow \infty} \rho_n(\varepsilon)^{1/n} = \text{Inf}_{n \in \mathbb{N}} (\rho_n(\varepsilon))^{1/n}$. It follows that, in order to show $\rho(\varepsilon) < 1$,

for ε small it suffices to show $\rho_{n_0}(\varepsilon) < 1$ for some n_0 . To do that, we choose n_0 such that

$$\sup_{x,y} \mathbb{E}^s \left(\text{Log} \frac{\delta(S_{n_0} \cdot x, S_{n_0} \cdot y)}{\delta(x, y)} \right) = c < 0$$

which is possible using the theorem, and we take ε sufficiently small so that $\varepsilon^2 I_{n_0} + \varepsilon c < 0$. Then we get :

$$\rho_{n_0}(\varepsilon) \leq 1 + \varepsilon^2 I_{n_0} + \varepsilon c < 1. \quad \square$$

Corollary 3.19

Let $H_\varepsilon(\mathbb{P}^{d-1})$ be the Banach space of ε -Hölder functions on \mathbb{P}^{d-1} with the norm $\varphi \rightarrow [\varphi]_\varepsilon + |\varphi|$ and assume $0 \leq s < s_\infty$, $\int |g|^s \gamma^\delta(g) d\mu(g)$ for some $\delta > 0$. Then for ε sufficiently small the operator Q^s (defined in Theorem 2.6) on $H_\varepsilon(\mathbb{P}^{d-1})$ satisfies the following Doeblin-Fortet inequality where $B \geq 0$, $n_0 \in \mathbb{N}$, $\rho^{n_0}(\varepsilon) < \rho'(\varepsilon) < 1$: $[(Q^s)^{n_0} \varphi]_\varepsilon \leq \rho'(\varepsilon)[\varphi]_\varepsilon + B|\varphi|$. In particular the operator P^s admits the following spectral decomposition in $H_\varepsilon(\mathbb{P}^{d-1})$:

$$P^s = k(s)(\nu^s \otimes e^s + U^s)$$

where U^s has spectral radius less than 1, and satisfies $U^s(\nu^s \otimes e^s) = (\nu^s \otimes e^s)U^s = 0..$

Proof

From Lemma 3.6, we know that $Q_x^s \leq c(s)Q^s$, hence, using Corollary 3.18, for $n \geq n_0$ sufficiently large and with $\rho'(\varepsilon) \in]\rho^{n_0}(\varepsilon), 1[$: $\sup_{x,y} \mathbb{E}_x^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)} \right) \leq \rho'(\varepsilon)$.

We can write :

$$\begin{aligned} (Q^s)^n \varphi(x) - (Q^s)^n \varphi(y) &= \mathbb{E}_x^s(\varphi(S_n \cdot x)) - \mathbb{E}_y^s(\varphi(S_n \cdot y)) = \mathbb{E}_x^s(\varphi(S_n \cdot x) - \varphi(S_n \cdot y)) + \\ &(\mathbb{E}_x^s - \mathbb{E}_y^s)(\varphi(S_n \cdot y)). \end{aligned}$$

The first term in the right hand side is bounded by $[\varphi]_\varepsilon \delta^\varepsilon(x, y) \sup_{x,y} \mathbb{E}_x^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)} \right)$ i.e by $[\varphi]_\varepsilon \delta^\varepsilon(x, y) \rho'(\varepsilon)$. Using Lemma 3.5, we know that the second term is bounded by $B|\varphi| \delta^\varepsilon(x, y)$. Hence with $\varepsilon < \bar{\varepsilon}$ we get the required inequality.

From Theorem 2.6 we know that Q^s has a unique stationary measure π^s and 1 is the unique eigenvalue with modulus one. Then the above Doeblin-Fortet inequality implies (see [34]) the relation $Q^s = \pi^s \otimes 1 + V^s$ where V^s commutes with the projection $\pi^s \otimes 1$ has spectral radius less than one and satisfies $V^s(\pi^s \otimes 1) = 0$, hence the required formula for P^s . \square

Corollary 3.20

With the notations and hypothesis of Corollary 3.19, the following Doeblin-Fortet inequality is valid, if $z = s + it$, $0 \leq s < s_\infty$:

$$[(Q^z)^{n_0} \varphi]_\varepsilon \leq \rho'(\varepsilon)[\varphi]_\varepsilon + (B + A_{n_0}(\varepsilon)|t|^\varepsilon)|\varphi|,$$

where $0 \leq A_{n_0}(\varepsilon) < \infty$. For $t \neq 0$, the spectral radius of Q^z is less than 1.

Furthermore $k(s)$ and the projection $\nu^s \otimes e^s$ are analytic on $]0, s_\infty[$, and 1 is a simple eigenvalue of Q^s .

Proof

By definition of $Q^z = Q^{s+it}$ we have $(Q^{s+it})^n \varphi(x) = \mathbb{E}_x^s(|S_n x|^{it} \varphi(S_n \cdot x))$, hence :

$$|(Q^z)^n \varphi(x) - (Q^z)^n \varphi(y)| \leq |(\mathbb{E}_x^s - \mathbb{E}_y^s)(|S_n x|^{it} \varphi(S_n \cdot x)) + |\mathbb{E}_y^s(|S_n x|^{it} \varphi(S_n \cdot x) - |S_n y|^{it} \varphi(S_n \cdot y))|.$$

Using Corollary 3.5 the first term is bounded by $B\delta^\varepsilon(x, y)\mathbb{E}^s(|\varphi|)$ i.e by $B|\varphi|\delta^\varepsilon(x, y)$. The second term is dominated by

$$\mathbb{E}_y^s(|(S_n x|^{it} - |S_n y|^{it})|\varphi| + \mathbb{E}_y^s(|\varphi(S_n \cdot x) - \varphi(S_n \cdot y)|)).$$

As in the proof of Corollary 3.19, for $n \geq n_0$:

$$\begin{aligned} \mathbb{E}_y^s(|\varphi(S_n \cdot x) - \varphi(S_n \cdot y)|) &\leq [\varphi]_\varepsilon \mathbb{E}_y^s(\delta^\varepsilon(S_n \cdot x, S_n \cdot y)) \\ &\leq [\varphi]_\varepsilon \delta^\varepsilon(x, y) \sup_{x, y} \mathbb{E}_y^s\left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)}\right) \leq [\varphi]_\varepsilon \delta^\varepsilon(x, y) \rho'(\varepsilon). \end{aligned}$$

On the other hand, using the relation :

$$||u|^{it} - |v|^{it}| \leq 2|t|^\varepsilon |Log|u| - Log|v||^\varepsilon \leq 2|t|^\varepsilon \sup\left(\frac{1}{|u|}, \frac{1}{|v|}\right)^\varepsilon ||u| - |v||^\varepsilon \text{ we get :}$$

$$|(|S_n x| - |S_n y|)^{it}| \leq 2|t|^\varepsilon \sup_{|v|=1} \frac{1}{|S_n v|^\varepsilon} |S_n(x - y)|^\varepsilon \leq 2|t|^\varepsilon \sup_{|v|=1} \frac{|S_n|^\varepsilon}{|S_n v|^\varepsilon} \delta^\varepsilon(x, y).$$

Since $|S_n v| \geq |S_n^{-1}|^{-1}$ we get :

$$\mathbb{E}_y^s(|S_n x|^{it} - |S_n y|^{it})| \leq 2c(s)|t|^\varepsilon \delta^\varepsilon(x, y) \mathbb{E}^s(\gamma^{2\varepsilon}(S_n)).$$

Since $\gamma(S_{m+n}) \leq \gamma(S_m)\gamma(S_n \circ \theta^m)$ and \mathbb{Q}^s is shift-invariant :

$$\mathbb{E}^s(\gamma^{2\varepsilon}(S_n)) \leq (\mathbb{E}^s(\gamma^{2\varepsilon}(S_1)))^n < \infty.$$

Then for n fixed and ε sufficiently small the hypothesis implies that $|\mathbb{E}_y^s(|S_n x|^{it} - |S_n y|^{it})|$ is bounded by $A_n(\varepsilon)|t|^\varepsilon \delta^\varepsilon(x, y)$. Finally for $n = n_0$:

$$[(Q^z)^{n_0} \varphi]_\varepsilon \leq \rho'(\varepsilon) [\varphi]_\varepsilon + (B + A_{n_0}(\varepsilon)|t|^\varepsilon) |\varphi|.$$

Then, using [34], one gets that the possible unimodular spectral values of Q^z are eigenvalues. Using Theorem 2.6, if $t \neq 0$, one get that no such eigenvalue exists, hence the spectral radius of Q^z is less than 1.

In order to show the analyticity of $k(s)$ and $\nu^s \otimes e^s$ on $]0, s_\infty[$, we consider the operator P^z for $z \in \mathbb{C}$ close to s . We begin by showing the holomorphy of P^z for $Re z \in]0, s_\infty[$. Let γ be a loop contained in the strip $Re z \in \overset{\circ}{I}_\mu$ and $\varphi \in H_\varepsilon(\mathbb{P}^{d-1})$. Then, since $z \rightarrow |gx|^z$ is holomorphic:

$$\int_\gamma P^z \varphi(x) dz = \int_{G \times \gamma} \varphi(g \cdot x) |gx|^z d\mu(g) dz = \int_G \varphi(g \cdot x) d\mu(g) \int_\gamma |gx|^z dz = 0,$$

On the other hand, the spectral gap property of the operator P^s implies that $k(s)$ is a simple pole of the function $\zeta \rightarrow (\zeta I - P^s)^{-1}$, hence by functional calculus if γ is a small cercle of center $k(s) \in \mathbb{C}$:

$$k(s) \nu^s \otimes e^s = \frac{1}{2i\pi} \int_\gamma (\zeta I - P^s)^{-1} d\zeta.$$

Since P^z depends continuously of z , the function $(\zeta I - P^z)^{-1}$ has a pole inside the small disk defined by γ , if z is close to s . Then by perturbation theory P^z has an isolated spectral value $k(z)$ close to $k(s)$. The corresponding projection $\nu^z \otimes e^z$ satisfies :

$$k(z) \nu^z \otimes e^z = \frac{1}{2i\pi} \int_\gamma (\zeta I - P^z)^{-1} d\zeta.$$

This formula and the holomorphy of P^z shows that $k(z)$ and $\nu^z \otimes e^z$ are holomorphic in a neighbourhood of s . The analyticity of $k(s)$ and $\nu^s \otimes e^s$ follow. The fact that 1 is a simple eigenvalue of Q^s follows from Theorem 2.6. \square

Corollary 3.21

Assume $\int |g|^s \gamma^\delta(g) d\mu(g) < \infty$ for some $\delta > 0$. Then given $\varepsilon > 0$ sufficiently small, for any $\varepsilon_0 > 0$ there exists $\delta_0 = \delta_0(\varepsilon_0)$, $n_0 = n_0(\varepsilon_0)$ such that if $x, y \in \mathbb{S}^{d-1}$ satisfy $\tilde{\delta}(x, y) \leq \delta_0$, then $\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y)) \leq \varepsilon_0 \tilde{\delta}^\varepsilon(x, y)$.

One has the following Doeblin-Fortet inequality with $D \geq 0$, $\rho_0 = \varepsilon_0 c(s) < 1$:

$$[(\tilde{Q}^s)^{n_0} \varphi]_\varepsilon \leq \rho_0 [\varphi]_\varepsilon + D |\varphi|, \text{ where } c(s) \text{ satisfies } \mathbb{Q}_x^s \leq c(s) \mathbb{Q}^s.$$

In particular the spectral value 1 is isolated and the corresponding finite dimensional projector depends analytically on $s \in]0, s_\infty[$.

In case I, the \tilde{Q}^s -invariant functions are constant. In case II, the space of \tilde{Q}^s -invariant functions is generated by p_+^s and p_-^s .

If $t \neq 0$ the spectral radius of \tilde{Q}^z is less than 1.

Furthermore, 1 is the unique unimodular eigenvalue of \tilde{Q}^s except in case I, where -1 is the unique non trivial possibility.

Proof

Assume ε is as in Corollary 3.18. We will use for any $n \in \mathbb{N}$, $t > 0$ the relation :

$$\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_n \cdot x, S_n \cdot y)) = \mathbb{E}^s(\tilde{\delta}^\varepsilon(S_n \cdot x, S_n \cdot y) 1_{\{\gamma(S_n) > t\}}) + \mathbb{E}^s(\tilde{\delta}^\varepsilon(S_n \cdot x, S_n \cdot y) 1_{\{\gamma(S_n) \leq t\}})$$

In view of Corollary 3.18 we have for some n_0 , any $\bar{x}, \bar{y} \in \mathbb{P}^{d-1}$, given $\varepsilon_0 > 0$:

$$\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0} \cdot \bar{x}, S_{n_0} \cdot \bar{y})) \leq \frac{\varepsilon_0}{2} \tilde{\delta}^\varepsilon(\bar{x}, \bar{y}).$$

Using Lemma 2.11 we have for $x, y \in \mathbb{S}^{d-1}$: $\tilde{\delta}(S_{n_0} \cdot x, S_{n_0} \cdot y) \leq 2\gamma^2(S_{n_0}) \tilde{\delta}(x, y)$, hence :

$$\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y) 1_{\{\gamma(S_{n_0}) > t\}}) \leq 2\mathbb{E}^s(\gamma^{2\varepsilon}(S_{n_0}) 1_{\{\gamma(S_{n_0}) > t\}}).$$

Since, as in the proof of Corollary 3.20 we have if ε is sufficiently small, $\mathbb{E}^s(\gamma^{2\varepsilon}(S_{n_0})) < \infty$, we can choose $t_0 > 0$ so that $\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y) 1_{\{\gamma(S_{n_0}) > t_0\}}) \leq \frac{\varepsilon_0}{2} \tilde{\delta}^\varepsilon(x, y)$. Then, on the set $\{\gamma(S_{n_0}) \leq t_0\}$ we have :

$$\tilde{\delta}(S_{n_0} \cdot x, S_{n_0} \cdot y) \leq 2\gamma^2(S_{n_0}) \tilde{\delta}(x, y) \leq 2t_0^2 \tilde{\delta}(x, y).$$

We observe that, if $\tilde{\delta}(u, v) \leq \sqrt{2}$ with $u, v \in \mathbb{S}^{d-1}$, then $\delta(\bar{u}, \bar{v}) = \tilde{\delta}(u, v)$. Hence, if $2t_0^2 \tilde{\delta}(x, y) \leq \sqrt{2}$, we get : $\tilde{\delta}(S_{n_0} \cdot x, S_{n_0} \cdot y) = \delta(S_{n_0} \cdot \bar{x}, S_{n_0} \cdot \bar{y})$ on the set $\{\gamma(S_{n_0}) \leq t_0\}$.

It follows, if $\tilde{\delta}(x, y) \leq \frac{\sqrt{2}}{2t_0^2} = \delta_0$:

$$\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y) 1_{\{\gamma(S_{n_0}) \leq t_0\}}) \leq \frac{\varepsilon_0}{2} \tilde{\delta}^\varepsilon(x, y)$$

Hence we get, if $\tilde{\delta}(x, y) \leq \delta_0$: $\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y)) \leq \varepsilon_0 \tilde{\delta}^\varepsilon(x, y)$.

Using $\mathbb{Q}_x^s < c(s) \mathbb{Q}^s$ we obtain : $\sup_{\tilde{\delta}(x, y) \leq \delta_0} \mathbb{E}_x^s\left(\frac{\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y)}{\tilde{\delta}^\varepsilon(x, y)}\right) \leq c(s) \varepsilon_0$.

On the other hand, if $\varphi \in H_\varepsilon(\mathbb{S}^{d-1})$:

$$(\tilde{Q}^s)^n \varphi(x) - (\tilde{Q}^s)^n \varphi(y) = \mathbb{E}_x^s(\varphi(S_n \cdot x) - \varphi(S_n \cdot y)) + (\mathbb{E}_x - \mathbb{E}_y)(\varphi(S_n \cdot y))$$

In view of Lemma 3.5, the second term is bounded by $B \tilde{\delta}^s(x, y) |\varphi|$. Then, for $\tilde{\delta}(x, y) \leq \delta_0$ we obtain, since $\varepsilon \leq \bar{s}$:

$$|(\tilde{Q}^s)^{n_0} \varphi(x) - (\tilde{Q}^s)^{n_0} \varphi(y)| \leq c(s) \varepsilon_0 [\varphi]_\varepsilon \tilde{\delta}^\varepsilon(x, y) + B |\varphi| \tilde{\delta}^\varepsilon(x, y)$$

If $\tilde{\delta}(x, y) \geq \delta_0$ we have trivially : $\tilde{\mathbb{E}}_x(|\varphi(S_{n_0} \cdot x) - \varphi(S_{n_0} \cdot y)|) \leq 2c(s) \frac{\tilde{\delta}^\varepsilon(x, y)}{\delta_0^\varepsilon} |\varphi|$

Finally on \mathbb{S}^{d-1} :

$$[(\tilde{Q}^s)^{n_0} \varphi]_\varepsilon \leq c(s) \varepsilon_0 [\varphi]_\varepsilon + (B + 2 \frac{c(s)}{\delta_0^\varepsilon}) |\varphi|$$

hence the result with $D = B + 2 \frac{c(s)}{\delta_0^\varepsilon}$. The structure of the space of \tilde{Q}^s -invariant functions is given by Theorem 2.17. It follows from [34] that 1 is an isolated spectral value of \tilde{Q}^s and the corresponding projector has finite rank. The same argument as in the proof of Corollary 3.20 gives the analyticity of this projector. Doeblin-Fortet inequality implies that the possible unimodular spectral values of \tilde{Q}^z are eigenvalues. Then, as in the end of proof of Theorem 2.7, one would have for some $\varphi \in H_\varepsilon(\mathbb{S}^{d-1})$ $e^{i\theta} \in \mathbb{C}$, and any $g \in \text{supp} \mu$: $|gx|^{it} \varphi(g.x) = e^{i\theta} \varphi(x)$. This would contradict Proposition 2.5 if $t \neq 0$.

The last assertion is a direct consequence of Corollary 2.19. \square

Proof of Theorem A

The spectral decomposition $P^s = k(s)(\nu^s \otimes e^s + U^s)$ is part of Corollary 3.19. The analyticity of $k(s)$ and $\nu^s \otimes e^s$ on $]0, s_\infty[$ is stated in Corollary 3.20. The strict convexity of $\text{Log}k(s)$ is stated in Theorem 2.6. The fact that the spectral radius of P^z is less than $k(s)$ follows from the corresponding assertion for Q^z in Corollary 3.20. \square

IV Renewal theorems for products of random matrices

In this section using the results of subsections 3.2, 3.3, we show that the renewal theorem of [36] can be applied to our situation and we give the corresponding statements for products of random matrices. In particular, under condition i-p, if the largest Lyapunov exponent is negative and there exists $\alpha > 0$ with $k(\alpha) = 1$, we get the matricial analogue of Cramer's estimate of ruin for random walk on \mathbb{R} , ([15]) *i.e* there exists $A > 0$ such that for any $x \in \mathbb{S}^{d-1}$:

$$\lim_{t \rightarrow \infty} t^\alpha \mathbb{P} \left\{ \sup_{n > 0} |S_n(\omega)x| > t \right\} = A e^\alpha(x) \text{ where } A > 0.$$

Hence, under i-p condition, we extend the results of [36], [39] to the general case. Theorems 4.7, 4.7' below will play an essential role in section 5.

1) The renewal theorem for fibered Markov chains

We begin by summarizing, with a few changes and comments, the basic notations of [37]. Let (S, δ) be a complete separable metric space, P (resp \bar{P}) a Markov kernel on $S \times \mathbb{R}$ (resp S) which preserves $C_b(S \times \mathbb{R})$ (resp $C_b(S)$). We assume that P commutes with the translations $(x, t) \rightarrow (x, t + a)$ on $S \times \mathbb{R}$, \bar{P} is the factor kernel of P on S , and π is a fixed \bar{P} -stationary probability measure on S . In our applications, we will have S compact and $S = \mathbb{P}^{d-1}$ or $S \subset \mathbb{S}^{d-1}$. Here we consider paths on $S \times \mathbb{R}$ starting from $(x, 0) \in S \times \{0\}$. Such a path can be written as $(X_n, V_n)_{n \in \mathbb{N}}$ with $V_0 = 0$, $X_0 = x$, $V_n - V_{n-1} = U_n$ ($n \geq 1$). The corresponding space of paths will be denoted ${}^a\Omega = S \times \prod_1^\infty (S \times \mathbb{R})$, the Markov measures on ${}^a\Omega$ will be denoted by ${}^a\mathbb{P}_x$, and the expectation symbol will be written ${}^a\mathbb{E}_x$. The space

of bounded measurable functions on a measurable space Y will be denoted $\mathcal{B}(Y)$. We observe that the Markov kernel P on $S \times \mathbb{R}$ is completely defined by the family of measures $F(du|x, y)$ ($x, y \in S$) where $F(du|x, y)$ is the conditional law of V_1 given $X_0 = x, X_1 = y$. The number $\int uF(du|x, y)\overline{P}(x, dy)d\pi(x)$ will be called the mean of P , if the corresponding integral $\int |u|F(du|x, y)\overline{P}(x, dy)d\pi(x)$ is finite. In that case, we say that P has a 1-moment. If $t \in \mathbb{R}_+$ we define the hitting time $N(t)$ of the interval $]t, \infty[$ by :

$N(t) = \text{Min}\{n \geq 1 ; V_n > t\}$ ($= +\infty$ if no such n exists).

On the event $N(t) < +\infty$ we take $W(t) = V_{N(t)} - t$, $Z(t) = X_{N(t)}$. Then $W(t)$ is the residual waiting time of the interval $]t, \infty[$ (see [15]). The law of $(Z(t), W(t))$ under \mathbb{P}_x is the hitting measure of $S \times]t, \infty[$ starting from $(x, 0)$.

One needs some technical definitions concerning direct Riemann integrability, non arithmeticity of the Markov chain defined by P on $S \times \mathbb{R}$, and the possibility of comparing ${}^a\mathbb{P}_x, {}^a\mathbb{P}_y$ in a weak sense for different points x, y in S . We add some comments as follows. Given a fibered Markov chain on $S \times \mathbb{R}$, we denote :

$$C_k = \{x \in S ; {}^a\mathbb{P}_x\{\frac{V_m}{m} > \frac{1}{k} \quad \forall m \geq k\} \geq \frac{1}{2}\}, C_0 = \phi.$$

For any $f \in \mathcal{B}({}^a\Omega)$ and $\varepsilon > 0$ we write :

$$f^\varepsilon(x_0, x_1, \dots, v_1 \dots) = \text{sup}\{f(y_0, y_1, \dots, w_1, \dots) ; \delta(x_i, y_i) + |v_i - w_i| < \varepsilon \text{ if } i \in \mathbb{N}\}.$$

Definition 4.1

A Borel function $\varphi \in \mathcal{B}(S \times \mathbb{R})$ is said to be *d.R.i* (for directly Riemann integrable) if :

$$\sum_0^\infty \sum_{\ell=-\infty}^{+\infty} (k+1) \text{sup}\{|\varphi(x, t)| ; x \in C_{k+1} \setminus C_k, \ell \leq t \leq \ell+1\} < +\infty$$

and for every fixed $x \in S$ and any $\beta > 0$, the function $t \rightarrow \varphi(x, t)$ is Riemann integrable on $[-\beta, \beta]$.

In our setting below we will have $C_k = S$ for some $k > 0$ and for some $\varepsilon > 0$ any $x \in S$ and m sufficiently large ${}^a\mathbb{P}_x(\frac{V_m}{m} \geq \varepsilon) \geq \frac{1}{2}$. Then the following stronger form of the above definition will be used

Definition 4.1'

$\varphi \in \mathcal{B}(S \times \mathbb{R})$ is said to be boundedly Riemann integrable (*b.R.i*) if

$$\sum_{\ell=-\infty}^{\ell=\infty} \text{sup}\{|\varphi(x, t)| ; x \in S, t \in [\ell, \ell+1]\} < \infty$$

and for any fixed $x \in S$, any $\beta > 0$, the function $t \rightarrow \varphi(x, t)$ is Riemann integrable on $[-\beta, \beta]$.

Remark

Definition 4.1' corresponds to $\text{sup}\{|\varphi(x, t)| ; x \in S\}$ directly Riemann integrable in the sense of [15]. If $C_k = S$ for some $k \in \mathbb{N}$, then condition *b.R.i* implies condition *d.R.i*

The following will help us to express the appropriate aperiodicity condition for (P, π) .

Definition 4.2

The kernel P , the space (S, δ) and the measure $\pi \in M^1(S)$ beeing as above we consider a point $(\zeta, \lambda, y) \in \mathbb{R} \times [0, 1] \times S$ and we say that (P, π) satisfies distortion (ζ, λ) at y if for any $\varepsilon > 0$, there exists $A \in \mathcal{B}(S)$ with $\pi(A) > 0$ and $m_1, m_2 \in \mathbb{N}$, $\tau \in \mathbb{R}$ such that for any $x \in A$:

$${}^a\mathbb{P}_x\{\delta(X_{m_1}, y) < \varepsilon, |V_{m_1} - \tau| \leq \lambda\} > 0, \quad {}^a\mathbb{P}_x\{\delta(X_{m_2}, y) < \varepsilon, |V_{m_2} - \tau - \zeta| \leq \lambda\} > 0.$$

Definition 4.3

We will say that the kernel P on $S \times \mathbb{R}$ is non-expanding if for each fixed $x \in S$, $\varepsilon > 0$, there exists $r_0 = r_0(x, \varepsilon)$ such that for all real valued $f \in \mathcal{B}({}^a\Omega)$ and for all y with $\delta(x, y) < r_0$ one has :

$${}^a\mathbb{P}_y(f) \leq {}^a\mathbb{P}_x(f^\varepsilon) + \varepsilon|f|, \quad {}^a\mathbb{P}_x(f) \leq {}^a\mathbb{P}_y(f^\varepsilon) + \varepsilon|f|.$$

We denote by I.1 -I.4 the following conditions

I.1 For every open set O with $\pi(O) > 0$, and ${}^a\mathbb{P}_x - a.e$, for each $x \in S$, we have :

$${}^a\mathbb{P}_x\{X_n \in O \text{ for some } n\} = 1.$$

I.2 P has a 1-moment and for all $x \in S$, ${}^a\mathbb{P}_x - a.e$:

$$\lim_{n \rightarrow \infty} \frac{V_n}{n} = L = \int uF(du|x, y)\overline{P}(x, dy)d\pi(x) > 0.$$

I.3 There exists a sequence $(\zeta_i)_{i \geq 1} \subset \mathbb{R}$ such that the group generated by ζ_i is dense in \mathbb{R} and such that for any $i \geq 1$ and $\lambda \in [0, 1]$, there exists $y = y(i, \lambda) \in S$ such that (P, π) satisfies distortion (ζ_i, λ) at y .

I.4 The kernel P on $S \times \mathbb{R}$ is non-expanding.

Then, the following extension of the classical renewal theorem (see [15]) is proved in [36].

