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Yves Guivarc'H, Emile Le Page

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Spectral gap properties and asymptotics of stationary measures for affine random walks.

Y. Guivarc'h, E. Le Page

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Abstract

Let $V = \mathbb{R}^d$ be the Euclidean d -dimensional space, μ (resp λ) a probability measure on the linear (resp affine) group $G = GL(V)$ (resp $H = Aff(V)$) and assume that μ is the projection of λ on G . We study asymptotic properties of the convolutions $\mu^n * \delta_v$ (resp $\lambda^n * \delta_v$) if $v \in V$, i.e asymptotics of the random walk on V defined by μ (resp λ), if the subsemigroup $T \subset G$ (resp $\Sigma \subset H$) generated by the support of μ (resp λ) is "large". We show spectral gap properties for the convolution operator defined by μ on spaces of homogeneous functions of degree $s \geq 0$ on V , which satisfy Hölder type conditions. As a consequence of our analysis we get precise asymptotics for the potential kernel $\sum_0^\infty \mu^k * \delta_v$, which imply its asymptotic homogeneity. Under natural conditions the H -space V is a λ -boundary ; then we use the above results to show that the unique λ -stationary measure ρ on V is "homogeneous at infinity" with respect to dilations $v \rightarrow tv(t > 0)$. It follows that ρ belongs to the domain of attraction of a stable law depending only of μ and Σ , up to a coefficient. Our proofs are based on the simplicity of the dominant Lyapunov exponent for certain products of Markov-dependant random matrices, on the use of a renewal theorem for Markov walks, and on the dynamical properties of a conditional λ -boundary dual to V .

I Introduction, statement of results

We consider the d -dimensional Euclidean space $V = \mathbb{R}^d$, endowed with the natural scalar product $(x, y) \rightarrow \langle x, y \rangle$, the linear group $G = GL(V)$, and the affine group $H = Aff(V)$. Let λ be a probability measure on H with projection μ on G , such that $supp\lambda$ has no fixed point in V . In this paper we are interested in the "shape at infinity" of λ -stationary measures on V , i.e probability measures ρ on V which satisfy the convolution equation $\lambda * \rho = \rho$. Under natural conditions, including negativity of the dominant Lyapunov exponent corresponding to μ , such a measure exists, is unique, has unbounded support and for any $v \in V$, $\lambda^n * \delta_v$ converges weakly to ρ . The study of deeper properties of ρ is of interest for various questions in Probability Theory and Mathematical Physics (see [12], [19], [20], [33], [36]) and also for the analysis of the H -space (V, ρ) as a λ -boundary and its dynamical consequences (see [16], [17], [30]). Our main geometrical hypothesis here is on $supp\mu$. We assume that the closed subsemigroup T generated by $supp\mu$ satisfies the so-called i-p condition, i.e T is strongly irreducible and contains at least one element with a unique simple dominant eigenvalue ; if $d = 1$, we assume furthermore that T is non arithmetic, i.e T is not contained in a subgroup of \mathbb{R}^* of the form $\{\pm a^n; n \in \mathbb{Z}\}$ for some $a > 0$. We observe that for $d > 1$ condition i-p is satisfied under very general conditions, in particular if the Zariski closure of T is a "large" subgroup of G (see [22], [41]). On the other hand, the set of probability measures μ on G such that the associated semigroup T satisfies condition i-p is open and dense in the weak topology. Also, if $d > 1$, an essential aperiodicity consequence of condition i-p is the density in the multiplicative group \mathbb{R}_+^* of the subgroup generated by the positive dominant eigenvalues of the elements of T (see [25], [28]).

We denote by g^* the transposed map of $g \in G$, i.e g^* is the linear map defined by the relation $\langle g^*x, y \rangle = \langle x, gy \rangle$, and μ^* is the push-forward of μ by $g \rightarrow g^*$. For a Radon measure η on $V \setminus \{0\}$ and $t > 0$, we write $t.\eta$ for the push-forward of η by the map $x \rightarrow tx$. For $v \in V \setminus \{0\}$ we denote $H_v^+ = \{x \in V ; \langle x, v \rangle > 1\}$ and if η is a probability measure on $V \setminus \{0\}$ we write $\hat{\eta}(v) = \eta(H_v^+)$. The asymptotic expansion of $\hat{\eta}(tv)$ at $t = o_+$ gives the tail of η in direction v ; also, using map $\eta \rightarrow \hat{\eta}$ and the affine structures we observe that the equation $\rho = \lambda * \rho$ implies Poisson-type equations on $V \setminus \{0\}$ and $E = (V \setminus \{0\}) \times \mathbb{R}$, for functions associated with $\hat{\rho}$. The space E is \mathbb{R}_+^* -fibered over the space W of affine oriented hyperplanes in V , and has a natural H -homogeneous space structure. Our analysis of the equation $\rho = \lambda * \rho$ will rely on the dynamical properties of "dual" λ -walks on the H -spaces $E, W, V \setminus \{0\}$ and on the study of Poisson-type equations satisfied by $\hat{\rho}$.

Since the dominant Lyapunov exponent of μ is negative, the random walk $S_n(\omega)v$ on V defined by μ converges to zero a.e ; however $S_n(\omega)v$ takes arbitrarily large values with a small probability which has a power law decay with degree $\alpha > 0$. As observed in [33], this fluctuation property of $S_n(\omega)v$ is closely related to the tail-homogeneity of ρ , i.e the existence of $\lim_{t \rightarrow 0_+} t^{-\alpha} \hat{\rho}(tv)$. We consider the more precise condition of "homogeneity at infinity", i.e existence and non triviality of $\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho)$ in the sense of vague convergence.

This property is proved here under the above hypothesis and the product structure of the limiting measure is described in terms of twisted μ -stationary measures on the unit sphere \mathbb{S}^{d-1} . This agrees with F. Spitzer's conjecture to the effect that ρ belongs to the domain of attraction of a stable law. For another general construction of such measures, in the context of winding of geodesic flow on surfaces see [1]. The cases where λ is concentrated on non negative matrices or λ has a density on H were studied in [33], tail homogeneity of ρ was proved and these results were extended to the general case in [37], under some restrictive conditions. If $d = 1$, tail homogeneity means homogeneity at infinity ; in this case, more information on the convergence of $t^{-\alpha}(t.\rho)$ and on the limiting measure was given in [19] where the more general quasi linear context was also considered. Furthermore, the case where T consists of similarities of V was studied in [8] under general conditions, homogeneity at infinity of ρ in an appropriate sense was proved, and the structure of the limiting measure was described. Here we complete and improve the results of [33], [37], which are simpler in a sense than those of [8]. We observe that the recent results of [2] allow to pass from tail-homogeneity to homogeneity at infinity if $\alpha \notin \mathbb{N}$. No such general result is valid for $\alpha \in \mathbb{N}$ and the situation is different (see [45]). Here the arguments depend on a detailed study of asymptotics for the potential kernel $\sum_0^\infty (\mu^*)^k * \delta_v$ and for certain Birkhoff sums on the H -space W ; the homogeneity at infinity of ρ is closely related to spectral gap properties for operators associated with μ , which are of independant interest. Hence we begin by a description of these properties in a general setting, if $d > 1$.

We denote by $|v|$ the length of $v \in V$, by $|g|$ the norm of $g \in G$ and we write $\gamma(g) = \sup(|g|, |g^{-1}|)$. We write $I_\mu = \{s \geq 0 ; \int |g|^s d\mu(g) < \infty\}$ and we denote $]0, s_\infty[$ the interior

of the interval I_μ . We consider the convolution action of μ on continuous functions on $V \setminus \{0\}$ which are homogeneous of degree $s \geq 0$, *i.e* functions f which satisfy : $f(tv) = |t|^s f(v)$ ($t \in \mathbb{R}$). This action reduces to the action of a certain positive operator P^s on $C(\mathbb{P}^{d-1})$, the space of continuous functions on the projective space \mathbb{P}^{d-1} . More precisely, if $f(v) = |v|^s \varphi(\bar{v})$ with $\varphi \in C(\mathbb{P}^{d-1})$, $\bar{v} \in \mathbb{P}^{d-1}$, then $P^s \varphi$ is given by :

$$P^s \varphi(x) = \int |gx|^s \varphi(g.x) d\mu(g),$$

where $x \in \mathbb{P}^{d-1}$, $x \rightarrow g.x$ denotes the projective action of g on x , and $|gx|$ is the norm of any vector gv with $|v| = 1$ and $\bar{v} = x$. Also for $z = s + it \in \mathbb{C}$ we write $P^z \varphi(x) = \int |gx|^z \varphi(g.x) d\mu(g)$. For $\varepsilon > 0$ let $H_\varepsilon(\mathbb{P}^{d-1})$ be the space of ε -Hölder functions on \mathbb{P}^{d-1} , with respect to a certain natural distance. We denote by ℓ^s (resp ℓ) the s -homogeneous (resp Lebesgue) measure on \mathbb{R}_+^* given by $\ell^s(dt) = \frac{dt}{t^{s+1}}$ (resp $\ell(dt) = \frac{dt}{t}$) and we write an s -homogeneous Radon measure η on $\dot{V} = \mathbb{P}^{d-1} \times \mathbb{R}_+^*$ as $\eta = \pi \otimes \ell^s$ where π is a bounded measure on \mathbb{P}^{d-1} . For $s \in I_\mu$ we define $k(s) = \lim_{n \rightarrow \infty} (\int |g|^s d\mu^n(g))^{1/n}$ where μ^n is the n -th convolution power of μ and we observe that $\text{Log}k(s)$ is a convex function on I_μ . A key tool in our analysis for $d > 1$ is the

Theorem A

Assume $d > 1$ and the subsemigroup $T \subset GL(V)$ generated by $\text{supp}\mu$ satisfies condition i-p. Then, for any $s \in I_\mu$ there exists a unique probability measure ν^s on \mathbb{P}^{d-1} , a unique positive continuous function $e^s \in C(\mathbb{P}^{d-1})$ with $\nu^s(e^s) = 1$ such that $P^s \nu^s = k(s) \nu^s$, $P^s e^s = k(s) e^s$. For $s \in I_\mu$, if $\int |g|^s \gamma^\delta(g) d\mu(g) < \infty$ for some $\delta > 0$ and if $\varepsilon > 0$ is sufficiently small, the action of P^s on $H_\varepsilon(\mathbb{P}^{d-1})$ has a spectral gap :

$$P^s = k(s)(\nu^s \otimes e^s + U_s),$$

where $\nu^s \otimes e^s$ is the projection on $\mathbb{C}e^s$ defined by ν^s, e^s and U_s is an operator with spectral radius less than 1 which commutes with $\nu^s \otimes e^s$. Furthermore the function $k(s)$ is analytic, strictly convex on $]0, s_\infty[$ and the spectral radius of P^z is less than $k(s)$ if $t = \text{Im}z \neq 0$.

We observe that, since condition i-p is open, the last property is robust under perturbation of μ in the weak topology. If $d = 1$, $k(s)$ is the Mellin transform of μ and the above statements are also valid if T is non arithmetic. However, the last property is not robust for $d = 1$. If $s = 0$, P^s reduces to the convolution operator by μ on \mathbb{P}^{d-1} and convergence to the unique μ -stationary measure $\nu^0 = \nu$ was studied in [17]. In this case, spectral gap properties for P^0 were first proved in [38]. Limit theorems of Probability Theory for the product $S_n = g_n \cdots g_1$ of the random i.i.d matrices g_k , distributed according to μ , are consequences of this result and of radial Fourier Analysis on $V \setminus \{0\}$ used in combination with boundary theory (see[4] [16], [21], [28], [38]). Also, for $s > 0$, the properties described in the theorem are basic ingredients for the study of precise large deviations of $S_n(\omega)v$ ([39]). The Radon measure $\nu^s \otimes \ell^s$ on \dot{V} satisfies the convolution equation $\mu * (\nu^s \otimes \ell^s) = k(s) \nu^s \otimes \ell^s$ and the support of ν^s is the unique T -minimal subset of \mathbb{P}^{d-1} , the so-called limit set $\Lambda(T)$ of T . The function e^s is an integral transform of the twisted μ^* -eigenmeasure ${}^* \nu^s$. For $s > 0$ and σ a probability measure on \mathbb{P}^{d-1} not concentrated on a proper subspace,

$|g|^s$ is comparable to $\int |gx|^s d\sigma(x)$. The uniqueness properties of e^s and ν^s are based on this geometrical fact. The proof of the spectral gap property depends on the simplicity of the dominant Lyapunov exponent for the product of random matrices $S_n = g_n \cdots g_1$ with respect to a natural shift-invariant Markov measure \mathbb{Q}^s on $\Omega = G^{\mathbb{N}}$, which is locally equivalent to the product measure $\mathbb{Q}^0 = \mu^{\otimes \mathbb{N}}$. As in [37] and [6] a martingale construction (here based on ${}^*\nu^s$) plays an essential role in the proof of simplicity and in the comparison of $|S_n(\omega)|$ with $|S_n(\omega)v|$ as in [28].

We observe that spectral gap properties for the family of operators $P^s (s \in \mathbb{R})$ play an important role in various problems, for example in the localisation problem for the Schrödinger operator with random potential on the line (See [4], [23]). In the paper [39], the spectral gap property for $P^s (s > 0)$ is not proved but the large deviations asymptotics stated there for $|S_n(\omega)v|$ can be justified with the use of Theorem A. If μ has a density with compact support, the above analysis is valid for any $s \in \mathbb{R}$. In general and for $d > 1$, it turns out that the function $k(s)$, as defined above, loses its analyticity at some $s < 0$. We endow $\Omega = G^{\mathbb{N}}$ with the shift-invariant measure $\mathbb{P} = \mu^{\otimes \mathbb{N}}$ (resp \mathbb{Q}^s). We know that if $\int \text{Log} \gamma(g) d\mu(g)$ (resp $\int |g|^s \text{Log} \gamma(g) d\mu(g)$) is finite the dominant Lyapunov exponent L_μ (resp $L_\mu(s)$) of $S_n = g_n \cdots g_1$ with respect to \mathbb{P} (resp \mathbb{Q}^s) exists and :

$$L_\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log} |S_n(\omega)| d\mathbb{P}(\omega), \quad L_\mu(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log} |S_n(\omega)| d\mathbb{Q}^s(\omega).$$

If $s \in]0, s_\infty[$, $k(s)$ has a derivative $k'(s)$ and $L_\mu(s) = \frac{k'(s)}{k(s)}$. By strict convexity of $\text{Log} k(s)$, if $\lim_{s \rightarrow s_\infty} k(s) \geq 1$ and $s_\infty > 0$, we can define $\alpha > 0$ by $k(\alpha) = 1$. We consider the potential

kernel U on \dot{V} defined by $U(v, \cdot) = \sum_0^\infty \mu^k * \delta_v$. Then we have the following multidimensional extensions of the classical renewal theorems of Probability theory (see [15]), which describes the asymptotic homogeneity of $U(v, \cdot)$:

Theorem B

Assume T satisfies condition i-p, $\int \text{Log} \gamma(g) d\mu(g) < \infty$ and $L_\mu > 0$. If $d = 1$ assume furthermore that μ is non-arithmetic. Then, for any $v \in \mathbb{P}^{d-1}$ we have the vague convergence:

$$\lim_{t \rightarrow 0_+} U(tv, \cdot) = \frac{1}{L_\mu} \nu \otimes \ell,$$

where ν is the unique μ -stationary measure on \mathbb{P}^{d-1} .

Theorem B'

Assume T satisfies condition i-p, $\int \text{Log} \gamma(g) d\mu(g) < \infty$, $L_\mu < 0$, $s_\infty > 0$ and there exists $\alpha > 0$ with $k(\alpha) = 1$, $\int |g|^\alpha \text{Log} \gamma(g) d\mu(g) < \infty$. If $d = 1$ assume furthermore that μ is non-arithmetic. Then for any $v \in \mathbb{P}^{d-1}$ we have the vague convergence on \dot{V} :

$$\lim_{t \rightarrow 0_+} t^{-\alpha} U(tv, \cdot) = \frac{e^\alpha(v)}{L_\mu(\alpha)} \nu^\alpha \otimes \ell^\alpha.$$

Up to normalization the Radon measure $\nu^\alpha \otimes \ell^\alpha$ is the unique α -homogeneous measure which satisfies the harmonicity equation $\mu * (\nu^\alpha \otimes \ell^\alpha) = \nu^\alpha \otimes \ell^\alpha$.

Corollary

With the notations of Theorem B', for any $v \in \mathbb{P}^{d-1}$, we have the convergence :

$$\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}\{\exists n \in \mathbb{N}; |S_n v| > t\} = Ae^\alpha(v) > 0.$$

The corollary is a matricial version of the famous Cramer estimate for the probability of ruin in collective risk theory (see [15]). Theorems B, B' are consequences of the arguments used in the proof of Theorem A and of a renewal theorem for a class of Markov walks on \mathbb{R} (see [34]). An essential role is played by the law of large numbers for $\text{Log}|S_n v|$ under $\mathbb{Q}^s(s = 0, \alpha)$; the comparison of $|S_n|$ and $|S_n v|$ follows from the finiteness of the limit of $(\text{Log}|S_n| - \text{Log}|S_n v|)$. For the sake of brevity, we have formulated these theorems in the context of \dot{V} instead of V . Corresponding statements where \mathbb{P}^{d-1} is replaced by \mathbb{S}^{d-1} are given in section 4. Also the above weak convergence can be extended to a larger class of functions. In [33], renewal theorems as above were obtained for non negative matrices, the extension of these results to the general case was an open problem and a partial solution was given in [37]. Theorems B and B' extend these results to a wider setting. In view of the interpretation of $U(v, \cdot)$ as a mean number of visits, Theorem B is a strong reinforcement of the law of large numbers for $S_n(\omega)v$, hence it can be used in some problems of dynamics for group actions on T -spaces. In this respect we observe that the asymptotic homogeneity of U stated in Theorem B has been of essential use in [30] for the description of the T -minimal subsets of the action of a large subsemigroup $T \subset SL(d, \mathbb{Z})$ of automorphisms of the torus \mathbb{T}^d . Also the convergence in Theorem B' gives the convergence of the μ -Martin kernel on \dot{V} to the point $\nu^\alpha \otimes \ell^\alpha$ of the μ -Martin boundary ; if $\int |g|^{\alpha\gamma^\delta} d\mu(g) < \infty$ for some $\delta > 0$, $\nu^\alpha \otimes \ell^\alpha$ is a minimal point. On the other hand Theorem B' gives a description of the fluctuations of a linear random walk on V with \mathbb{P} -a.e exponential convergence to zero, under condition i-p and the existence in T of a matrix with spectral radius greater than one. These fluctuation properties are responsible for the homogeneity at infinity of stationary measures for affine random walks on V , that we discuss now.

Let λ be a probability measure on the affine group H of V , μ its projection on G , Σ the closed subsemigroup of H generated by $\text{supp}\lambda$. As above we assume that the semigroup T generated by $\text{supp}\mu$ satisfies condition i-p, and $\text{supp}\lambda$ has no fixed point in V . If $d = 1$ we assume that μ is non-arithmetic. We consider the affine stochastic recursion :

$$x_{n+1} = A_{n+1}x_n + B_{n+1},$$

where (A_n, B_n) are λ -distributed i.i.d random variables. From a heuristic point of view, the corresponding Markov chain can be considered as a superposition of an additive random walk, governed by B_n and a multiplicative random walk, governed by A_n . Here, as it appears below in Theorem C, which is reminiscent of Theorem B', the non trivial multiplicative part A_n plays a dominant role, while the additive part B_n has a stabilizing effect since ρ is finite if B_n is not zero. If $\mathbb{E}(\text{Log}|A_n|) + \mathbb{E}(\text{Log}|B_n|) < \infty$ and the dominant Lyapunov exponent L_μ for the product $\Pi_n = A_1 \cdots A_n$ is negative, then $R_n = \sum_0^{n-1} A_1 \cdots A_k B_{k+1}$ converges $\lambda^{\otimes \mathbb{N}} - a.e.$ to R , the law ρ of R is the unique λ -stationary measure on $V, (V, \rho)$

is a λ -boundary and $\text{supp}\rho = \Lambda_a(\Sigma)$ is the unique Σ -minimal subset of V . If T contains at least one matrix with spectral radius greater than one and $I_\mu = [0, \infty[$, then $\Lambda_a(\Sigma)$ is unbounded, there exists $\alpha > 0$ with $k(\alpha) = 1$ and we can inquire about the "shape at infinity" of ρ . According to a conjecture of F. Spitzer the measure ρ should belong to the domain of attraction of a stable law with index α if $\alpha \in [0, 2[$ or a Gaussian law if $\alpha \geq 2$. Here we prove a multidimensional precise form of this conjecture, i.e α -homogeneity at infinity of ρ , where we assume that μ satisfies the conditions of Theorem B', λ satisfies moment conditions and $\text{supp}\lambda$ has no fixed point in V . The main idea is to express $\hat{\rho}$ as a μ^* -potential of a "small" implicit function on $V \setminus \{0\}$, to use the asymptotics given in Theorem B' and to complete the argument by a study of the ladder heights for certain Birkhoff sums along the dual λ -walk on W and a Choquet-Deny lemma, for translation-bounded μ -harmonic measures.

In order to state the result we need further notations related to the unit sphere \mathbb{S}^{d-1} of V and to limit sets. We denote by $\tilde{\Lambda}(T)$ the inverse image in \mathbb{S}^{d-1} of the limit set $\Lambda(T) \subset \mathbb{P}^{d-1}$. If T leaves invariant a proper convex cone in V , then $\tilde{\Lambda}(T)$ splits into two T -minimal subsets $\Lambda_+(T)$ and $\Lambda_-(T) = -\Lambda_+(T)$; otherwise $\tilde{\Lambda}(T)$ is T -minimal. We denote by $\tilde{\nu}^\alpha$ the unique symmetric measure on \mathbb{S}^{d-1} with projection ν^α on \mathbb{P}^{d-1} . If $\tilde{\Lambda}(T)$ is not T -minimal we write $\tilde{\nu}^\alpha = \frac{1}{2}(\nu_+^\alpha + \nu_-^\alpha)$ where ν_+^α and ν_-^α are the normalized restrictions of $\tilde{\nu}^\alpha$ to $\Lambda_+(T)$ and $\Lambda_-(T)$ respectively. Then it follows that the α -homogeneous measures $\tilde{\nu}^\alpha \otimes \ell^\alpha$, $\nu_+^\alpha \otimes \ell^\alpha$ and $\nu_-^\alpha \otimes \ell^\alpha$ are μ -harmonic. If $\tilde{\Lambda}(T)$ is T -minimal, the probability measure $\tilde{\nu}^\alpha$ is uniquely defined by the equation $\mu * (\tilde{\nu}^\alpha \otimes \ell^\alpha) = \tilde{\nu}^\alpha \otimes \ell^\alpha$. If not, any α -homogeneous harmonic measure is a linear combination of $\nu_+^\alpha \otimes \ell^\alpha$ and $\nu_-^\alpha \otimes \ell^\alpha$. We compactify V by adding at infinity the unit sphere \mathbb{S}_∞^{d-1} in the usual way and we denote by $\Lambda^\infty(T)$, $\Lambda_+^\infty(T)$, $\Lambda_-^\infty(T)$ the subsets of \mathbb{S}_∞^{d-1} corresponding to $\tilde{\Lambda}(T)$, $\Lambda_+(T)$, $\Lambda_-(T)$ respectively. The "shape at infinity" of ρ depends of the relative positions of the sets $\Lambda^\infty(T)$, $\Lambda_+^\infty(T)$, $\Lambda_-^\infty(T)$ with respect to the closure $\overline{\Lambda_a(\Sigma)}$ of $\Lambda_a(\Sigma)$ in $V \cup \mathbb{S}_\infty^{d-1}$ as described below.

Theorem C

With the above notations we assume that T satisfies condition i-p, $\text{supp}\lambda$ has no fixed point in V , $s_\infty > 0$, $L_\mu < 0$ and $\alpha \in]0, s_\infty[$ satisfies $k(\alpha) = 1$. If $d = 1$ we assume also μ is non-arithmetic. Then, if $\mathbb{E}(|B|^{\alpha+\delta}) < \infty$ and $\mathbb{E}(|A|^{\alpha}\gamma^\delta(A)) < \infty$ for some $\delta > 0$, the unique λ -stationary measure ρ on V satisfies the following vague convergence on $V \setminus \{0\}$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho) = C\sigma^\alpha \otimes \ell^\alpha,$$

where $C > 0$, σ^α is a probability measure on $\tilde{\Lambda}(T)$ and $\sigma^\alpha \otimes \ell^\alpha$ is a μ -harmonic Radon measure supported on $\mathbb{R}^*\tilde{\Lambda}(T)$. If T has no proper convex invariant cone in V , we have $\sigma^\alpha = \tilde{\nu}^\alpha$. In the opposite case, then $C\sigma^\alpha = C_+\nu_+^\alpha + C_-\nu_-^\alpha$, with $C_+ = 0$ (resp $C_- = 0$) if and only if $\overline{\Lambda_a(\Sigma)} \cap \Lambda_+^\infty(T) = \emptyset$ (resp $\overline{\Lambda_a(\Sigma)} \cap \Lambda_-^\infty(T) = \emptyset$). The above convergence is also valid on any Borel function f such that the set of discontinuities of f is $\sigma^\alpha \otimes \ell^\alpha$ -negligible and such that for some $\varepsilon > 0$ the function $|v|^{-\alpha}|\text{Log}|v||^{1+\varepsilon}|f(v)|$ is bounded.

This statement gives the homogeneity at infinity of ρ and the measure $C\sigma^\alpha \otimes \ell^\alpha$ can be

interpreted as the "tail measure" of ρ . In the context of [36], the convergence stated in the theorem says that ρ has "multivariate regular variation". If the moment condition on $|A|$ is replaced by $\mathbb{E}(|A|^\alpha \text{Log}\gamma(A)) < \infty$, then convergence remains valid on the sets H_v^+ . Theorem C was stated in [26]. Since the expression of $\hat{\rho}$ as a μ^* -potential depends on signed and implicit quantities, an important point is the discussion of positivity for C, C_+ and C_- . For $d = 1$ positivity of $C = C_+ + C_-$ was proved in [19] using Levy's symmetrisation argument, positivity of C_+, C_- was tackled in [26] by a complex analytic method introduced in [12]. Here our main tool is a detailed analysis of a conditional λ -walk on the H -space $E = (V \setminus \{0\}) \times \mathbb{R}$ considered as a \mathbb{R}_+^* -fibered space over the space W of affine oriented hyperplanes. The space $E = (V \setminus \{0\}) \times \mathbb{R}$ has a right H -homogeneous space structure given by $h(v, r) = (g^*v, r + \langle v, b \rangle)$ where $h = (g, b) \in H = G \ltimes V$; also the orientation preserving affine group D of the line acts on E by $d(v, r) = (tv, tr + x)$ with $d = (t, x) \in D$, this action commutes with the H -action and the quotient space is \mathbb{S}^{d-1} . Then the λ -random walk on E defines a D -valued cocycle of \mathbb{Z} through the G -action on \mathbb{S}^{d-1} ; also the map $\eta \rightarrow \hat{\eta}$ commutes with the H -actions on V and W . Then, the idea is to study Poisson-type equations on E satisfied by $\hat{\rho}$, and to express C_+ in terms of the ladder process of $\langle R_n, v \rangle$ under a conditional measure. The process $\langle R_n, v \rangle$ is closely related to a certain \mathbb{R}^* -valued \mathbb{Z} -cocycle over the natural λ -walk on the space W and this space is a 2-covering of a conditional λ -boundary. Moment estimations of the corresponding stationary measure allows to show, using Kac's recurrence theorem (see [44], p.195) and a suitable stationary measure, that the ladder index and ladder height of $\langle R_n, v \rangle$ have finite expectations, hence to establish positivity of C_+ . For a direct approach to this problem, based on singularity analysis of Mellin transforms of positive measures see [8], if T consists of similarities. For the homogeneity properties of ρ we observe that Theorem B' gives the convergence of $t^{-\alpha}(t.\rho)$ towards $C(\sigma^\alpha \otimes \ell^\alpha)$ on the sets H_v^+ . Then, if $\alpha \notin \mathbb{N}$, the recent results of [2] on homogeneity for Radon transforms of positive measures allow to deduce the vague convergence of $t^{-\alpha}(t.\rho)$ ($t \rightarrow 0_+$) towards $C(\sigma^\alpha \otimes \ell^\alpha)$. If $\alpha \in \mathbb{N}$, as general counterexamples show, the argument breaks down. Here we use the μ -harmonicity and boundedness properties of the cluster values of $t^{-\alpha}(t.\rho)$ ($t \rightarrow 0_+$), we apply radial Fourier Analysis on $V \setminus \{0\}$ and the spectral gap property stated in Theorem A, in order to prove a strong form of Choquet-Deny lemma for μ -harmonic measures on $V \setminus \{0\}$. We observe that theorem C gives a natural construction for a large class of probability measures in the domain of attraction of an α -stable law. Using also the weak dependance properties of the process x_n , Theorem C allows to prove convergence to α -stable laws for the normalized Birkhoff sums along the affine λ -walk on V (see [18]). If $d > 1$ and in contrast to [9], this convergence is robust under perturbations of λ in the weak topology. These convergences to stable laws are connected with the study of random walk in a random medium on the line or the strip (see [20]). On the other hand the study of the extremal value behaviour of the process x_n can be fully developed on the basis of Theorem C and the above weak dependance properties of x_n (see [36]). The arguments developed in the proof of homogeneity at infinity for ρ can also be used in the study of some quasi-linear

equations which occur in Mathematical Statistics (see [19]), Fractal analysis (see [39]) or in Statistical Mechanics (see [10], [12], [13]).

For recent information on the role of spectral gap properties in limit theorems for Probability theory and Ergodic theory we refer to [11], [27]. For information on products of random matrices we refer to [4], [16], [24]. Theorem A (resp B, B' and C) is proved in sections 2,3 (resp 4 and 5). Some auxiliary tools are developed in the Appendix. Preliminary partial versions of the above results, have appeared in [29], [37]. We thank Ch.M. Goldie, I. Melbourne and D. Petritis for useful informations on stochastic recursions.