Theorem 4.4

Assume conditions I.1-I.4 are satisfied for P . Then there exists a positive measure ψ on S such that for any $x \in S$ and $\varphi \in C_b(S \times]0, \infty[)$:

$$\lim_{t \rightarrow \infty} {}^a\mathbb{E}_x\varphi(Z(t), W(t)) = \frac{1}{L} \int \varphi(z, s)1_{[s, \infty[}(t) {}^a\mathbb{P}_y\{X_{N(0)} \in dz, V_{N(0)} \in dt\}d\psi(y)ds = H(\varphi)$$

Moreover, if $\varphi \in C_b(S \times \mathbb{R})$ is *d.R.i*, then, for any $x \in S$:

$$\lim_{t \rightarrow \infty} {}^a\mathbb{E}_x\left(\sum_0^\infty \varphi(X_n, V_n - t)\right) = \frac{1}{L} \int \varphi(y, s)d\pi(y)ds$$

Furthermore ψ is an invariant measure for the Markov chain on S with kernel ${}^aP(y, dz) = {}^a\mathbb{P}_y(X_{N(0)} \in dz)$ and $\int \mathbb{E}_y(N(0))d\psi(y) = 1$, $\int \mathbb{E}_y(V_{N(0)})d\psi(y) = L$.

Remark

If S is compact, condition I.1 is a consequence of uniqueness of the \bar{P} -stationary measure π . This follows from the law of large numbers for Markov chains with a unique stationary measure [7] : for any continuous function with $0 \leq f \leq 1$, $\pi(f) > 0$ we have ${}^a\mathbb{P}_x - a.e$ for

all $x \in S$: $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_0^{n-1} f(X_k) = \pi(f) > 0$. This implies condition I.1.

The construction of ψ in [37] is based on Kac's recurrence theorem and implies the absolute continuity of ψ with respect to π , hence the probability H is independant of x and absolutely continuous with respect to $\pi \otimes \ell$.

2) Conditions I1-I4 are valid for linear random walks.

We verify conditions I1-I4 in four related situations. Here \mathbb{R} is identified with \mathbb{R}_+^* through the map $t \rightarrow e^t$. If $d > 1$ we use condition i-p. If $d = 1$ we use non arithmeticity of μ .

The first and simpler one, corresponds to $S = \mathbb{P}^{d-1}$, $S \times \mathbb{R} = \dot{V}$, $P(v, \cdot) = \mu * \delta_v$ where P is the operator denoted \dot{P} in section 2. Also we write on \mathbb{P}^{d-1} : $\bar{P}(x, \cdot) = \mu * \delta_x$ if $x \in \mathbb{P}^{d-1}$.

We will begin the verifications by this case and show how to modify the arguments in the other cases.

In the second case, $S \times \mathbb{R} \subset V \setminus \{0\}$, S is a compact subset of \mathbb{S}^{d-1} and P (resp \tilde{P}) will be the restriction to $S \times \mathbb{R}$ (resp S) of the kernel already denoted P (resp \tilde{P}) in section 2. Since, for any $t \in \mathbb{R}_+^*$ and $g \in G$, we have $g(tv) = tg(v)$, the kernels P and \tilde{P} will satisfy the commutation condition with the \mathbb{R} -action required in the above paragraph. As shown at the end of section 2, we need to consider two cases for \tilde{P} , depending of the fact that P preserves a proper convex cone (case II) or not (case I). In case I (resp II) we will have $S = \mathbb{S}^{d-1}$ (resp $S = Co(\Lambda_+([supp\mu])$). With these choices, there exists a unique \tilde{P} -stationary measure on S , as follows from Theorem 2.17. We denote by $\alpha \in I_\mu$ the positive number (if it exists) such that $k(\alpha) = 1$, where $k(s) = \lim_{n \rightarrow \infty} (\int |g|^s d\mu^n(g))^{1/n}$.

We know from section 3, that for any $s \in I_\mu$, there are two Markov kernels Q^s on \mathbb{P}^{d-1} and \tilde{Q}^s on \mathbb{S}^{d-1} , naturally associated with the operator P^s considered in section 2. We are here mainly interested in the cases $s = \alpha$ and $s = 0$, with $Q^0 = \bar{P}$ and $\tilde{Q}^0 = \tilde{P}$, but we observe that our considerations are valid for any $s \in I_\mu$. In what follows the notation S will be used in all the cases above.

We denote by ${}^a\mathbb{Q}_x^s$ the natural Markov measures on the path space ${}^a\Omega$. If $s = 0$ we use the notation ${}^a\mathbb{P}_x$. We write $V_k = Log|S_k x|$, $X_k = S_k x$ and we denote by Δ_x the map from $\Omega = G^{\mathbb{N}}$ to ${}^a\Omega$ given by $(g_1, g_2, \dots) \rightarrow (x, X_1, V_1, X_2, V_2, \dots)$. Clearly ${}^a\mathbb{Q}_x^\alpha$ (resp ${}^a\mathbb{P}_x$) is the push-forward of \mathbb{Q}_x^α (resp \mathbb{P}) by Δ_x , hence we can translate the results of section 3 in the new setting.

The validity of condition I.1, in all cases, is a direct consequence of the remark following the theorem, since by Theorem 2.6 (see also Theorem 2.17) the kernels \bar{P} , Q^α , \tilde{P} , \tilde{Q}^α have unique stationary measures on S .

In order to verify I.2 we begin by $S = \mathbb{P}^{d-1}$, $S \times \mathbb{R} = \dot{V}$, $\bar{P}(x, \cdot) = \mu * \delta_x$, $\dot{P}(v, \cdot) = \mu * \delta_v$.

Then $F(A|x, y) = \mu\{g \in G, \text{Log}|gx| \in A, g.x = y\}$ for any Borel set $A \subset \mathbb{R}$, and $\pi = \nu$ with $\mu * \nu = \nu$. We observe that $|\text{Log}|gx|| \leq \text{Log}\gamma(g)$ if $|x| = 1$ and $\gamma(g) = \text{sup}(|g|, |g^{-1}|)$. The finiteness of $\int |u|F(du|x, y)\bar{P}(x, dy)d\pi(x)$ follows since $\text{Log}\gamma(g)$ is μ -integrable. Also $\int uF(du|x, y)\bar{P}(x, dy)d\pi(x) = \int \text{Log}|gx|d\mu(g)d\nu(x) = L_\mu$. Then the relation ${}^a\mathbb{P}_x = \Delta_x(\mathbb{P})$ and Theorem 3.10 imply, for every $x \in \mathbb{P}^{d-1}$ (case $s = 0$), and \mathbb{P} -a.e :

$$L_\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n x| = \int \text{Log}|gx|d\mu(g)d\nu(x).$$

This is condition I.2 in the first case. For $S \subset \mathbb{S}^{d-1}$, the result is the same, since the involved quantities depend only on $|gx|$ with $x \in \mathbb{P}^{d-1}$, and \tilde{P} has a unique stationary measure on S .

In the cases of Q^α and \tilde{Q}^α it suffices also to consider the case $S = \mathbb{P}^{d-1}$.

The 1-moment condition of I.2 follows from $\int |g|^\alpha |\text{Log}\gamma(g)|d\mu(g) < +\infty$. The convergence part follows from Theorem 3.10 with $L_\mu(\alpha) = \int q^\alpha(x, g)\text{Log}|gx|d\pi^\alpha(x)d\mu(g) = \frac{k'(\alpha)}{k(\alpha)} > 0$.

We show I.3 as follows.

If $d > 1$, since the semigroup $T = \bigcup_{n \geq 0} (\text{supp}\mu)^n$ satisfies (i-p), we know using Proposition 2.5 that the set $\Delta = \{\text{Log}|\lambda_h| ; h \in T^{\text{prox}}\}$ is dense in \mathbb{R} .

The same is true of $2\Delta = \{\text{Log}\lambda_{h^2} ; h \in T^{\text{prox}}\}$.

If $d = 1$, the same properties follow from the non arithmeticity of μ .

We take for ζ_i ($i \in \mathbb{N}$) a dense countable subset of 2Δ . Let $\zeta_i = \text{Log}\lambda_g \in 2\Delta$, with $\lambda_g > 0$, $g = h^2$, $h = u_1 \cdots u_n$, $u_i \in \text{supp}\mu$ ($1 \leq i \leq n$) and $y = y(\zeta_i, \lambda) = \bar{v}_g \in \mathbb{P}^{d-1} = S$. We observe that, if ε is sufficiently small and $B_\varepsilon = \{x \in \mathbb{P}^{d-1}; \delta(x, \bar{v}_g) \leq \varepsilon\}$, then $g.B_\varepsilon \subset B_{\varepsilon'}$, with $\varepsilon' < \varepsilon$ and g as above.

Also, $\lambda > 0$ being fixed, and ε sufficiently small, we have $|\text{Log}\lambda_g - \text{Log}|gx|| < \lambda$ if $x \in B_\varepsilon$. These relations remain valid for g' instead of g if g' is sufficiently close to g .

Then we have for $x \in B_\varepsilon$, and $S_{2n} = g \in (\text{supp}\mu)^{2n}$ as above :

$$\mathbb{P}\{\delta(S_{2n}.x, \bar{v}_g) < \varepsilon, |\text{Log}|S_{2n}x| - \text{Log}\lambda_g| < \lambda\} > 0.$$

With $\zeta_i = \text{Log}\lambda_g$, $y = \bar{v}_g$, $A = B_\varepsilon$, $\tau = 0$, $m_1 = 0$, $m_2 = 2n$, this implies condition I.3 for the probability ${}^a\mathbb{P}_x = \Delta_x(\mathbb{P})$.

The definition of \mathbb{Q}_x^α shows its equivalence to \mathbb{P} on the σ -algebra of the sets depending of the first n coordinates. Then the relation ${}^a\mathbb{Q}_x^\alpha = \Delta_x(\mathbb{Q}_x^\alpha)$ implies with g as above :

$${}^a\mathbb{Q}_x^\alpha\{\delta(S_{2n}.x, \bar{v}_g) < \varepsilon, |\text{Log}|S_{2n}x| - \text{Log}\lambda_g| < \lambda\} > 0.$$

Hence condition I.3 is valid for ${}^a\mathbb{Q}_x^\alpha$ also.

If we consider \mathbb{S}^{d-1} instead of \mathbb{P}^{d-1} , i.e $S = \mathbb{S}^{d-1}$ or $S = \text{Co}(\Lambda_+([\text{supp}\mu]))$, and the metric $\tilde{\delta}$ on S , the above geometrical argument remains valid with $g = h^2$, $y = \bar{v}_g \in \Lambda_+([\text{supp}\mu])$ in the second case, $\lambda_g > 0$ and ε sufficient small. This shows I.3 in this setting.

Condition I.4 follows from the proof of Proposition 1 of [36]. The proof of the corresponding part of this proposition is a consequence of the condition :

$${}^a\mathbb{P}_x\{\exists C > 0 \text{ with } |S_n x| \geq C|S_n| \text{ for all } n\} = 1$$

for all $x \in S$, which implies that $|S_n x|$ and $|S_n y|$ are comparable if x and y are close. For the proof of the above condition, we observe that if $s \in I_\mu$, in particular if $s = 0$ or α , this condition has been proved in the stronger form $\lim_{n \rightarrow \infty} \frac{|S_n x|}{|S_n|} = |\langle z^*(\omega), x \rangle|$, $\mathbb{Q}_x^s - a.e.$, since as shown in Theorem 3.2 $|\langle z^*(\omega), x \rangle| > 0$, $\mathbb{Q}_x^s - a.e$ for any fixed $x \in \mathbb{P}^{d-1}$. Hence condition I.4 is valid in all the cases under consideration.

3) Direct Riemann integrability

In case of the spaces $S = \mathbb{P}^{d-1}$ or $S \subset \mathbb{S}^{d-1}$ considered above, under condition i-p for $[supp\mu]$, the *d.R.i* condition takes the simple form given by Lemma 4.5 below, in multiplicative notation.

We assume now that the hypothesis of Theorem 3.10 is satisfied, use the corresponding notations, and C_k is as in Definition 4.1.

Lemma 4.5

Assume $\varphi \in C_b(\mathbb{S}^{d-1} \times \mathbb{R}_+^*)$ is *b.R.i*, i.e φ is locally Riemann integrable and satisfies :

$$\sum_{\ell=-\infty}^{\ell=+\infty} \sup\{|\varphi(x, t)| ; x \in \mathbb{S}^{d-1}, t \in [2^\ell, 2^{\ell+1}]\} < +\infty.$$

Then $C_k = \mathbb{S}^{d-1}$ for k large, hence φ is *d.R.i* with respect to \mathbb{P} and \mathbb{Q}^α .

Proof

We consider first ${}^a\mathbb{P}_x$. Using Theorem 3.10 for $s = 0$, we get for any fixed x :

$$\lim_{n \rightarrow +\infty} \frac{\text{Log}|S_n(\omega)x|}{n} = L_\mu, \quad \mathbb{P} - a.e.$$

We observe that for any $x, y \in \mathbb{S}^{d-1}$, $\|S_n y| - |S_n x|\| \leq |S_n| \tilde{\delta}(x, y) \leq 2|S_n|$.

It follows :

$$\left| \frac{|S_n y|}{|S_n x|} - 1 \right| \leq 2 \frac{|S_n|}{|S_n x|}, \quad |\text{Log}|S_n y| - \text{Log}|S_n x|| \leq 2 \frac{|S_n|}{|S_n x|},$$

Using Theorem 3.2, we get that the sequence $\frac{|S_n|}{|S_n x|}$ converges $\mathbb{P} - a.e$ to $\frac{1}{|\langle z(\omega), x \rangle|} < \infty$, hence the sequence $\frac{1}{n} \frac{|S_n|}{|S_n x|}$ converges $\mathbb{P} - a.e$ to zero.

It follows that $\frac{1}{n} \text{Log}|S_n x| - \frac{2}{n} \frac{|S_n|}{|S_n x|}$ converges $\mathbb{P} - a.e$ to L_μ .

Hence there exists $m_0 > 0$ such that $\mathbb{P}\{\frac{1}{n} \text{Log}|S_n x| - \frac{2}{n} \frac{|S_n|}{|S_n x|} > \frac{1}{2} L_\mu \text{ for all } n \geq m_0\} \geq 1/2$.

In view of the inequality $\frac{1}{n} \text{Log}|S_n y| \geq \frac{1}{n} \text{Log}|S_n x| - \frac{2}{n} \frac{|S_n|}{|S_n x|}$, we have for any $y \in \mathbb{S}^{d-1}$,

$$\mathbb{P}\{\frac{1}{n} \text{Log}|S_n y| > L_\mu/2 \text{ for all } n \geq m_0\} > \frac{1}{2}.$$

This implies $C_k = \mathbb{S}^{d-1}$, $C_{k+1} \setminus C_k = \emptyset$ if $\frac{1}{k} \leq \text{Inf}\left(\frac{1}{m_0}, \frac{L_\mu}{2}\right)$. Then, φ is *d.R.i*.

If $s = \alpha$, the argument is the same with \mathbb{P} replaced by \mathbb{Q}^α and the relation $\mathbb{Q}_x^\alpha \leq c(\alpha)\mathbb{Q}^\alpha$ is used as follows.

$$\mathbb{Q}^\alpha\{\frac{1}{n} \text{Log}|S_n y| > L_\mu(\alpha)/2 \text{ for all } n \geq m_0\} > 1 - \frac{1}{2c(\alpha)}.$$

Since $\mathbb{Q}_y^\alpha \leq c(\alpha)\mathbb{Q}^\alpha$, this gives for any $y \in \mathbb{P}^{d-1}$:

$$\mathbb{Q}_y^\alpha \left\{ \frac{1}{n} \text{Log}|S_n y| > L_\mu(\alpha)/2 \text{ for all } n \geq m_0 \right\} > \frac{1}{2}.$$

Then also $C_k = \mathbb{S}^{d-1}$ for $\frac{1}{k} \leq \text{Inf} \left(\frac{1}{m_0}, \frac{L_\mu(\alpha)}{2} \right)$. Hence we conclude as above

4) The renewal theorems for linear random walks.

We consider $V \setminus \{0\} = \mathbb{S}^{d-1} \times \mathbb{R}_+^*$, $\dot{V} = \mathbb{P}^{d-1} \times \mathbb{R}_+^*$ and we study the asymptotics of the potential kernels of the corresponding random walks defined by μ . We denote :

$$\dot{V}_1 = \{v \in \dot{V} ; |v| > 1\}, \quad V_1 = \{v \in V ; |v| > 1\}$$

and we consider also the entrance measures $H_1(v, \cdot)$ or $\dot{H}_1(v, \cdot)$ of $S_n v$ in V_1 or \dot{V}_1 , starting from $v \neq 0$. Since conditions I are valid, their behaviours for v small are given by Theorem 4.4, and we will state them below. We denote by $\dot{\Lambda}([\text{supp}\mu])$ the inverse image of $\Lambda([\text{supp}\mu])$ in \dot{V} . Also we denote :

$$\dot{\Lambda}_1([\text{supp}\mu]) = \{v \in \dot{V} ; \bar{v} \in \Lambda([\text{supp}\mu]), |v| \geq 1\}$$

$$\Lambda_1([\text{supp}\mu]) = \{v \in V ; \bar{v} \in \Lambda([\text{supp}\mu]), |v| \geq 1\}.$$

As shown below these closed sets support the limits of $\dot{H}_1(v, \cdot)$ and $H_1(v, \cdot)$.

The results will take two forms according as $L_\mu > 0$ or $L_\mu < 0$. Also, each of these situations leads to two results depending on the geometrical case \dot{V} or V , under consideration.

Theorem 4.6

Assume $\mu \in M^1(G)$ is such that the semigroup $[\text{supp}\mu]$ satisfies condition i-p, if $d > 1$ or μ is non arithmetic if $d = 1$, $\text{Log}\gamma(g)$ is μ -integrable and $L_\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|g| d\mu^n(g) > 0$.

Then if $v \in \dot{V}$, $\sum_0^\infty \mu^k * \delta_v$ is a Radon measure on \dot{V} such that on $C_c(\dot{V})$ we have the vague convergence :

$$\lim_{v \rightarrow 0} \sum_0^\infty \mu^k * \delta_v = \frac{1}{L_\mu} \nu \otimes \ell$$

where $\nu \in M^1(\Lambda([\text{supp}\mu]))$ is the unique \bar{P} -invariant measure on \mathbb{P}^{d-1} .

This convergence is valid on any bounded continuous function f which satisfy on \dot{V} :

$$\sum_{-\infty}^{+\infty} \sup\{|f(v)| ; 2^\ell \leq |v| \leq 2^{\ell+1}\} < \infty.$$

Furthermore we have the weak convergence :

$$\lim_{v \rightarrow 0} \dot{H}_1(v, \cdot) = \dot{H}_1 \in M^1(\dot{\Lambda}_1([\text{supp}\mu])).$$

Proof

In view of the verifications of conditions I in subsections 2, 3 and of Lemma 4.5, this is a direct consequence of Theorem 4.4 applied in the case $S = \mathbb{P}^{d-1}$, $S \times \mathbb{R} = \dot{V} \setminus \{0\}$.

The fact that \dot{H}_1 is supported on $\dot{\Lambda}_1([supp\mu])$ follows from the $supp\mu$ -invariance of $\dot{\Lambda}([supp\mu])$, the existence of the limit and the fact that, if $\bar{v} \in \dot{\Lambda}([supp\mu])$ then $\dot{H}_1(v, \cdot) \in M^1(\dot{\Lambda}_1([supp\mu]))$.

□

Theorem 4.6'

Assume μ is as in Theorem 4.6. Then there are 2 cases as in Theorem 2.17 :

Case I : No proper convex cone in V is $supp\mu$ -invariant.

Then, in vague topology :

$$\lim_{v \rightarrow 0} \sum_0^\infty \mu^k * \delta_v = \frac{1}{L_\mu} \tilde{\nu} \otimes \ell$$

where $\tilde{\nu}$ is the unique μ -stationary measure on \mathbb{S}^{d-1}

Case II : Some proper convex cone in V is $supp\mu$ -invariant.

Then, for any $u \in \mathbb{S}^{d-1}$, in vague topology,

$$\lim_{t \rightarrow 0_+} \sum_0^\infty \mu^k * \delta_{tu} = \frac{1}{L_\mu} (p_+(u)\nu_+ \otimes \ell + p_-(u)\nu_- \otimes \ell)$$

where ν_+ is the unique μ -stationary measure on $\Lambda_+([supp\mu])$, ν_- is symmetric of ν_+ , $p_+(u)$ is the entrance probability of $S_n.u$ in $Co(\Lambda_+([supp\mu]))$, $p_-(u) = 1 - p_+(u)$.

In the two cases these convergences are also valid on any bounded continuous functions f

on $V \setminus \{0\}$ such that $\sum_{-\infty}^{+\infty} \sup\{|f(v)| ; 2^\ell \leq |v| < 2^{\ell+1}\} < \infty$.

In addition, for any $u \in \mathbb{S}^{d-1}$, in weak topology :

$$\lim_{t \rightarrow 0_+} H_1(tu, \cdot) = H_{1,u} \in M^1(\Lambda_1([supp\mu])).$$

In case I, $H_{1,u}$ is independent of u .

In case II, with $\Lambda_{1,+}([supp\mu]) = \{v \in V ; \bar{v} \in \Lambda_+([supp\mu]), |v| \geq 1\}$ we have :

$$H_{1,u} = p_+(u)H_{1,+} + p_-(u)H_{1,-},$$

where $H_{1,+} \in M^1(\Lambda_{1,+}(supp\mu))$, and $H_{1,-}$ is symmetric of $H_{1,+}$

Proof

In case I, the proof is the same as for Theorem 4.6 with $S = \mathbb{S}^{d-1}$ instead of \mathbb{P}^{d-1} .

In case II, we take $S = Co(\Lambda_+([supp\mu]))$ and we observe that $S \times \mathbb{R}_+^*$ is a $supp\mu$ -invariant convex cone with non zero interior to which we can apply Theorem 4.4.

If $u \in S$ (resp $u \in -S$) we have :

$$\lim_{t \rightarrow 0_+} \sum_0^\infty \mu^k * \delta_{tu} = \frac{1}{L_\mu} (\nu_+ \otimes \ell) \quad (\text{resp } \lim_{t \rightarrow 0_+} \sum_0^\infty \mu^k * \delta_{tu} = \frac{1}{L_\mu} (\nu_- \otimes \ell))$$

If $u \in V \setminus \{0\}$, we denote by $p_+(u, dv)$ (resp $p_-(u, dv)$) the entrance measure of $S_n.u$ in the cone $\Phi = S \times \mathbb{R}_+^*$ (resp $-\Phi$). Clearly the mass of $p_+(u, dv)$ is $p_+(u)$, and $p_+(u, dv)$ is

supported on Φ .

We denote also by $U(v, \cdot) = \sum_0^\infty \mu^k * \delta_v$ the potential kernel of the linear random walk $S_n(\omega)v$ starting from v in $V \setminus \{0\}$. Then, for any $\varphi \in C_c(\Phi \cup -\Phi)$, $U(tu, \varphi) = \int U(v, \varphi)(p_+(tu, dv) + p_-(tu, dv))$.

Clearly the kernel $p_+(x, dv)$ commutes with the scaling $x \rightarrow tx$ ($t > 0$). Then it follows from above that, on $C_c(\Phi \cup -\Phi)$:

$$\lim_{t \rightarrow 0_+} U(tu, \cdot) = \frac{1}{L_\mu} (p_+(u)\nu_+ \otimes \ell + p_-(u)\nu_- \otimes \ell).$$

If $\varphi \in C_c(V \setminus \{0\})$ vanishes on $\Phi \cup -\Phi$, Theorem 4.6 implies $\lim_{t \rightarrow 0_+} U(tu, \varphi) = 0$.

Finally we have $\lim_{t \rightarrow 0_+} \sum_0^\infty \mu^k * \delta_{tu} = \frac{1}{L_\mu} (p_+(u)\nu_+ \otimes \ell + p_-(u)\nu_- \otimes \ell)$. The existence of $H_{1,u}$ follow from the first formula in Theorem 4.4. In particular the right hand side of this formula is independant of $x \in S$. Hence, in case I, $H_{1,u}$ is independent of u . In case II, we use $S = C_0(\Lambda_+([supp\mu]))$ and we argue as above in order to obtain the formula $H_{1,u} = p_+(u)H_{1,+} + p_-(u)H_{1,-}$ where $H_{1,+} = H_{1,u}$ for $u \in C_0(\Lambda_+([supp\mu]))$ and $H_{1,-} = H_{1,u}$ for $u \in C_0(\Lambda_-([supp\mu]))$. \square

Theorem 4.7

Assume that $\mu \in M^1(G)$ is such that $[supp\mu]$ satisfies i-p, if $d > 1$ or μ is non arithmetic if $d = 1$. Assume $L_\mu < 0, \alpha > 0$ exists with $k(\alpha) = 1$, $\int |g|^\alpha \text{Log} \gamma(g) d\mu(g) < \infty$ and write :

$$L_\mu(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \int |g|^\alpha \text{Log} |g| d\mu^n(g) = \frac{k'(\alpha)}{k(\alpha)}.$$

Then $L_\mu(\alpha) > 0$ and for any $u \in \mathbb{P}^{d-1}$, we have the vague convergence in \dot{V} :

$$\lim_{t \rightarrow 0} t^{-\alpha} \sum_0^\infty \mu^k * \delta_{tu} = \frac{e^\alpha(u)}{L_\mu(\alpha)} \nu^\alpha \otimes \ell^\alpha$$

where $\nu^\alpha \in M^1(\mathbb{P}^{d-1})$ (resp $e^\alpha \in C(\mathbb{P}^{d-1})$, $\nu^\alpha(e^\alpha) = 1$) is the unique solution of $P^\alpha \nu^\alpha = \nu^\alpha$ (resp $P^\alpha e^\alpha = e^\alpha$) and ν^α has support $\Lambda([supp\mu])$.

Furthermore, on $C_b(\dot{V}_1)$ and for any $u \in \mathbb{P}^{d-1} \subset \dot{V}$ we have the vague convergence :

$$\lim_{t \rightarrow 0} t^{-\alpha} \dot{H}_1(tu, \cdot) = e^\alpha(u) \dot{H}_1^\alpha,$$

where \dot{H}_1^α is a positive measure supported on $\dot{\Lambda}_1([supp\mu])$.

In particular, for some $A > 0$ and any $u \in \mathbb{P}^{d-1}$: $\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}\{\sup_{n \geq 1} |S_n u| > t\} = A e^\alpha(u)$.

The first convergence is valid on any continuous function f which satisfies :

$$\sum_{-\infty}^{+\infty} 2^{-\ell\alpha} \sup\{|f(v)| ; 2^\ell \leq |v| \leq 2^{\ell+1}\} < \infty.$$

Proof

We observe that the function $e^\alpha \otimes h^\alpha$ on \dot{V} satisfies $\dot{P}(e^\alpha \otimes h^\alpha) = e^\alpha \otimes h^\alpha$, hence we can

consider the associated Markov operator \dot{Q}_α on \dot{V} defined by

$$\dot{Q}_\alpha(f) = \frac{1}{e^\alpha \otimes h^\alpha} \dot{P}(f e^\alpha \otimes h^\alpha).$$

Then the potential kernel of \dot{Q}_α is given by

$$\sum_0^\infty (\dot{Q}_\alpha)^k(f) = \frac{1}{e^\alpha \otimes h^\alpha} \sum_0^\infty \dot{P}^k(f e^\alpha \otimes h^\alpha).$$

Clearly \dot{Q}_α commutes with dilations, hence defines a fibered Markov kernel on \dot{V} .