II Ergodic properties of transfer operators on projective spaces

1) Notations, preliminary results

Let $V = \mathbb{R}^d$ be the Euclidean space endowed with the scalar product $\langle x, y \rangle = \sum_1^d x_i y_i$

and the norm $|x| = \left(\sum_1^d |x_i|^2 \right)^{1/2}$, \dot{V} the factor space of V by the finite group $\{\pm Id\}$. We

denote by \mathbb{P}^{d-1} (resp \mathbb{S}^{d-1}), the projective space (resp unit sphere) of V and by \bar{v} (resp \tilde{v}) the projection of $v \in V$ on \mathbb{P}^{d-1} (resp \mathbb{S}^{d-1}). The linear group $G = GL(V)$ acts on V, \dot{V} by $(g, v) \rightarrow gv$. If $v \in V \setminus \{0\}$, we write $g.v = \frac{gv}{|gv|}$ and we observe that G acts on \mathbb{S}^{d-1} by $(g, x) \rightarrow gx$. We will also write the action of $g \in G$ on $x \in \mathbb{P}^{d-1}$ by gx ; we observe that $|gx|$ is well defined and equal to $|g\tilde{x}|$ if $x \in \mathbb{P}^{d-1}$ and $\tilde{x} \in \mathbb{S}^{d-1}$ has projection $x \in \mathbb{P}^{d-1}$. Also, if $x, y \in \mathbb{P}^{d-1}$, $|\langle x, y \rangle|$ is well defined and equal to $|\langle \tilde{x}, \tilde{y} \rangle|$ where $\tilde{x}, \tilde{y} \in \mathbb{S}^{d-1}$ have projections x, y . Corresponding notations will be taken when convenient. For a subset $A \subset \mathbb{S}^{d-1}$ the convex envelope $Co(A)$ of A is defined as the intersection with \mathbb{S}^{d-1} of the closed convex cone generated by A in V . We denote by $O(V)$ the orthogonal group of V and by m the $O(V)$ -invariant measure on \mathbb{P}^{d-1} . A positive measure η on \mathbb{P}^{d-1} will be said to be proper if $\eta(U) = 0$ for every proper projective subspace $U \neq \mathbb{P}^{d-1}$.

Let P be a positive kernel on a Polish space E and let e be a positive function on E which satisfies $Pe = ke$ for some $k > 0$. Then we can define a Markov kernel Q_e on E by $Q_e \varphi = \frac{1}{ke} P(\varphi e)$. This procedure will be used frequently here; in particular if $e = e^s$ depends of a parameter s we will write $Q_e = Q_s$. For a Polish G -space E we denote by $M^1(E)$ the space of probability measures on E . If $\nu \in M^1(E)$, and P is as above, ν will be said to be P -stationary if $P\nu = \nu$, i.e for any Borel function $\varphi : \nu(P\varphi) = \nu(\varphi)$. We will write $C(E)$ (resp $C_b(E)$) for the space of continuous (resp bounded continuous) functions on E . If E is a locally compact G -space, $\mu \in M^1(G)$, and ρ is a Radon measure on E , we recall that the convolution $\mu * \rho$ is defined as a Radon measure by $\mu * \rho = \int \delta_{gx} d\mu(g) d\rho(x)$, where δ_y is the Dirac measure at $y \in E$. A μ -stationary measure on E will be a probability measure $\rho \in M^1(E)$ such that $\mu * \rho = \rho$. In particular, if $E = V$ or \dot{V} and $\mu \in M^1(G)$ we will consider the Markov kernel P on V (resp \dot{P} on \dot{V}) defined by $P(v, \cdot) = \mu * \delta_v$, (resp

$\dot{P}(v, \cdot) = \mu * \delta_v$). On \mathbb{P}^{d-1} (resp \mathbb{S}^{d-1}) we will write $\bar{P}(x, \cdot) = \mu * \delta_x$ (resp $\tilde{P}(x, \cdot) = \mu * \delta_x$). If $u \in \text{End}V$, we denote $u^* \in \text{End}V$ its adjoint map, i.e $\langle u^*x, y \rangle = \langle x, uy \rangle$ if $x, y \in V$. If $\mu \in M^1(G)$ we will write μ^* for its push forward by the map $g \rightarrow g^*$. For $s \geq 0$ we denote ℓ^s (resp h^s) the s -homogeneous measure (resp function) on $\mathbb{R}_+^* = \{t \in \mathbb{R} ; t > 0\}$ given by $\ell^s(dt) = \frac{dt}{t^{s+1}}$ (resp $h^s(t) = t^s$). For $s = 0$ we write $\ell(dt) = \frac{dt}{t}$. Using the polar decomposition $V \setminus \{0\} = \mathbb{S}^{d-1} \times \mathbb{R}_+^*$, every s -homogeneous measure η (resp function ψ) on $V \setminus \{0\}$ can be written as :

$$\eta = \pi \otimes \ell^s \quad (\text{resp } \psi = \varphi \otimes h^s)$$

where π (resp φ) is a measure (resp function) on \mathbb{S}^{d-1} . Similar decompositions are valid on $\dot{V} = \mathbb{P}^{d-1} \times \mathbb{R}_+^*$. If $g \in G$, and $\eta = \pi \otimes \ell^s$ (resp $\psi = \varphi \otimes h^s$) the directional component of $g\eta$ (resp $\psi \circ g$) is given by :

$$\rho^s(g)(\eta) = \int |gx|^s \delta_{g.x} d\eta(x) \quad (\text{resp } \rho_s(g)(\psi)(x) = |gx|^s \psi(g.x))$$

The representations ρ^s and ρ_s extend to measures on G by the formulas :

$$\rho^s(\mu)(\eta) = \int |gx|^s \delta_{g.x} d\mu(g) d\pi(x) \quad , \quad \rho_s(\mu)(\psi)(x) = \int |gx|^s \psi(g.x) d\mu(g)$$

We will write, for $\varphi \in C(\mathbb{P}^{d-1})$ (resp $\psi \in C(\mathbb{S}^{d-1})$) :

$$P^s \varphi = \rho_s(\mu)(\varphi), \quad (\text{resp } \tilde{P}^s \psi = \rho_s(\mu)(\psi)), \quad *P^s \varphi = \rho_s(\mu^*)(\varphi) \quad , \quad (\text{resp } *\tilde{P}^s \psi = \rho_s(\mu^*)(\psi)).$$

We endow \mathbb{S}^{d-1} (resp \mathbb{P}^{d-1}) with the distance $\tilde{\delta}$ (resp δ) defined by $\tilde{\delta}(x, y) = |x - y|$ (resp $\delta(\bar{x}, \bar{y}) = \text{Inf}\{|x - y|; |x| = |y| = 1\}$).

For $\varepsilon > 0$, $\varphi \in C(\mathbb{P}^{d-1})$ (resp $\psi \in C(\mathbb{S}^{d-1})$), we denote :

$$[\varphi]_\varepsilon = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\delta^\varepsilon(x, y)} \quad (\text{resp } [\psi]_\varepsilon = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{\tilde{\delta}^\varepsilon(x, y)}),$$

$$|\varphi| = \sup\{|\varphi(x)|; x \in \mathbb{P}^{d-1}\} \quad (\text{resp } |\psi| = \sup\{|\psi(x)|; x \in \mathbb{S}^{d-1}\}),$$

and we write $H_\varepsilon(\mathbb{P}^{d-1}) = \{\varphi \in C(\mathbb{P}^{d-1}); [\varphi]_\varepsilon < \infty\}$,

(resp $H_\varepsilon(\mathbb{S}^{d-1}) = \{\psi \in C(\mathbb{S}^{d-1}); [\psi]_\varepsilon < \infty\}$).

The set of positive integers will be denoted by \mathbb{N} .

Definition 2.1

If $s \in [0, \infty[$ we denote :

$$k(s) = k_\mu(s) = \lim_{n \rightarrow \infty} \left(\int |g|^s d\mu^n(g) \right)^{1/n}, \quad I_\mu = \{s \geq 0; k_\mu(s) < +\infty\}.$$

We observe that the above limit exists, since by subadditivity of $g \rightarrow \text{Log}|g|$, the quantity $u_n(s) = \int |g|^s d\mu^n(g)$ satisfies $u_{m+n}(s) \leq u_m(s)u_n(s)$. Also $k_\mu(s) = \text{Inf}_{n \in \mathbb{N}} (u_n(s))^{1/n}$, which

implies $I_\mu = \{s \geq 0; \int |g|^s d\mu(g) < +\infty\}$. Furthermore, Hölder inequality implies that I_μ is an interval of the form $[0, s_\infty[$ or $[0, s_\infty]$, and $\text{Log}k_\mu(s)$ is convex on I_μ . Also $k_{\mu^*} = k_\mu$ since $|g| = |g^*|$. If μ and μ' commute and $c \in [0, 1]$, $\mu'' = \mu + (1-c)\mu'$ then $k_{\mu''}(s) = ck_\mu(s) + (1-c)k_{\mu'}(s)$, if $s \in I_\mu \cap I_{\mu'}$.

Definition 2.2

a) An element $g \in \text{End}V$ is said to be proximal if g has a unique and simple eigenvalue $\lambda_g \in \mathbb{R}$ such that $|\lambda_g| = \lim_{n \rightarrow \infty} |g^n|^{1/n}$.

b) A semigroup $T \subset G$ is said to be strongly irreducible if no finite union of proper subspaces is T -invariant.

Proximality of g means that we can write $V = \mathbb{R}v_g \oplus V_g^<$ with $gv_g = \lambda_g v_g$, $gV_g^< \subset V_g^<$ and the restriction of g to $V_g^<$ has spectral radius less than $|\lambda_g|$. In this case $\lim_{n \rightarrow +\infty} g^n \cdot \bar{x} = \bar{v}_g$ if $x \notin V_g^<$ and we say that λ_g is the dominant eigenvalue of g . If $E \subset G$ we denote by E^{prox} the set of proximal elements of E . The closed subsemigroup (resp group) generated by E will be denoted $\langle E \rangle$ (resp $\langle E \rangle$). In particular we will consider below the case $E = \text{supp}\mu$ where $\text{supp}\mu$ is the support of $\mu \in M^1(G)$.

Definition 2.3

A semigroup $T \subset G$ is said to satisfy condition i-p if T is strongly irreducible and $T^{prox} \neq \phi$.

As shown in ([22], [41]) this property is satisfied if it is satisfied by $Zc(T)$, the Zariski closure of T . It can be proved that condition i-p is valid if and only if the connected component of the closed subgroup $Zc(T)$ is the product of a similarity group and a semi-simple real Lie group without compact factor which acts proximally and irreducibly on \mathbb{P}^{d-1} . In this sense T is "large". For example, if T is a countable subgroup of G which satisfies condition i-p then T contains a free subgroup with two generators.

We recall that in \mathbb{C}^n , the Zariski closure of $E \subset \mathbb{C}^n$ is the set of zeros of the set of polynomials which vanish on E . The group $G = GL(V)$ can be considered as a Zariski-closed subset of \mathbb{R}^{d^2+1} . If T is a semigroup, then $Zc(T)$ is a closed subgroup of G with a finite number of connected components. If $d = 1$, condition i-p is always satisfied. Hence, when using condition i-p, $d > 1$ will be understood.

Remark

The above definitions will be used below in the analysis of laws of large numbers and renewal theorems. A corresponding analysis has been developed in [33] for the case of non-negative matrices. We observe that proximality of an element in G as defined above is closely related to the Perron-Frobenius property for a positive matrix. Hence condition i-p above is also closely related to the non degeneracy conditions used in ([33] Theorem A) for non negative matrices.

Definition 2.4

Assume T is a subsemigroup of G which satisfies condition i-p. Then the closure of the set $\{\bar{v}_g; g \in T^{prox}\}$ will be called the limit set of T and will be denoted $\Lambda(T)$.

With these definitions we have the

Proposition 2.5

Assume $T \subset G$ is a subsemigroup which satisfies i-p and $S \subset T$ generates T . Then $\Lambda(T)$ is the unique T -minimal subset of \mathbb{P}^{d-1} . If $\mu \in M^1(G)$ is such that $T = [\text{supp}\mu]$ satisfies i-p, there exists a unique μ -stationary measure ν on \mathbb{P}^{d-1} . Also $\text{supp}\nu = \Lambda(T)$ and ν is proper. Furthermore, if $d > 1$, the subgroup of \mathbb{R}_+^* generated by the set $\{|\lambda_g|; g \in T^{prox}\}$ is dense

in \mathbb{R}_+^* . Also, if $\varphi \in C(\Lambda(T))$ satisfies for some $t \in \mathbb{R}$, $|e^{i\theta}| = 1 : \varphi(g.x) |gx|^{it} = e^{i\theta} \varphi(x)$ for any $g \in S$, $x \in \Lambda(T)$ then $t = 0$, $e^{i\theta} = 1$, $\varphi = \text{constant}$.

Remark

The first part of the above statement is essentially due to H. Furstenberg ([17], Propositions 4.8, 7.4). The second part is proved in ([28], Proposition 3). For another proof and extensions of this property see [25]. This property plays an essential role in the renewal theorems of section 4 as well as in section 5 for $d > 1$. In the context of non negative matrices a modified form is also valid ; in ([33], Theorem A) its conclusion is assumed as an hypothesis. If $d = 1$, we will need to assume it, i.e we will assume that T is non-arithmetic. If $T = [supp\mu]$ satisfies this condition, we say that μ is non-arithmetic.

2) Uniqueness of eigenfunctions and eigenmeasures on \mathbb{P}^{d-1}

Here we consider $s \in I_\mu$ and the operator P^s (resp $*P^s$) on \mathbb{P}^{d-1} defined by :

$$P^s \varphi(x) = \int |gx|^s \varphi(g.x) d\mu(g) \quad (\text{resp } *P^s \varphi(x) = \int |gx|^s \varphi(g.x) d\mu^*(g)).$$

For $z = s + it \in \mathbb{C}$ we will also write $P^z \varphi(x) = \int |gx|^z \varphi(g.x) d\mu(g)$

Below we study existence and uniqueness for eigenfunctions or eigenmeasures of P^s . We show equicontinuity properties of the normalized iterates of P^s and \tilde{P}^s .

Theorem 2.6

Assume $\mu \in M^1(G)$ is such that the semigroup $[supp\mu]$ satisfies (i-p) and let $s \in I_\mu$. If $d = 1$ we assume that μ is non arithmetic. Then the equation $P^s \varphi = k(s)\varphi$ has a unique continuous solution $\varphi = e^s$, up to normalization. The function e^s is positive and \bar{s} -Hölder with $\bar{s} = Inf(1, s)$.

Furthermore there exists a unique $\nu^s \in M^1(\mathbb{P}^{d-1})$ such that $P^s \nu^s$ is proportional to ν^s . One has $P^s \nu^s = k(s)\nu^s$ and $supp\nu^s = \Lambda([supp\mu])$. If $*\nu^s \in M^1(\mathbb{P}^{d-1})$ satisfies $*P^s(*\nu^s) = k(s)*\nu^s$, and e^s is normalized by $\nu^s(e^s) = 1$, one has $p(s)e^s(x) = \int | \langle x, y \rangle |^s d*\nu^s(y)$ where $p(s) = \int | \langle x, y \rangle |^s d\nu^s(x) d*\nu^s(y)$.

The map $s \rightarrow \nu^s$ (resp $s \rightarrow e^s$) is continuous in the weak topology (resp uniform topology) and the function $s \rightarrow Logk(s)$ is strictly convex.

The Markov operator Q^s on \mathbb{P}^{d-1} defined by $Q^s \varphi = \frac{1}{k(s)e^s} P^s(\varphi e^s)$ has a unique stationary measure π^s given by $\pi^s = e^s \nu^s$ and we have for any $\varphi \in C(\mathbb{P}^{d-1})$ the uniform convergence of $(Q^s)^n \varphi$ towards $\pi^s(\varphi)$. If Q^z is defined by $Q^z \varphi = \frac{1}{k(s)e^s} P^z(e^s \varphi)$, the equation $Q^z \varphi = e^{i\theta} \varphi$ with $\varphi \in C(\mathbb{P}^{d-1})$ $\varphi \neq 0$ implies $e^{i\theta} = 1$, $t = 0$, $\varphi = \text{constant}$.