Also the mean of \dot{Q}_α is $L_\mu(\alpha) > 0$. Then, taking $f = \frac{\varphi}{e^\alpha \otimes h^\alpha}$, since conditions I are valid, the result follows from Theorem 4.4. Cramer estimation for $\mathbb{P}\{|S_n u| > t\}$ follows with $A = H_1^\alpha(\dot{V}_1) > 0$. \square

Theorem 4.7'

Assume μ and α are as in Theorem 4.7. Then for any $u \in \mathbb{S}^{d-1}$ we have the vague convergence : $\lim_{t \rightarrow 0_+} t^{-\alpha} \sum_0^\infty \mu^k * \delta_{tu} = \frac{e^\alpha(u)}{L_\mu(\alpha)} \tilde{\nu}_u^\alpha \otimes \ell^\alpha$, where $\tilde{\nu}_u^\alpha \in M^1(\tilde{\Lambda}(T))$ is \tilde{P}^α -invariant.

These are 2 cases like in Theorem 4.6'.

Case I : $\tilde{\nu}_u^\alpha = \tilde{\nu}^\alpha$ has support $\tilde{\Lambda}(T)$

Case II : $\tilde{\nu}_u^\alpha = p_+^\alpha(u) \nu_+^\alpha + p_-^\alpha(u) \nu_-^\alpha$ where $p_+^\alpha(u)$ (resp $p_-^\alpha(u)$) denotes the entrance probability under \mathbb{Q}_u^α of $S_n \cdot u$ in the convex envelope of $\Lambda_+(T)$ (resp $\Lambda_-(T)$).

The above convergences are valid on any continuous function f which satisfies :

$$\sum_{\ell=-\infty}^{\ell=+\infty} 2^{-\ell\alpha} \sup\{|f(v)| ; 2^\ell \leq |v| \leq 2^{\ell+1}\} < \infty.$$

Furthermore, on $C_b(V_1)$ for any $u \in \mathbb{S}^{d-1}$, we have the vague convergence :

$$\lim_{t \rightarrow 0_+} t^{-\alpha} H_1(tu, \cdot) = e^\alpha(u) (p_+^\alpha(u) H_{1,+}^\alpha + p_-^\alpha(u) H_{1,-}^\alpha).$$

In case I, $H_{1,+}^\alpha = H_{1,-}^\alpha = H_1^\alpha$ is a positive measure supported on $\Lambda_1([\text{supp}\mu])$. In case II, $H_{1,+}^\alpha$ is a positive measure supported on $\Lambda_{1,+}([\text{supp}\mu])$ and $H_{1,-}^\alpha$ is symmetric of $H_{1,+}^\alpha$.

Proof

The proof is a simple combination of the proofs of Theorems 4.7, 4.6'. \square

For the existence of $\alpha > 0$ such that $k(\alpha) = 1$, we have the following sufficient condition where we denote by $r(g)$ the spectral radius of $g \in G$.

Proposition 4.8

Let $\mu \in M^1(G)$ and assume that $k(s) = \lim_{n \rightarrow \infty} (\int |g|^s d\mu^n(g))^{1/n}$ is finite for any $s > 0$. For any $p \in \mathbb{N}$ and $g \in (\text{supp}\mu)^p$ we have :

$$\lim_{s \rightarrow \infty} \frac{\text{Log} k(s)}{s} \geq p^{-1} \text{Log} r(g).$$

In particular if some $g \in [\text{supp}\mu]$ satisfies $r(g) > 1$, then $k(s) > 1$ for s sufficiently large.

The proof is based on the following elementary lemma which we state without proof.

Lemma 4.9

Let $g \in G$. Then for any $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ and a neighbourhood $V(\varepsilon)$ of g such for any sequence $g_k \in V(\varepsilon)$ one has $|g_n \cdots g_1| \geq c(\varepsilon) r^n(g)(1 - \varepsilon)^n$

Proof of the proposition

The convexity of $\text{Log } k(s)$ implies that $\lim_{s \rightarrow \infty} \frac{\text{Log } k(s)}{s}$ exists. Let $g \in \text{supp}\mu$, hence given $\varepsilon > 0$ there exists a neighbourhood $V(\varepsilon)$ of g such that $\mu(V(\varepsilon)) = C(\varepsilon) > 0$. From the lemma we have, with $k(s) = \lim_{n \rightarrow \infty} \left(\int |g_n \cdots g_1|^s d\mathbb{P}(\omega) \right)^{1/n}$:

$$k(s) \geq \lim_{n \rightarrow \infty} (c^s(\varepsilon) r^{ns}(g)(1 - \varepsilon)^{ns} C^n(\varepsilon))^{1/n} = r^s(g)(1 - \varepsilon)^s C(\varepsilon)$$

$$\frac{\text{Log } k(s)}{s} \geq \text{Log}(1 - \varepsilon) + \text{Log } r(g) + \frac{\text{Log } C(\varepsilon)}{s}, \quad \lim_{s \rightarrow \infty} \frac{\text{Log } k(s)}{s} \geq \text{Log } r(g)$$

We observe that if μ is replaced by μ^p , then $k(s)$ is replaced by $k^p(s)$. Hence for $g \in (\text{supp}\mu)^p$ we have from above the required inequality.

If $g \in [\text{supp}\mu]$ then we can assume $g \in (\text{supp}\mu)^p$ for some $p \in \mathbb{N}$; since $r(g) > 1$, we have $\text{Log } r(g) > 0$, hence $\lim_{s \rightarrow \infty} \frac{\text{Log } k(s)}{s} > 0$. \square

Proofs of Theorems B, B' and Corollary

Theorem B (resp B') is a direct consequence of Theorem 4.6 (resp 4.7). The assertion of the corollary is part of Theorem 4.7. \square

V The tails of an affine stochastic recursion.

1) Notations and main result

Let H be the affine group of the d -dimensional Euclidean space V , i.e the set of maps f of V into itself of the form $f(x) = gx + b$ where $g \in GL(V) = G, b \in V$. Let λ be a probability measure on H , μ its projection on G . We consider the affine random walk on $V = \mathbb{R}^d$ defined by λ , i.e the Markov chain on V described by the stochastic recursion :

$$X_{n+1}^x = A_{n+1}X_n^x + B_{n+1}, \quad X_0^x = x \in V,$$

where (A_n, B_n) are H -valued i.i.d random variables with law λ . We denote $\widehat{\Omega} = H^{\mathbb{N}}$ and we endow $\widehat{\Omega}$ with the shift $\widehat{\theta}$ and the product measure $\widehat{\mathbb{P}} = \lambda^{\otimes \mathbb{N}}$; by abuse of notation the expectation symbol with respect to $\widehat{\mathbb{P}}$ will be denoted by \mathbb{E} . We have :

$$X_n^x = A_n \cdots A_1 x + \sum_1^n A_n \cdots A_{k+1} B_k.$$

We are interested in the case where $R_n = \sum_1^n A_1 \cdots A_{k-1} B_k$ converges $\widehat{\mathbb{P}} - a.e$ to a random variable R and X_n^x converges in law to R . We observe that $X_n^x - A_n \cdots A_1 x$ and R_n have the same law. In that case we have :

$$R = \sum_0^\infty A_1 \cdots A_k B_{k+1},$$

hence R satisfies the following equation : (S) $R = ARo\hat{\theta} + B$,
and the law ρ of R satisfies the convolution equation $\rho = \lambda * \rho = \int h\rho d\lambda(h)$. Also, if R is unbounded, we will be interested in the tail of R in direction u , *i.e* the asymptotics ($t \rightarrow \infty$) of $\hat{\mathbb{P}}\{\langle R, u \rangle > t\}$ (resp $\hat{\mathbb{P}}\{|\langle R, u \rangle| > t\}$ where $u \in \mathbb{S}^{d-1}$ (resp $u \in \mathbb{P}^{d-1}$). We are mainly interested in the "shape at infinity" of ρ *i.e* the asymptotics ($t \rightarrow 0_+$) of the measure $t.\rho$ where $t.\rho$ is the push-forward of ρ by the dilation $v \rightarrow tv$ in $V(t > 0)$. It turns out that this "shape at infinity" depends only of the semigroups T and Σ ; the main burden occurs when T preserves a convex cone but Σ does not (case II' below). A basic role will be played by the top Lyapunov exponent L_μ of the product of random matrices $S_n = A_n \cdots A_1$, and μ will be assumed to satisfy $\int \text{Log}\gamma(g)d\mu(g) < \infty$ where $\gamma(g) = \sup(|g|, |g^{-1}|)$. The main hypothesis will be on μ , which is always assumed to satisfy $L_\mu < 0$ and condition i-p of section 2 if $d > 1$, or μ non arithmetic if $d = 1$. We recall that the function $k(s)$ is defined on the interval $I_\mu \subset [0, \infty[$ by $k(s) = \lim_{n \rightarrow \infty} (\int |g|^s d\mu^n(g))^{1/n}$ and $\text{Log}k(s)$ is strictly convex. It is natural to assume that $\text{supp}\lambda$ has no fixed point in V since otherwise the affine recursion reduces to a linear recursion. Let Σ (resp T) be the closed subsemigroup of H (resp G) generated by $\text{supp}\lambda$ (resp $\text{supp}\mu$). We denote by $\Delta_a(\Sigma)$ the set of fixed attractive points of the elements of Σ , *i.e* fixed points $h^+ \in V$ of elements $h = (g, b) \in \Sigma$ such that $\lim_{n \rightarrow \infty} |g^n|^{1/n} < 1$. For $v \in V \setminus \{0\}$ we denote $H_v^+ = \{x \in V; \langle v, x \rangle > 1\}$ and, for a bounded measure ξ on V we consider its Radon transform $\hat{\xi}$, *i.e* the function on $V \setminus \{0\}$ defined by $\hat{\xi}(v) = \xi(H_v^+) = \hat{\mathbb{P}}\{\langle R, u \rangle > t\}$. We also write $u = tv$ with $u \in \mathbb{S}^{d-1}$, $t > 0$ and $\hat{\xi}(u, t) = \hat{\xi}(\frac{u}{t})$. In particular, the directional tails of ξ are described by the function $\hat{\xi}(v)$ ($v \rightarrow 0$). We start with the basic :

Proposition 5.1

Assume $L_\mu < 0$ and $\mathbb{E}(\text{Log}|B|) < \infty$. Then R_n converges $\hat{\mathbb{P}} - a.e$ to $R = \sum_1^\infty A_1 \cdots A_{k-1} B_k$,

and for any $x \in V$, X_n^x converges in law to R . If $\beta \in I_\mu$ satisfies $k(\beta) < 1$ and $\mathbb{E}(|B|^\beta) < \infty$, then $\mathbb{E}(|R|^\beta) < \infty$.

The law ρ of R is the unique λ -stationary measure on V . The closure $\overline{\Delta_a(\Sigma)} = \Lambda_a(\Sigma)$ is the unique Σ -minimal subset in V and is equal to $\text{supp}\rho$. If T contains an element g with $\lim_{n \rightarrow \infty} |g^n|^{1/n} > 1$ and T has no fixed point then $\text{supp}\rho$ is unbounded.

If T satisfies condition i-p and $\text{supp}\lambda$ has no fixed point in V , then $\rho(W) = 0$ for any affine subspace W .

Proof

Under the conditions $L_\mu < 0$ and $\mathbb{E}(\text{Log}|B|) < \infty$ the $\hat{\mathbb{P}} - a.e$ convergence of R_n is well known as well as the moment condition $\mathbb{E}(|R|^\beta) < \infty$ if $k(\beta) < 1$ (see for example [5]). We complete the argument by observing that, since $L_\mu < 0$, we have $\lim_{n \rightarrow \infty} |A_n \cdots A_1 x| = 0$ hence, since $X_n^x = A_n \cdots A_1 x$ has the same law as R_n , the convergence in law of X_n^x to R

for any x follows. In particular, if $x \in V$ is distributed according to $\xi \in M^1(V)$, the law of X_n^x is $\lambda^n * \xi = \int \lambda^n * \delta_x d\xi(x)$, hence has limit ρ at $n = \infty$. If ξ is λ -stationary, we have $\lambda^n * \xi = \xi$, hence $\xi = \rho$.

Since $L_\mu < 0$, there exists $h = (g, b) \in \Sigma$, such that $|g| < 1$, hence $\lim_{n \rightarrow \infty} |g^n|^{1/n} < 1$.

If $h = (g, b) \in \Sigma$ satisfies $\lim_{n \rightarrow \infty} |g^n|^{1/n} < 1$, then $I - g$ is invertible, hence the unique fixed point h^+ of h satisfies $(I - g)h^+ = b$, and for any $x \in V$, $h^n x - h^+ = g^n(x - h^+)$, hence $\lim_{n \rightarrow \infty} h^n x = h^+$. Taking x in $\text{suppp}\rho$ we get $h^+ \in \text{suppp}\rho$, since $\text{suppp}\rho$ is h -invariant.

Furthermore, for any $x \in V$ and $h' \in \Sigma$ we have $\lim_{n \rightarrow \infty} h'^n x = h'(h^+)$ and $h'h^n \in \Sigma$ satisfies

$\lim_{n \rightarrow \infty} |g'h^n| = 0$, hence the unique fixed point x_n of $h'h^n$ satisfies $\lim_{t \rightarrow \infty} x_n = h'(h^+)$. Then

$\overline{\Delta_a(\Sigma)} = \Lambda_a(\Sigma)$ is a Σ -invariant non trivial, closed subset of $\text{suppp}\rho$.

On the other hand, for $x \in \Delta_a(\Sigma)$ we have : $\lim_{n \rightarrow \infty} \lambda^n * \delta_x = \rho$, $(\lambda^n * \delta_x)(\Lambda_a(\Sigma)) = 1$ for all n hence $\rho(\Lambda_a(\Sigma)) = 1$, i.e $\Lambda_a(\Sigma) = \text{suppp}\rho$. The Σ -minimality of $\Lambda_a(\Sigma)$ follows from the fact that, for any $x \in V$ and $h = (g, b)$ with $|g| < 1$, one has $\lim_{t \rightarrow \infty} h^t x = h^+ \in \Lambda_a(\Sigma)$ hence

$\overline{\Sigma x} \supset \Lambda_a(\Sigma)$. This implies also the uniqueness of the Σ -minimal set.

We observe that, if $\text{suppp}\rho$ is bounded, then $Co(\text{suppp}\rho)$ is compact. Also any $h \in \Sigma$ preserves $\text{suppp}\rho$ and $Co(\text{suppp}\rho)$. Then Markov-Kakutani theorem implies that the affine map h has a fixed point h^0 in $Co(\text{suppp}\rho)$. If $h = (g, b) \in \Sigma$ satisfies $\lim_{n \rightarrow \infty} |g^n|^{1/n} > 1$:

$$(I - g)h^0 = b \quad \text{and} \quad h^n x - h^0 = g^n(x - h^0),$$

hence if $x \neq h^0$, we have $\lim_{n \rightarrow \infty} |h^n x| = \infty$. Then $\text{suppp}\rho$ is unbounded since if $x \in \text{suppp}\rho \neq \delta_{h^0}$ $h^n x$ belongs to $\text{suppp}\rho$.

Let $\mathcal{W} = \{W_i ; i \in I\}$ be the set of affine subspaces of minimal dimension with $\rho(W_i) > 0$. Since $\dim(W_i \cap W_j) < \dim W_i$ if $i \neq j$, we have $\rho(W_i \cap W_j) = 0$, hence $\sum_{i \in I} \rho(W_i) \leq 1$. It

follows that, for any $\varepsilon > 0$, the set $\{W_j ; j \in I \text{ and } \rho(W_j) \geq \varepsilon\}$ has cardinality at most $\frac{1}{\varepsilon}$, hence $\rho(W_i)$ reaches its maximum on a finite set $\{W_j ; j \in J \subset I\}$ of affine subspaces.

Then the stationarity equation $\lambda * \rho = \rho$ gives on such a subspace W_j :

$$\rho(W_j) = \int \rho(h^{-1}W_j) d\lambda(h).$$

Since $\rho(h^{-1}W_j) \leq \rho(W_j)$ we get, for any $h \in \text{supp}\lambda$, $j \in J$:

$$\rho(h^{-1}W_j) = \rho(W_j), \quad \text{i.e } h^{-1}W_j = W_i, \text{ for some } i \in J.$$

In other words the set $\{W_j ; j \in J\}$ is $\text{supp}\lambda$ -invariant. If $\dim W_j > 0$, one gets that the set of directions $\overline{W_j}(j \in J)$ is a $\text{suppp}\mu$ -invariant finite set of subspaces of V , which contradicts condition i-p. Hence each $W_j(j \in J)$ is reduced to a point w_j . Then the barycenter of the finite set $\{w_j ; j \in J\}$ is invariant under $\text{supp}\lambda$, which contradicts the hypothesis. Hence $\rho(W) = 0$ for any affine subspace W . \square

In order to state the main result of this section we consider the compactification $V \cup \mathbb{S}_\infty^{d-1}$, the sets $\tilde{\Lambda}^\infty(T)$, $\Lambda_+^\infty(T)$, $\Lambda_-^\infty(T)$ defined in section 1. The closure $\overline{\Lambda_a(\Sigma)}$ of $\Lambda_a(\Sigma)$ in the

compact space $V \cup \mathbb{S}_\infty^{d-1}$ is T -invariant hence $\overline{\Lambda_a(\Sigma)} \cap \mathbb{S}_\infty^{d-1} = \Lambda_a^\infty(\Sigma)$, which is non void if $supp\rho = \Lambda_a(\Sigma)$ is unbounded, is a closed T -invariant subset of \mathbb{S}_∞^{d-1} .

Then Proposition 2.15 applied to $\Lambda_a^\infty(\Sigma) \subset \mathbb{S}_\infty^{d-1}$ gives the following trichotomy, since condition i-p is satisfied by T :

case I : T has no invariant proper convex cone and $\Lambda_a^\infty(\Sigma) \supset \tilde{\Lambda}^\infty(T)$

case II' : T has an invariant proper convex cone and $\Lambda_a^\infty(\Sigma) \supset \tilde{\Lambda}^\infty(T)$

case II'' : T has an invariant proper convex cone and $\Lambda_a^\infty(\Sigma)$ contains only one of the sets $\Lambda_+^\infty(T), \Lambda_-^\infty(T)$, say $\Lambda_+^\infty(T)$, hence $\Lambda_a^\infty(\Sigma) \cap \Lambda_-^\infty(T) = \phi$.

As in Theorem 4.7', we consider the \tilde{P}^α -invariant measures $\tilde{\nu}^\alpha, \nu_+^\alpha, \nu_-^\alpha$.

In case $\alpha \in]0, s_\infty[$ exists with $k(\alpha) = 1$ (see Proposition 4.8 for a sufficient condition). The following corresponds to Theorem C of the introduction and describes the asymptotics of the probability measure $t.\rho(t \rightarrow 0_+)$:

Theorem 5.2

With the above notations assume $L_\mu < 0$, Σ has no fixed point in V , T satisfies condition i-p, there exists $\alpha \in]0, s_\infty[$ such that $k(\alpha) = 1$ and $\mathbb{E}(|B|^{\alpha+\delta}) < \infty$, $\mathbb{E}(|A|^\alpha \gamma^\delta(A)) < \infty$ for some $\delta > 0$. If $d = 1$ assume also that μ is non arithmetic.

Then $supp\rho$ is unbounded and we have the following vague convergence on $V \setminus \{0\}$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho) = \Lambda = C(\sigma^\alpha \otimes \ell^\alpha),$$

where $C > 0$, $\sigma^\alpha \in M^1(\tilde{\Lambda}(T))$ and the measure $\Lambda = C(\sigma^\alpha \otimes \ell^\alpha)$ satisfies $\mu * \Lambda = \Lambda$.

In case I, we have $\sigma^\alpha = \tilde{\nu}^\alpha$.

In case II', there exists $C_+, C_- > 0$ with $C\sigma^\alpha = C_+\nu_+^\alpha + C_-\nu_-^\alpha$

In case II'', $\sigma^\alpha = \nu_+^\alpha$

In case I the Radon measure $\tilde{\nu}^\alpha \otimes \ell^\alpha$ on $V \setminus \{0\}$ is a minimal μ -harmonic measure, and Λ is symmetric.

In cases II, $\nu_+^\alpha \otimes \ell^\alpha$ and $\nu_-^\alpha \otimes \ell^\alpha$ are minimal μ -harmonic measures on $V \setminus \{0\}$.

Remarks

In general $supp(\sigma^\alpha \otimes \ell^\alpha) \cap \mathbb{S}_\infty^{d-1}$ is smaller than $\Lambda_a^\infty(\Sigma)$ and $supp\sigma^\alpha$ has a fractal structure.

In the context of extreme value theory for the process X_n , the convergence stated in the theorem plays a basic role and implies that ρ has "multivariate regular variation".

As the proof below shows, if the moment condition on $|A|$ is replaced by $\mathbb{E}(|A|^\alpha \text{Log}\gamma(A)) < \infty$, the convergence remains valid on the sets H_v^+ . Using [3], if $\alpha \notin \mathbb{N}$, the weak convergence of $t^{-\alpha}(t.\rho)$ follows under this weaker moment condition. Actually, using the properties of Λ , [3] gives also the weak convergence for any α (resp $\alpha \notin 2\mathbb{N}$) in case II'' (resp case I).

This is valid too for $\alpha \notin 2\mathbb{N}$ if $C_+ = C_-$ in case II', for example if ρ is symmetric.

2) Asymptotics of directional tails

We apply Theorem 4.7' to μ^* -potentials of suitable functions ; we pass, using the map $\eta \rightarrow \hat{\eta}$, from the convolution equation $\lambda * \rho = \rho$ to a Poisson type equation on $V \setminus \{0\}$ which involves μ^* and $\hat{\rho}$ and we note that $t^\alpha(t^{-1}.\rho)(H_u^+) = t^\alpha \hat{\rho}(u, t)$. The corresponding

convergences will play an essential role in the proof of Theorem 5.2. We denote by ρ_1 the law of $R - B$ and we consider the signed measure $\rho_0 = \rho - \rho_1$, hence $\rho_0(V) = 0$. Also we show that ρ_0 is "small at infinity", we define $C, C_+, C_-, \sigma^\alpha$.

Proposition 5.3

With the hypothesis of Theorem 5.2, we denote by ${}^*\tilde{\nu}_u^\alpha$ the positive kernel on \mathbb{S}^{d-1} given by Theorem 4.7' and associated with μ^* . Then one has the equations on $V \setminus \{0\}$:

$$\rho = \sum_0^\infty \mu^k * (\rho - \rho_1) \quad , \quad \widehat{\rho}(v) = \sum_0^\infty ((\mu^*)^k * \delta_v)(\widehat{\rho} - \widehat{\rho}_1).$$

For $u \in \mathbb{S}^{d-1}$, the function $t \rightarrow t^{\alpha-1} \widehat{\rho}_0(u, t)$ is Riemann-integrable in generalised sense on $]0, \infty[$ and, one has with $r_\alpha(u) = \int_0^\infty t^{\alpha-1} \widehat{\rho}_0(u, t) dt$:

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\rho}(u, t) = \frac{{}^*e^\alpha(u)}{L_\mu(\alpha)} {}^*\tilde{\nu}_u^\alpha(r_\alpha) = C(\sigma^\alpha \otimes \ell^\alpha)(H_u^+)$$

where $C = 2 \frac{{}^*\tilde{\nu}^\alpha(r_\alpha)\alpha}{L_\mu(\alpha)p(\alpha)} \geq 0$ and $\sigma^\alpha \in M^1(\widetilde{\Lambda}(T))$ satisfies $\mu * (\sigma^\alpha \otimes \ell^\alpha) = \sigma^\alpha \otimes \ell^\alpha$.

Furthermore $supp\rho$ is unbounded and,

In case I : $\sigma^\alpha = \tilde{\nu}^\alpha$

In case II : ${}^*\tilde{\nu}^\alpha(r_\alpha)\sigma^\alpha = \frac{1}{2}({}^*\nu_+^\alpha(r_\alpha)\nu_+^\alpha + {}^*\nu_-^\alpha(r_\alpha)\nu_-^\alpha)$ where ${}^*\nu_+^\alpha(r_\alpha) \geq 0, {}^*\nu_-^\alpha(r_\alpha) \geq 0$.

The proof will follow from a series of Lemmas.

We start with the following simple Tauberian Lemma (see [20]).

Lemma 5.4

For a non negative and non increasing function f on \mathbb{R}_+^* and $s \geq 0$, we denote : $f^s(t) = \frac{1}{t} \int_0^t x^s f(x) dx$. Then the condition $\lim_{t \rightarrow \infty} f^s(t) = c$ implies $\lim_{t \rightarrow \infty} t^s f(t) = c$.

Proof

Let b be a positive real number with $b > 1$ and let us observe that, since f is non increasing :

$$\frac{1}{t} \int_t^{bt} x^s f(x) dx \leq f(t) \frac{1}{t} \int_t^{bt} x^s dx = \frac{t^s}{s+1} (b^{s+1} - 1) f(t).$$

It follows :

$$\frac{b^{s+1}-1}{s+1} t^s f(t) \geq b f^s(bt) - f^s(t).$$

Then the hypothesis gives :

$$\liminf_{t \rightarrow \infty} \frac{b^{s+1}-1}{s+1} t^s f(t) \geq (b-1)c$$

Using the relation $\lim_{b \rightarrow 1} \frac{b^{s+1}-1}{(s+1)(b-1)} = 1$ we get $\liminf_{t \rightarrow \infty} f(t^s f(t)) \geq c$. An analogous argument gives $\limsup_{t \rightarrow \infty} f(t^s f(t)) \leq c$. It follows $\lim_{t \rightarrow \infty} t^s f(t) = c$. \square

We will use the multiplicative structure of the group $\mathbb{R}_+^* =]0, \infty[$, and we recall that Lebesgue measure ℓ on the multiplicative group \mathbb{R}_+^* is given by $\frac{dt}{t}$.

Lemma 5.5

Assume that the V -valued random variable R satisfies equation (S), and $\mathbb{E}(|B|^{\alpha+\delta}) < \infty$, with $\delta > 0$. For $u \in \mathbb{S}^{d-1}$ and $t, x > 0$ we write : $r^\alpha(u, t) = \frac{1}{t} \int_0^t x^\alpha \widehat{\rho}_0(u, x) dx$.

Then $|r^\alpha(u, t)| \leq \frac{t^\alpha}{\alpha+1}$. For any $\delta' \in]0, \frac{\delta}{\delta+\alpha+1}[$ there exists $C(\delta') > 0$ such that if $t \geq 1$, $|r^\alpha(u, t)| \leq C(\delta') t^{-\delta'}$. In particular the function $r^\alpha(u, t)$ is *b.R.i* on $\mathbb{S}^{d-1} \times \mathbb{R}_+^*$.

Proof

The inequality $|r^\alpha(u, t)| \leq \frac{t^\alpha}{\alpha+1}$ follows from $|\widehat{\rho}_0(u, t)| \leq 1$.