Remarks

- a) If $s = 0$, then $e^s = 1$, and $\nu^s = \nu$ is the unique μ -stationary measure [17]. The fact that ν is proper is of essential use in [38], [4] and [28], for the study of limit theorems.
- b) In section 3 we will also, as in [17], construct a suitable measure-valued martingale which allows to prove that ν^s is proper (see Theorem 3.2) if $s \in I_\mu$. We note that analyticity of $k(s)$ is proved in Corollary 3.20 below. Continuity of the derivative of k will be essential in sections 3,4 and is proved in Theorem 3.10.

The proof of the theorem depends of a proposition and the following lemmas improving corresponding results for positive matrices in [33].

Lemma 2.7

Assume $\sigma \in M^1(\mathbb{P}^{d-1})$ is not supported by a hyperplane. Then, there exists a constant $c_s(\sigma) > 0$ such that, for any u in $EndV$:

$$\int |ux|^s d\sigma(x) \geq c_s(\sigma)|u|^s$$

Proof

Clearly it suffices to show the above inequality if $|u| = 1$. The fonction $u \rightarrow \int |ux|^s d\sigma(x)$ is continuous on $EndV$, hence its attains its infimum $c_s(\sigma)$ on the compact subset of $EndV$ defined by $|u| = 1$. If $c_s(\sigma) = 0$, then for some $u \in EndV$ with $|u| = 1$, we have $\int |ux|^s d\sigma(x) = 0$ hence, $ux = 0$, $\sigma - a.e$. In other words, $supp\sigma \subset Ker(u)$, which contradicts the hypothesis on σ . Hence $c_s(\sigma) > 0$. \square

Lemma 2.8

If $s \in I_\mu$ there exists $\sigma \in M^1(\mathbb{P}^{d-1})$ such that $P^s\sigma = k\sigma$ for some $k > 0$. For any such σ , we have $k = k(s)$ and σ is not supported on a hyperplane.

Furthermore for every $n \in \mathbb{N}$:

$$\int |g|^s d\mu^n(g) \geq k^n(s) \geq c_s(\sigma) \int |g|^s d\mu^n(g)$$

Proof

We consider the non-linear operator \hat{P}^s on $M^1(\mathbb{P}^{d-1})$ defined by $\hat{P}^s\sigma = \frac{P^s\sigma}{(P^s\sigma)(1)}$.

Since $\int |g|^s d\mu(g) < +\infty$, this operator is continuous in the weak topology. Since $M^1(\mathbb{P}^{d-1})$ is compact and convex, Schauder-Tychonov theorem implies the existence of $k > 0$ and $\sigma \in M^1(\mathbb{P}^{d-1})$ with $P^s\sigma = k\sigma$, hence $k = (P^s\sigma)(1)$.

For such a σ , the equation

$$k\sigma(\varphi) = \int \varphi(g.x)|gx|^s d\mu(g)d\sigma(x)$$

implies that if $x \in supp\sigma$, then $g.x \in supp\sigma \mu - a.e$.

Then for any $g \in supp\mu$: $g.supp\sigma \subset supp\sigma$. In particular the projective subspace H generated by $supp\mu$ satisfies $[supp\mu].H = H$. Since $[supp\mu]$ satisfies i-p, we have $H = \mathbb{P}^{d-1}$. Then Lemma 2.7 gives :

$$\int |gx|^s d\sigma(x) \geq c_s(\sigma)|g|^s.$$

The relation $(P^s)^n\sigma = k^n\sigma$ implies $k^n = \int |gx|^s d\mu^n(g)d\sigma(x)$, hence using Lemma 2.7 : $c_s(\sigma) \int |g|^s d\mu^n(g) \leq k^n \leq \int |g|^s d\mu^n(g)$. It follows $k = \lim_{n \rightarrow +\infty} \left(\int |g|^s d\mu^n(g) \right)^{1/n} = k(s)$. \square

Assume $e \in C(\mathbb{P}^{d-1})$ is positive and satisfies $P^s e = k(s)e$. Then we can define the Markov kernel Q_e^s by $Q_e^s \varphi(x) = \frac{1}{k(s)} \int \varphi(g.x) \frac{e(g.x)}{e(x)} |gx|^s d\mu(g)$. In view of the cocycle property of $\theta_e^s(x, g) = |gx|^s \frac{e(g.x)}{e(x)}$ we can calculate the iterate $(Q_e^s)^n$ by the formula :

$$(Q_e^s)^n \varphi(x) = \int \varphi(g.x) q_{e,n}^s(x, g) d\mu^n(g)$$

with $q_{e,n}^s(x, g) = \frac{1}{k^n(s)} \frac{e(g.x)}{e(x)} |gx|^s$, and $\int q_{e,n}^s(x, g) d\mu^n(g) = 1$.

Lemma 2.9

Assume e is as above, $f \in C(\mathbb{P}^{d-1})$ is real valued and satisfies $Q_e^s f \leq f$. Then, on $\Lambda([\text{supp}\mu])$, f is constant and equal to its infimum on \mathbb{P}^{d-1} .

Proof

Let $M^- = \{x \in \mathbb{P}^{d-1} ; f(x) = \text{Inf}\{f(y); y \in \mathbb{P}^{d-1}\}$. The relation $f(x) \geq \int q_e^s(x, g) f(g.x) d\mu(g)$ implies that if $x \in M^-$ then $g.x \in M^-$, μ -a.e. Hence $[\text{supp}\mu].M^- \subset M^-$. Since $\Lambda([\text{supp}\mu])$ is the unique $[\text{supp}\mu]$ -minimal subset of \mathbb{P}^{d-1} , we get $\Lambda([\text{supp}\mu]) \subset M^-$, i.e :

$$f(x) = \text{Inf}\{f(y); y \in \mathbb{P}^{d-1}\}, \text{ if } x \in \Lambda([\text{supp}\mu]) \quad \square$$

The following shows the existence of $e \in C(\mathbb{P}^{d-1})$ with $P^s e = k(s)e$, using Lemma 2.8 applied to μ^* .

Lemma 2.10

Assume $\sigma \in M^1(\mathbb{P}^{d-1})$ and $k > 0$ satisfy ${}^*P^s \sigma = k\sigma$. Then the function $\hat{\sigma}^s$ on \mathbb{P}^{d-1} defined by

$$\hat{\sigma}^s(x) = \int | \langle x, y \rangle |^s d\sigma(y)$$

satisfies $P^s \hat{\sigma}^s = k \hat{\sigma}^s$. Furthermore $\hat{\sigma}^s$ is positive and Hölder of order $\bar{s} = \text{Inf}(1, s)$.

Proof

We have $|gx|^s \hat{\sigma}^s(g.x) = \int | \langle x, g^*.y \rangle |^s |g^*.y|^s d\sigma(y)$ and ${}^*P^s \sigma = k\sigma$, hence :

$$P^s \hat{\sigma}^s(x) = k \int | \langle x, z \rangle |^s d\sigma(z) = k \hat{\sigma}^s(x).$$

If $\hat{\sigma}^s(x) = 0$ for some x , then $| \langle x, y \rangle | = 0$ σ -a.e, hence :

$$\text{supp}\sigma \subset \{y \in \mathbb{P}^{d-1}; \langle x, y \rangle = 0\}.$$

This contradicts Lemma 2.8, since $[\text{supp}\mu]$ satisfies i-p. Hence $\hat{\sigma}^s$ is positive.

In order to show the Hölder property of $\hat{\sigma}^s$, we use the inequality $|a^s - b^s| \leq \hat{s}|a - b|^{\bar{s}}$ where $a, b \in [0, 1]$ $\hat{s} = \text{sup}(s, 1)$, $\bar{s} = \text{Inf}(s, 1)$. Then

$$| | \langle x, y \rangle |^s - | \langle x', y \rangle |^s | \leq \hat{s} |x - x'|^{\bar{s}} |y|^{\bar{s}}, |\hat{\sigma}^s(x) - \hat{\sigma}^s(x')| \leq \hat{s} \hat{\delta}^{\bar{s}}(x, x'). \quad \square$$

Lemma 2.11

Let e be a positive and \bar{s} -Hölder function on \mathbb{P}^{d-1} with $\bar{s} = \text{Inf}(s, 1)$, $s > 0$. Then for any $(x, y) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$, $g \in G$ and a constant $b_s > 0$:

$$||gx|^s - |gy|^s| \leq (s+1)|g|^s \tilde{\delta}^{\bar{s}}(x, y), \tilde{\delta}(g.x, g.y) \leq 2 \frac{|g|}{|gx|} \tilde{\delta}(x, y), |\theta_e^s(x, g) - \theta_e^s(y, g)| \leq b_s |g|^s \tilde{\delta}^{\bar{s}}(x, y).$$

Proof

We use the inequality $|a^s - b^s| \leq |a - b|^s$ if $a, b \geq 0$, $s \leq 1$.

We get : $||gx|^s - |gy|^s| \leq ||gx| - |gy||^s \leq |g(x-y)|^s \leq |g|^s|x-y|^s$.

Hence $||gx|^s - |gy|^s| \leq |g|^s \tilde{\delta}^{\bar{s}}(x, y)$.

If $s > 1$, we use $\frac{1}{s}|a^s - b^s| \leq \sup(a, b)^{s-1}|a - b|$ if $a, b \geq 0$. We get :

$$||gx|^s - |gy|^s| \leq s|g|^{s-1}||gx| - |gy|| \leq s|g|^{s-1}|g(x-y)| \leq s|g|^s|x-y|.$$

Hence we get the first inequality. Furthermore :

$$\tilde{\delta}(g.x, g.y) = \left| \frac{gx}{|gx|} - \frac{gy}{|gy|} \right| \leq \frac{|g(x-y)|}{|gx|} + |gy| \left| \frac{1}{|gx|} - \frac{1}{|gy|} \right| \leq 2 \frac{|g||x-y|}{|gx|}$$

Hence $\tilde{\delta}(g.x, g.y) \leq 2 \frac{|g|}{|gx|} \tilde{\delta}(x, y)$.

We write :

$$|\theta_e^s(x, g) - \theta_e^s(y, g)| \leq ||gx|^s - |gy|^s| \frac{e(g.x)}{e(x)} + |gy|^s \frac{|e(g.x) - e(g.y)|}{e(x)} + |gy|^s e(g.y) \left| \frac{1}{e(x)} - \frac{1}{e(y)} \right|$$

In view of the first inequality the first term satisfies the required bound. The last term also satisfies it, since $\frac{1}{e(x)}$ is \bar{s} - Hölder and $|gy|^s \leq |g|^s$. For the second term we write :

$$|e(g.x) - e(g.y)| \leq [e]_{\bar{s}} \tilde{\delta}^{\bar{s}}(g.x, g.y) \leq 2^{\bar{s}} [e]_{\bar{s}} \left(\frac{|g|}{|gy|} \right)^{\bar{s}} \tilde{\delta}^{\bar{s}}(x, y).$$

Hence this term is bounded by :

$$2^{\bar{s}} [e]_{\bar{s}} \tilde{\delta}^{\bar{s}}(x, y) |gy|^{s-\bar{s}} |g|^{\bar{s}} \leq 2^{\bar{s}} [e]_{\bar{s}} |g|^s \tilde{\delta}^{\bar{s}}(x, y).$$

The inequalities imply the lemma. \square

The following is well known (see for example [42], Theorem 6)

Lemma 2.12

Let X be a compact metric space, Q a Markov operator on X , which preserves $C(X)$. Assume that all the Q -invariant continuous functions are constant and for any $\varphi \in C(X)$, the sequence $Q^n \varphi$ is equicontinuous. Then Q has a unique stationary measure. Furthermore if the equation $Q\psi = e^{i\theta}\psi$, $\psi \in C(X)$ implies $e^{i\theta} = 1$, then for any $\varphi \in C(X)$ $Q^n \varphi$ converges.

Proposition 2.13

Let $\mu \in M^1(G)$ and assume that the semigroup $[supp\mu]$ satisfies i-p.

Let $s \in I_\mu$, $s > 0$ $\varepsilon \in]0, \bar{s}]$ with $\bar{s} = \text{Inf}(1, s)$, $e \in C(\mathbb{P}^{d-1})$ is positive, \bar{s} -Hölder with $P^s e = k(s)e$.

Then there exists $a_s \geq 0$ such that for any $n \in \mathbb{N}$ and ε -Hölder function φ on \mathbb{P}^{d-1} :

$$[(Q_e^s)^n \varphi]_\varepsilon \leq a_s |\varphi| + \rho_{n,s}(\varepsilon) [\varphi]_\varepsilon$$

where $\rho_{n,s}(\varepsilon) = \text{Sup}_{x,y} \int q_{e,n}^s(x, g) \frac{\delta^\varepsilon(g.x, g.y)}{\delta^\varepsilon(x, y)} d\mu^n(g)$ is bounded independantly of n .

In particular for any $\psi \in C(\mathbb{P}^{d-1})$ the sequence $(Q_e^s)^n \psi$ is equicontinuous. Furthermore any continuous Q_e^s -invariant function is constant and Q_e^s has a unique stationary measure π_e^s . If $s = 0$ and $\varphi \in C(\mathbb{P}^{d-1})$ we have the uniform convergence : $\lim_{n \rightarrow \infty} \overline{P}^n \varphi = \nu(\varphi)$.

Proof

The definition of Q_e^s gives for any ε -Hölder function φ :

$$|(Q_e^s)^n \varphi(x) - (Q_e^s)^n \varphi(y)| \leq |\varphi| \int |q_{e,n}^s(x, g) - q_{e,n}^s(y, g)| d\mu^n(g) + [\varphi]_\varepsilon \int q_{e,n}^s(y, g) \delta^\varepsilon(g.x, g.y) d\mu^n(g).$$

Lemma 2.11 shows that the first integral is dominated by $\frac{b_s}{k^n(s)} \delta^s(x, y) \int |g|^s d\mu^n(g)$ and Lemma 2.8 gives $k^n(s) \geq c_s \int |g|^s d\mu^n(g)$. Hence : $[(Q_e^s)^n \varphi]_\varepsilon \leq a_s |\varphi| + \rho_{n,s}(\varepsilon) [\varphi]_\varepsilon$ with $a_s = b_s/c_s$.