We write $\widehat{\rho}_0(u, x) = r_1(u, x) - r_2(u, x)$ with :

$$r_1(u, x) = \widehat{\mathbb{P}}\{x - \langle B, u \rangle < \langle R - B, u \rangle \leq x\},$$

$$r_2(u, x) = \widehat{\mathbb{P}}\{x < \langle R - B, u \rangle \leq x - \langle B, u \rangle\}$$

and $r_1^\alpha(u, t) = \frac{1}{t} \int_0^t x^\alpha r_1(u, x) dx$, $r_2^\alpha(u, t) = \frac{1}{t} \int_0^t x^\alpha r_2(u, x) dx$.

In order to estimate r_1^α , we choose $\varepsilon \in]0, 1[$ with $\varepsilon > \frac{\alpha}{\alpha+\delta}$ and write for $t \geq 2$:

$$r_1^\alpha(u, t) \leq \frac{1}{t} \int_2^t x^\alpha \widehat{\mathbb{P}}\{\langle B, u \rangle \geq x^\varepsilon\} dx + \frac{1}{t} \int_2^t x^\alpha \widehat{\mathbb{P}}\{x - x^\varepsilon < \langle R - B, u \rangle \leq x\} dx + \frac{2^\alpha}{t}.$$

Then Markov's inequality gives : $\widehat{\mathbb{P}}\{\langle B, u \rangle \geq x^\varepsilon\} \leq x^{-(\alpha+\delta)\varepsilon} \mathbb{E}(|B|^{\alpha+\delta})$.

Hence the first term $I_1^\varepsilon(t)$ in the above equality satisfies :

$$I_1^\varepsilon(t) \leq \mathbb{E}(|B|^{\alpha+\delta}) \frac{1}{t} \int_2^t x^{\alpha-\varepsilon(\alpha+\delta)} dx \leq \mathbb{E}(|B|^{\alpha+\delta}) t^{\alpha-\varepsilon(\alpha+\delta)}.$$

For $t - t^\varepsilon \geq 2$, the second term $I_2^\varepsilon(t)$ satisfies :

$$I_2^\varepsilon(t) = \frac{1}{t} \int_2^t x^\alpha \widehat{\mathbb{P}}\{\langle R - B, u \rangle > x - x^\varepsilon\} dx - \frac{1}{t} \int_2^{t-t^\varepsilon} x^\alpha \widehat{\mathbb{P}}\{\langle R - B, u \rangle > x\} dx.$$

In the second integral above we use the change of variables $x \rightarrow x - x^\varepsilon$ and we get :

$$I_2^\varepsilon(t) \leq \frac{1}{t} \int_2^t [x^\alpha - (x - x^\varepsilon)^\alpha (1 - \varepsilon x^{\varepsilon-1})] \widehat{\mathbb{P}}\{\langle R - B, u \rangle > x - x^\varepsilon\} dx + \frac{k_0(\varepsilon)}{t}$$

with $0 < k_0(\varepsilon) < \infty$.

We observe that there exists $k_1(\varepsilon) < \infty$ such that for any $x \geq 2$:

$$x^\alpha - (x - x^\varepsilon)^\alpha (1 - \varepsilon x^{\varepsilon-1}) \leq k_1(\varepsilon) x^{\alpha+\varepsilon-1}$$

For any $\beta \in]0, \alpha[$, Proposition 5.1 implies that $\mathbb{E}(|R|^\beta) < \infty$. Also R satisfies equation (S) and $A, R\widehat{\theta}$ are independant. Hence Markov's inequality gives :

$$\widehat{\mathbb{P}}\{\langle R - B, u \rangle > x\} \leq x^{-\beta} \mathbb{E}(|A|^\beta) \mathbb{E}(|R|^\beta) \leq k_2(\beta) x^{-\beta} \text{ with } k_2(\beta) < \infty.$$

It follows that for any t with $t - t^\varepsilon > 2$:

$$I_2^\varepsilon(t) \leq \frac{k_0(\varepsilon)}{t} + k_1(\varepsilon) k_2(\beta) \frac{1}{t} \int_2^t \frac{x^{\alpha+\varepsilon-1}}{(x-x^\varepsilon)^\beta} dx \leq k_3(\varepsilon, \beta) t^{\alpha-\beta+\varepsilon-1}$$

We can choose β very close to $\frac{\alpha+1}{\alpha+1+\delta} > \frac{\alpha}{\alpha+\delta-1}$ in $]\alpha+\varepsilon-1, \alpha[$ such that for $\delta' = \text{Inf}(\delta, \frac{\delta}{\delta+\alpha})$ we have $\sup\{t^{\delta'} |r_1^\alpha(u, t)|; t \geq 1\} < \infty$, as follows from the above estimations of $I_1^\varepsilon(t)$ and $I_2^\varepsilon(t)$.

Hence, there exists $k_3 < \infty$ such that for $t \geq 1$: $r_1^\alpha(u, t) \leq k_3 t^{-\delta'}$.

The same argument is valid for r_2^α , hence for $\delta' < \frac{\delta}{\delta+\alpha+1}$ and $t \geq 1$, we have : $r^\alpha(u, t) \leq C(\delta') t^{-\delta'}$, with $C(\delta') < \infty$. Furthermore, for $t \in]0, 1[$ we have $|r^\alpha(u, t)| \leq \frac{t^\alpha}{\alpha+1}$, hence the function $r^\alpha(u, t)$ is *b.R.i.* on $\mathbb{S}^{d-1} \times \mathbb{R}_+^*$ \square

Lemma 5.6

We denote by r the finite measure on \mathbb{R}_+^* defined by $r(dx) = 1_{]0,1[}(x) x^\alpha dx$ and we write $\rho_0 = \rho - \rho_1$. Then the function h_α on $V \setminus \{0\}$ defined by

$h_\alpha(v) = |v|^{-\alpha}(r * \widehat{\rho}_0)(v) = \frac{1}{t} \int_0^t x^\alpha \widehat{\rho}_0(u, x) dx$, $u = tv$ ($t > 0, |u| = 1, v \in V \setminus \{0\}$) is b.R.i and one has $(\delta_u \otimes \ell^\alpha)(r * \widehat{\rho}_0) = \int_0^\infty t^{\alpha-1} \widehat{\rho}_0(u, t) dt = r_\alpha(u)$ where $t \rightarrow t^{\alpha-1} \widehat{\rho}_0(u, t)$ is Riemann-integrable on $]0, \infty[$ in generalised sense.

Proof

By definition :

$$h_\alpha(v) = |v|^{-\alpha}(r * \widehat{\rho}_0)(v) = t^\alpha(r * \widehat{\rho}_0)\left(\frac{u}{t}\right) = \frac{1}{t} \int_0^t y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy = r^\alpha(u, t),$$

$$(\delta_u \otimes \ell^\alpha)(r * \widehat{\rho}_0) = \int_0^\infty t^{\alpha-1} \frac{dt}{t^{\alpha+1}} \int_0^t y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy = \lim_{T \rightarrow \infty} \int_0^T \frac{dt}{t^2} \int_0^t y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy.$$

Lemma 5.5 implies that $h_\alpha(v)$ is b.R.i and has limit 0 at $|v| = \infty$. Integration by parts in the above formula gives :

$$\int_0^T \frac{dt}{t^2} \int_0^t y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy = -\frac{1}{T} \int_0^T y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy + \int_0^T t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right) dt = -r^\alpha(u, T) + \int_0^T t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right) dt.$$

Since, using Lemma 5.5 $\lim_{T \rightarrow \infty} r^\alpha(u, T) = 0$, it follows that $\int_0^T t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right) dt$ has a finite limit at $T = \infty$, hence $t \rightarrow t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right)$ is Riemann-integrable on \mathbb{R}_+^* in generalised sense and, for $u \in \mathbb{S}^{d-1}$:

$$(\delta_u \otimes \ell^\alpha)(r * \widehat{\rho}_0) = \int_0^\infty t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right) dt = r_\alpha(u). \quad \square$$

Proof of Proposition 5.3

If $u = tv$ with $u \in \mathbb{S}^{d-1}$, $t > 0$ the function $\widehat{\rho}(v) = \widehat{\mathbb{P}}\{\langle R, u \rangle > t\}$ is bounded, right continuous and non increasing. Since equation (S) can be written as $R - B = ARo\widehat{\theta}$ and $A, Ro\widehat{\theta}$ are independant we have : $\rho_1 = \mu * \rho$, $\rho - \mu * \rho = \rho - \rho_1 = \rho_0$. Furthermore :

$$\rho = \sum_0^n \mu^k * \rho_0 + \mu^{n+1} * \rho, \quad \widehat{\rho}(v) = \sum_0^n ((\mu^*)^k * \delta_v)(\widehat{\rho}_0) + ((\mu^*)^{n+1} * \delta_v)(\widehat{\rho}).$$

Also if r^- denote the push-forward of r by $x \rightarrow x^{-1}$:

$$r^- * \rho = \sum_0^n \mu^k * (r^- * \rho_0) + \mu^{n+1} * (r^- * \rho).$$

Since $L_\mu < 0$ the subadditive ergodic theorem applied to $\text{Log}|S_n(\omega)|$ gives the convergence of $S_n(\omega)v$ to 0. In particular, for $\xi \in M^1(V)$, the sequence $\mu^n * \xi$ converges in law to δ_0 hence : $\lim_{n \rightarrow \infty} \widehat{\mu^n * \xi}(v) = (\mu^*)^n * \delta_v(\widehat{\xi}) = 0$.

From the above convergence on V : $\rho - \delta_0 = \sum_0^\infty \mu^k * \rho_0$, $\widehat{\rho}(v) = \sum_0^\infty ((\mu^*)^k * \delta_v)(\widehat{\rho}_0)$.

But, by Proposition 5.1, $\rho(\{0\}) = 0$, hence the stated vague convergence of $\sum_0^\infty \mu^k * \rho_0$.

Since the sequence $(\mu^{n+1} * (r^- * \rho))(\psi)$ converges to zero for any bounded Borel function ψ on V such that $\lim_{v \rightarrow 0} \psi(v) = 0$, we have on such functions : $r^- * \rho = \sum_0^\infty (\mu^*)^k * (r^- * \rho_0)$.

We observe that, for any bounded measure ξ , we have $(\mu^k * \widehat{\xi})(v) = ((\mu^*)^k * \delta_v)(\widehat{\xi})$ and $\widehat{r^- * \xi} = r * \widehat{\xi}$. It follows from the above equality that the potential $\sum_0^\infty ((\mu^*)^k * \delta_v)(r * \widehat{\rho}_0)$ is finite and equal to $(r * \widehat{\rho})(v)$.

We have observed in Lemma 5.6 that the function $v \rightarrow |v|^{-\alpha}(r * \widehat{\rho}_0)(v)$ is b.R.i, hence the renewal Theorem 4.7' applied to μ^* and to the function $r * \widehat{\rho}_0$ gives for $u \in \mathbb{S}^{d-1}$:

$$\lim_{t \rightarrow \infty} t^\alpha (r * \hat{\rho})(u, t) = \frac{{}^*e^\alpha(u)}{L_\mu(\alpha)} ({}^*\tilde{\nu}_u^\alpha \otimes \ell^\alpha)(r * \hat{\rho}_0).$$

Since for fixed u , $\hat{\rho}(u, x) = \hat{\mathbb{P}}\{< R, u >> x\}$ is non increasing, Lemma 5.4 gives :

$$\lim_{t \rightarrow \infty} t^\alpha \hat{\rho}(u, t) = \frac{{}^*e^\alpha(u)}{L_\mu(\alpha)} {}^*\tilde{\nu}_u^\alpha(r_\alpha).$$

In particular, we have ${}^*\tilde{\nu}_u^\alpha(r_\alpha) \geq 0$. In case I this gives ${}^*\tilde{\nu}^\alpha(r_\alpha) \geq 0$ since ${}^*\tilde{\nu}_u^\alpha = {}^*\tilde{\nu}^\alpha$.

In case II, taking $u \in \Lambda_+(T^*)$ and using ${}^*p_+^\alpha(u) = 1$, this gives ${}^*\nu_+^\alpha(r_\alpha) \geq 0$. Also, in the same way ${}^*\nu_-^\alpha(r_\alpha) \geq 0$. Furthermore, in case II, using Theorem 4.7' :

$${}^*\tilde{\nu}_u^\alpha(r_\alpha) = {}^*p_+^\alpha(u) {}^*\nu_+^\alpha(r_\alpha) + {}^*p_-^\alpha(u) {}^*\nu_-^\alpha(r_\alpha).$$

If ${}^*\tilde{\nu}^\alpha(r_\alpha) > 0$, in case II we can define a probability measure σ^α on $\tilde{\Lambda}(T)$ by :

$${}^*\tilde{\nu}^\alpha(r_\alpha) \sigma^\alpha = \frac{1}{2} ({}^*\nu_+^\alpha(r_\alpha) \nu_+^\alpha + {}^*\nu_-^\alpha(r_\alpha) \nu_-^\alpha)$$

while in case I : $\sigma^\alpha = \tilde{\nu}^\alpha$. If ${}^*\tilde{\nu}^\alpha(r_\alpha) = 0$, we have also ${}^*\nu_+^\alpha(r_\alpha) = {}^*\nu_-^\alpha(r_\alpha) = 0$, hence we can leave σ^α with projection ν^α on \mathbb{P}^{d-1} undefined in the above formulas. In any case $\sigma^\alpha \otimes \ell^\alpha$ is μ -harmonic.

We get another expression for the above limit, by using the formulas for ${}^*e^\alpha(u)$, ${}^*p_+^\alpha(u)$, ${}^*p_-^\alpha(u)$ of section 2, for case II as follows (see Theorem 2.17) :

$${}^*e^\alpha(u) {}^*p_+^\alpha(u) p(\alpha) = \int < u, u' >_+^\alpha d\nu_+^\alpha(u') = \alpha(\nu_+^\alpha \otimes \ell^\alpha)(H_u^+).$$

From above, we get :

$${}^*e^\alpha(u) {}^*\tilde{\nu}_u^\alpha(r_\alpha) = \frac{\alpha}{p(\alpha)} (({}^*\nu_+^\alpha(r_\alpha) \nu_+^\alpha + {}^*\nu_-^\alpha(r_\alpha) \nu_-^\alpha) \otimes \ell^\alpha)(H_u^+).$$

Hence, with $C = 2 \frac{{}^*\tilde{\nu}^\alpha(r_\alpha) \alpha}{L_\mu(\alpha) p(\alpha)}$, σ^α as above and $C_+ = \frac{{}^*\tilde{\nu}_+^\alpha(r_\alpha) \alpha}{L_\mu(\alpha) p(\alpha)}$, $C_- = \frac{{}^*\tilde{\nu}_-^\alpha(r_\alpha) \alpha}{L_\mu(\alpha) p(\alpha)}$

$$\lim_{t \rightarrow \infty} t^\alpha \hat{\rho}(u, t) = C(\sigma^\alpha \otimes \ell^\alpha)(H_u^+), \quad C\sigma^\alpha = C_+ \nu_+^\alpha + C_- \nu_-^\alpha.$$

In case I we get the corresponding formula.

In order to show that $\text{supp} \rho$ is unbounded, in view of Proposition 5.1, it suffices to show that there exists $g \in T$ with $\lim_{n \rightarrow \infty} |g^n|^{1/n} > 1$. If not, then the trace $\text{Tr} g$ of g is bounded by d on T . On the other hand, condition i-p implies the irreducibility of the action of T on $V \otimes \mathbb{C}$, as shown now. Let $W \subset V \otimes \mathbb{C}$ be a proper T -invariant subspace of $V \otimes \mathbb{C}$ and \overline{W} its complex conjugate. Then $W \cap \overline{W}$ and $W + \overline{W}$ are complexified subspaces of subspaces of V which are also T -invariant. Using irreducibility of T we get $W \cap \overline{W} = \{0\}$, $W + \overline{W} = V \otimes \mathbb{C}$ hence $V \otimes \mathbb{C} = W \oplus \overline{W}$. Let $g \in T^{\text{prox}}$ and $v \in V \setminus \{0\}$ with $gv = \lambda_g v$ and write $v = w + \bar{w}$ with $w \in W$, hence since W, \overline{W} are g -invariant $gw = \lambda_g w$, $g\bar{w} = \lambda_g \bar{w}$. Since λ_g is a simple eigenvalue we get $w = \bar{w}$ i.e $W \cap \overline{W} \neq \{0\}$ which gives a contradiction with condition i-p, hence T acts irreducibly on $V \otimes \mathbb{C}$. Then Burnside's density theorem implies that $\text{End}(V \otimes \mathbb{C})$ is generated as an algebra by T , i.e there exists a base $g_i \in T$ ($i = 1, \dots, d^2$) of $\text{End} V$. Then the linear forms $u \rightarrow \text{Tr}(ug_i)$ ($i = 1, \dots, d^2$) form a basis of the dual space of $\text{End} V$. In particular, for some constant $c > 0$ we have for any $g \in T$, $|g| \leq c \sum_{i=1}^{d^2} |\text{Tr}(gg_i)| \leq cd^3$. Then for any $n \in \mathbb{N}$, and $s > 0$: $\int |g|^s d\mu^n(g) \leq cd^{3s}$, $k(s) \leq 1$. Since $k'(\alpha) > 0$ with $\alpha \in]0, s_\infty[$, this contradicts the convexity of $\text{Log} k(s)$. \square

Corollary 5.7

For any $v \in V \setminus \{0\}$, we have :

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{| < R, v > | > t\} = C \frac{p(\alpha)}{\alpha} (*e^\alpha \otimes h^\alpha)(v)$$

In particular, there exists $b > 0$ such that $\widehat{\mathbb{P}}\{|R| > t\} \leq bt^{-\alpha}$.

Proof

By definition of $*\tilde{\nu}_u^\alpha$ and since $*p_+^\alpha(u) = *p_-^\alpha(-u)$ we have : $\frac{1}{2}(*\nu_u^\alpha + *\nu_{-u}^\alpha) = *\tilde{\nu}^\alpha$. Hence, using the proposition :

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{| < R, u > | > t\} = 2 \frac{*e^\alpha(u)}{L_\mu(\alpha)} *\tilde{\nu}^\alpha(r_\alpha) = C \frac{p(\alpha)}{\alpha} *e^\alpha(u).$$

The formula in the corollary follows by α -homogeneity.

We take a base $u_i \in V$ ($1 \leq i \leq d$) and write $|R| \leq b' \sum_{i=1}^d | < R, u_i > |$ with $b' > 0$. For

t large : $\widehat{\mathbb{P}}\{|R| > t\} \leq \sum_{i=1}^d \widehat{\mathbb{P}}\{| < R, u_i > | > \frac{t}{b'}\} \leq (C' + \varepsilon)t^{-\alpha} b'^\alpha \sum_{i=1}^d *e^\alpha(u_i)$, with $\varepsilon > 0$,

$C' = C \frac{p(\alpha)}{\alpha}$, hence the result. \square

3) A dual Markov walk and the positivity of directional tails

Here we go over from the process $< R_n, v >$ to its Markovian extension $(v_n, r_n) = (S'_n v, r + < R_n, v >) \in V \times \mathbb{R}$, we denote by $*\widehat{P}$ its corresponding extended kernel on $E = (V \setminus \{0\}) \times \mathbb{R}$ and we study the corresponding generalized ladder process (see [15], p 391). The properties of this process allow to give a useful new expression for $\widehat{\rho}(v) = \widehat{\mathbb{P}}\{< R, u > > t\}$ as a potential of a non negative function on E .

Let M be a T^* -minimal subset of \mathbb{S}^{d-1} and $X = M \times \mathbb{R}$, hence $M = \widetilde{\Lambda}(T^*)$ in case I and $M = \Lambda_+(T^*)$ (or $\Lambda_-(T^*)$) in cases II. We denote by $\Lambda_a^*(\Sigma)$ the set of $u \in \mathbb{S}^{d-1}$ such that the projection of ρ on the line $\mathbb{R}u$ has unbounded support in direction u .

The following says that $\Lambda_a^*(\Sigma)$ is "large"

Lemma 5.8

In cases I or II' : $\Lambda_a^*(\Sigma) = \mathbb{S}^{d-1}$. In case II'' : $\Lambda_a^*(\Sigma) \supset \Lambda_+(T^*)$

Proof

Let $\Lambda_a^\infty(\Sigma) = \overline{\Lambda_a(\Sigma)} \cap \mathbb{S}_\infty^{d-1}$ and $u \in \mathbb{S}^{d-1}$, $u'_\infty \in \Lambda_a^\infty(\Sigma)$ corresponds to $u' \in \mathbb{S}^{d-1}$. If $< u', u > > 0$, then $u \in \Lambda_a^*(\Sigma)$. Hence the complement of $\Lambda_a^*(\Sigma)$ in \mathbb{S}^{d-1} is contained in the set $\{u \in \mathbb{S}^{d-1} ; < u, u' > \leq 0 \forall u'_\infty \in \Lambda_a^\infty(\Sigma)\}$. From the discussion at the beginning of this section we know that $\Lambda_a^\infty(\Sigma) \neq \phi$ is T -invariant and closed, hence contains $\widetilde{\Lambda}^\infty(T)$ in cases I, II' or only $\Lambda_+^\infty(T)$ in case II'' with $\Lambda_+^\infty(T) \cap \Lambda_a^\infty(\Sigma) = \phi$.

Since $\widetilde{\Lambda}^\infty(T)$ is symmetric and condition i-p is valid it follows $\Lambda_a^*(\Sigma) = \mathbb{S}^{d-1}$ in cases I, II'.

In case II'', we know from the end of proof of Theorem 2.17 that the complement of $\Lambda_a^*(\Sigma)$ is contained in $\widehat{\Lambda}_+(T^*) = \{u \in \mathbb{S}^{d-1} ; \forall u' \in \Lambda_+(T), < u, u' > > 0\}$. Since $\Lambda_+(T^*) \cap -\widehat{\Lambda}_+(T^*) = \phi$ we get $\Lambda_+(T^*) \subset \Lambda_a^*(\Sigma)$. \square

The following will play an essential role in the discussion of positivity for C, C_+, C_- . Its proof is based on the study of the ladder process on E mentioned above.

Proposition 5.9

With the hypothesis of Theorem 5.2 if $M \subset \mathbb{S}^{d-1}$ is T^* -minimal and $\Lambda_a^*(\Sigma) \supset M$, then for any $u \in M$:

$$C(u) = \lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{\langle R, u \rangle > t\} > 0.$$

We begin by introducing notations for the proof. We observe that $R_n = \sum_0^{n-1} A_1 \cdots A_k B_{k+1}$ satisfies the relation $\langle R_{n+1}, v \rangle = \langle R_n, v \rangle + \langle B_{n+1}, S'_n v \rangle$ where $S'_n = (A_1 \cdots A_n)^*$. Also $h = (g, b) \in H$ acts on $E = (V \setminus \{0\}) \times \mathbb{R}$ according to the formula $h(v, r) = (g^*v, r + \langle b, v \rangle)$, hence the pair $(S'_n v, r + \langle R_n, v \rangle)$ is a random walk on the right homogeneous H -space E . Actually, $V \times \mathbb{R}$ is a vector space and the above formula for $h(v, r)$ defines a right linear representation of H in $V \times \mathbb{R}$ which leaves invariant $E \subset V \times \mathbb{R}$. In particular, using the radial \mathbb{R}_+^* -fibration of this vector space, we see that the map $(v, r) \rightarrow (u, p)$ with $v = |v|u$, $r = p|v|$ defines an H -equivariant projection from E to $\mathbb{S}^{d-1} \times \mathbb{R}$ hence $(S'_n v, r + \langle R_n, v \rangle)$ is also a \mathbb{R}_+^* -fibered Markov chain over $\mathbb{S}^{d-1} \times \mathbb{R}$. The action of $h = (g, b)$ on $\mathbb{S}^{d-1} \times \mathbb{R}$ is given by $h(u, p) = (g^*.u, h^u p)$ with $h^u p = \frac{1}{|g^*u|}(p + \langle b, u \rangle)$. We observe that, if H_u^p is the affine hyperplane defined by $\langle y, u \rangle + p = 0$, and $h \in H$, the hyperplane $h^{-1}(H_u^p)$ is defined by $\langle y, g^*.u \rangle + h^u p = 0$, hence the formula $h(u, p) = (g^*.u, h^u p)$ corresponds to the action of h^{-1} on the space $W = \mathbb{S}^{d-1} \times \mathbb{R}$ of affine oriented hyperplanes in V .

The random walk $(v_n, r_n) = (S'_n v, r + \langle R_n, v \rangle)$ has projection $S'_n v$ on V , the kernel ${}^*\widehat{P}$ has projection *P already defined in section 2, hence the homogeneous function ${}^*e^\alpha \otimes h^\alpha$ on E satisfies ${}^*\widehat{P}({}^*e^\alpha \otimes h^\alpha) = {}^*e^\alpha \otimes h^\alpha$, and we can consider the new relativized fibered Markov kernel ${}^*\widehat{P}_\alpha$ on E . If $(u, p) \in M \times \mathbb{R}$, the projection $x_n = (u_n, p_n)$ of (v_n, r_n) on $X = M \times \mathbb{R}$ depends on the kernel ${}^*\widehat{Q}^\alpha$ given by ${}^*\widehat{Q}^\alpha \varphi(u, p) = \int \varphi(g^*.u, h^u p) {}^*q^\alpha(u, g) d\lambda(h)$ and ${}^*q^\alpha$ corresponds to q^α as in section 3. It will turn out that ${}^*\widehat{Q}^\alpha$ and its extension ${}^*\widehat{Q}^\alpha$ to $M \times \mathbb{R}$ have similar stochastic properties.

For the analysis of ${}^*\widehat{Q}^\alpha$ we consider on $\widehat{\Omega}$ the projective limit ${}^*\widehat{Q}_u^\alpha$ of the system ${}^*q_n^\alpha(u, \cdot) \lambda^{\otimes n}$ ($n \in \mathbb{N}$) and, by abuse of notation, the corresponding expectation will be written \mathbb{E}_u^α . Given a ${}^*\widehat{Q}^\alpha$ -stationary measure $\widetilde{\pi}_M^\alpha$, we write ${}^*\widehat{Q}^\alpha = \int \delta_u \otimes {}^*\widehat{Q}_u^\alpha d\widetilde{\pi}_M^\alpha(u)$ and we denote by \mathbb{E}^α the corresponding expectation symbol. We denote by $\widehat{\theta}^\#$ the map of $M \times H^\mathbb{Z}$ into itself defined by $\widehat{\theta}^\#(u, \widehat{\omega}) = (g_1^*.u, \widehat{\theta}\widehat{\omega})$ where $\widehat{\theta}$ is the bilateral shift on $H^\mathbb{Z}$, and ${}^*\widehat{Q}^\alpha$ will again denote the natural $\widehat{\theta}^\#$ -invariant measure on $M \times H^\mathbb{Z}$. Also we extend $S'_n(\omega)$ as a G -valued \mathbb{Z} -cocycle. If η is a probability measure on X , the associated Markov measure on ${}^a\widehat{\Omega} = X \times \widehat{\Omega}$, is denoted by ${}^*\widehat{Q}_\eta^\alpha$, and the extended shift by ${}^a\widehat{\theta}$ where ${}^a\widehat{\theta}(x, \widehat{\omega}) = (h_1 x, \widehat{\theta}\widehat{\omega})$. It will be convenient to use the functions $a(g, u), b(h, u)$ defined by $h^u p = a(g, u)p + b(h, u)$, and the random variables a_k, b_k ($k \in \mathbb{Z}$) defined by $a_k(\widehat{\omega}, u) = a(g_k, S'_{k-1}.u)$, $b_k(\widehat{\omega}, u) = b(h_k, S'_{k-1}.u)$. Then we can express the action of $h_n \cdots h_1 \in H$ on X as : $u_n = S'_n.u$, $y_n^p(u) = (h_n \cdots h_1)^u p$,

where :

$$y_n^p(u) = a(S'_n, u)p + y_n^o(u) \text{ and } y_n^o(u) = \sum_1^n a_{k+1}^n(u)b_k(u)$$

with $a_k^n(u) = a(g_n \cdots g_k, S'_{k-1} \cdot u)$. The random variables a_k, b_k are ${}^*\widehat{Q}^\alpha$ -stationary and y_n^o has the same law as $p_n^o(\widehat{\omega}, u) = \sum_0^{n-1} a_{-1} \cdots a_{-k} b_{-k-1}$.