Lemma 2.4 allows to bound $\rho_{n,s}(\varepsilon)$:

$$\delta^\varepsilon(g.x, g.y) \leq 2^\varepsilon \frac{|g|^\varepsilon}{|gx|^\varepsilon} \delta^\varepsilon(x, y), \quad \rho_{n,s}(\varepsilon) \leq \frac{2^\varepsilon}{k^n(s)} \sup_x \int \frac{e(g.x)}{e(x)} |gx|^{s-\varepsilon} |g|^\varepsilon d\mu^n(g).$$

We denote $c = \sup_{g,x} \frac{e(g.x)}{e(x)} < \infty$ hence using $s \geq \varepsilon$, $|gx|^{s-\varepsilon} \leq |g|^{s-\varepsilon}$ we get :

$$\frac{e(g.x)}{e(x)} |gx|^{s-\varepsilon} |g|^\varepsilon \leq c |g|^s, \quad \rho_{n,s}(\varepsilon) \leq \frac{2^\varepsilon}{k^n(s)} \int |g|^s d\mu^n(g) \leq c \frac{2^\varepsilon}{c_s}.$$

Assume that $\varphi \in C(\mathbb{P}^{d-1})$ satisfies $Q_e^s \varphi = \varphi$ and denote :

$$M^+ = \{x \in \mathbb{P}^{d-1}; \varphi(x) = \sup_{y \in \mathbb{P}^{d-1}} \varphi(y)\}, \quad M^- = \{x \in \mathbb{P}^{d-1}; \varphi(x) = \inf_{y \in \mathbb{P}^{d-1}} \varphi(y)\}.$$

Then, as in the proof of Lemma 2.9, $\text{supp} \mu.M^+ \subset M^+$, $\text{supp} \mu.M^- \subset M^-$, hence by minimality of $\Lambda([\text{supp} \mu])$ we have $\Lambda([\text{supp} \mu]) \subset M^+ \cap M^-$.

It follows $M^+ \cap M^- \neq \emptyset$, $\varphi = \text{cte}$.

If $\psi \in C(\mathbb{P}^{d-1})$ is ε -Hölder the above inequality gives for any $x, y \in \mathbb{P}^{d-1}$:

$$|(Q_e^s)^n \varphi(x) - (Q_e^s)^n \varphi(y)| \leq (a_s |\varphi| + \rho_{n,s}(\varepsilon)) [\varphi]_\varepsilon \delta^\varepsilon(x, y).$$

Since $\rho_{n,s}(\varepsilon)$ is bounded this shows that the sequence $(Q_e^s)^n \psi$ is equicontinuous. By density this remains valid for any $\psi \in C(\mathbb{P}^{d-1})$.

Hence we can apply Lemma 2.12 to $Q = Q_e^s$: there is a unique Q_e^s -stationary measure. If $s = 0$, we have $\varepsilon = 0$, hence the above inequality do not show the equicontinuity of $\overline{P}^n \varphi$. In this case the equicontinuity follows from Theorem 3.2 in the next section : we have for

$\varepsilon > 0$ $\lim_{n \rightarrow \infty} \int \delta^\varepsilon(g.x, g.y) d\mu^n(g) = 0$, which implies for $\varphi \in H_\varepsilon(\mathbb{P}^{d-1})$:

$$\begin{aligned} |(\overline{P}^n \varphi)(x) - \overline{P}^n \varphi(y)| &\leq [\varphi]_\varepsilon \int \delta^\varepsilon(g.x, g.y) d\mu^n(g). \\ |\overline{P}^n \varphi(x) - \nu(\varphi)| &\leq [\varphi]_\varepsilon \int \delta^\varepsilon(g.x, g.y) d\nu(y) d\mu^n(g), \quad \lim_{n \rightarrow \infty} |\overline{P}^n \varphi(x) - \nu(\varphi)| = 0. \quad \square \end{aligned}$$

Remarks

- a) If $s < s_\infty$, $\varepsilon < s_\infty - \bar{s}$, $e = e^s$ it is shown in Corollary 3.18 below, if for some $\delta > 0$ $\int |g|^\delta \gamma^\delta(g) d\mu(g) < \infty$ then $\lim_{n \rightarrow \infty} \rho_{n,s}(\varepsilon) = 0$, hence $\rho_{n,s}(\varepsilon) < 1$ for some $n = n_0$. Hence $(Q_e^s)^{n_0}$ satisfies a so called Doeblin-Fortet inequality (see [27]).
- b) Let \tilde{Q}_e^s be the Markov kernel on \mathbb{S}^{d-1} defined by $\tilde{Q}_e^s \varphi = \frac{1}{k(s)e} \tilde{P}^s(\varphi e)$ where e still denotes the function on \mathbb{S}^{d-1} corresponding to $e \in C(\mathbb{P}^{d-1})$. Then the inequality and its proof remain valid for \tilde{Q}_e^s instead of Q_e^s . In particular for any $\psi \in C(\mathbb{S}^{d-1})$ the sequence $(\tilde{Q}_e^s)^n \psi$ is equicontinuous. This fact will be used in the next paragraph

Proof of the theorem

As in the proof of Lemma 2.8, we consider the non linear operator $*\hat{P}^s$ on $M^1(\mathbb{P}^{d-1})$ defined by $*\hat{P}^s \sigma = \frac{*P^s \sigma}{(*P^s \sigma)(1)}$. The same argument gives the existence of k and $\sigma \in M^1(\mathbb{P}^{d-1})$ such that $*P^s \sigma = k\sigma$, with $k = (*P^s \sigma)(1) > 0$. We consider only the case $d > 1$.

Since $[supp\mu]^* = [supp\mu]^*$ satisfies i-p, Lemma 2.8 applied to μ^* gives $k = k(s)$ and σ is not supported by a hyperplane. Then Lemma 2.10 implies that $\hat{\sigma}^s(x) = \int | \langle x, y \rangle |^s d\sigma(y)$ satisfies $P^s \hat{\sigma}^s = k(s) \hat{\sigma}^s$ and is positive, Hölder continuous of order $\bar{s} = Inf(1, s)$. Hence we can apply Proposition 2.13 with $e = \hat{\sigma}^s$; then we get existence and uniqueness of e^s with $P^s e^s = k(s) e^s$, $\nu^s(e^s) = 1$ and e^s satisfies $p(s) e^s(x) = \int | \langle x, y \rangle |^s d^* \sigma(y)$. Also $Q^s = Q_{e^s}^s$ has a unique stationary measure π^s . The uniqueness of $\nu^s \in M^1(\mathbb{P}^{d-1})$ with $P^s \nu^s = k(s) \nu^s$ follows. Also $\sigma = *\nu^s$ by the same proof.

Lemma 2.8 implies that if some $\eta \in M^1(\mathbb{P}^{d-1})$ satisfies $P^s \eta = k\eta$, then $k = k(s)$. Since $supp\nu^s$ is $[supp\mu]$ -invariant and $\Lambda([supp\mu])$ is minimal we get $supp\nu^s \supset \Lambda([supp\mu])$. We can again use Schauder-Tychonoff theorem in order to construct $\sigma' \in M^1(\mathbb{P}^{d-1})$ with $supp\sigma' \subset \Lambda([supp\mu])$, $P^s \sigma' = k\sigma'$. Since $\sigma' = \nu^s$, we get finally $supp\nu^s = \Lambda([supp\mu])$.

In order to show the continuity of $s \rightarrow \nu^s$, $s \rightarrow e^s$ we observe that, from above, ν^s is uniquely defined by : $P^s \nu^s = k(s) \nu^s$, $\nu^s \in M^1(\mathbb{P}^{d-1})$. Also, by convexity, $k(s)$ is continuous. On the other hand, the uniform continuity of $(x, s) \rightarrow |gx|^s$ and the fact that $|gx|^s \leq |g|^s$ is bounded by the μ -integrable function $Sup(1, |g|^{s_2})$ on $[s_1, s_2] \subset I_\mu$ implies the uniform continuity of $P^s \varphi$ if φ is fixed. Then we consider a sequence $s_n \in I_\mu$, $s_0 \in I_\mu$ with $\lim_{s_n \rightarrow s_0} \nu^{s_n} = \eta \in M^1(\mathbb{P}^{d-1})$. We have

$$P^{s_n} \nu^{s_n}(\varphi) = \nu^{s_n}(P^{s_n} \varphi), \quad \lim_{s_n \rightarrow s_0} P^{s_n} \nu^{s_n}(\varphi) = \lim_{s_n \rightarrow s_0} k(s_n) \nu^{s_n}(\varphi) = k(s_0) \eta(\varphi).$$

Then the uniform continuity in (s, x) of $P^s \varphi(x)$ implies $P^{s_0} \eta = k(s_0) \eta$. The uniqueness of ν^{s_0} implies $\nu^{s_0} = \eta$, and the arbitrariness of s_n gives the continuity of $s \rightarrow \nu^s$ at s_0 . The same property is true for the operator $*P^s$ and the measure $*\nu^s$ defined by $*P^s (*\nu^s) = k(s) (*\nu^s)$, $*\nu^s \in M^1(\mathbb{P}^{d-1})$.

Lemma 2.10, implies $p(s) e^s(x) = \int | \langle x, y \rangle |^s d^* \nu^s(y)$, and since the set of functions $x \rightarrow | \langle x, y \rangle |^s$ ($y \in \mathbb{P}^{d-1}$, $s \in I_\mu$) is locally equicontinuous : $\lim_{s \rightarrow s_0} |e^s - e^{s_0}| = 0$.

In order to show the strict convexity of $Logk(s)$ we take $s, t \in I_\mu$, $p \in [0, 1]$ and we observe that from Hölder inequality, $P^{ps+(1-p)t}[(e^s)^p (e^t)^{1-p}] \leq k^p(s) k^{1-p}(t) (e^s)^p (e^t)^{1-p}$.

We denote $f = (e^s)^p (e^t)^{1-p}$ and assume $k(ps + (1-p)t) = k^p(s)k^{1-p}(t)$ for some $s \neq t$. Then Lemma 2.9 can be used with $e = e^{ps+(1-p)t}$ and $Q_e^{pq+(1-p)t}\varphi = \frac{1}{k(ps+(1-p)t)e} P^{ps+(1-p)t}(\varphi e)$.

It gives on $\Lambda([supp\mu]) : f = ce^{ps+(1-p)t}$ for some constant $c > 0$.

Hence, on $\Lambda([supp\mu])$ we have :

$$P^{ps+(1-p)t}[(e^s)^p (e^t)^{1-p}] = k^p(s)k^{1-p}(t)(e^s)^p (e^t)^{1-p}.$$

This means that there is equality in the above Hölder inequality. It follows that, for some positive function $c(x)$ and any x in $\Lambda([supp\mu])$, $g \in supp\mu$:

$$|gx|^s \frac{e^s(g.x)}{e^s(x)} = c(x)|gx|^t \frac{e^t(g.x)}{e^t(x)}.$$

Integration with respect to μ gives : $c(x) = \frac{k(s)}{k(t)}$. Since $s \neq t$, we get, for some constant $c > 0$ and $\varphi \in C(\mathbb{P}^{d-1})$, positive, for any (x, g) as above : $|gx| = c \frac{\varphi(g.x)}{\varphi(x)}$. It follows, if $g \in (supp\mu)^n$ and $x \in \Lambda([supp\mu])$, $|gx| = c^n \frac{\varphi(g.x)}{\varphi(x)}$. If $g \in [supp\mu]^{prox}$, we get $|\lambda_g| \in c^{\mathbb{N}}$. This contradicts Proposition 2.5.

In order to show the convergence of $(Q^s)^n \varphi$, since by Proposition 2.13 the family $(Q^s)^n \varphi$ is equicontinuous, it suffices to show in view of Lemma 2.12 that the relation $Q^s \varphi = e^{i\theta} \varphi$ with $\varphi \in C(\mathbb{P}^{d-1})$, $|e^{i\theta}| = 1$ implies $e^{i\theta} = 1$, $\varphi = cte$.

Taking absolute values we get $|\varphi| \leq Q^s |\varphi|$. As in Lemma 2.9, we get that for any x in $\Lambda([supp\mu]) : |\varphi(x)| = \sup\{|\varphi(y)|; y \in \mathbb{P}^{d-1}\}$.

Hence we can assume $|\varphi(x)| = 1$ on $\Lambda([supp\mu])$. Now we can use the equation

$$e^{i\theta} \varphi(x) = \int q^s(x, g) \varphi(g.x) d\mu(g)$$

where $q^s(x, g) = \frac{1}{k(s)} \frac{e^s(g.x)}{e^s(x)} |gx|^s$, hence $\int q^s(x, g) d\mu(g) = 1$. Strict convexity gives : $e^{i\theta} \varphi(x) = \varphi(g.x)$, for any $x \in \Lambda([supp\mu])$, $g \in supp\mu$.

We know, from Proposition 2.13 that $\bar{P}^n \varphi$ converges uniformly to $\nu(\varphi)$ where ν is the unique \bar{P} -stationary measure on \mathbb{P}^{d-1} . Furthermore, on $\Lambda([supp\mu])$ we have $\bar{P}^n \varphi = e^{in\theta} \varphi$. The above convergence gives $e^{i\theta} = 1$, since $\varphi \neq 0$ on $\Lambda([supp\mu])$. The fact that φ is constant follows from Proposition 2.13.

In order to show the last assertion in case $d > 1$ we write $Q^z \varphi(x)$ as

$$Q^z \varphi(x) = \int |gx|^{it} q^s(x, g) \varphi(g.x) d\mu(g).$$

We observe that the absolute value of the function $Q^z \varphi$ is bounded by the function $(Q^s)|\varphi|$. Hence, from above, the equation $Q^z \varphi = e^{i\theta} \varphi$ gives $Q^s |\varphi| \geq |\varphi|$, hence $Q^s |\varphi| = |\varphi|$ and $|\varphi| = cte$. Then the equation $Q^z \varphi = e^{i\theta} \varphi$ gives for any x and $g \in supp\mu$ since $\int q^s(x, g) d\mu(g) = 1 : |gx|^{it} \frac{\varphi(g.x)}{\varphi(x)} = e^{i\theta} \mu - a.e.$

This contradicts Proposition 2.5 if $t \neq 0$. If $t = 0$ we have $e^{i\theta} = 1$, $\varphi = cte$ from above.

□

3) Eigenfunctions and eigenmeasures on \mathbb{S}^{d-1}

Here we study the operator \tilde{P}^s on \mathbb{S}^{d-1} defined by $\tilde{P}^s\varphi(x) = \int \varphi(g.x)|gx|^s\mu(dg)$. We show that there are 2 cases, depending of the existence of a $[supp\mu]$ -invariant proper convex cone in V or not. We still denote by e^s the function on \mathbb{S}^{d-1} lifted from $e^s \in C(\mathbb{P}^{d-1})$. We denote \tilde{Q}^s the operator on \mathbb{S}^{d-1} defined by $\tilde{Q}^s\varphi = \frac{1}{k(s)e^s}\tilde{P}^s(\varphi e^s)$.

We already know, using the remark which follows Proposition 2.13, that for $s > 0$ and any given $\varphi \in C(\mathbb{S}^{d-1})$, the sequence $(\tilde{Q}^s)^n\varphi$ is equicontinuous. For any subsemigroup T of G satisfying condition i-p, we denote by $\tilde{\Lambda}(T)$ the inverse image of $\Lambda(T)$ in \mathbb{S}^{d-1} . We begin by considering the dynamics of T on \mathbb{S}^{d-1} . For analogous results in more general situations see [25].

Proposition 2.15

Assume $T \subset G$ is a subsemigroup which satisfies condition i-p. If $d = 1$, we assume that T is non-arithmetic. Then the action of T on \mathbb{S}^{d-1} has one or two minimal sets whose union is $\tilde{\Lambda}(T)$:

Case I There is no T -invariant proper convex cone in V .
Then $\tilde{\Lambda}(T)$ is the unique T -minimal subset of \mathbb{S}^{d-1} .

Case II T preserves a closed proper convex cone $C \subset V$.
Then the action of T on \mathbb{S}^{d-1} has two and only two minimal subsets $\Lambda_+(T)$, $\Lambda_-(T)$ with $\Lambda_-(T) = -\Lambda_+(T)$, $\Lambda_+(T) \subset \mathbb{S}^{d-1} \cap C$. The convex cone generated by $\Lambda_+(T)$ is T -invariant.