Also if η is ${}^*\widehat{Q}^\alpha$ -stationary we will consider the bilateral associated system $(\Omega^\#, {}^a\widehat{\theta}, \eta^\#)$ where $\Omega^\# = X \times H^\mathbb{Z}$, ${}^a\widehat{\theta}$ is the bilateral shift and $\eta^\#$ is the unique ${}^a\widehat{\theta}$ -invariant measure with projection ${}^*\widehat{Q}_\eta^\alpha$ on $X \times \widehat{\Omega}$. The Birkhoff sum $\text{Log}(|S'_n u| |\frac{p_n}{p}|)$ occurs below and can be extended as an \mathbb{R} -valued \mathbb{Z} -cocycle on X , again denoted by the same formula. The following lemma is a fibered version of a well known fact in the context of generalized autoregressive processes.

Lemma 5.10

Let M be a T^* -minimal subset of \mathbb{S}^{d-1} , π_M^α the unique ${}^*\widehat{Q}^\alpha$ -stationary measure on M . With the above notations, we consider the Markov chain $x_n = (u_n, p_n)$, on $M \times \mathbb{R}$ given by :

$$u_{n+1} = g_{n+1}^* \cdot u_n, \quad p_{n+1} = \frac{p_n + \langle b_{n+1}, u_n \rangle}{|g_{n+1}^* u_n|}, \quad p_0 = p, \quad u_0 = u,$$

where (g_n, b_n) are distributed according to ${}^*\widehat{Q}_u^\alpha$. Then, for any $p \in \mathbb{R}$, x_n converges in ${}^*\widehat{Q}^\alpha$ -law to the unique ${}^*\widehat{Q}^\alpha$ -stationary measure κ , the projection of κ on M is π_M^α , $\kappa(M \times \{p\}) = 0$ and $\int |p|^\varepsilon d\kappa(u, p) < \infty$ for ε small. We have $\kappa^\# - a.e$:

$$\limsup_{n \rightarrow \infty} |S'_n u| |p_n| = \infty, \quad \lim_{n \rightarrow \infty} |S'_{-n} u| |p_{-n}| = 0.$$

If $\Lambda_a^*(\Sigma) \supset M$, then $\kappa(M \times]0, \infty[) > 0$ and $\limsup_{n \rightarrow \infty} |S'_n u| |p_n| = \infty$, ${}^*\widehat{Q}_\kappa^\alpha - a.e$.

Proof

We estimate $\mathbb{E}^\alpha(|p_n^o|^\varepsilon)$ for $0 < \varepsilon < \delta$ and ε small, where $p_n^o(\widehat{\omega}, u) = \sum_0^{n-1} a_{-1} \cdots a_{-k} b_{-k-1}$. Since $(a_k \cdots a_1)(\omega, u) = a(S'_k(\omega), u)$ we get $\mathbb{E}^\alpha(|a_{-1} \cdots a_{-k}|^\varepsilon) = \mathbb{E}^\alpha(|S'_k u|^{-\varepsilon})$. Hence Corollary 3.18 gives $\lim_{k \rightarrow \infty} (\mathbb{E}^\alpha(|a_{-1} \cdots a_{-k}|^\varepsilon))^{1/k} < 1$ since $k'(\alpha) > 0$ and $\mathbb{E}(|A|^{\alpha+\delta}) < \infty$. Also for ε small,

$$\mathbb{E}^\alpha(|b_k|^\varepsilon) = \mathbb{E}^\alpha(| < \frac{b_1, u}{|g^* u}| > |^\varepsilon) \leq \mathbb{E}(|B_1|^\varepsilon \gamma^\varepsilon(A)) < \infty,$$

using ${}^*\widehat{Q}^\alpha$ -stationarity, Hölder inequality and the condition $\mathbb{E}(|B_1|^{\alpha+\delta}) + \mathbb{E}(|A_1|^{\alpha+\delta}) < \infty$.

Since $|p_n^o(\widehat{\omega}, u)|^\varepsilon \leq \sum_0^{n-1} |a_{-1} \cdots a_{-k}|^\varepsilon |b_{-k-1}|^\varepsilon$, we get that $\mathbb{E}^\alpha(|p_n^o|^\varepsilon/2)$ is bounded. The

${}^*\widehat{Q}^\alpha - a.e$ convergence of the partial sum $p_n^o(\widehat{\omega}, u)$ to $p(\widehat{\omega}, u) = \sum_0^\infty a_{-1} \cdots a_{-k} b_{-k-1}$ and

the finiteness of $\mathbb{E}^\alpha(|p|^\varepsilon)$ follows. By definition, $p(\widehat{\omega}, u)$ satisfies the functional equation $p \circ {}^a\widehat{\theta} = ap + b$ where p and (a, b) are independant. It follows that the probability measure κ on $M \times \mathbb{R}$ given by the formula $\kappa = \int \delta_u \otimes \delta_{p(\widehat{\omega}, u)} d {}^*\widehat{Q}^\alpha(\widehat{\omega}, u)$ is ${}^*\widehat{Q}^\alpha$ -invariant.

As observed above, the ${}^*\widehat{Q}^\alpha$ -laws of y_n^o and p_n^o are the same. Since the product of $\tilde{\pi}_M^\alpha$ with

the law of y_n° is $(\widehat{Q}^\alpha)^n (\pi_M^\alpha \otimes \delta_\circ)$ we have in weak topology : $\lim_{n \rightarrow \infty} (\widehat{Q}^\alpha)^n (\pi_M^\alpha \otimes \delta_\circ) = \kappa$. Since $|y_n^p(\widehat{\omega}, u) - y_n^{p'}(\widehat{\omega}, u)| = a(S'_n(\omega), u)|p - p'|$ and $a(S'_n(\omega), u) = |S'_n u|^{-1}$ converges ${}^*\widehat{Q}_u^\alpha - a.e$ to zero, we get the convergence of $(\widehat{Q}^\alpha)^n (\pi_M^\alpha \otimes \delta_p)$ to κ , for any p . On the other hand, if η' is a ${}^*\widehat{Q}^\alpha$ -stationary measure on $M \times \mathbb{R}$, its projection on M is ${}^*\widehat{Q}^\alpha$ -stationary, hence equal to π_M^α , since M is T^* -minimal. Then, from above $(\widehat{Q}^\alpha)^n \eta'$ converges to κ , hence $\eta' = \kappa$. The ${}^*\widehat{Q}^\alpha$ -ergodicity of κ implies the ${}^a\widehat{\theta}$ -ergodicity of ${}^*\widehat{Q}_\kappa^\alpha$ and $\kappa^\#$. Assume $\kappa(M \times]0, \infty[) = 0$, i.e the ${}^*\widehat{Q}^\alpha$ -invariant set $supp \kappa$ is contained in $M \times]-\infty, 0]$. Then, for any $(u, p) \in supp \kappa : p+ < R_n, u > \leq 0$ ${}^*\widehat{Q}_u^\alpha$ -a.e, i.e $< R_n, u > \leq -p$ ${}^*q_n^\alpha(u, \cdot)$ -a.e for any $n \in \mathbb{N}$, for some $p \leq 0$, $u \in M$. It follows $< R_n, u > \leq -p$, $\lambda^{\otimes n}$ -a.e, and $< R, u > \leq -p$, $\widehat{\mathbb{P}}$ -a.e. This implies that the support of the projection of ρ on $\mathbb{R}u$ is bounded in direction u , hence contradicts the condition $\Lambda_a^*(\Sigma) \supset M$. Furthermore, arguments as in the proof of Proposition 5.1, using that $supp \lambda$ has no fixed point in V , show $\kappa(M \times \{p\}) = 0$ for any $p \in \mathbb{R}$. From Theorem 3.10 we know that $\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} |S'_n u| = L_\mu(\alpha) > 0$, ${}^*\widehat{Q}_u^\alpha$ -a.e. Furthermore since κ is ${}^*\widehat{Q}^\alpha$ -ergodic and $\kappa(M \times \{0\}) = 0$ we have $\limsup_{n \rightarrow \infty} |p_n| > 0$ ${}^*\widehat{Q}_\kappa^\alpha - a.e$. Then we get $\limsup_{n \rightarrow \infty} |S'_n u| |p_n| = \infty$ ${}^*\widehat{Q}_\kappa^\alpha$ -a.e. If $\Lambda_a^*(\Sigma) \supset M$, we have $\kappa(M \times]0, \infty[) > 0$, and again using ergodicity : $\limsup_{n \rightarrow \infty} p_n > 0$. Since $\lim_{n \rightarrow \infty} |S'_n u| = \infty$ ${}^*\widehat{Q}_u^\alpha - a.e$, it follows $\limsup_{n \rightarrow \infty} |S'_n u| p_n = \infty$ ${}^*\widehat{Q}_\kappa^\alpha$ -a.e. Using Theorem 3.2, we have also , for any $u \in M$ and ${}^*\widehat{Q}_u^\alpha$ -a.e : $\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} |S'_{-n} u| = -L_\mu(\alpha) < 0$. The condition $\int |p|^\varepsilon d\kappa(u, p) < \infty$ implies : $\limsup_{n \rightarrow \infty} \frac{\text{Log} |p_n|}{|n|} \leq 0$ $\kappa^\#$ -a.e. Then we get : $\lim_{n \rightarrow \infty} |S'_{-n} u| |p_{-n}| = 0$ $\kappa^\#$ -a.e. \square

Remark

The moment condition on A in the proposition can be replaced by the finiteness of the expectation of $|A|^\alpha \text{Log} \gamma(A)$, with change of $|p|^\varepsilon$ into $\text{Log}^+ |p|$ in the conclusion. The proof uses Theorem 3.10 instead of Corollary 3.18

For the analysis of ${}^*\widehat{P}_\alpha, \widehat{\rho}$ we consider the optional time τ given for $p \neq 0$ by :

$$\tau = \text{Inf} \{n > 0 ; p^{-1} < R_n, u > > 0\}, \tau = \infty \text{ if } p^{-1} < R_n, u > \leq 0 \text{ for every } n.$$

We observe that τ is independant of p as long as $p > 0$ or $p < 0$. By definition of p_n , $p+ < R_n, u > = p_n |S'_n u|$, hence $\tau = \text{Inf} \{n > 0 ; p^{-1} p_n |S'_n u| > 1\}$, in particular $p^{-1} p_\tau > 0$. Also we define $\tau_n = \tau \circ ({}^a\widehat{\theta})^{\tau_{n-1}}$, so that τ can be interpreted as the first ladder epoch and $p^{-1} p_\tau |S'_\tau u|$ as the first ladder height of the \mathbb{Z} -cocycle $p^{-1} p_n |S'_n u|$ which is well defined $\kappa^\# - a.e$ since $\kappa(M \times \{0\}) = 0$. The random times τ_n can be seen as the successive times of increase for $r^{-1} r_k = p^{-1} p_k |S'_k u|$ along the random walk (v_n, r_n) . These two descriptions of τ will be used below. On the other hand, by Poincaré recurrence theorem we have, $\kappa^\#$ -a.e $\limsup_{n \rightarrow \infty} p^{-1} p_n > 0$; since $\lim_{n \rightarrow \infty} |S'_n u| = \infty$, τ, τ_n are finite ${}^*\widehat{Q}_\kappa^\alpha$ -a.e and $p^{-1} p_{\tau_n} > 0$.

In view of the above discussion it is natural to use (u, p, t') coordinates with $t' = r^{-1} = p^{-1}t \in \mathbb{R}^*$, $p|v| = r \in \mathbb{R}$, $|v|^{-1} = t$ so that the λ -random walk on E can be written as :

$$u_{n+1} = g_{n+1}^* \cdot u_n, \quad p_{n+1} = \frac{p_{n+1} + \langle b_{n+1}, u_n \rangle}{|g_{n+1}^* u_n|}, \quad t'_{n+1} = t'_n |g_{n+1}^* u_n|^{-1} p_n p_{n+1}^{-1}$$

hence $v_n = S'_n v$, $t'_n = t'(|S'_n u| \frac{p_n}{p})^{-1}$, $p_n = \frac{p + \langle R_n, u \rangle}{|S'_n u|}$.

We denote $X_+ = M \times]0, \infty[$, hence $x \in X_+$ implies $x_\tau \in X_+$.

We consider also the stopped operator ${}^* \widehat{P}^\tau$ on $V \setminus \{0\} \times \mathbb{R}$. In (u, p, t') coordinates with $x = (u, p)$ the associated process at time n starting from $(x, t') \in X_+ \times \mathbb{R}_+^*$ is $(x_{\tau_n}, t' |S'_{\tau_n} u|^{-1} p_{\tau_n}^{-1} p)$, hence ${}^* \widehat{P}^\tau$ is a \mathbb{R}_+^* -fibered kernel on $X_+ \times \mathbb{R}_+^* \subset E$. Since ${}^* \widehat{P}({}^* e^\alpha \otimes h^\alpha) = {}^* e^\alpha \otimes h^\alpha$ and τ is finite ${}^* \mathbb{Q}_\kappa^\alpha$ -a.e, the kernel ${}^* \widehat{P}_\alpha^\tau$ given by

$${}^* \widehat{P}_\alpha^\tau \varphi = ({}^* e^\alpha \otimes h^\alpha)^{-1} \widehat{P}^\tau ({}^* e^\alpha \otimes h^\alpha \varphi)$$

is a positive fibered kernel on $X_+ \times \mathbb{R}_+^*$ which satisfies ${}^* \widehat{P}_\alpha^\tau 1 = 1, \kappa \otimes \ell$ a.e.

The following lemma expresses the function $\widehat{\rho}(p^{-1}v) = \widehat{\mathbb{P}}\{p^{-1} < R, u \gg t\} = \psi(v, p)$ on E as a ${}^* \widehat{P}^\tau$ -potential of a non negative function. Its asymptotics will give the positivity of $C(u)$ in Proposition 5.8.

Lemma 5.11

With $tv = u \in \mathbb{S}^{d-1}$, $t > 0$ and $p \neq 0$, we write : $\tau = \text{Inf}\{n > 0; p^{-1} < R_n, u \gg 0\}$
 $\psi(v, p) = \widehat{\mathbb{P}}\{p^{-1} < R, u \gg t\}$, $\psi_\tau(v, p) = \widehat{\mathbb{P}}\{t < p^{-1} < R, u \gg \leq t + p^{-1} < R_\tau, u \gg; \tau < \infty\}$
 where $R_\tau = \sum_0^{\tau-1} A_1 \cdots A_k B_{k+1}$, $\psi^\alpha = ({}^* e^\alpha \otimes h^\alpha)^{-1} \psi$, $\psi_\tau^\alpha = ({}^* e^\alpha \otimes h^\alpha)^{-1} \psi_\tau$. Then :

$$\psi = \sum_0^\infty ({}^* \widehat{P}^\tau)^k \psi_\tau, \quad \psi^\alpha = \sum_0^\infty ({}^* \widehat{P}_\alpha^\tau)^k \psi_\tau^\alpha.$$

Proof

We write $\langle R - R_n, v \rangle = \langle R^n, S'_n v \rangle$, hence if $\tau < \infty$: $\langle R - R_\tau, v \rangle = \langle R^\tau, S'_\tau v \rangle$. By definition of τ :

$$\psi_\tau(v, p) = \psi(v, p) - \widehat{\mathbb{P}}\{\langle R - R_\tau, u \rangle p^{-1} > t; \tau < \infty\}$$

On the other hand, since $p^{-1} p_\tau > 0$:

$$\widehat{\mathbb{P}}\{\langle R - R_\tau, u \rangle p^{-1} > t; \tau < \infty\} = \widehat{\mathbb{P}}\{\langle R^\tau, u_\tau \rangle p^{-1} > \frac{t}{|S'_\tau u|}; \tau < \infty\} = {}^* \widehat{P}^\tau \psi(v, p).$$

It follows $\psi_\tau = \psi - {}^* \widehat{P}^\tau \psi$, $\psi = \sum_0^{n-1} ({}^* \widehat{P}^\tau)^k \psi_\tau + {}^* \widehat{P}^{\tau_n} \psi$ with :

$${}^* \widehat{P}^{\tau_n} \psi(v, p) = \widehat{\mathbb{P}}\{t |S'_{\tau_n} u|^{-1} < p^{-1} < R^{\tau_n}, u_{\tau_n} \gg; \tau_n < \infty\}.$$

For $(x, \widehat{\omega}) \in M \times \widehat{\Omega}$ we have either $\tau_n(x, \widehat{\omega}) = \infty$ for some n or $\lim_{n \rightarrow \infty} \tau_n(x, \widehat{\omega}) = \infty$. In the second case, $\lim |S'_{\tau_n} u|^{-1} = \infty$ and, since R_{τ_n} is bounded, $\lim_{n \rightarrow \infty} {}^* \widehat{P}^{\tau_n} \psi = 0$, $\psi = \sum_0^\infty ({}^* \widehat{P}^\tau)^k \psi_\tau$.

The last relation follows from the definitions of ψ_τ^α and ${}^* \widehat{P}_\alpha^\tau$, since ψ_τ^α is non negative. \square

Let τ be as above, $\Lambda_a^*(\Sigma) \supset M$ and let ${}^* \widehat{Q}^{\alpha, \tau}(x, \cdot)$ be the law of x_τ under ${}^* \widehat{Q}_x^\alpha$. Since $\tau < \infty$, ${}^* \mathbb{Q}^\alpha - a.e$ and $p^{-1} p_\tau > 0$ we have $p_\tau > 0$ if $p > 0$, hence the operator ${}^* \widehat{Q}^{\alpha, \tau}$ preserves $X_+ = M \times]0, \infty[$ and satisfies ${}^* \widehat{Q}^{\alpha, \tau} 1 = 1, \kappa - a.e$. We will also consider the first

return time to X_+ and the corresponding Markov operator ${}^*\widehat{Q}_+^\alpha$ on X_+ . Following closely ([37], Lemma 2) we can construct a stationary measure for ${}^*\widehat{Q}_+^{\alpha,\tau}$ on X_+ , and show the finiteness of the corresponding expectation of τ .

Lemma 5.12

Assume $\Lambda_a^*(\Sigma) \supset M$ and write :

$$\tau = \text{Inf}\{n > 0 ; p^{-1}p_n|S'_n u| > 1\} = \text{Inf}\{n > 0 ; p^{-1} < R_n, u \gg 0\}.$$

Then the stopped operator ${}^*\widehat{Q}_+^{\alpha,\tau}$ preserves X_+ and admits a stationary ergodic measure κ^τ on X_+ which is absolutely continuous with respect to κ .

The integral $\mathbb{E}_0^\alpha(\tau) = \int \mathbb{E}_u^\alpha(\tau) d\kappa^\tau(u, p)$ is finite.

If $\gamma_\tau^\alpha = L_\mu(\alpha)\mathbb{E}_0^\alpha(\tau)$ one has $\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}(|S'_{\tau_n} u| \frac{p_{\tau_n}}{p}) = \gamma_\tau^\alpha \in]0, \infty[$, ${}^*\widehat{Q}_+^{\alpha,\tau}$ -a.e.

Also $V_\tau = \text{Log}(p_\tau p^{-1}|S'_\tau u|)$ has finite expectation with respect to ${}^*\widehat{Q}_+^{\alpha,\tau}$.

Proof

Since $\Lambda_a^*(\Sigma) \supset M$, Lemma 5.10 gives $\kappa(X_+) > 0$. In order to deal only with positive values of the \mathbb{Z} -cocycle $|S'_n u| p_n p^{-1}$ it is convenient to consider the two sided Markov chain $x_{n_k} (k \in \mathbb{Z}, x_0 \in X_+)$, induced on X_+ by x_n hence n_1 is the first return time of x_n to X_+ and $n_1 \leq \tau$ since $p^{-1}p_\tau > 0$. We note that the normalized restriction κ_+ of κ is a stationary ergodic measure for x_{n_k} . Also the Markov kernel ${}^*\widehat{P}_\alpha$ on E induces a fibered Markov kernel ${}^*\widehat{P}_{\alpha,+}$ on $(X_+ \times \mathbb{R}^*) \cap E$ with projection ${}^*\widehat{Q}_+^\alpha$ on X_+ . The corresponding bilateral Markov chain is (x_{n_k}, t'_{n_k}) with $t'_{n_k} = t' p p_{n_k}^{-1} |S'_{n_k} u|^{-1}$. We observe that, if $t' > 0, p > 0$, then $t' p_{n_k} p^{-1} |S'_{n_k} u| > 0$, hence ${}^*\widehat{P}_{\alpha,+}$ preserves $X_+ \times \mathbb{R}_+^* \subset E$ and ${}^*\widehat{P}_{\alpha,+}(\kappa_+ \otimes \ell) = \kappa_+ \otimes \ell$. We denote by $\Omega_+^\#$ the subset of $\Omega^\#$ defined by the conditions $x \in X_+, x_n \in X_+$ infinitely often for $n > 0$ and $n < 0$, by $\kappa_+^\#$ the normalized restriction of $\kappa^\#$ to $\Omega_+^\#$ and by ${}^a\widehat{\theta}_+$ the induced shift. Also let $\Omega_0^\#$ be the subset of $\Omega_+^\#$ defined by the conditions $x \in X_+, \sup_{k>0} (p_{n-k} p^{-1} |S'_{n-k} u|) < 1$. From Lemma 5.10, we know that $\kappa_+^\# - a.e$:

$$\lim_{n \rightarrow \infty} |S'_{-n} u| p_{-n} p^{-1} = 0, \text{ hence } \lim_{k \rightarrow \infty} |S'_{n-k} u| p_{n-k} p^{-1} = 0, |S'_{n-k} u| p_{n-k} p^{-1} > 0.$$

Then the index $-\nu_0 \leq 0$ of the strict last maximum of the sequence $p_{n-k} p^{-1} |S'_{n-k} u| = V_{-k}$ ($k \geq 0$) is finite $\kappa_+^\# - a.e$. We have $\Omega_0^\# = \{\nu_0 = 0\}$ and, using Lemma 5.10 : $\lim_{k \rightarrow \infty} V_{-k} = 0$.

It follows :

$$1 = \sum_{k=0}^{\infty} \kappa_+^\# \{\nu_0 = k\} \leq \sum_{k=0}^{\infty} \kappa^\# \{V_{-k} > \sup_{j < -k} V_j\} = \sum_{j=0}^{\infty} \kappa^\# \{V_0 > \sup_{j < 0} V_j\} = \sum_{j=0}^{\infty} q,$$

where $q = \kappa^\#(\Omega_0^\#)$. Hence $\kappa^\#(\Omega_0^\#) > 0$. Furthermore if $\omega^\# \in \Omega_0^\#$ we see that $\tau(\omega^\#)$ is the first return time of $({}^a\widehat{\theta}_+)^n(\omega^\#)$ to $\Omega_0^\#$, hence ${}^a\widehat{\theta}_+^\tau$ is the transformation on $\Omega_0^\#$ induced by ${}^a\widehat{\theta}_+$ or ${}^a\widehat{\theta}_+$ and $\tau_n = \tau \circ ({}^a\widehat{\theta}_+^\tau)^n$ on $\Omega_0^\#$. This allows to proceed as in ([37] Lemma 2) with the \mathbb{R}_+^* -valued \mathbb{Z} -cocycle $|S'_{n_k} u| p_{n_k} p^{-1}$. Since $\kappa_+^\#$ is ${}^a\widehat{\theta}_+$ -invariant and $\kappa_+^\#(\Omega_0^\#) > 0$ we can apply Kac's recurrence theorem to $\Omega_0^\#, {}^a\widehat{\theta}_+^\tau$ and $\Omega^\#$ (see [49]), hence the normalized

restriction $\kappa_0^\#$ of $\kappa^\#$ to $\Omega_0^\#$ is ${}^a\widehat{\theta}^\tau$ -ergodic and stationary, the return time τ has finite expectation $\mathbb{E}_0^\alpha(\tau)$ and $\lim_{n \rightarrow \infty} \frac{\tau_n}{n} = \mathbb{E}_0^\alpha(\tau)$, $\kappa_0^\# - a.e.$ Since $\kappa_0^\#$ is absolutely continuous with respect to $\kappa^\#$, Theorem 3.10 gives :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} |S'_{\tau_n} u| = \left(\lim_{n \rightarrow \infty} \frac{1}{\tau_n} \text{Log} |S'_{\tau_n} u| \right) \left(\lim_{n \rightarrow \infty} \frac{\tau_n}{n} \right) = \mathbb{E}_0^\alpha(\tau) L_\mu(\alpha) = \gamma_\tau^\alpha > 0, \quad \kappa_0^\# - a.e.$$

Using Birkhoff's theorem for the non-negative increments of $W_{\tau_n} = \text{Log} \left(\frac{p_{\tau_n}}{p} |S'_{\tau_n} u| \right)$ and ${}^a\widehat{\theta}^\tau$, we get the $\kappa_0^\#$ -a.e convergence of $\frac{1}{n} W_{\tau_n}$. Since $\kappa_0^\#$ is ${}^a\widehat{\theta}^\tau$ -invariant $\frac{1}{n} \text{Log} \frac{p_{\tau_n}}{p}$ converges to zero in $\kappa_0^\#$ -measure, hence using the $\kappa_0^\# - a.e$ convergence of $\frac{1}{n} \text{Log} |S'_{\tau_n} u|$, we get the $\kappa_0^\# - a.e$ convergence of $\frac{1}{n} W_{\tau_n}$ to γ_τ^α . In particular $\mathbb{E}_0^\alpha(W_\tau) = \gamma_\tau^\alpha \in]0, \infty[$.

In order to relate $\kappa_0^\#$ and the kernel ${}^*\widehat{Q}^{\alpha, \tau}$ we consider the Markov kernel adjoint to ${}^*\widehat{Q}_+^\alpha(x, \cdot)$ with respect to κ_+ , and we denote by ${}^*\widehat{Q}_-^\alpha \otimes \delta_x$ the corresponding Markov measure on $H^{\mathbb{Z}^-} \times X_+$ with $\mathbb{Z}^- = -\mathbb{N} \cup \{0\}$. Also we write ${}^*\widehat{Q}_-^\alpha = \delta_x \otimes {}^*\widehat{Q}_+^\alpha$ where ${}^*\widehat{Q}_+^\alpha$ is supported on $H^{\mathbb{N}}$ and $\kappa_+^\# = \int {}^*\widehat{Q}_-^\alpha \otimes \delta_x \otimes {}^*\widehat{Q}_+^\alpha d\kappa_+(x)$, i.e $\kappa^\#(\Omega_0^\#) \kappa_0^\# = \int (1_{\Omega_0^\#}) {}^*\widehat{Q}_-^\alpha \otimes \delta_x \otimes {}^*\widehat{Q}_+^\alpha d\kappa_+(x)$ with $\Omega_0^\# = \Omega_0^- \times H^{\mathbb{N}}$ and $\Omega_0^- \subset H^{\mathbb{Z}^-} \times X_+$. We denote by κ^τ the projection of $\kappa_0^\#$ on X_+ , hence κ^τ has density $u(x)$ given by $\kappa^\#(\Omega_0^\#)u(x) = ({}^*\widehat{Q}_-^\alpha \otimes \delta_x)(\Omega_0^-)$ with respect to κ_+ . It follows that the projection of $\kappa_0^\#$ on $X \times \widehat{\Omega}$ can be expressed as :

$$\int u(x) \delta_x \otimes {}^*\widehat{Q}_+^\alpha d\kappa_+(x) = \int \delta_x \otimes {}^*\widehat{Q}_+^\alpha d\kappa^\tau(x) = {}^*\widehat{Q}_{\kappa^\tau}^\alpha.$$

Since $\kappa_0^\#$ is invariant and ergodic with respect to the bilateral shift ${}^a\widehat{\theta}^\tau$, the same is valid for ${}^*\widehat{Q}_{\kappa^\tau}^\alpha$ with respect to the associated unilateral shift ${}^a\widehat{\theta}^\tau$. Since the kernel $x \rightarrow {}^*\widehat{Q}_x^\alpha$ commutes with ${}^a\widehat{\theta}^\tau$ and ${}^*\widehat{Q}^{\alpha, \tau}$, the ${}^*\widehat{Q}^{\alpha, \tau}$ -invariance and ergodicity of κ^τ follows. Also we have $\mathbb{E}_0^\alpha(\tau) = \int \mathbb{E}_x^\alpha(\tau) d\kappa^\tau(x)$ and the above convergences are valid ${}^*\widehat{Q}_{\kappa^\tau}^\alpha - a.e$ \square

Remark

If $S = X_+$, $P = {}^*\widehat{P}_{\alpha, +}$, $\pi = \kappa_+$ the measure κ^τ is closely connected with the measure ψ of Theorem 4.4. The measure κ^τ can be characterized as the unique ${}^*\widehat{Q}^{\alpha, \tau}$ -stationary measure which is absolutely continuous with respect to κ_+ . However, in our context, the function $\text{Log}|p|$ is not known to be κ -integrable, but we know that $\lim_{n \rightarrow \infty} |S'_{-n} u| p_{-n} p^{-1} = 0$.

Proof of Proposition 5.9

Assume that for some $u \in M$, $C(u) = \lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{\langle R, u \rangle > t\} = 0$.

For $p > 0$, with the notations of Lemma 5.11, this means $\lim_{t \rightarrow \infty} \psi^\alpha(v, p) = 0$. Using Proposition 5.3 we know that this implies $\lim_{t \rightarrow \infty} \psi^\alpha(u, p) = 0$ for any $u = tv \in M$ ($t > 0$). Also, from Lemma 5.12, we have, since $\Lambda_a^*(\Sigma) \supset M$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}(|S'_{\tau_n} u| \frac{p_{\tau_n}}{p}) = \gamma_\tau^\alpha > 0, \quad {}^*\widehat{Q}_{\kappa^\tau}^\alpha - a.e.$$

Since the canonical Markov measure associated with κ^τ and ${}^*\widehat{Q}^{\alpha,\tau}$ is a push-forward of ${}^*\widehat{Q}_{\kappa^\tau}^\alpha$, this convergence is also valid with respect to this canonical measure.

Then, using Lemma 5.11 and $\psi_\tau^\alpha \geq 0$, we can apply Lemma A.1 to the Markov kernel ${}^*\widehat{P}_\alpha^\tau$ on $X_+ \times \mathbb{R}_+^*$, the potential $\sum_0^\infty ({}^*\widehat{P}_\alpha^\tau)^k \psi_\tau^\alpha$ of the non negative function $\psi_\tau^\alpha \leq ({}^*e^\alpha \otimes h^\alpha)^{-1}$ and

the ${}^*\widehat{Q}^{\alpha,\tau}$ stationary measure κ^τ : we get $\psi_\tau^\alpha = 0$, $\kappa^\tau \otimes \ell - a.e$, hence :

$$\widehat{\mathbb{P}}\{t < p^{-1} < R, u > \leq t + p^{-1} < R_\tau, u >, \tau < \infty\} = 0.$$

Since $p^{-1} < R_\tau, u > > 0$, this gives $p^{-1} < R, u > \leq 0$ $\kappa^\tau \otimes \widehat{\mathbb{P}} - a.e$ on $\{\tau < \infty\}$, in particular for some $(u, p) \in X_+$, $p^{-1} < R, u > \leq 0$ i.e $< R, u > \leq 0$ $\widehat{\mathbb{P}} - a.e$ on $\{\tau < \infty\}$. But, since $\Lambda^*(\Sigma) \supset M$, Lemma 5.10 implies that, for any $u \in M$ the set $\{< R, u > > 0, \tau < \infty\} = \{< R, u > > 0\}$ is not $\widehat{\mathbb{P}}$ -negligible, hence the required contradiction. \square

The following improves Corollary 5.7.

Corollary 5.13

For any $u \in \mathbb{S}^{d-1}$:

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{|< R, u > | > t\} = C \frac{p(\alpha)}{\alpha} {}^*e^\alpha(u) > 0$$

In cases I, for any $u \in \mathbb{S}^{d-1}$:

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{< R, u > > t\} = \frac{1}{2} C \frac{p(\alpha)}{\alpha} {}^*e^\alpha(u) > 0$$

In case II, for any $u \in \Lambda_+(T^*)$, if $\Lambda_a(\Sigma) \supset \Lambda_+^\infty(T)$:

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{< R, u > > t\} = \frac{p(\alpha)}{\alpha} C_+ {}^*e^\alpha(u) > 0.$$

Proof

This a trivial consequence of Proposition 5.3, Corollary 5.7, Proposition 5.9 and Lemma 5.8. \square

The following is a corollary of the proof of Proposition 5.9.

Corollary 5.14

With the above notations we write :

$\gamma_\tau^\alpha = L_\mu(\alpha) \mathbb{E}_0^\alpha(\tau)$, $\psi_\tau^\alpha(v, p) = \widehat{\mathbb{P}}\{t < p^{-1} < R, u > \leq t + p^{-1} < R_\tau, u >, \tau < \infty\} {}^*e^\alpha(u)^{-1} t^\alpha$ and we denote by κ^τ the ${}^*\widehat{Q}^{\alpha,\tau}$ -stationary measure on X_+ , given by Lemma 5.12. Then, if $\Lambda_a(\Sigma) \supset \Lambda_+^\infty(T)$:

$$C_+ \geq \frac{\alpha}{p(\alpha)\gamma_\tau^\alpha} \int_{]0, \infty[\times X_+} \psi_\tau^\alpha(v, p) t^{-1} d\kappa^\tau(u, p) dt > 0.$$

Proof

With the notations of Lemma 5.11, we have $\psi^\alpha = \sum_0^\infty ({}^*\widehat{P}_\alpha^\tau)^k \psi_\tau^\alpha$ where ${}^*\widehat{P}_\alpha^\tau$ is a fibered Markov kernel on $X_+ \times \mathbb{R}_+^*$ which satisfies the conditions of Lemma A.1 and ψ_τ^α is bounded by $({}^*e^\alpha \otimes h^\alpha)^{-1}$. Hence, as in the proof of the proposition, if $\overline{\psi}_\tau^\alpha$ is a Borel function with compact support, bounded by ψ_τ^α :

$$\lim_{t \rightarrow \infty} \psi^\alpha(v, p) \geq \limsup_{t \rightarrow \infty} \sum_0^\infty {}^*\widehat{P}_\alpha^\tau \overline{\psi}_\tau^\alpha(v, p),$$

and, using Corollary 5.13, since $\lim_{t \rightarrow \infty} \psi^\alpha(v, p)$ is constant on X_+ :

$$\lim_{t \rightarrow \infty} \psi^\alpha(v, p) \geq \frac{1}{\gamma_\tau^\alpha} \int_{]0, \infty[\times X_+} \bar{\psi}_\tau^\alpha(v', p) d\kappa^\tau(u', p) \frac{dt}{t}.$$

Hence, approximating from below ψ_τ^α by $\bar{\psi}_\tau^\alpha$:

$$\begin{aligned} \lim_{t \rightarrow \infty} \psi^\alpha(v, p) &\geq \frac{1}{\gamma_\tau^\alpha} \int_{]0, \infty[\times X_+} \psi_\tau^\alpha(v', p) \frac{dt}{t} d\kappa^\tau(u', p), \\ C(u) &= \lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{\langle R, u \rangle > t\} = {}^*e^\alpha(u) \lim_{t \rightarrow \infty} \psi^\alpha(v, p), \\ C(u) &\geq \frac{{}^*e^\alpha(u)}{\gamma_\tau^\alpha} \int_{]0, \infty[\times X_+} \psi_\tau^\alpha(v', p) t^{-1} dt d\kappa^\tau(u', p). \end{aligned}$$

The final formula follows from Corollary 5.13. \square

Remark

We observe that, if $d = 1$, and A, B are positive, a formula of this type for $C = C_+$, with equality, is given in [14]. We don't know if such an equality is valid in our setting. However if λ is non singular, then theorem 4.4 can be used instead of Lemma A.1, since from the corollary ψ_τ^α is $\kappa^\tau \otimes \ell$ -integrable, hence equality in the above formula is valid.

4) Homogeneity at infinity of the stationary measure

For the proof of Theorem 5.2 we prepare the following lemmas. If $\alpha \notin \mathbb{N}$, it follows from [3] that Theorem 5.2 is a consequence of Propositions 5.3 and 5.9. If $\alpha \in \mathbb{N}$, as follows from [50] the situation is different in general. Here, we will need to use the Choquet-Deny type results of Appendix 2.

Lemma 5.15

For any compact subset K of $V \setminus \{0\}$, there exists a constant $C(K) > 0$ such that $\sup_{t > 0} t^{-\alpha}(t.\rho)(K) \leq C(K)$. In particular the family $\rho_t = t^{-\alpha}(t.\rho)$ is relatively compact for the topology of vague convergence and any cluster value η of the family ρ_t satisfies $\sup_{t > 0} t^{-\alpha}(t.\eta)(K) \leq C(K)$, hence $\sup_{t > 0} t \cdot ((e^\alpha \otimes h^\alpha)\eta)(K) \leq C'(K)$ with $C'(K) > 0$.

Proof

For some $\delta > 0$ we have $K \subset \{x \in V ; |x| > \delta\}$, hence using Corollary 5.7, $t^{-\alpha}\mathbb{P}\{|R| > \frac{\delta}{t}\} \leq \frac{b}{\delta^\alpha} = C(K)$. The relative compactness of the family ρ_t follows. Also :

$$(tt_n)^{-\alpha}(tt_n.\rho)(K) \leq C(K), \quad t^{-\alpha}(t.\eta)(K) = \lim_{n \rightarrow \infty} (tt_n)^{-\alpha}(tt_n.\rho)(K) \leq C(K),$$

Hence $\sup_{t > 0} t^{-\alpha}(t.\eta)(K) \leq C(K)$.

Since $e^\alpha \otimes h^\alpha$ is α -homogeneous : $t \cdot ((e^\alpha \otimes h^\alpha)\eta) = (e^\alpha \otimes h^\alpha)(t^{-\alpha}(t.\eta))$.

With $C_K = \sup_{v \in K} (e^\alpha \otimes h^\alpha)(v)$, we get :

$$t \cdot ((e^\alpha \otimes h^\alpha)\eta)(K) \leq C_K t^{-\alpha}(t.\eta)(K) \leq C_K C(K) = C'(K). \quad \square$$

Lemma 5.16

Assume η is the vague limit of $t_n^{-\alpha}(t_n.\rho)$ ($t_n \rightarrow 0_+$). Then η is μ -harmonic, i.e $\mu * \eta = \eta$.

Proof

Let φ be ε -Hölder continuous on V with compact support contained in the set $\{x \in V; |x| \geq \delta\}$ with $\delta > 0$, and let us show $\lim_{t \rightarrow 0_+} t^{-\alpha} I_t(\varphi) = 0$ where $I_t(\varphi) = (t.\rho - t.(\mu * \rho))(\varphi)$.

By definition :

$I_t(\varphi) = \mathbb{E}(\varphi(tR) - \varphi(tA_1 R o \widehat{\theta}))$ with $\varphi(tR) = 0$ if $|tR| < \delta$ and $\varphi(tA_1 R o \widehat{\theta}) = 0$ if $|tA_1 R o \widehat{\theta}| < \delta$. Hence : $I_t(\varphi) \leq [\varphi]_\varepsilon t^\varepsilon \mathbb{E}(|B_1|^\varepsilon 1_{\{|tR| > \delta\}} + |B_1|^\varepsilon 1_{\{|tA_1 R o \widehat{\theta}| > \delta\}})$.

We write :

$$I_t^1 = t^{\varepsilon - \alpha} \mathbb{E}(|B_1|^\varepsilon 1_{\{|tR| > \delta\}}), \quad I_t^2 = t^{\varepsilon - \alpha} \mathbb{E}(|B_1|^\varepsilon 1_{\{|tA_1 R o \widehat{\theta}| > \delta\}})$$

and we estimate I_t^1, I_t^2 as follows. We have $I_t^1 \leq \delta^{\varepsilon - \alpha} \mathbb{E}(|B_1|^\varepsilon |R|^{\alpha - \varepsilon} 1_{\{|tR| > \delta\}})$. Since $|R|^{\alpha - \varepsilon} \leq c(|A_1 R o \widehat{\theta}|^{\alpha - \varepsilon} + |B_1|^{\alpha - \varepsilon})$, using independance of $R o \widehat{\theta}$ and $|B_1|^\varepsilon |A_1|^{\alpha - \varepsilon}$:

$$\mathbb{E}(|B_1|^\varepsilon |R|^{\alpha - \varepsilon}) \leq c \mathbb{E}(|B_1|^\alpha) + c \mathbb{E}(|A_1|^{\alpha - \varepsilon} |B_1|^\varepsilon) \mathbb{E}(|R|^{\alpha - \varepsilon}).$$

Using Hölder inequality we get $\mathbb{E}(|A_1|^{\alpha - \varepsilon} |B_1|^\varepsilon) < \infty$. Also using Proposition 5.1, we get $\mathbb{E}(|R|^{\alpha - \varepsilon}) < \infty$. It follows that $|B_1|^\varepsilon |R|^{\alpha - \varepsilon} 1_{\{|tR| > \delta\}}$ is bounded by the integrable function $|B_1|^\varepsilon |R|^{\alpha - \varepsilon}$. Then by dominated convergence : $\lim_{t \rightarrow 0} I_t^1 = 0$.

In the same way we have :

$$I_t^2 \leq \delta^{\varepsilon - \alpha} \mathbb{E}(|B_1|^\varepsilon |A_1 R o \widehat{\theta}|^{\alpha - \varepsilon} 1_{\{|tA_1 R o \widehat{\theta}| > \delta\}})$$

Also, using independance and Hölder inequality :

$$\mathbb{E}(|B_1|^\varepsilon |A_1 R o \widehat{\theta}|^{\alpha - \varepsilon}) \leq \mathbb{E}(|B_1|^\varepsilon |A_1|^{\alpha - \varepsilon}) \mathbb{E}(|R|^{\alpha - \varepsilon}) < \infty.$$

Then by dominated convergence $\lim_{t \rightarrow 0_+} I_t^2 = 0$. Hence $\lim_{t \rightarrow 0_+} t^{-\alpha} I_t(\varphi) = 0$.

By definition of η we have, for any $g \in G$: $\lim_{t_n \rightarrow 0_+} t_n^{-\alpha} (t_n.g\rho)(\varphi) = (g\eta)(\varphi)$.

Furthermore we have $|\varphi(x)| \leq |\varphi| 1_{\{x \in V; |x| \geq \delta\}}$ and,

$$|(g\eta)(\varphi)| \leq |\varphi| \eta \{x \in V; |gx| \geq \delta\} \leq |\varphi| \lim_{n \rightarrow \infty} t_n^{-\alpha} \mathbb{P}\{|R| > \frac{\delta}{|g|t_n}\}.$$

Using Corollary 5.7, we get : $|(g\eta)(\varphi)| \leq \frac{b}{\delta^\alpha} |\varphi| |g|^\alpha$.

Since $\int |g|^\alpha d\mu(g) < \infty$ and for any $g \in G$ $\lim_{t_n \rightarrow 0_+} t_n^{-\alpha} (t_n.g\rho)(\varphi) = g\eta(\varphi)$, we have by dominated convergence : $\lim_{t_n \rightarrow 0_+} t_n^{-\alpha} (t_n.(\mu * \rho))(\varphi) = (\mu * \eta)(\varphi)$. Then the property $\lim_{t \rightarrow 0_+} t^{-\alpha} I_t = 0$ implies $(\mu * \eta)(\varphi) = \eta(\varphi)$, hence $\mu * \eta = \eta$. \square

Lemma 5.17

Assume η and $\sigma \otimes \ell^\alpha$ are μ -harmonic Radon measures on $V \setminus \{0\}$ with $\sigma \in M^1(\mathbb{S}^{d-1})$. Assume also that for any $v \in V \setminus \{0\}$, $\eta(H_v^+) = (\sigma \otimes \ell^\alpha)(H_v^+)$. Then we have $\eta = \sigma \otimes \ell^\alpha$

Proof

As in the proof of Corollary 5.7, we observe that the condition $\eta(H_v^+) = (\sigma \otimes \ell^\alpha)(H_v^+)$ implies for any $\delta > 0$:

$$\sup_{t > 0} t^{-\alpha} (t.\eta) \{x \in V; |x| > \delta\} < \infty$$

Hence, as in Lemma 5.14, for any compact $K \subset V \setminus \{0\}$, with $\eta^\alpha = (e^\alpha \otimes h^\alpha)\eta$:

$$\sup_{t>0} t^{-\alpha}(t.\eta)(K) \leq C(K), \quad \sup_{t>0} (t.\eta^\alpha)(K) \leq C'(K).$$

It follows that η^α is dilation-bounded.

On the other hand, the projection $\bar{\sigma} \otimes \ell^\alpha$ (resp $\dot{\eta}$) of $\sigma \otimes \ell^\alpha$ (resp η) on \dot{V} satisfy :
 $\mu * (\bar{\sigma} \otimes \ell^\alpha) = \bar{\sigma} \otimes \ell^\alpha$ (resp $\mu * \dot{\eta} = \dot{\eta}$), hence $\dot{Q}_\alpha(e^\alpha \bar{\sigma} \otimes \ell) = e^\alpha \bar{\sigma} \otimes \ell$, $\dot{Q}_\alpha(\dot{\eta}^\alpha) = \dot{\eta}^\alpha$.

We observe that the fibered Markov operator \dot{Q}_α satisfies condition D of the appendix, in view of Corollary 3.20 and of the moment condition on A .

Then Theorem 2.6 implies $\bar{\sigma} = \nu^\alpha$. Also, in view of Corollary 3.20 for $s = \alpha$ and the above observations, we can apply the second part of Theorem A.4 to $\dot{\eta}^\alpha$ with $P = \dot{Q}_\alpha$, hence $\dot{\eta}^\alpha$ is proportional to $\pi^\alpha \otimes \ell$, i.e $\dot{\eta}$ is proportional to $\nu^\alpha \otimes \ell^\alpha$. Since $\bar{\sigma} = \nu^\alpha$ and $\eta(H_v^+) = (\sigma \otimes \ell^\alpha)(H_v^+)$ we get $\dot{\eta} = \nu^\alpha \otimes \ell^\alpha$.

We denote for $v \in \mathbb{S}^{d-1}$: $\lambda_v = |\langle v, \cdot \rangle|^\alpha \sigma \otimes \ell^\alpha$, $\eta_v = |\langle v, \cdot \rangle|^\alpha \eta$. Since $\sigma \otimes \ell^\alpha$ and η are μ -harmonic, we have $\int g \lambda_{g^*v} d\mu(g) = \lambda_v$, $\int g \eta_{g^*v} d\mu(g) = \eta_v$. The projections $\dot{\lambda}_v$ and $\dot{\eta}_v$ on \dot{V} satisfy the same equation hence are equal. As in section 3, we get that the sequences of Radon measures $g_1 \cdots g_n \lambda_{g_n^* \cdots g_1^* v}$ and $g_1 \cdots g_n \eta_{g_n^* \cdots g_1^* v}$ are vaguely bounded ${}^* \mathbb{Q}_v^\alpha$ -martingales. On \dot{V} we get, using Theorem 3.2, for some $z(\omega) \in \mathbb{P}^{d-1}$ and ${}^* \mathbb{Q}_v^\alpha - a.e$:
 $\lim_{n \rightarrow \infty} g_1 \cdots g_n \dot{\lambda}_{g_n^* \cdots g_1^* v} = \lim_{n \rightarrow \infty} g_1 \cdots g_n \dot{\eta}_{g_n^* \cdots g_1^* v} = \delta_{z(\omega)} \otimes \ell$.

Let $z_+(\omega)$ and $z_-(\omega)$ be opposite points on \mathbb{S}^{d-1} with projection $z(\omega)$ on \mathbb{P}^{d-1} . The martingale convergence on $V \setminus \{0\}$ gives that $g_1 \cdots g_n \lambda_{g_n^* \cdots g_1^* v}$ (resp $g_1 \cdots g_n \eta_{g_n^* \cdots g_1^* v}$) converges vaguely to $p(\omega) \delta_{z_+(\omega)} + q(\omega) \delta_{z_-(\omega)}$ (resp $p'(\omega) \delta_{z_+(\omega)} + q'(\omega) \delta_{z_-(\omega)}$) with $p(\omega) + q(\omega) = p'(\omega) + q'(\omega) = 1$. The condition $\eta(H_v^+) = (\sigma \otimes \ell^\alpha)(H_v^+)$ implies in the limit :

$$p(\omega) \delta_{z_+(\omega)} + q(\omega) \delta_{z_-(\omega)} = p'(\omega) \delta_{z_+(\omega)} + q'(\omega) \delta_{z_-(\omega)}.$$

Hence, taking expectations we get $\eta = \sigma \otimes \ell^\alpha$. \square

Proof of Theorem 5.2

The convergence of $\rho_t = t^{-\alpha}(t.\rho)$ to $C(\sigma^\alpha \otimes \ell^\alpha)$ on the sets H_v^+ and the positivity properties of C, C_+, C_- follows from Corollary 5.13. For the vague convergence of ρ_t we observe that Lemma 5.15 gives the vague compactness of ρ_t . If $\eta = \lim_{t_n \rightarrow 0_+} t_n^{-\alpha}(t_n.\rho)$, Lemma

5.16 gives the μ -harmonicity of η . Since $\eta(H_v^+) = C(\sigma^\alpha \otimes \ell^\alpha)(H_v^+)$, Lemma 5.17 gives $\Lambda = C(\sigma^\alpha \otimes \ell^\alpha) = \eta$, hence the vague convergence of ρ_t to Λ . The detailed form of Λ follows from Proposition 5.3.

For the final minimality assertions one uses the second part of Theorem A.4 and we replace η by $(e^\alpha \otimes h^\alpha)\eta$. We verify condition D for \tilde{Q}^α or operators associated with \tilde{Q}^α as follows. Condition D_1 follows directly from Corollary 3.21. In case I, \tilde{Q}^α satisfies D_2 by Corollary 3.21, 1 is a simple eigenvalue of \tilde{Q}^α and, if θ is positive Radon measure with $\mu * \theta = \theta$, $\theta \leq \tilde{\nu}^\alpha \otimes \ell^\alpha$, then hence θ is proportional to $\tilde{\nu}^\alpha \otimes \ell^\alpha$. In case II, one restricts \tilde{Q}^α to the convex cone generated to $\Lambda_+(T)$, so as to achieve the simplicity of 1 as an eigenvalue of \tilde{Q}^α and the absence of other unimodular eigenvalue. Then Corollary 3.21 shows that condition

D_2 is satisfied for the corresponding operator hence the above argument is also valid for $\nu_+^\alpha \otimes \ell^\alpha$. Condition D_3 follows from the moment conditions assumed on λ . \square

Corollary 5.18

Let $\mathcal{B}_{\varepsilon,\alpha}$ be the set of locally bounded Borel functions on $V \setminus \{0\}$ such that the set of discontinuities of f is Λ -negligible and for $\varepsilon > 0$:

$$K_f(\varepsilon) = \sup\{|v|^{-\alpha} |\text{Log} v|^{1+\varepsilon} |f(v)| ; v \neq 0\} < \infty$$

Then for any $f \in \mathcal{B}_{\varepsilon,\alpha}$: $\lim_{t \rightarrow 0_+} t^{-\alpha} (t.\rho)(f) = \Lambda(f)$.