The proof depends of the following Lemma.

Lemma 2.16

Let $V_i (1 \leq i \leq r)$ be vector subspaces of V . If condition i-p is valid, then there exists $g \in T^{prox}$ such that the hyperplane $V_g^<$ does not contain any $V_i (1 \leq i \leq r)$.

Proof

The dual semigroup T^* of T satisfies also condition i-p hence we can also consider its limit set $\Lambda(T^*) \subset \mathbb{P}(V^*)$. Let $\bar{v}(g^*)$ be the point of $\mathbb{P}(V^*)$ corresponding to a dominant eigenvector of g^* . Observe that the condition that an hyperplane contains V_i defines a subspace of V^* . If for any $g^* \in (T^*)^{prox}$ $\bar{v}(g^*)$ contains some V_i then by density any $x \in \Lambda(T^*)$ contains some V_i . Then the T^* -invariance of $\Lambda(T^*)$ implies that T^* leaves invariant a finite union of subspaces of $\mathbb{P}(V^*)$, which contradicts condition i-p. \square

Proof of the proposition

Let $x \in \tilde{\Lambda}(T)$ and $S = \overline{T.x}$. We observe that if $y \in \mathbb{S}^{d-1}$, then $\overline{T.y}$ contains x or $-x$, since the projection of $\overline{T.x}$ in \mathbb{P}^{d-1} contains $\Lambda(T)$. Assume first $-x \notin \overline{T.x}$. If $y \in \overline{T.x}$, then $\overline{T.y} \subset S$, hence $x \in \overline{T.y}$. This shows the T -minimality of S . The same argument shows that $-y \notin S$, hence $S \cap -S = \phi$. Since the projection of S in \mathbb{P}^{d-1} is $\Lambda(T)$, we see that the projection of \mathbb{S}^{d-1} on \mathbb{P}^{d-1} gives a T -equivariant homeomorphism of S on $\Lambda(T)$. Since

$-x \notin S$, there are two T -minimal sets, S and $-S$. Since for any $y \in \mathbb{S}^{d-1}$, $\overline{T.y}$ contains S or $-S$, these sets are the unique minimal sets.

Assume now $-x \in \overline{T.x}$, hence $S = -S$. Since the projection of S in \mathbb{P}^{d-1} is $\Lambda(T)$, we see that $S = \tilde{\Lambda}(T)$.

Assume now that C is a T -invariant closed proper convex cone, that we can suppose closed. Then $C \cap \mathbb{S}^{d-1}$ is T -invariant and closed, hence $C \cap \mathbb{S}^{d-1} \supset \Lambda_+(T)$ or $\Lambda_-(T)$ in the first situation, ($-x \notin \overline{T.x}$). In the second situation C cannot exist, since $C \cap \mathbb{S}^{d-1}$ would contain $\tilde{\Lambda}(T)$, which is symmetric.

It remains to show that, in the first situation, there exists a T -invariant closed proper convex cone. Let C be the convex cone generated by $\Lambda_+(T)$ and let us show $C \cap -C = \{0\}$. Assume $C \cap -C \neq \{0\}$; then we can find $y_1, \dots, y_p \in C$, $z_1, \dots, z_q \in -C$ and convex combinations $y = \sum_1^p \alpha_i y_i$, $z = \sum_1^q \beta_j z_j$ with $y = z$. The lemma shows that there exists $g \in T^{prox}$ such that $y_i (1 \leq i \leq p)$ and $z_j (1 \leq j \leq q)$ do not belong to $V_g^<$. Hence, with $n \in 2\mathbb{N}$:

$$\lim_{n \rightarrow +\infty} \frac{g^n y}{|g^n|} = \lim_{n \rightarrow \infty} \sum_1^p \alpha_i \frac{g^n y_i}{|g^n y_i|} \frac{|g^n y_i|}{|g^n|} = \left(\sum_1^p \alpha_i u_i \right) v_g$$

where $u_i = \lim_{n \rightarrow \infty} \frac{|g^n y_i|}{|g^n|} > 0$ and $v_g \in \Lambda_+(T)$ is the unique dominant eigenvector of g in $\Lambda_+(T)$. In the same way:

$$\lim_{n \rightarrow \infty} \frac{g^n z}{|g^n|} = - \left(\sum_1^q \beta_j u'_j \right) v_g$$

with $u'_j > 0$. Since $y = z$ we have a contradiction. Hence we have the required dichotomy. The last assertion follows. \square

Theorem 2.17

Let $\mu \in M^1(G)$, $s \in I_\mu$ and assume $[supp\mu]$ satisfies i-p. If $d = 1$ we assume that μ is non-arithmetic. Then for any $\varphi \in C(\mathbb{S}^{d-1})$, $x \in \mathbb{S}^{d-1}$, we have the uniform convergence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_1^n (\tilde{Q}^s)^n \varphi(x) = \tilde{\pi}^s(x)(\varphi),$$

where, $\tilde{\pi}^s(x) \in M^1(\mathbb{S}^{d-1})$ is supported on $\tilde{\Lambda}([supp\mu])$ and is \tilde{Q}^s -stationary.

Furthermore there are 2 cases given by Proposition 2.15.

Case I, \tilde{Q}^s has a unique stationary measure $\tilde{\pi}^s$ with $supp\tilde{\pi}^s = \tilde{\Lambda}([supp\mu])$ and $\tilde{\pi}^s(x) = \tilde{\pi}^s$ for any $x \in \mathbb{S}^{d-1}$. The \tilde{Q}^s -invariant functions are constant.

Case II, \tilde{Q}^s has two and only two extremal stationary measures π_+^s, π_-^s . We have $supp\pi_+^s = \Lambda_+([supp\mu])$ and π_-^s is symmetric of π_+^s . If $\pi_+^s = e^s \nu_+^s$, then $\tilde{P}^s \nu_+^s = k(s) \nu_+^s$.

Also, there are 2 minimal \tilde{Q}^s -invariant continuous functions p_+^s, p_-^s and we have:

$$\tilde{\pi}^s(x) = p_+^s(x) \pi_+^s + p_-^s(x) \pi_-^s.$$

Furthermore $p_+^s(x)$ is equal to the entrance probability in the set $Co(\Lambda_+([supp\mu]))$ for the Markov chain defined by \tilde{Q}^s . In particular $p_+^s(x) = 1$ if $x \in \Lambda_+([supp\mu])$.

If ${}^* \nu_+^s \in M^1(\Lambda_+(T^*))$ satisfies ${}^* P^s {}^* \nu_+^s = k(s) {}^* \nu_+^s$, we have for $u \in \mathbb{S}^{d-1}$,
 $p_+^s(u) e^s(u) = \int \langle u, u' \rangle_+^s d {}^* \nu_+^s(u')$ with $\langle u, u' \rangle_+ = \sup(0, \langle u, u' \rangle)$.
The space of \tilde{Q}^s -invariant functions is generated by p_+^s and p_-^s .

For $s = 0$ we will need the following lemma, which uses results of section 3.

Lemma 2.18

For $u \in \mathbb{S}^{d-1}$, we denote $\Delta_u^t = \{y \in \mathbb{P}^{d-1} ; |\langle u, y \rangle| < t\}$. Then, for any $\varepsilon, t > 0$, $x, y \in \mathbb{S}^{d-1}$:

$$\limsup_{n \rightarrow \infty} \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g) \leq \frac{2^\varepsilon}{t^\varepsilon} \tilde{\delta}^\varepsilon(x, y) + 2^\varepsilon \nu(\Delta_x^t).$$

In particular, for any $\varphi \in H_\varepsilon(\mathbb{S}^{d-1})$ the sequence $\tilde{P}^n \varphi$ is equicontinuous.

Proof

We write :

$$\int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g) = \int 1_{[1/t, \infty[}(\frac{|S_n|}{|S_n x|}) \tilde{\delta}^\varepsilon(S_n.x, S_n.y) d\mathbb{P}(\omega) + \int 1_{]0, 1/t]}(\frac{|S_n|}{|S_n x|}) \tilde{\delta}^\varepsilon(S_n.x, S_n.y) d\mathbb{P}(\omega).$$

Using Lemma 2.11 we have :

$$\tilde{\delta}^\varepsilon(S_n.x, S_n.y) \leq (2 \frac{|S_n|}{|S_n x|} \tilde{\delta}(x, y))^\varepsilon.$$

On the other hand, using Theorem 3.2, we know that : $\lim_{n \rightarrow \infty} \frac{|S_n|}{|S_n x|} = \frac{1}{|\langle z^*(\omega), x \rangle|}$

where $z^*(\omega) \in \mathbb{P}^{d-1}$ has law ν . Hence :

$$\limsup_{n \rightarrow \infty} \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g) \leq \frac{2^\varepsilon}{t^\varepsilon} \tilde{\delta}^\varepsilon(x, y) + 2^\varepsilon \mathbb{P}\{|\langle z^*(\omega), x \rangle| < t\} = \frac{2^\varepsilon}{t^\varepsilon} \tilde{\delta}^\varepsilon(x, y) + 2^\varepsilon \nu(\Delta_x^t).$$

We have $|\tilde{P}^n \varphi(x) - \tilde{P}^n \varphi(y)| \leq [\varphi]_\varepsilon \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g)$. From Theorem 3.2, we know that ν is proper, hence $\lim_{t \rightarrow 0} \nu(\Delta_x^t) = 0$. Then, for x fixed we use the above estimation of

$\limsup_{n \rightarrow \infty} \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g)$ to choose t sufficiently small in order to get the continuity of

$\limsup_{n \rightarrow \infty} \int \tilde{\delta}^\varepsilon(g.x, g.y) d\mu^n(g)$. This gives that $|\tilde{P}^n \varphi(x) - \tilde{P}^n \varphi(y)|$ depends continuously of y ,

hence the equicontinuity of the sequence $\tilde{P}^n \varphi$. \square

Proof of the theorem

As observed in remark *b* after Proposition 2.13, if $s > 0$ for any $\varphi \in C(\mathbb{S}^{d-1})$ the set of functions $\{(\tilde{Q}^s)^n \varphi ; n \in \mathbb{N}\}$ is equicontinuous. In view of Lemma 2.18, this is also valid if $s = 0$. Hence we can use here Lemma 2.12 and the results of [43]. This gives the first convergence. Since $\tilde{\pi}^s(x)$ is \tilde{Q}^s -stationary, its projection on \mathbb{P}^{d-1} is equal to the unique Q^s -stationary measure π^s , hence $\text{supp} \tilde{\pi}^s(x) \subset \tilde{\Lambda}([\text{supp} \mu])$. On the other hand $\text{supp} \tilde{\pi}^s(x)$ is $[\text{supp} \mu]$ -invariant, hence contains a $[\text{supp} \mu]$ -minimal set.

In case I, $\tilde{\Lambda}([\text{supp} \mu])$ is the unique minimal set, hence $\text{supp} \tilde{\pi}^s(x) = \tilde{\Lambda}([\text{supp} \mu])$. Furthermore, if $\varphi \in C(\mathbb{S}^{d-1})$ is \tilde{Q}^s -invariant, the sets :

$$M^- = \{x ; \varphi(x) = \text{Inf}\{\varphi(y); y \in \mathbb{P}^{d-1}\}\}, \quad M^+ = \{x ; \varphi(x) = \text{sup}\{\varphi(y); y \in \mathbb{P}^{d-1}\}\},$$

are closed and $[\text{supp}\mu]$ -invariant, hence contains minimal sets. Since $\tilde{\Lambda}([\text{supp}\mu])$ is the unique minimal set, $M^+ \cap M^- \supset \tilde{\Lambda}([\text{supp}\mu]) \neq \phi$, hence $\varphi = \text{cte}$.

Then, using Proposition 2.13 and Lemma 2.12, we get that there exists a unique stationary measure $\tilde{\pi}^s$. It follows that $\tilde{\pi}^s$ is symmetric with projection π^s on \mathbb{P}^{d-1} .

In case II, the restriction to $Co(\Lambda_+([\text{supp}\mu])) = \Phi$ of the projection on \mathbb{P}^{d-1} is a $[\text{supp}\mu]$ -equivariant homeomorphism. If we denote by i_+ its inverse, we get that $i_+(\pi^s)$ is the unique \tilde{Q}^s -stationary measure supported in Φ . Hence $i_+(\pi^s) = \pi_+^s$. Then π_+^s and π_-^s are extremal \tilde{Q}^s -stationary measures.

Since the projection of $\tilde{\pi}^s(x)$ on \mathbb{P}^{d-1} is π^s , we can write :

$$\tilde{\pi}^s(x) = \int (p_+^s(x, y)\delta_y + p_-^s(x, y)\delta_{-y})d\pi^s(y) = p_+^s(x)\pi_+^s + p_-^s(x)\pi_-^s$$

where $p_+^s(x) = p_+^s(x, \cdot)$ and $p_-^s(x) = p_-^s(x, \cdot)$ are Borel functions of $y \in \tilde{\Lambda}([\text{supp}\mu])$ such that $p_+^s(x) + p_-^s(x) = 1$. Then $p_+^s(x)\pi_+^s$ is the restriction of $\tilde{\pi}^s(x)$ to $\Lambda_+([\text{supp}\mu])$, hence is a \tilde{Q}^s -invariant measure. In view of the uniqueness of the stationary measure of \tilde{Q}^s restricted to $\Lambda_+([\text{supp}\mu])$, we get that $p_+^s(x)\pi_+^s$ is proportional to π_+^s , i.e $p_+^s(x)$ is independant of y π_+^s - a.e.

Hence, the first assertion of the theorem implies that the only extremal \tilde{Q}^s -stationary measures are π_+^s and π_-^s .

Also the operator defined by $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_0^{n-1} (\tilde{Q}^s)^k$ is the projection on the space of \tilde{Q}^s -invariant functions and is equal to $p_+^s(x)\pi_+^s + p_-^s(x)\pi_-^s$. The continuity and the extremality of the \tilde{Q}^s -invariant functions $p_+^s(x)$ and $p_-^s(x)$ follows.

If we restrict the convergence of $\frac{1}{n} \sum_0^{n-1} (\tilde{Q}^s)^k(\delta_x)$ to $x \in \Phi$, in view of the fact that the restriction to Φ of the projection on \mathbb{P}^{d-1} is a homeomorphism onto its image, we get : $p_+^s(x) = 1, p_-^s(x) = 0$ if $x \in \Phi$.

Let us denote by τ the entrance time of $S_n(\omega).x$ in $\Phi \cup -\Phi$ and by ${}^a\mathbb{E}_x^s$ the expectation symbol associated with the Markov chain $S_n(\omega).x$ defined by \tilde{Q}^s . Using theorem 2.6 we get ${}^a\mathbb{E}_x^s(1_{\Phi \cup -\Phi}(S_\tau.x)) = 1$. Since $p_+^s(x)$ is a \tilde{Q}^s -invariant function $p_+^s(S_n.x)$ is a martingale, hence $p_+^s(x) = {}^a\mathbb{E}_x^s(p_+^s(S_\tau.x))$. Since $p_+^s(x) = 1$ on Φ and $p_+^s(x) = 0$ on $-\Phi$ we get $p_+^s(x) = {}^a\mathbb{E}_x^s(1_\Phi(S_\tau.x))$, hence the stated interpretation for $p_+^s(x)$.