The proof depends of two lemmas in which we will use the norm $\|v\| = \sup_{1 \leq i \leq d} | \langle v, e_i \rangle |$ instead of $|v|$ where $e_i (1 \leq i \leq d)$ is a basis of V . Also for $\delta > 0$ and $0 < \delta_1 < \delta_2$ we write

$$B_\delta = \{v \in V ; \|v\| \leq \delta\}, B_{\delta_1, \delta_2} = B_{\delta_2} \setminus B_{\delta_1}, B'_\delta = V \setminus B_\delta.$$

Lemma 5.19

For any $f \in \mathcal{B}_{\varepsilon,\alpha}$, $0 < \delta_1 < \delta_2$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha} (t.\rho)(f 1_{B_{\delta_1, \delta_2}}) = \Lambda(f 1_{B_{\delta_1, \delta_2}}).$$

Proof

From the fact that ν^α gives measure zero to any projective subspace and the homogeneity of $\Lambda = \sigma^\alpha \otimes \ell^\alpha$, we know that Λ gives measure zero to any affine hyperplane, hence the boundary of B_{η_1, η_2} is Λ -negligible. Then the proof follows from the vague convergence of $t^{-\alpha} (t.\rho)$ to Λ and the hypothesis of Λ -negligibility of the discontinuity set of f . \square

Lemma 5.20

a) There exists $C > 0$ such that for any $f \in \mathcal{B}_{\varepsilon,\alpha}$, $t > 0$ and $\delta_2 > e$:

$$|t^{-\alpha} (t.\rho)(f 1_{B_{\delta_2}})| \leq CK_f(\varepsilon) |\text{Log} \delta_2|^{-\varepsilon}$$

b) Then exists $C(\varepsilon) > 0$ such that for any $f \in \mathcal{B}_{\varepsilon,\alpha}$, $t > 0$, $\delta_1 < e^{-1}$:

$$|t^{-\alpha} (t.\rho)(f 1_{B_{\delta_1}})| \leq C(\varepsilon) K_f(\varepsilon) |\text{Log} \delta_1|^{-\varepsilon}$$

Proof

a) Let $\varphi_\varepsilon(x)$ be the function on $\mathbb{R}_+ \setminus \{1\}$ given by $\varphi_\varepsilon(x) = x^\alpha |\text{Log} x|^{-1-\varepsilon}$. For $x \geq e$ we have $\varphi'_\varepsilon(x) \leq \alpha x^{\alpha-1} |\text{Log} x|^{-1-\varepsilon}$. We denote $F_t(x) = \mathbb{P}(\|tR\| > x)$ and we observe that, using Proposition 5.1, the non increasing function F_t is continuous. We have :

$$|t^{-\alpha} (t.\rho)(f 1_{B'_{\delta_2}})| \leq t^{-\alpha} K_f(\varepsilon) \int_{\delta_2}^\infty \varphi_\varepsilon(x) dF_t(x)$$

Integrating by parts, we get :

$$|t^{-\alpha} (t.\rho)(f 1_{B'_{\delta_2}})| \leq t^{-\alpha} K_f(\varepsilon) [\varphi_\varepsilon(x) F_t(x)]_{\delta_2}^\infty + t^{-\alpha} K_f(\varepsilon) \int_{\delta_2}^\infty \varphi'_\varepsilon(x) F_t(x) dx.$$

From Corollary 5.7 we know that, for some $C > 0$, $F_t(x) \leq C t^\alpha x^{-\alpha}$. Then, using the above estimation of $\varphi'_\varepsilon(x)$, we get :

$$|t^{-\alpha} (t.\rho)(f 1_{B'_{\delta_2}})| \leq CK_\varepsilon(f) \int_{\delta_2}^\infty \frac{\alpha}{|\text{Log} x|^{1+\varepsilon}} \frac{dx}{x} \leq CK_\varepsilon(f) |\text{Log} \delta_2|^{-\varepsilon}$$

b) The proof follows the same lines and uses the estimation of $|\varphi'_\varepsilon(x)|$ by

$$(\alpha + 1 + \varepsilon) x^{\alpha-1} |\text{Log} x|^{-1-\varepsilon} \text{ for } x \leq e^{-1}. \quad \square$$

Proof of the Corollary

For $\delta \geq e$ and with $D > 0$ we have $\int_{B'_\delta} \varphi_\varepsilon(\|v\|)d\Lambda(v) \leq D \int_\delta^\infty \frac{x^\alpha}{(\text{Log}x)^{1+\varepsilon}} \frac{dx}{x^{\alpha+1}} = \frac{D}{|\text{Log}\delta|^\varepsilon}$ hence

$\lim_{\delta \rightarrow \infty} \int_{B'_\delta} \varphi_\varepsilon(\|v\|)d\Lambda(v) = 0$. Also for $\delta < 1$:

$$\int_{B'_\delta} \varphi_\varepsilon(\|v\|)d\Lambda(v) \leq D \int_0^\delta \frac{x^\alpha}{|\text{Log}x|^{1+\varepsilon}} \frac{dx}{x^{\alpha+1}} = \frac{D}{|\text{Log}\delta|^\varepsilon},$$

hence $\lim_{\delta \rightarrow \infty} \int_{B'_\delta} \varphi_\varepsilon(\|v\|)d\Lambda(v) = 0$.

Then the corollary follows of the lemmas. \square

Proof of Theorem C

Except for the last assertion, Theorem C is a reformulation of Theorem 5.2. The last assertion is the content of Corollary 5.18. \square

A) Appendix

Here we give the proofs of two general results used in section 5. We give also an analytic proof of the tail-homogeneity of ρ if λ has compact support.

1) A weak renewal theorem for Markov walks

Let (X, ν) be a complete separable metric space, where ν is a probability measure. As in section 4 we consider a general Markov chain on $X \times \mathbb{R}$ with kernel P , we assume that P commutes with the \mathbb{R} -translations and we denote Lebesgue measure on \mathbb{R} by ℓ . We assume that the measure $\nu \otimes \ell$ is P -invariant. We write a path of this Markov chain as (x_n, V_n) where $x_n \in X$ and $V_n \in \mathbb{R}$, we denote by ${}^a\mathbb{P}_x$ the Markov measure on the paths starting from $x \in X$ and we write ${}^a\mathbb{P}_\nu = \int {}^a\mathbb{P}_x d\nu(x)$, ${}^a\mathbb{E}_x$ for the corresponding expectation symbol. In this context the following weak analogue of the renewal theorem holds.

Lemma A.1

With the above notations, assume that ψ is a compactly supported bounded non negative Borel function on $X \times \mathbb{R}$, the potential $U\psi = \sum_0^\infty P^k \psi$ is bounded on $X \times [-c, c]$ for any $c > 0$ and we have for any $\varepsilon > 0$:

$$\lim_{n \rightarrow \infty} {}^a\mathbb{P}_\nu \left\{ \left| \frac{V_n}{n} - \gamma \right| > \varepsilon \right\} = 0, \text{ with } \gamma > 0.$$

Then : $\lim_{t \rightarrow \infty} \frac{1}{t} \int_{-t}^0 ds \int_X U\psi(x, s) d\nu(x) = \frac{1}{\gamma} (\nu \otimes \ell)(\psi)$.

Furthermore if ψ is a non negative Borel function on $X \times \mathbb{R}$ and $\lim_{t \rightarrow -\infty} U\psi(x, t) = 0$, $\nu - a.e$ then $\psi = 0$, $\nu \otimes \ell - a.e$.

If ψ is a Borel function on $X \times \mathbb{R}$ which satisfies

$$|\psi|_b = \sum_{\ell=-\infty}^{\ell=\infty} \sup \{ |\psi(x, s)| ; x \in X, s \in [\ell, \ell + 1] \} < \infty,$$

then the above convergence is valid.

Proof

We observe that the maximum principle implies $|U\psi| = \sup_{x,t} |U\psi|(x,t) < \infty$, since $U\psi$ is locally bounded. For $\varepsilon > 0$, $t > 0$ we denote $n_1 = n_1(t) = \lfloor \frac{1}{\gamma}\varepsilon t \rfloor$, $n_2 = n_2(t) = \lfloor \frac{1}{\gamma}(1 + \varepsilon)t \rfloor$ where $\lfloor t \rfloor$ denotes integer part of $t > 0$.

We write : $\sum_0^\infty P^k \psi = U\psi$, $\sum_0^{n-1} P^k \psi = U_n \psi$, $\sum_{n+1}^\infty P^k \psi = U^n \psi$, $\sum_n^m P^k \psi = U_n^m \psi$

and $I(t) = \frac{1}{t} \int_{-t}^0 ds \int_X (U\psi)(x,s) d\nu(x) = \sum_1^3 I_k(t) - I_4(t)$ where :

$$I_1(t) = \frac{1}{t} \int_X d\nu(x) \int_{-\infty}^\infty U_{n_1}^{n_2} \psi(x,s) ds, \quad I_2(t) = \frac{1}{t} \int_X d\nu(x) \int_{-t}^0 U_{n_1} \psi(x,s) ds$$

$$I_3(t) = \frac{1}{t} \int_X d\nu(x) \int_{-t}^0 U^{n_2} \psi(x,s) ds \quad I_4(t) = \frac{1}{t} \int_X d\nu(x) \int_{\mathbb{R} \setminus [-t,0]} U_{n_1}^{n_2} \psi(x,s) ds.$$

We estimate each term $I_k(t)$.

We have since the measure $\nu \otimes \ell$ is P -invariant :

$$I_1(t) = \frac{n_2 - n_1 + 1}{t} (\nu \otimes \ell)(\psi), \text{ hence } \lim_{t \rightarrow \infty} I_1(t) = \frac{1}{\gamma} (\nu \otimes \ell)(\psi).$$

Furthermore :

$$|I_4(t)| \leq \frac{|\psi|}{t} (n_2 - n_1 + 1) \sup_{n_1 \leq n \leq n_2} \int_X ({}^a \mathbb{P}_x \{V_n \leq a\} + {}^a \mathbb{P}_x \{V_n \geq t - a\}) d\nu(x).$$

Since $n^{-1}V_n$ converges to $\gamma > 0$ in probability, the above integral has limit zero, hence $\lim_{t \rightarrow \infty} I_4(t) = 0$.

We have also : $|I_2(t)| \leq \frac{\varepsilon}{\gamma} \nu \otimes \ell(\psi)$.

In order to estimate $I_3(t)$ we denote for $n \in \mathbb{N}$, $s > 0$: $\rho_n^s = \text{Inf}\{k \geq n ; -a \leq V_n - s \leq a\}$, where ψ is supported on $[-a, a]$, and we use the interpretation of $U^n \psi$ as the expected number of visits to ψ after time n :

$$U^n \psi(x, s) \leq |U\psi|^a \mathbb{P}_x \{\rho_n^s < \infty\}$$

Taking $n = \lfloor \frac{(1+\varepsilon)t}{\gamma} \rfloor = n_2$ we get

$$I_3(t) \leq |U\psi| \int_X {}^a \mathbb{P}_x \{V_k - t \leq a \text{ for some } k \geq \lfloor \frac{(1+\varepsilon)t}{\gamma} \rfloor\} d\nu(x).$$

Since $\frac{V_n}{n}$ converges to $\gamma > 0$ in probability, we get $\lim_{t \rightarrow \infty} I_3(t) = 0$.

Since ε is arbitrary we get finally : $\lim_{t \rightarrow \infty} I(t) = \frac{1}{\gamma} (\nu \otimes \ell)(\psi)$.

The second conclusion follows by restriction and truncation of ψ on $X \times [-a, a]$.

For the proof of the last assertion we observe that for any $\ell \in \mathbb{Z}$

$$\Delta = |U1_{X \times [0,1]}| = |U1_{X \times [\ell, \ell+1]}| < \infty. \text{ Writing } \psi = \sum_{\ell=-\infty}^{\ell=\infty} \psi 1_{X \times [\ell, \ell+1]} \text{ we get } |U\psi| \leq \Delta |\psi|_b.$$

The quantity $\psi \rightarrow |\psi|_b$ is a norm on the space \mathcal{H} of Borel functions on $X \times \mathbb{R}$ such that $|\psi|_b < \infty$. Since the set of bounded functions supported on $X \times [-c, c]$ for some $c > 0$ is dense in \mathcal{H} and $\psi \rightarrow (\nu \otimes \ell)(\psi)$ is a continuous functional on \mathcal{H} , the above relation extends by density to any $\psi \in \mathcal{H}$. \square

2) A Choquet-Deny type property

Here, as in section 4, we consider a fibered Markov chain on $S \times \mathbb{R}$ but we reinforce the hypothesis on the Markov kernel P . Hence S is a compact metric space, P commutes with \mathbb{R} -translations and acts continuously on $C_b(S \times \mathbb{R})$. We define for $t \in \mathbb{R}$, the Fourier operator P^{it} on $C(S)$ by :

$$P^{it}\varphi(x) = P(\varphi \otimes e^{it\cdot})(x, 0)$$

For $t = 0$ $P^{it} = P^0$ is equal to \overline{P} , the factor operator on S defined by P . We assume that for each $t \in \mathbb{R}$, P^{it} preserves the space $H_\varepsilon(S)$ of ε -Hölder functions and is a bounded operator therein. Moreover we assume that $P^{it}(t \in \mathbb{R})$, and P satisfies the following condition D (compare [31]) :

- 1) For any $t \in \mathbb{R}$, one can find $n_0 \in \mathbb{N}$, $\rho(t) \in [0, 1[$ and $C(t) > 0$ for which

$$[(P^{it})^{n_0}\varphi]_\varepsilon \leq \rho(t)[\varphi]_\varepsilon + C(t)|\varphi|.$$
- 2) For any $t \in \mathbb{R}$, the equation $P^{it}\varphi = e^{i\theta}\varphi$, $\varphi \in H_\varepsilon(S)$, $\varphi \neq 0$ has only the trivial solution $e^{i\theta} = 1$, $t = 0$, $\varphi = \text{cte}$.
- 3) For some $\delta > 1$: $M_\delta = \sup_{x \in S} \int |a|^\delta P((x, 0), d(y, a)) < \infty$.

Conditions 1,2 above imply that \overline{P} has a unique stationary measure π and the spectrum of \overline{P} in $H_\varepsilon(S)$ is of the form $\{1\} \cup \Delta$ where Δ is a compact subset of the open unit disk (see [34]). They imply also that, for any $t \neq 0$, the spectral radius of P^{it} is less than one. With the notations of section 4, condition 3 above will allow to estimate ${}^a\mathbb{E}(|V_n|^p)$ for $p \leq \delta$ and to show the continuity of $t \rightarrow |P^{it}|$.

The following is a simple consequence of conditions 1,2,3 above.

Lemma A.2

With the above notations, let $I \subset \mathbb{R}$ be a compact subset of $\mathbb{R} \setminus \{0\}$. Then there exists $D > 0$ and $\sigma \in [0, 1[$ such that for any $n \in \mathbb{N}$: $\sup_{t \in I} |(P^{it})^n| \leq D\sigma^n$

Proof

Conditions 1 and 2 for $P^{it}(t \neq 0)$ imply that the spectral radius r_t of P^{it} satisfies $r_t < 1$ (see [30]). Hence there exists $C_t > 0$ such that for any $n \in \mathbb{N}$: $|(P^{it})^n| \leq C_t(\frac{1}{2} + \frac{r_t}{2})^n$. On the other hand $t \rightarrow |P^{it}|$ is continuous as the following calculation shows. For $a, t, t' \in \mathbb{R}$, $\delta' \in [0, 1]$, we have $|e^{iat} - e^{iat'}| \leq 2|a|^{\delta'} |t - t'|^{\delta'}$ hence :

$$|P^{it}\varphi(x) - P^{it'}\varphi(x)| \leq 2|\varphi| |t - t'|^{\delta'} \int |a|^{\delta'} P((x, 0), d(y, a)), \quad |P^{it} - P^{it'}| \leq 2M_{\delta'} |t - t'|^{\delta'}$$

For each $t \in I$ we fix $n_t \in \mathbb{N}$ such that $|(P^{it})^{n_t}| \leq \frac{1}{3}$. Then the above continuity of P^{it} , hence of $(P^{it})^{n_t}$, gives for t' sufficiently close to t : $|(P^{it'})| \leq \frac{1}{2}$; Then, using compactness of I we find n_1, \dots, n_k such that one of the inequalities $|(P^{it})^{n_j}| \leq \frac{1}{2}$ ($1 \leq j \leq k$) is valid at any given point of I . Then, since $|P^{it}| \leq 1$ with $n_0 = n_1 \cdots n_k$ we get $|P^{it})^{n_0}| \leq \frac{1}{2}$. Using Euclidean division of n by n_0 , we get the required inequality. \square

We are interested in the action of P^n on functions on $S \times \mathbb{R}$ which are of the form $u \otimes f$ where $u \in C(S)$, $f \in \mathbb{L}^1(\mathbb{R})$ and also in P -harmonic Radon measures which satisfy boundedness

conditions. In some proofs, since $H_\varepsilon(S)$ is dense in $C(S)$ it will be convenient to assume $u \in H_\varepsilon(S)$.

Definition A.3

We say that the Radon measure θ on $S \times \mathbb{R}$ is translation-bounded if for any compact subset K of $S \times \mathbb{R}$, any $a \in \mathbb{R}$, there exists $C(K) > 0$ such that $|\theta(a + K)| \leq C(K)$ where $a + K$ is the compact subset of $S \times \mathbb{R}$ obtained from K using translation by $a \in \mathbb{R}$.

We are led to consider a positive function ω on \mathbb{R}^d which satisfies $\omega(x + y) \leq \omega(x)\omega(y)$. For example, if $p \geq 0$, such a function ω_p is defined by $\omega_p(a) = (1 + |a|)^p$.

We denote $\mathbb{L}_\omega^1(\mathbb{R}) = \{f \in \mathbb{L}^1(\mathbb{R}) ; \omega f \in \mathbb{L}^1(\mathbb{R})\}$ and we observe that $f \rightarrow \|\omega f\|_1 = \|f\|_{1,\omega}$ is a norm under which $\mathbb{L}_\omega^1(\mathbb{R})$ is a Banach algebra. The dual space of $\mathbb{L}_\omega^1(\mathbb{R})$ is the space $\mathbb{L}_\omega^\infty(\mathbb{R})$ of measurable functions g such that $g\omega^{-1} \in \mathbb{L}^\infty(\mathbb{R})$ and the duality is given by $\langle g, f \rangle = \int g(a)f(a)da$. The Fourier transform \widehat{f} of $f \in \mathbb{L}_\omega^1(\mathbb{R})$ is well defined by $\widehat{f}(t) = \int f(a)e^{ita}da$. We denote by J^c the ideal of $\mathbb{L}^1(\mathbb{R})$ which consists of functions $f \in \mathbb{L}^1(\mathbb{R})$ such that \widehat{f} has a compact support not containing 0 and we write $J_\omega^c = \mathbb{L}_\omega^1(\mathbb{R}) \cap J^c$. Also we denote by $\mathbb{L}_0^1(\mathbb{R})$ the ideal of $\mathbb{L}^1(\mathbb{R})$ defined by the condition $\widehat{f}(0) = 0$. It is well known that J^c is dense in $\mathbb{L}_0^1(\mathbb{R})$, hence for $\omega = \omega_p$ $J_{\omega_p}^c$ is dense in $\mathbb{L}_\omega^1(\mathbb{R})$ (see [33] p 187).

Theorem A.4

With the above notations, assume that the family $P^{it}(t \in \mathbb{R})$ satisfies conditions D and let $\omega = \omega_p$ with $p < \delta$. Then for any $f \in \mathbb{L}_\omega^1(\mathbb{R}) \cap J^c$, $u \in C(S)$, we have the convergence :

$$\lim_{n \rightarrow \infty} \sup_{x \in S} \|P^n(u \otimes f)(x, \cdot)\|_{1,\omega} = 0.$$

If θ is a P -harmonic Radon measure which is translation-bounded, then θ is proportional to $\pi \otimes \ell$. In particular $\pi \otimes \ell$ is a minimal P -harmonic Radon measure.

The proof follows from the above considerations and the following lemmas.

Lemma A.5

Assume π_n is a sequence of bounded measures on \mathbb{R} , ω is a positive Borel function on \mathbb{R} such that for any $x, y \in \mathbb{R}$, $\omega(x + y) \leq \omega(x)\omega(y)$ and assume that the total variation measures $|\pi_n|$ of π_n satisfy $\sup\{|\pi_n|(\omega) ; n \in \mathbb{N}\} < \infty$. Let $f \in \mathbb{L}_\omega^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$ and assume A_n, B_n are sequences of Borel subsets of \mathbb{R} such that, with $A'_n = \mathbb{R} \setminus A_n$, $B'_n = \mathbb{R} \setminus B_n$:

- a) $\lim_{n \rightarrow \infty} |\pi_n|(\omega) \|f 1_{B'_n}\|_{1,\omega} = 0$
- b) $\lim_{n \rightarrow \infty} |\pi_n|(\omega 1_{A'_n}) = 0$
- c) $\lim_{n \rightarrow \infty} \|\pi_n * f\|_2 \|\omega^2 1_{A_n+B_n}\|_1^{1/2} = 0.$

Then we have $\lim_{n \rightarrow \infty} \|\pi_n * f\|_{1,\omega} = 0$. Furthermore, if the measures π_n depend of a parameter λ and if the convergences in a), b), c) are uniform in λ , then the convergence of $\|\pi_n * f\|_{1,\omega}$ is also uniform in λ .

Proof

Let η, η' be two bounded measures on \mathbb{R} and let A, B be Borel subsets with complements A', B' in \mathbb{R} . Observe that, since $0 \leq \omega(x+y) \leq \omega(x)\omega(y)$:

$$\omega(x+y) \leq \omega(x+y)1_{A+B}(x+y) + (\omega 1_A)(x)(\omega 1_{B'})(y) + (\omega 1_{A'})(x)\omega(y).$$

It follows

$$|(\eta * \eta')(\omega) \leq |\eta * \eta'|(\omega 1_{A+B}) + |\eta|(\omega) |\eta'|(\omega 1_{B'}) + |\eta|(\omega 1_{A'})|\eta'|(\omega).$$

Then we take $\eta = \pi_n, \eta' = f(a)da, A = A_n, B = B_n$ and we get :

$$\|\pi_n * f\|_{1,\omega} \leq \int |\pi_n * f|(a)|\omega 1_{A_n+B_n}|(a)da + |\pi_n|(\omega 1_{A_n})\|f 1_{B'_n}\|_{1,\omega} + |\pi_n|(\omega 1_{A'_n})\|f\|_{1,\omega}.$$

Conditions a, b imply that the two last terms in the above inequality have limits zero.

Using condition c and Schwarz inequality we see that the first term has also limit zero.

If π_n depends of a parameter λ , the uniformity of the convergence of $\|\pi_n * f\|_{1,\omega}$ follows directly from the bound for $\|\pi_n * f\|_{1,\omega}$ given above. \square

The following lemma is an easy consequence of condition 3 on the Markov kernel P and of Hölder inequality

Lemma A.6

For any $p \in [1, \delta]$, there exists $C_p > 0$ such that : $\sup_x \mathbb{E}_x((|V_n|)^p) \leq C_p n^p$

In particular, for any $L > 0$: $\sup_{x,n} \mathbb{P}_x\{|V_n| > nL\} \leq \frac{C_p}{L^p}$.

We leave to the reader the proof of the well known first inequality. The second one follows from Markov's inequality.

For a Radon measure θ on $S \times \mathbb{R}$ and $b \in \mathbb{R}$, we denote by $\theta * \delta_b$ the Radon measure defined by $(\theta * \delta_b)(\varphi) = \int \varphi(x, a+b)d\theta(x, a)$ and for $\varphi \in C_c(S \times \mathbb{R})$, θ translation-bounded we write $|\theta|_\varphi = \sup\{|\theta * \delta_b(\varphi)| ; b \in \mathbb{R}\}$. For such measures and any bounded measure r on \mathbb{R} , $\theta * r$ is well defined by $(\theta * r)(\varphi) = \int (\theta * \delta_b)(\varphi)dr(b)$ and we have $|\theta * r|_\varphi \leq \|r\|\theta_\varphi$ where $\|r\|$ is the total variation of r . In particular, $f \in \mathbb{L}^1(\mathbb{R})$ can be identified with the measure $r_f = f(a)da$, we can define $\theta * f = \theta * r_f$ and if $f_n \in \mathbb{L}^1(\mathbb{R})$ converges in \mathbb{L}^1 -norm to $f \in \mathbb{L}^1(\mathbb{R})$, then $\theta * f_n$ converges to $\theta * f$ in the vague topology. On the other hand, if r has compact support and θ is a Radon measure on \mathbb{R} , $\theta * r$ is well defined as a Radon measure.

Lemma A.7

With the above notations, assume that θ is a translation-bounded non negative Radon measure on $S \times \mathbb{R}$. Let r be a non negative continuous function on \mathbb{R} with compact support containing 0. Then for $p > 1$, there exists a non negative bounded measure $\bar{\theta}$ on S such that $\theta * r \leq (1_S \otimes \omega_p)(\bar{\theta} \otimes \ell)$.

Proof

For simplicity of notations, assume $r > 0$ on $[0, 1]$. We denote by $\theta_k * \delta_k$ the restriction of θ to $S \times [k, k+1[$ ($k \in \mathbb{Z}$) and we write $\theta = \sum_{k \in \mathbb{Z}} \theta_k * \delta_k$ with $\text{supp } \theta_k \subset S \times [0, 1]$. We observe that, since θ is translation-bounded, the mass of θ_k is bounded for $k \in \mathbb{Z}$, hence

$\dot{\theta} = \sum_{k \in \mathbb{Z}} (1 + |k|)^{-p} \theta_k$ is a bounded measure supported on $S \times [0, 1]$. We have clearly

$$\theta_k \leq (1 + |k|)^p \dot{\theta}, \quad \theta \leq \dot{\theta} * \sum_{k \in \mathbb{Z}} (1 + |k|)^p \delta_k, \quad \theta * r \leq \dot{\theta} * (r * \sum_{k \in \mathbb{Z}} (1 + |k|)^p \delta_k)$$

But, by definition of ω_p and since $\text{supp}(r)$ is compact, we have $r * \sum_{k \in \mathbb{Z}} (1 + |k|)^p \delta_k \leq c\omega_p$ for

some $c > 0$ and it follows, $\theta * r \leq c \dot{\theta} * \omega_p$. We desintegrate the bounded measure $\dot{\theta}$ as $\dot{\theta} = \int \delta_x \otimes \theta^x d\tilde{\theta}(x)$ where $\tilde{\theta}$ is the projection of $\dot{\theta}$ on S and θ^x is a probability measure. Hence

$\dot{\theta} * \omega_p = \int \delta_x \otimes (\theta^x * \omega_p) d\tilde{\theta}(x)$. But, since θ^x is supported on $[0, 1] : \theta^x * \omega_p \leq 2^p \theta^x([0, 1])\omega_p$.

Hence $\dot{\theta} \leq 2^p (1_S \otimes \omega_p)(\tilde{\theta} \otimes \ell)$ and finally $\theta * r \leq 2^p c (1_S \otimes \omega_p)(\tilde{\theta} \otimes \ell) = (1_S \otimes \omega_p)(\bar{\theta} \otimes \ell)$ with $\bar{\theta} = 2^p c \tilde{\theta}$. \square

Proof of the theorem

We fix $p \in [1, \delta[$, $\omega = \omega_p$, $u \in H_\varepsilon(S)$, $u \geq 0$ and for $x \in S$, we define the positive measure π_n^x on \mathbb{R} by $\pi_n^x(\varphi) = P^n(u \otimes \varphi)(x, 0)$ where φ is a non negative Borel function on \mathbb{R} . We observe that $\pi_n^x(1) \leq |u|$, and for $f \in C_b(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R})$:

$$P^n(u \otimes f)(x, a) = \pi_n^x(f * \delta_a) = (\pi_n^x * f^*)(a)$$

where $f^*(a) = \underline{f(-a)}$. It follows $\|P^n(u \otimes f)(x, \cdot)\|_1 \leq |u| \|f\|_1$ and also for $f \in \mathbb{L}_\omega^1(\mathbb{R})$, $\|P^n(u \otimes f)(x, \cdot)\|_{1, \omega} \leq \|f\|_{1, \omega} \sup\{\pi_n^x(\omega) ; x \in S\}$. From condition 3, since $p < \delta$:

$$\pi_n^x(\omega) \leq |u|^a \mathbb{E}_x(1 + |V_n|)^p \leq 2^p C_p n^p$$

where we have used Lemma A.6 in order to bound ${}^a\mathbb{E}_x(1 + |V_n|)^p$.

We fix $\delta > 1$, we denote $B_n = \{a \in \mathbb{R} ; |a| \leq n^{1+\delta}\}$, $A_n = \{a \in \mathbb{R} ; |a| \leq n^{c+1}\}$ with $c > 0$ to be defined later and we verify the conditions a, b, c of Lemma A.5 for $\pi_n = \pi_n^x$, uniformly in $x \in S$ for $f \in \mathbb{L}_{\omega^2}^1(\mathbb{R})$.