As in Lemma 2.10, we verify easily that the function $\varphi(u) = p(s) \int \langle u, u' \rangle_+^s d^* \nu_+^s(u')$ on \mathbb{S}^{d-1} satisfies $\tilde{P}^s \varphi = k(s)\varphi$, hence the function $\frac{\varphi}{e^s}$ satisfies $\tilde{Q}^s(\frac{\varphi}{e^s}) = \frac{\varphi}{e^s}$. By duality, cases II for μ and μ^* are the same, hence there are two minimal T^* -invariant subsets $\Lambda_+(T^*)$ and $\Lambda_-(T^*) = -\Lambda_+(T^*)$. On the other hand the set :

$$\hat{\Lambda}_+(T) = \{u \in \mathbb{S}^{d-1} ; \text{for any } u' \text{ in } \Lambda_+(T^*), \langle u, u' \rangle \geq 0\}$$

is non trivial, closed, T -invariant and has non zero interior, hence $\hat{\Lambda}_+(T)$ contains either $\Lambda_+(T)$ or $\Lambda_-(T)$ and has trivial intersection with one of them. We can assume $\hat{\Lambda}_+(T) \supset \Lambda_+(T)$. Then, for $u \in \Lambda_+(T)$ and any $u' \in \Lambda_+(T^*)$, we have $\langle u, u' \rangle_+ = \langle u, u' \rangle$, hence :

$$\varphi(u) = p(s) \int \langle u, u' \rangle |d^* \nu_+^s(u') = e^s(u),$$

i.e $\frac{\varphi}{e^s} = 1$ on $\Lambda_+(T)$. Also we have $\langle u, u' \rangle_+ = 0$ for $u \in \Lambda_-(T)$, $u' \in \Lambda_+(T^*)$. Since $\tilde{Q}(\frac{\varphi}{e^s}) = \frac{\varphi}{e^s} \geq 0$ and $\frac{\varphi}{e^s} = 1$ on $\Lambda_+(T)$, we conclude from above that $p_+^s = \frac{\varphi}{e^s}$, hence we get the last formula. \square

From above we know that if $s \geq 0$ and $\varphi \in C(\mathbb{S}^{d-1})$ the sequence $(\tilde{Q}^s)^n \varphi$ is equicontinuous. Lemma 2.12 reduces the discussion of the behaviour of $(\tilde{Q}^s)^n \varphi$ to the existence of eigenvalues z of \tilde{Q}^s with $|z| = 1$. In this direction we have the following.

Corollary 2.19

For $s \in I_\mu$, the equation $\tilde{Q}^s \varphi = e^{i\theta} \varphi$ with $e^{i\theta} \neq 1$, $\varphi \in C(\mathbb{S}^{d-1})$ has a non trivial solution only in case I. In that case $e^{i\theta} = -1$, φ is antisymmetric, satisfies on $supp\mu \times \tilde{\Lambda}(T)$ $\varphi(g.x) = -\varphi(x)$ and is uniquely defined up to a coefficient.

Proof

We observe that, since φ satisfies $\tilde{Q}^s \varphi = e^{i\theta} \varphi$, the function φ' defined by $\varphi'(x) = \varphi(-x)$ satisfies also $\tilde{Q}^s \varphi' = e^{i\theta} \varphi'$. Then $\varphi + \varphi'$ is symmetric and defines a function $\bar{\varphi}$ in $C(\mathbb{P}^{d-1})$ with $Q^s \bar{\varphi} = e^{i\theta} \bar{\varphi}$. If $e^{i\theta} \neq 1$, Theorem 2.6 gives $\bar{\varphi} = 0$, i.e φ is antisymmetric.

Furthermore, in case II, the restriction of φ to $\tilde{\Lambda}_+(T)$ satisfies the same equation and the projection of $\tilde{\Lambda}_+(T)$ on $\Lambda(T)$ is an equivariant homomorphism. Then Theorem 2.6 gives a contradiction. Hence if $\varphi \in C(\mathbb{S}^{d-1})$ satisfies $\tilde{Q}^s \varphi = e^{i\theta} \varphi$, then we are in case I. Also, passing to absolute values as in the proof of the theorem we get $\tilde{Q}^s |\varphi| = |\varphi|$, $|\varphi| = \text{cte}$. Furthermore by strict convexity we have on $supp\mu \times \tilde{\Lambda}(T)$, $\varphi(g.x) = e^{i\theta} \varphi(x)$, hence $\varphi^2(g.x) = e^{2i\theta} \varphi^2(x)$. Since φ^2 is symmetric and satisfies $\tilde{Q}^s \varphi^2 = e^{2i\theta} \varphi^2$, we get $e^{2i\theta} = 1$, i.e $e^{i\theta} = -1$; in particular $\varphi(g.x) = -\varphi(x)$ on $supp\mu \times \tilde{\Lambda}(T)$. If $\varphi' \in C(\mathbb{S}^{d-1})$ satisfies also $\tilde{Q}^s \varphi' = -\varphi'$, we get from above $\tilde{Q}^s \frac{\varphi'}{\varphi} = \frac{\varphi'}{\varphi}$, $\frac{\varphi'}{\varphi}(-x) = \frac{\varphi'}{\varphi}(x)$, hence φ' is proportional to φ . \square

III Laws of large numbers and spectral gaps.

1) Notations

As in section 2, we assume that condition i-p is valid for $[supp\mu]$. If $d = 1$ we assume that μ is non arithmetic. For $s \in I_\mu$ we consider the functions q^s and q_n^s ($n > 0$) on $\mathbb{P}^{d-1} \times G$, defined by :

$$q^s(x, g) = \frac{1}{k(s)} \frac{e^s(g.x)}{e^s(x)} |gx|^s = q_1^s(x, g), \quad q_n^s(x, g) = \frac{1}{k^n(s)} \frac{e^s(g.x)}{e^s(x)} |gx|^s,$$

hence by definition of e^s : $\int q_n^s(x, g) d\mu^n(g) = 1$.

We denote by \mathbb{P}^{d-1} the flag manifold of planes and by $\mathbb{P}_{1,2}^{d-1}$ the manifold of contact elements on \mathbb{P}^{d-1} . Such a plane is defined up to normalisation by a 2-vector $x \wedge y \in \wedge^2 V$ and we can assume $|x \wedge y| = 1$. Also a contact element ξ is defined by its origin $x \in \mathbb{P}^{d-1}$ and a line through x . Hence we can write $\xi = (x, x \wedge y)$ where $|x| = |x \wedge y| = 1$. The following additive cocycles of the actions of G on \mathbb{P}^{d-1} , \mathbb{P}_2^{d-1} , $\mathbb{P}_{1,2}^{d-1}$ will play an essential role :

$$\sigma_1(g, x) = \text{Log}|gx|, \quad \sigma_2(g, x \wedge y) = \text{Log}|g(x \wedge y)|, \quad \sigma(g, \xi) = \text{Log}|g(x \wedge y)| - 2\text{Log}|gx|.$$

In addition to the norm of g we will need to use the quantity $\gamma(g) = \sup(|g|, |g^{-1}|) \geq 1$. Clearly, for any $x \in V$, with $|x| = 1$, we have $-\text{Log} \gamma(g) \leq \text{Log}|gx| \leq \text{Log} \gamma(g)$.

For a finite sequence $\omega = (g_1, g_2, \dots, g_n)$ we write :

$$S_n(\omega) = g_n \cdots g_1 \in G, \quad q_n^s(x, \omega) = \prod_{k=1}^n q^s(S_{k-1}.x, g_k).$$

We denote by Ω_n the space of finite sequences $\omega = (g_1, g_2, \dots, g_n)$ and we write $\Omega = G^{\mathbb{N}}$.

We observe that $\theta^s(x, g) = |gx|^s \frac{e^s(g.x)}{e^s(x)}$ satisfies the cocycle relation $\theta^s(x, gg') = \theta^s(g, g'.x)\theta^s(g', x)$, hence $q_n^s(x, \omega) = \frac{1}{k^n(s)}\theta^s(x, S_n(\omega))$.

Definition 3.1

We denote $\mathbb{Q}_x^s \in M^1(\Omega)$ the limit of the projective system of probability measures $q_k^s(x, \cdot)\mu^{\otimes k}$ on Ω_k . We write $\mathbb{Q}^s = \int \mathbb{Q}_x^s d\pi^s(x)$ where π^s is the unique Q^s -stationary measure on \mathbb{P}^{d-1} . The corresponding expectation symbol will be written \mathbb{E}_x^s and the shift on Ω will be denoted by θ . We write also $\mathbb{E}^s(\varphi) = \int \mathbb{E}_x^s(\varphi) d\pi^s(x)$.

The path space of the Markov chain defined by Q^s is a factor space of ${}^a\Omega = \mathbb{P}^{d-1} \times \Omega$, and the corresponding shift on ${}^a\Omega$ will be written ${}^a\theta : {}^a\theta(x, \omega) = (g_1(\omega).x, \theta\omega)$. Hence $(\mathbb{P}^{d-1} \times \Omega, {}^a\theta)$ is a skew product over (Ω, θ) . The projection on Ω of the Markov measure ${}^a\mathbb{Q}_x^s = \delta_x \otimes \mathbb{Q}_x^s$ is \mathbb{Q}_x^s , hence ${}^a\mathbb{Q}^s = \int \delta_x \otimes \mathbb{Q}_x^s d\pi^s(x)$ projects on \mathbb{Q}^s . The uniqueness of the Q^s -stationary measure π^s implies the ergodicity of the ${}^a\theta$ -invariant measure ${}^a\mathbb{Q}^s$, hence \mathbb{Q}^s is also θ -invariant and ergodic.

If $s = 0$, the random variables $g_k(\omega)$ are i.i.d with law μ and $\mathbb{Q}^0 = \mathbb{P} = \mu^{\otimes \mathbb{N}}$. Here, under condition (i-p), we extend the results of [28] to the case $s \geq 0$, in particular we construct a suitable measure-valued martingale with contraction properties as in [17]. This will allow us to prove strong forms of the law of large numbers for $S_n(\omega)$ and to compare the measures \mathbb{Q}_x^s when x varies. Then we can deduce the simplicity of the dominant Lyapunov exponent of $S_n(\omega)$ under the θ -invariant probability \mathbb{Q}^s for $s \geq 0$. Spectral gap properties for twisted convolution operators on the projective space and on the unit sphere will follow.

2) A martingale and the equivalence of \mathbb{Q}_x^s to \mathbb{Q}^s

When convenient we identify $x \in \mathbb{P}^{d-1}$ with one of its representants in \mathbb{S}^{d-1} . We recall that the Markov kernel ${}^*Q^s$ is defined by ${}^*Q^s\varphi = \frac{1}{k(s){}^*e^s}{}^*P^s(\varphi{}^*e^s)$ where ${}^*P^s\varphi(x) = \int \varphi(g.x)|gx|^s d\mu^*(g)$, ${}^*P^s({}^*e^s) = k(s)({}^*e^s)$ and ${}^*Q^s$ has a unique stationary measure ${}^*\pi^s$. Furthermore we have ${}^*\pi^s = {}^*e^s{}^*\nu^s$ where ${}^*\nu^s \in M^1(\mathbb{P}^{d-1})$ is the unique solution of ${}^*P^s({}^*\nu^s) = k(s)({}^*\nu^s)$. We denote by m the unique rotation invariant probability measure on \mathbb{P}^{d-1} .

Theorem 3.2

Let $\Omega' \subset \Omega$ be the (shift-invariant) Borel subset of elements $\omega \in \Omega$ such that $S_n^*(\omega).m$ converges to a Dirac measure $\delta_{z^*}(\omega)$. Then $g_1^*.z^*(\theta\omega) = z^*(\omega)$, $\mathbb{Q}^s(\Omega') = 1$, the law of $z^*(\omega)$ under \mathbb{Q}^s is ${}^*\pi^s$ and ${}^*\pi^s$ is proper.

In particular if $\omega \in \Omega'$ and $|\langle x, z^*(\omega) \rangle \langle y, z^*(\omega) \rangle| \neq 0$, then :

$$\lim_{n \rightarrow \infty} \delta(S_n(\omega).x, S_n(\omega).y) = 0.$$

If $\omega \in \Omega'$ and $\xi = (x, x \wedge y) \in \mathbb{P}_{1,2}^{d-1}$

$$\lim_{n \rightarrow \infty} \frac{|S_n(\omega)x|}{|S_n(\omega)|} = |\langle z^*(\omega), x \rangle|, \quad \lim_{n \rightarrow \infty} S_n^*.m = \delta_{z^*(\omega)}.$$

In particular, if $\langle z^*(\omega), x \rangle \neq 0$ then $\lim_{n \rightarrow \infty} \sigma(S_n, \xi) = -\infty$

Also for any $x \in \mathbb{P}^{d-1}$ \mathbb{Q}_x^s is equivalent to \mathbb{Q}^s and $\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}_y^s}(\omega) = \left| \frac{\langle z^*(\omega), x \rangle}{\langle z^*(\omega), y \rangle} \right|^s \frac{e^s(y)}{e^s(x)}$.

The proof of Theorem 3.2 is based on the following lemmas, in particular on the study of a measure-valued martingale.

Lemma 3.3

Assume $z^* \in \mathbb{P}^{d-1}$ and $u_n \in G$ is a sequence such that : $\lim_{n \rightarrow \infty} u_n^*.m = \delta_{z^*}$. Then, for any $x, y \in \mathbb{P}^{d-1}$ with $|\langle z^*, x \rangle \langle z^*, y \rangle| \neq 0$, $\lim_{n \rightarrow \infty} \delta(u_n.x, u_n.y) = 0$. If $\xi = (x, x \wedge y) \in \mathbb{P}_{1,2}^{d-1}$ and $\langle z^*, x \rangle \neq 0$:

$$\lim_{n \rightarrow \infty} \frac{|u_n x|}{|u_n|} = |\langle z^*, x \rangle|, \quad \lim_{n \rightarrow \infty} \sigma(u_n, \xi) = -\infty.$$

These convergences are uniform on the compact subsets on which $\langle z^*, x \rangle$ do not vanish.

Proof

We denote by $e_i (1 \leq i \leq d)$ an orthonormal basis of V , by \bar{e}_i the projection of e_i in \mathbb{P}^{d-1} , by \bar{A}^+ the set of diagonal matrices $a = \text{diag}(a^1, a^2, \dots, a^d)$ and $a^1 \geq a^2 \geq \dots \geq a^d > 0$. We write $u_n = k_n a_n k_n'$ with $a_n \in \bar{A}^+$, $k_n, k_n' \in O(d)$. Then, for $x \in \mathbb{P}^{d-1}$:

$$|u_n x|^2 = |a_n k_n' x|^2 = \sum_1^d (a_n^i)^2 |\langle k_n' x, e_i \rangle|^2.$$

Also, $u_n^*.m = (k_n')^{-1} a_n .m$ converges to $z^* \in \mathbb{P}^{d-1}$, which implies :

$$\lim_{n \rightarrow \infty} a_n .m = \delta_{\bar{e}_1}, \quad \lim_{n \rightarrow \infty} (k_n')^{-1} .\bar{e}_1 = z^*.$$

In particular, if $i > 1$, we have $a_n^i = o(a_n^1)$ and

$$\lim_{n \rightarrow \infty} |\langle k_n' x, e_1 \rangle| = |\langle z^*, x \rangle| \neq 0.$$

It follows that $|u_n x| \sim a_n^1 |\langle z^*, x \rangle|$. Since $|u_n| = a_n^1$, we get : $\lim_{n \rightarrow \infty} \frac{|u_n x|}{|u_n|} = |\langle z^*, x \rangle|$, as asserted. We get also, if $|\langle y, z^* \rangle| \neq 0$, $|u_n y| \sim a_n^1 |\langle z^*, y \rangle|$.