Since $f\omega^2 \in \mathbb{L}^1(\mathbb{R})$, Markov's inequality gives $\|f1_{B_n}\|_{1, \omega} \leq \|f\|_{1, \omega^2} n^{-p(1+\delta)}$. Then, using the bound of $\pi_n^x(\omega)$ given above :

$$\pi_n^x(\omega) \|f1_{B_n}\|_{1, \omega} \leq 2^p C_p n^{-p\delta} \|f\|_{1, \omega^2}.$$

Hence condition a) is satisfied.

We write $\pi_n^x(\omega 1_{A_n'}) \leq |u|^a \mathbb{E}_x(\omega(V_n) 1_{A_n'}(V_n))$ and use Hölder inequality for $p' > 1$, $\frac{1}{q'} = 1 - \frac{1}{p'}$ and $p'p < \delta$:

$$\pi_n^x(\omega 1_{A_n'}) \leq C' 2^p n^p \mathbb{P}\{|V_n| \geq n^{c+1}\}^{1/q'} \leq C'' n^{p-cp'/q'}$$

where we have used the fact that $\sup_x {}^a\mathbb{E}_x(|U_1|^{pp'}) < \infty$ for $pp' < \delta$ and Lemma A.6. If

we take $c > q' = \frac{p'}{p'-1}$, we see that $\lim_{n \rightarrow \infty} \pi_n^x(\omega 1_{A_n'}) = 0$ uniformly, hence condition b) is satisfied.

In order to verify condition c) we observe that $\hat{\pi}_n^x(t) = (P^{it})^n u(x)$. For $f \in J^c \subset \mathbb{L}^2(\mathbb{R})$ we denote $Y = \text{supp} \hat{f} \subset \mathbb{R} \setminus \{0\}$ and we know from Lemma A.2, that there exists $D > 0$ and $\sigma \in [0, 1[$ such that for any $t \in Y$, $n \in \mathbb{N} : |(P^{it})^n| \leq D\sigma^n$.

From Plancherel formula, we get,

$$\|\pi_n^x * f\|_2 = \left(\int |\hat{\pi}_n^x(t)|^2 |\hat{f}(t)|^2 dt \right)^{1/2} \leq \|f\|_2 |u| \sup_{t \in Y} |(P^{it})^n| \leq D |u| \sigma^n \|f\|_2.$$

On the other hand, $\|\omega^2 1_{A_n+B_n}\|_1$ is bounded by a polynomial in n . Since $\sigma < 1$, condition c) is satisfied. Hence, Lemma A.5 gives : $\lim_{n \rightarrow \infty} \|\pi_n^x * f\|_{1,\omega} = 0$ uniformly. By density, the same relation is valid for all $f \in \mathbb{L}_\omega^1(\mathbb{R})$

Now, let us choose $p \in]1, \delta[$ and $\omega = \omega_p$. Since θ is translation-bounded we can assume θ to be non negative and translation-bounded. Then Lemma A.7 gives for any r as in the lemma : $\theta * r \leq (1_S \otimes \omega)(\bar{\theta} \otimes \ell)$. Taking $r = r_n$ as an approximate identity we have $\theta = \lim_{n \rightarrow \infty} \theta * r_n$ in the weak sense. Hence we can assume $\theta \leq (1_S \otimes \omega)(\bar{\theta} \otimes \ell)$ where $\bar{\theta}$ is a bounded measure on S . Let $f \in \mathbb{L}_\omega^1(\mathbb{R}) \cap \mathcal{J}^c(\mathbb{R})$, $u \in H_\varepsilon(S)$ be as above, hence f, u satisfy for every $n \in \mathbb{N}$:

$$\begin{aligned} \theta(u \otimes f) &= (P^n \theta)(u \otimes f) = \theta(P^n(u \otimes f)) = \int P^n(u \otimes f)(x, a) d\theta(x, a), \\ |\theta(u \otimes f)| &\leq \int d\bar{\theta}(x) \int |P^n(u \otimes f)(x, a)| \omega(a) da \leq \|\bar{\theta}\| \sup_x \|P^n(u \otimes f)(x, \cdot)\|_{1,\omega}. \end{aligned}$$

From the first part of the proof we get $\theta(u \otimes f) = 0$ for any $u \in H_\varepsilon(S)$, $f \in \mathcal{J}_\omega^c = \mathbb{L}_\omega^1(\mathbb{R}) \cap \mathcal{J}^c$. This relation remains valid for f in the ideal I_ω^c of $\mathbb{L}_\omega^1(\mathbb{R})$ generated by \mathcal{J}_ω^c . Using regularisation on Fourier transforms we see that the closure in $\mathbb{L}^1(\mathbb{R})$ of \mathcal{J}_ω^c contains \mathcal{J}^c , hence the unique Fourier exponential which vanish on I_ω^c is 1. Then using classical Fourier Analysis (see [33] p.187) we get that I_ω^c is dense in $\mathbb{L}_0^1(\mathbb{R})$. As observed above, since θ is translation-bounded, this implies $\theta(u \otimes f) = 0$ for any $f \in \mathbb{L}_0^1(\mathbb{R})$. Since $H_\varepsilon(S)$ is dense in $C(S)$, we get that θ is invariant by \mathbb{R} -translation. Then we have $\theta = \bar{\theta} \otimes \ell$ where $\bar{\theta}$ is a positive measure on S which satisfies $\bar{P} \bar{\theta} = \bar{\theta}$. Using conditions D.1, D.2, this implies that $\bar{\theta}$ is proportional to π , hence θ is proportional to $\pi \otimes \ell$.

For the final assertion we observe that, if θ is a P -harmonic positive Radon measure with $\theta \leq c \pi \otimes \ell$, for some $c > 0$, then θ is translation bounded. Hence as above, θ is proportional to $\pi \otimes \ell$. \square

3) Analytic proof of tail-homogeneity

Under the hypothesis of compact support for λ and density for μ , an analytic proof of tail-homogeneity of ρ is given below. The full hypothesis is only used in the study of positivity properties of C, C_+, C_- , hence we split the presentation into two parts according to the hypothesis at hand on λ . We note that Wiener-Ikehara theorem ([51], p.233) has the following consequence for the tail of a positive measure ν on $[0, \infty[$. Assume that $f(s) = \int_0^\infty x^s d\nu(x)$ is finite for $s < \alpha$, extends as a function \tilde{f} meromorphic in an open set $D \supset \{0 < \operatorname{Re} z \leq \alpha\}$ and \tilde{f} has only a possible unique simple pole at $z = \alpha$, with $\lim_{s \rightarrow \alpha^-} (\alpha - s)f(s) = A$; then $\lim_{t \rightarrow \infty} t^\alpha \nu(t, \infty) = \alpha^{-1} A$. The use of this result will give the tail-homogeneity of ρ . On the other hand, if β denotes the convergence abscissa of the Mellin transform $f(s) = \int_0^\infty x^s d\nu(x)$, a lemma of E. Landau (see [51] p 58) says that f cannot be extended holomorphically in a neighbourhood of β . This will allow to show $A > 0$.

The connection with the spectral gap properties obtained in section 3 depends of the following. The hypothesis is as in Corollary 3.21.

Lemma A.8

There exists an open set $D \subset \mathbb{C}$ which contains the set $\{Rez \in]0, \alpha]\}$ such that $(I - \tilde{P}^z)^{-1}$ is meromorphic in D with a unique simple pole at $z = \alpha$. We have :

In case I, $\lim_{z \rightarrow \alpha} (\alpha - z)(I - \tilde{P}^z)^{-1} = k'(\alpha)^{-1}(\tilde{\nu}^\alpha \otimes e^\alpha)$.

In cases II, $\lim_{z \rightarrow \alpha} (\alpha - z)(I - \tilde{P}^z)^{-1} = k'(\alpha)^{-1}(\nu_+^\alpha \otimes e_+^\alpha + \nu_-^\alpha \otimes e_-^\alpha)$.

Proof

We restrict to case I, since the proof is similar in cases II. The operator \tilde{P}^z on $H_\varepsilon(\mathbb{S}^{d-1})$ where $z = s + it$, is defined by the formula $\tilde{P}^z \varphi(x) = \int |gx|^z \varphi(g.x) d\mu(g)$ and $\frac{1}{k(s)} \tilde{P}^z$ is conjuguate to the operator \tilde{Q}^z considered in Corollary 3.21. From this corollary we deduce that \tilde{Q}^α satisfies a Doeblin-Fortet condition and $r(\tilde{P}^{\alpha+it}) = r(\tilde{Q}^{\alpha+it}) < 1$ for $t \neq 0$. On the other hand the function $z \rightarrow \tilde{P}^z$ is holomorphic in the set $\{0 < Rez < s_\infty\}$ since for any loop γ in this set we have $\int_\gamma \tilde{P}^z dz = \int \varphi(g.x) d\mu(g) \int_\gamma |gx|^z dz = 0$. It follows that there exists $\varepsilon > 0$ such that for $|z - \alpha| < \varepsilon$ there exists a holomorphic function $k(z)$ such that $k(z)$ is a simple dominant eigenvalue of \tilde{P}^z with $k(z) = 1 + k'(\alpha)(z - \alpha) + o(z - \alpha)$. Since $k'(\alpha) \neq 0$, we have $k(z) \neq 1$ for $z \neq \alpha$ and $|z - \alpha|$ small. Also for $|z - \alpha|$ small, we have in case I the decomposition $\tilde{P}^z = k(z)\tilde{\nu}^z \otimes e^z + U(z)$ where $\tilde{\nu}^z \otimes e^z$ is a projector on the line $\mathbb{C}e^z$, $U(z)$ satisfies $U(z)(\tilde{\nu}^z \otimes e^z) = (\tilde{\nu}^z \otimes e^z)U(z) = 0$, $r(U(z)) < 1$, and $\tilde{\nu}^z \otimes e^z$, $U(z)$ depend holomorphically on z . We consider also the projection $p^z = I - \tilde{\nu}^z \otimes e^z$ and we write $I - \tilde{P}^z = (1 - k(z))(\tilde{\nu}^z \otimes e^z) + p^z(I - U(z))$. Hence, for $|z - \alpha|$ small we have $(I - \tilde{P}^z)^{-1} = (1 - k(z))^{-1}(\tilde{\nu}^z \otimes e^z) + p^z(I - U(z))^{-1}$.

In particular : $\lim_{z \rightarrow \alpha} (\alpha - z)(I - \tilde{P}^z)^{-1} = k'(\alpha)^{-1}(\tilde{\nu}^\alpha \otimes e^\alpha)$, and $(I - \tilde{P}^z)^{-1}$ is meromorphic in a disk B_0 centered at α with radius $\varepsilon' \leq \varepsilon$, with unique pole at $z = \alpha$. For $z = \alpha + it$ with $|t| \geq \varepsilon'$, we get from above that there exists a disk B_t centered at $\alpha + it$ such that $r(\tilde{P}^z) < 1$ for $z \in B_t$, hence $(I - \tilde{P}^z)^{-1}$ is a bounded operator depending holomorphically on z for $z \in B_t$. If $Rez \in]0, \alpha[$, then $r(\tilde{P}^z) \leq r(\tilde{P}^s) < 1$ hence $(I - \tilde{P}^z)$ is invertible and the function $z \rightarrow (I - \tilde{P}^z)^{-1}$ is holomorphic in the domain $\{0 < Rez < \alpha\}$. Then the open set $D = (\cup_t B_t) \cup \{Rez \in]0, \alpha]\}$, where $t = 0$ or $|t| \geq \varepsilon'$ satisfies the conditions of the lemma and the formula for $\lim_{z \rightarrow \alpha} (\alpha - z)(I - \tilde{P}^z)^{-1}$ is valid. \square

We denote, for $u \in \mathbb{S}^{d-1}$ and $Rez = s < \alpha$:

$$\bar{f}_z(u) = \mathbb{E}(\langle R, u \rangle_+^z) \quad , \quad \bar{d}_z(u) = \mathbb{E}(\langle R, u \rangle_+^z - \langle R - B, u \rangle_+^z).$$

Proposition A.9

With the notations and hypothesis of Theorem 5.2, we have the convergence :

$$\lim_{t \rightarrow 0_+} t^{-\alpha} (t.\rho)(H_u^+) = C(\sigma^\alpha \otimes \ell^\alpha)(H_u^+) = C(u) = \alpha^{-1} \lim_{s \rightarrow \alpha_-} (\alpha - s) \bar{f}_s(u),$$

where $C \geq 0$ and $\sigma^\alpha \in M^1(\tilde{\Lambda}(T))$ satisfies $\mu * (\sigma^\alpha \otimes \ell^\alpha) = \sigma^\alpha \otimes \ell^\alpha$.

In case I, σ^α is symmetric, $\text{supp} \sigma^\alpha = \tilde{\Lambda}(T)$ and $C(u) = (\alpha k'(\alpha))^{-1} * \tilde{\nu}^\alpha(\bar{d}_\alpha)^* e^\alpha(u)$.

In cases II, $C(u) = (\alpha k'(\alpha))^{-1} [* \nu_+^\alpha(\bar{d}_\alpha)^* e_+^\alpha(u) + * \nu_-^\alpha(\bar{d}_\alpha)^* e_-^\alpha(u)]$

Proof

We write equation (S) of section 5 in the form : $R - B = AR \circ \widehat{\theta}$.

For any $v \in V \setminus \{0\}$, $Rez = s \in [0, \alpha[$ we define :

$$f_z(v) = \mathbb{E}(\langle R, v \rangle_+^z), \quad f_z^1(v) = \mathbb{E}(\langle R - B, v \rangle_+^z).$$

Then equation (S) implies : $\langle R - B, v \rangle_+ = \langle R \circ \widehat{\theta}, A^*v \rangle_+$, hence $*Pf_z = f_z^1$, $(I - *P)f_z = f_z - f_z^1 = d_z$ with $d_z(v) = \mathbb{E}(\langle R, v \rangle_+^z - \langle R - B, v \rangle_+^z)$. We write a continuous z -homogeneous function f on $V \setminus \{0\}$ as $f = \bar{f} \otimes h^z$ with $\bar{f} \in C(\mathbb{S}^{d-1})$, and we recall that, as in section 2 : $*Pf = *\tilde{P}^z \bar{f} \otimes h^z$. Then, since f_z and d_z are z -homogeneous and continuous, equation (S) gives : $(I - *\tilde{P}^z)\bar{f}_z = \bar{d}_z$.

For $u \in \mathbb{S}^{d-1}$ and $\varepsilon(z) = |z|1_{[1, s_\infty[}(s)$, $\varepsilon'(z) = 1_{[0, 1]}(s)$, $\bar{d}_z(u)$ is dominated by :

$$|\mathbb{E}(\langle R, u \rangle_+^z - \langle R - B, u \rangle_+^z)| \leq \varepsilon'(z)\mathbb{E}(|B|^s) + \varepsilon(z)\mathbb{E}(|B|(|B| + \langle R, u \rangle_+)^{s-1}).$$

Hence using Hölder inequality and the moment hypothesis we get that for u fixed, $\bar{d}_z(u)$ is a holomorphic function in the domain $Rez \in]0, \alpha + \delta[$. On the other hand, Lemma A.8 for μ^* implies that the operator-valued function $(I - *\tilde{P}^z)^{-1}$ is meromorphic in an open set D which contains the set $\{Rez \in]0, \alpha]\}$, with unique simple pole at $\alpha \in D$. The above estimation of $\bar{d}_z(u)$ shows that the same meromorphy property is valid for $\bar{f}_z = (I - *\tilde{P}^z)^{-1}\bar{d}_z$. If we denote by ρ_u the law of $\langle R, u \rangle_+$, we have $\bar{f}_s(u) = \int x^s d\rho_u(x)$, hence $\bar{f}_s(u)$ is the Mellin transform of the positive measure ρ_u . Then we can apply Wiener-Ikehara theorem to $\bar{f}_s(u)$, ρ_u and obtain the tail of ρ_u in the form : $\lim_{t \rightarrow \infty} t^\alpha \rho_u(t, \infty) =$

$\lim_{s \rightarrow \alpha_-} \alpha^{-1}(\alpha - s)\bar{f}_s(u)$. Hence using Lemma A.8 :

$$\text{In case I, } \lim_{s \rightarrow \alpha_-} (\alpha - s)\bar{f}_s(u) = \frac{1}{k'(\alpha)} *\tilde{\nu}^\alpha(\bar{d}_\alpha) *e^\alpha(u).$$

$$\text{In case II, } \lim_{s \rightarrow \alpha_-} (\alpha - s)\bar{f}_s(u) = \frac{1}{k'(\alpha)} [* \nu_+^\alpha(\bar{d}_\alpha) *e_+^\alpha(u) + * \nu_-^\alpha(\bar{d}_\alpha) *e_-^\alpha(u)].$$

Using the expressions of $*e^\alpha(u)$, $*e_+^\alpha(u)$, $*e_-^\alpha(u)$ given by Theorem 2.17 we obtain in the two cases :

$$\lim_{s \rightarrow \alpha_-} (\alpha - s)\bar{f}_s(u) = \alpha C(\sigma^\alpha \otimes \ell^\alpha)(H_u^+) = \alpha C(u)$$

with certain constants $C \geq 0$, $C(u) \geq 0$, , a certain $\sigma^\alpha \in M^1(\mathbb{S}^{d-1})$ which satisfies

$$\mu * (\sigma^\alpha \otimes \ell^\alpha) = \sigma^\alpha \otimes \ell^\alpha.$$

In case I, we see that σ^α is symmetric and $\text{supp}\sigma^\alpha = \widetilde{\Lambda}(T)$.

In cases II, the detailed expression of σ^α shows that $C\sigma^\alpha = C_+\sigma_+^\alpha + C_-\sigma_-^\alpha$ where $\text{supp}\sigma_+^\alpha = \Lambda_+(T)$, $\text{supp}\sigma_-^\alpha = \Lambda_-(T)$. As a result we have $\lim_{t \rightarrow \infty} t^\alpha \rho_u(t, \infty) = C(\sigma^\alpha \otimes \ell^\alpha)(H_u^+) = C(u)$.

□

Proposition A.10

With the notations and hypothesis of Proposition A.9, assume futhermore that $\text{suppl}\lambda$ is compact and μ has a density on G . Then $C > 0$. In cases II, we have $C(u) > 0$ if $u \in \Lambda_+(T^*)$.

Proof

Assume $C = 0$ and observe that Proposition A.9 gives $C(u) = 0$ for any $u \in \tilde{\Lambda}(T^*)$. The equation $(I - {}^*P^s)\bar{f}_s = \bar{d}_s$ which occurs in the proof of Proposition A.9 gives by integration against ${}^*\tilde{\nu}^s$, since ${}^*P^s({}^*\tilde{\nu}^s) = k(s){}^*\tilde{\nu}^s$:

$$(1 - k(s)){}^*\tilde{\nu}^s(\bar{f}_s) = {}^*\tilde{\nu}^s(\bar{d}_s) \quad (s < \alpha).$$

Here, the estimation of \bar{d}_z in the proof of Proposition A.9 gives the analyticity of ${}^*\tilde{\nu}^z(\bar{d}_z)$ in an open set containing $]0, \alpha + \delta[$ where $\tilde{\nu}^z$ is defined in the proof of Lemma A.8 by perturbation. Since $C(u) = 0$, the function \bar{f}_z considered in the proof of Proposition A.9 is meromorphic with no pole at α , hence Landau's lemma mentioned above gives that the convergence abscissa of $\int x^s d\rho_u(x)$ is larger than α and $\bar{f}_s(u) < \infty$ if $0 < s < \alpha + \delta$.

Now we consider the case $s \geq \alpha + \delta$. We observe that in case I, the density hypothesis on μ implies that $\text{supp}{}^*\tilde{\nu}^s = \text{supp}{}^*\tilde{\nu} = \mathbb{S}^{d-1}$ and the compactness hypothesis of $\text{suppl}\lambda$ implies that ${}^*P^z$ defines a compact operator on $C(\mathbb{S}^{d-1})$. Let $\Delta \subset \{\text{Re}z > 0\}$ be the set where $I - {}^*P^z$ is not invertible and observe that, since $r({}^*P^z) < 1$ if $\text{Re}z \in]0, \alpha[$, we have $\Delta \cap \{0 < \text{Re}z < \alpha\} = \emptyset$. Since the function $z \rightarrow {}^*P^z$ is holomorphic, the extension of Riesz-Schauder theory given in [35] implies that Δ is discrete without any accumulation point and $(I - {}^*P^z)^{-1}$ is meromorphic in the domain $\{0 < \text{Re}z\}$, with possible poles in Δ only. The same property is valid for the function $\bar{f}_z = (I - {}^*P^z)^{-1}(\bar{d}_z)$ in any domain $D \subset \{\text{Re}z > 0\}$ where \bar{d}_z extends \bar{d}_s holomorphically. We define :

$$\beta = \sup\{s > 0 ; \mathbb{E}(\langle R, u \rangle_+^s) < \infty \text{ for } u \in \mathbb{S}^{d-1}\},$$

hence from above, $\beta \geq \alpha + \delta$, and we show below $\beta = \infty$. Assume $\beta < \infty$ and let us show $\lim_{s \rightarrow \beta_-} {}^*\tilde{\nu}^s(\bar{f}_s) < \infty$. We observe that \bar{f}_z (resp \bar{d}_z) is well defined and holomorphic in the

domain $\{0 < \text{Re}z < \beta\}$ (resp $D = \{0 < \text{Re}z < \beta + 1\}$), because $\text{suppl}\lambda$ is compact and the estimation of \bar{d}_z given in the proof of Proposition A.9. Hence the function $z \rightarrow \bar{f}_z = (I - {}^*P^z)^{-1}(\bar{d}_z)$ is a meromorphic extension of \bar{f}_z to D . It follows that ${}^*\tilde{\nu}^z(\bar{f}_z)$ extends meromorphically ${}^*\tilde{\nu}^s(\bar{f}_s)$ to a possibly smaller domain $D' \subset D$ which contains $]0, \beta + 1[$. Also the equation $(1 - k(s)){}^*\tilde{\nu}^s(\bar{f}_s) = {}^*\tilde{\nu}^s(\bar{d}_s)$ extends meromorphically to D' , with $k(z)$, ${}^*\tilde{\nu}^z$ defined as in the proof of Lemma A.8. Here, since $\text{suppl}\mu$ is compact we have $s_\infty = \infty$. Since $k(s) > 1$ for $s > \alpha$ and \bar{d}_z is holomorphic in D' , the function ${}^*\tilde{\nu}^z(\bar{f}_z) = (1 - k(z))^{-1}{}^*\tilde{\nu}^z(\bar{d}_z)$ is holomorphic in a domain which contains the interval $[\alpha + \delta, \beta + 1[$, hence $\lim_{s \rightarrow \beta_-} {}^*\tilde{\nu}^s(\bar{f}_s) < \infty$.

On the other hand, the meromorphy of $\bar{f}_z = (I - {}^*P^z)^{-1}(\bar{d}_z)$ in D implies :

$$\bar{f}_z = \sum_1^m \varphi_j(\beta - z)^{-j} + \psi_z \text{ with } \psi_z, \varphi_j \in C(\mathbb{S}^{d-1}), \varphi_m = \lim_{s \rightarrow \beta_-} (\beta - s)^m \bar{f}_s. \text{ Since } f_s(u) \geq 0,$$

we have $\varphi_m(u) \geq 0$. Also :

$$*\tilde{\nu}^\beta(\varphi_m) = \lim_{s \rightarrow \beta_-} *\tilde{\nu}^s(\varphi_m) = \lim_{s \rightarrow \beta_-} (\beta - s)^m *\tilde{\nu}^s(\bar{f}_s).$$

From above we know that this limit is zero, hence $*\tilde{\nu}^\beta(\varphi_m) = 0$. Since φ_m is non negative we get $\varphi_m(u) = 0$ *a.e* and the continuity of φ_m implies $\varphi_m(u) = 0$ for $u \in \text{supp} *\tilde{\nu}^\beta = \mathbb{S}^{d-1}$. By induction we get $\varphi_j = 0$ for any $j \geq 1$, hence the function $z \rightarrow \bar{f}_z$ is holomorphic in a domain D_ε which contains $]0, \beta + \varepsilon[$ for some $\varepsilon > 0$ depending on the possible poles of $(I - *\tilde{P}^z)^{-1}(\bar{d}_z)$, in \mathbb{R}_+^* .

Then, as above, Landau's lemma gives that $\bar{f}_s(u)$ is finite for $s < \beta + \varepsilon$ and $u \in \mathbb{S}^{d-1}$. Hence $\mathbb{E}(\langle R, u \rangle_+^s)$ is finite for $u \in \mathbb{S}^{d-1}$, $s < \beta + \varepsilon$ and this gives the required contradiction.

Hence we have for $s > 0$: $(1 - k(s))*\tilde{\nu}^s(\bar{f}_s) = *\tilde{\nu}^s(\bar{d}_s)$. We observe also that $(I - *\tilde{P}^z)^{-1}(\bar{d}_z)$ is well defined and holomorphic in a domain which contains $]0, \infty[$.

It follows for $s > 0$: $(k(s)*\tilde{\nu}^s(\bar{f}_s))^{1/s} = (*\tilde{\nu}^s(\bar{f}_s^1))^{1/s}$. Since $\bar{f}_s^1(u) = \mathbb{E}(\langle R - B, u \rangle_+^s)$ we have for $s > 1$:

$$(*\tilde{\nu}^s(\bar{f}_s^1))^{1/s} \leq \mathbb{E}(|B|^s)^{1/s} + (*\tilde{\nu}^s(\bar{f}_s))^{1/s}, \quad (k(s)^{1/s} - 1)(*\tilde{\nu}^s(\bar{f}_s))^{1/s} \leq \mathbb{E}(|B|^s)^{1/s}.$$

Since λ has compact support we have $\lim_{s \rightarrow \infty} \mathbb{E}(|B|^s)^{1/s} = d < \infty$; also Proposition 4.8 gives

$$\lim_{s \rightarrow \infty} k(s)^{1/s} = c > 1. \quad \text{Hence } \lim_{s \rightarrow \infty} (*\tilde{\nu}^s(\bar{f}_s))^{1/s} \leq (c - 1)^{-1}d < \infty.$$

By definition of \bar{f}_s it follows $\text{sup}\{\langle R, u \rangle_+; (\omega, u) \in \widehat{\Omega} \times \widetilde{\Lambda}(T^*)\} < \infty$; using Lemma 5.8, this contradicts the fact that $\langle R, u \rangle_+$ is unbounded on $\widehat{\Omega} \times \widehat{\Lambda}(T^*)$.

In cases II, the above argument can easily be modified, $*\tilde{\nu}^s$ replaced by $*\nu_+^s$ and \mathbb{S}^{d-1} by $\text{supp} *\nu_+$. Then we get that $\langle R, u \rangle_+$ is bounded on $\widehat{\Omega} \times \Lambda_+(T^*)$, which contradicts Lemma 5.8.

Remark

If $\text{supp} \lambda$ is compact and μ has a density, Corollary 3.21 and the use of [35], allow to avoid the use of the renewal theorem of section 4 and of Kac's formula in the proof of Theorem 5.2. Furthermore, if $\alpha \notin \mathbb{N}$, Theorem 5.2 then follows from the properties of Radon transforms of positive Radon measures (see [3]). However, in the general case one needs to use Lemma 5.12 and Theorem A.4. On the other hand if $d = 1$, the density assumption on μ is not necessary for the validity of Proposition A.10 as shown in Theorem 5.2. Also, in this case, Theorem 5.2 is valid with $\delta = 0$.

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