On the exterior product space $\wedge^2 V$ there exists an $O(d)$ -invariant scalar product such that on a decomposable 2-vector $x \wedge y$:

$$|x \wedge y|^2 = |x|^2|y|^2 - |\langle x, y \rangle|^2.$$

For $x, y \in \mathbb{P}^{d-1}$ and corresponding $\tilde{x}, \tilde{y} \in \mathbb{S}^{d-1}$ we write $|x \wedge y| = |\tilde{x} \wedge \tilde{y}|$. Then on \mathbb{P}^{d-1} , there is an associated distance δ_1 given by $\delta_1(x, y) = |x \wedge y|$ and we have $\frac{1}{2}\delta \leq \delta_1 \leq \delta$. We observe that $\delta_1(u_n \cdot x, u_n \cdot y) = \frac{|u_n x \wedge u_n y|}{|u_n x| |u_n y|}$.

Also $|u_n x \wedge u_n y|^2 = \sum_{i>j} (a_n^i a_n^j)^2 |\langle k_n(\tilde{x} \wedge \tilde{y}), e_i \wedge e_j \rangle|^2 \leq \frac{d(d-1)}{2} (a_n^1 a_n^2 |\tilde{x} \wedge \tilde{y}|)^2$.

It follows

$$\delta_1(u_n \cdot x, u_n \cdot y) \leq \left(\frac{d(d-1)}{2} \right)^{1/2} \frac{a_n^1 a_n^2}{|u_n x| |u_n y|} |\tilde{x} \wedge \tilde{y}|.$$

Since $|u_n x| \sim a_n^1 |\langle z^*, x \rangle|$, $|u_n y| \sim a_n^1 |\langle z^*, y \rangle|$ and $a_n^2 = o(a_n^1) < z^*, x \rangle < z^*, y \rangle \neq 0$, we get :

$$\lim_{n \rightarrow \infty} \delta_1(u_n \cdot x, u_n \cdot y) = 0.$$

It follows, for any $x, y \in \mathbb{P}^{d-1}$: $\lim_{n \rightarrow \infty} \delta(u_n \cdot x, u_n \cdot y) = 0$.

Also, since $a_n^2 = o(a_n^1)$, and $\langle z^*, x \rangle \neq 0$ we get $\lim_{n \rightarrow \infty} \sigma(u_n, \xi) = -\infty$.

The above calculations imply the uniformities in the convergences. \square

Lemma 3.4

Assume $\nu_n \in M^1(\mathbb{P}^{d-1})$ is a sequence such that ν_n is relatively compact in variation, and each ν_n is proper. Let $u_n \in G$ be a sequence such that $u_n^* \cdot \nu_n$ converges weakly to δ_{z^*} ($z^* \in \mathbb{P}^{d-1}$). Then for any proper $\rho \in M^1(\mathbb{P}^{d-1})$, $u_n^* \cdot \rho$ converges weakly to δ_{z^*} .

Proof

We can assume, in variation, $\lim_{n \rightarrow \infty} \nu_n = \nu_0$ where ν_0 is proper. Also we can assume going to subsequences that u_n^* converges to a quasi-projective map of the form u^* , defined and continuous outside a projective subspace $H \subset \mathbb{P}^{d-1}$. Let $\varphi \in C(\mathbb{P}^{d-1})$ and denote :

$$I_n = (u_n^* \cdot \nu_n)(\varphi) - (u^* \cdot \nu_0)(\varphi) = (\nu_n - \nu_0)(\varphi \circ u_n^*) - \nu_0(\varphi \circ u_n^* - \varphi \circ u^*)$$

The first term is bounded by $|\varphi| \|\nu_n - \nu_0\|$, hence it converges to zero. Since $\nu_0(H) = 0$ and $\varphi \circ u_n^*$ converges to $\varphi \circ u^*$ outside H , we can use dominated convergence for the second term : $\lim_{n \rightarrow \infty} \nu_0(\varphi \circ u_n^* - \varphi \circ u^*) = 0$, hence $\lim_{n \rightarrow \infty} I_n = 0$. Then $u_n^* \cdot \nu_n$ converges to $u^* \cdot \nu_0$ weakly. In particular $u^* \cdot \nu_0 = \delta_{z^*}$, hence $u^* \cdot y = z^*$ $\nu_0 - a.e.$ Since $\nu_0(H) = 0$, we have $u^* \cdot y = z^*$ on $\mathbb{P}^{d-1} \setminus H$.

Since ρ is proper $u^* \cdot \rho = \delta_{z^*}$, hence $\lim_{n \rightarrow \infty} u_n^* \cdot \rho = \delta_{z^*}$. \square

Lemma 3.5

For $x, y \in \mathbb{P}^{d-1}$ the total variation measure of $\mathbb{Q}_x^s - \mathbb{Q}_y^s$ is bounded by $B\delta^{\bar{s}}(x, y)\mathbb{Q}^s$. Then exists $c(s) > 0$ such that, for any $x \in \mathbb{P}^{d-1}$, $\mathbb{Q}_x^s \leq c(s) \mathbb{Q}^s$

Proof

We write $q_n^s(g) = \int q_n^s(x, g)d\pi^s(x)$ and we observe that for any measurable φ depending on the first n coordinates :

$$\int \varphi(\omega)d\mathbb{Q}^s(\omega) = \int q_n^s(S_n(\omega))\varphi(\omega)d\mu^{\otimes n}(\omega).$$

Also : $|(\mathbb{Q}_x^s - \mathbb{Q}_y^s)(\varphi)| \leq \int |q_n^s(x, S_n) - q_n^s(y, S_n)| |\varphi(\omega)|d\mu^{\otimes n}$. Using Lemma 2.11 we have for any $g \in G$:

$$|q_n^s(x, g) - q_n^s(y, g)| \leq b_s \frac{|g|^s}{k^n(s)} \delta^{\bar{s}}(x, y).$$

Using Theorem 2.6 for some $b > 0$:

$$q_n^s(g) \geq b \frac{|g|^s}{k^n(s)}, \text{ hence } |q_n^s(x, g) - q_n^s(y, g)| \leq \frac{b_s}{b} q_n^s(g) \delta^{\bar{s}}(x, y).$$

It follows :

$$|(\mathbb{Q}_x^s - \mathbb{Q}_y^s)(\varphi)| \leq \frac{b_s}{b} \delta^{\bar{s}}(x, y) \int |\varphi(\omega)|d\mathbb{Q}^s(\omega)$$

hence the first conclusion with $B = \frac{b_s}{b}$.

Integrating with respect to π we get, since $\delta(x, y) \leq \sqrt{2}$: $\mathbb{Q}_x^s \leq (1 + B(\sqrt{2})^{\bar{s}})\mathbb{Q}^s$ hence the second formula with $c(s) = 1 + B(\sqrt{2})^{\bar{s}}$. \square

Lemma 3.6

We consider the positive kernel ν_x^s from \mathbb{P}^{d-1} to \mathbb{P}^{d-1} given by $\nu_x^s = \frac{|\langle x, \cdot \rangle|^s}{e^s(x)} * \nu^s$. Then:

$\int g^* \cdot \nu_{g,x}^s q^s(x, g)d\mu(g) = \nu_x^s$, $\nu_x^s(1) = \frac{1}{e^s(x)} \int |\langle x, y \rangle|^s d^* \nu^s(y) = p(s) \in]0, 1]$ and $x \rightarrow \nu_x^s$ is continuous in variation.

In particular $S_n^* \cdot \nu_{S_n \cdot x}^s$ is a bounded martingale with respect to \mathbb{Q}_x^s and the natural filtration.

Proof

We consider the s -homogeneous measure λ^s on \dot{V} defined by $\lambda^s = * \nu^s \otimes \ell^s$. By definition of $* \nu^s$, $\int g^* \lambda^s d\mu(g) = k(s) \lambda^s$. Then the Radon measure λ_v^s defined by $\lambda_v^s = |\langle v, \cdot \rangle|^s \lambda^s$ satisfies $\int g^* \lambda_{gv}^s d\mu(g) = k(s) \lambda_v^s$. This can be written, by definition of ν_x^s and $q^s(x, g)$:

$$\int g^* \cdot \nu_{g,x}^s q^s(x, g)d\mu(g) = \nu_x^s.$$

The martingale property of $S_n^* \cdot \nu_{S_n \cdot x}^s$ follows.

Furthermore since $p(s)e^s(x)$ is equal to $\int |\langle x, y \rangle|^s d^* \nu^s(y)$, Lemma 2.10 gives :

$$\nu_x^s(1) = \frac{1}{e^s(x)} \int |\langle x, y \rangle|^s d^* \nu^s(y) = p(s) \in]0, 1].$$

The continuity in variation of $x \rightarrow \nu_x^s$ follows from the definition.

Lemma 3.7

Let $\rho \in M^1(\mathbb{P}^{d-1})$, \mathcal{H} be the set of projective subspaces H of minimal dimension such that $\rho(H) > 0$. Then the subset of elements $H \in \mathcal{H}$ such that $\rho(H) = \sup\{\rho(L) ; L \in \mathcal{H}\}$ is finite and non void. Furthermore, there exists $\varepsilon_\rho > 0$ such that for any $H \in \mathcal{H}$:

$$\rho(H) = c_\rho \text{ or } \rho(H) \leq c_\rho - \varepsilon_\rho,$$

where $c_\rho = \sup\{\rho(L) ; L \in \mathcal{H}\}$.

Proof

If $H, H' \in \mathcal{H}$, $H \neq H'$, then $\dim H \cap H' < \dim H$, hence $\rho(H \cap H') = 0$. Then, for any $\beta > 0$, the cardinality of the set of elements $H \in \mathcal{H}$ with $\rho(H) \geq \beta$ is bounded by $\frac{1}{\beta}$. The first assertion follows. Assume the second assertion is false. Then there exists a sequence $H_n \in \mathcal{H}$ with $\frac{c_\rho}{2} < \rho(H_n) < c_\rho$, $\lim_{n \rightarrow \infty} \rho(H_n) = c_\rho$, and $\rho(H_n) \neq \rho(H_m)$ if $n \neq m$. This contradicts the fact that the cardinality of the sequence H_n is at most $\frac{2}{c_\rho}$.

Lemma 3.8

Assume that the Markovian kernel $x \rightarrow \nu_x \in M^1(\mathbb{P}^{d-1})$ is continuous in variation and satisfies :

$$\nu_x = \int q^s(x, g) g^* \cdot \nu_{g \cdot x} d\mu(g)$$

Let $\mathcal{H}_{p,r}$ the set of finite unions of r distinct subspaces of dimension p and let h be the function $h(x) = \sup\{\nu_x(W) ; W \in \mathcal{H}_{p,r}\}$. Then h is continuous and the set

$$X = \{x ; h(x) = \sup h(y), y \in \mathbb{P}^{d-1}\}$$

is closed and $[supp\mu]$ -invariant

Proof

If $W \in \mathcal{H}_{p,r}$ is fixed the function $x \rightarrow \nu_x(W)$ is continuous since $|\nu_x(W) - \nu_y(W)| \leq \|\nu_x - \nu_y\|$. This implies $|h(x) - h(y)| \leq \|\nu_x - \nu_y\|$, hence the continuity of h .

We have for any $W \in \mathcal{H}_{p,r}$:

$$\nu_x(W) = \int q^s(x, g) \nu_{g \cdot x}((g^*)^{-1}W) d\mu(g).$$

Hence : $h(x) \leq \int q^s(x, g) h(g \cdot x) d\mu(g)$. Then, as in Lemma 2.9, X is $[supp\mu]$ -invariant and closed.

Lemma 3.9

Let ν_x be as in Lemma 3.8. Then for any $x \in \mathbb{P}^{d-1}$, ν_x is proper.

Proof

We write $\pi_x = \frac{\nu_x}{\nu_x(1)}$, denote by \mathcal{H}_k be the set of projective subspaces of dimension k and : $\mathcal{H} = \bigcup_{k \geq 0} \mathcal{H}_k$, $d(x) = \text{Inf}\{\dim H ; H \in \mathcal{H}, \pi_x(H) > 0\}$,

$$m(x) = \sup\{\pi_x(H) ; H \in \mathcal{H}, \dim H = d(x)\}, \mathcal{W}(x) = \{H \in \mathcal{H} ; \pi_x(H) = m(x)\}.$$

Lemma 3.7 implies that the set $\mathcal{W}(x)$ has finite cardinality $n(x) > 0$. Also we denote $p = \text{Inf}\{d(x); x \in \mathbb{P}^{d-1}\}$, $h_p(x) = \text{sup}\{\pi_x(H); H \in \mathcal{H}_p\}$.

Lemma 3.8 shows that $h_p(x)$ reaches its maximum β on a closed $[\text{supp}\mu]$ -invariant subset $X \subset \mathbb{P}^{d-1}$. Hence on $\Lambda([\text{supp}\mu])$ we have $h_p(x) = \beta = m(x)$. It follows $d(x) = p$ on $\Lambda([\text{supp}\mu])$. The relation $n(x)m(x) \leq 1$ implies $n(x) \leq \frac{1}{\beta}$ on $\Lambda([\text{supp}\mu])$.

Let $r = \text{sup}\{n(x); x \in \Lambda([\text{supp}\mu])\}$ and denote $h_{p,r}(x) = \text{sup}\{\pi_x(W); W \in \mathcal{H}_{p,r}\}$. Then Lemma 3.8 implies : $h_{p,r}(x) = r\beta$ on $\Lambda([\text{supp}\mu])$. Since $m(x) = \beta$, this relation implies $n(x) = r$ on $\Lambda([\text{supp}\mu])$. Let $W(x) = \cup\{H; H \in \mathcal{W}(x)\}$ and let us show the local constancy of the function $W(x)$. Using Lemma 3.7 we get,

$$\beta(x) = \text{sup}\{\pi_x(H); H \in \mathcal{H}_p, H \notin \mathcal{W}(x)\} < \beta.$$

Let $x \in \Lambda([\text{supp}\mu])$, $U_x = \{y; \|\pi_y - \pi_x\| < \beta - \beta(x)\}$ and $H_y \in \mathcal{H}_p$ with $\pi_y(H_y) = \beta$. Then:

$$\beta - \pi_x(H_y) = \pi_y(H_y) - \pi_x(H_y) \leq \|\pi_y - \pi_x\| < \beta - \beta(x).$$

Hence $\pi_x(H_y) > \beta(x)$ and, using Lemma 3.8, we get $H_y \in \mathcal{W}(x)$ for any $y \in U_x$. Since π_x is continuous in variation, U_x is a neighbourhood of x , hence $W(x)$ is locally constant. Since $\Lambda([\text{supp}\mu])$ is compact, $W = \bigcup_{x \in \Lambda([\text{supp}\mu])} W(x)$ is a finite union of subspaces.

On the other hand, the relations :

$$r\beta = \pi_x(W(x)) = \int q^s(x, g)g^*.\pi_{g.x}(W(x))d\mu(g) \quad , \quad r\beta \geq (g^*.\pi_{g.x})(W(x))$$

imply that, for any $x \in \Lambda([\text{supp}\mu])$, $r\beta = g^*.\pi_{g.x}(W(x)) \mu - a.e.$

By definition of $W(g.x)$, we get : $(g^*)^{-1}W(x) = W(g.x) \mu - a.e.$

Hence, for any $g \in \text{supp}\mu$: $(g^*)^{-1}(W(x)) = W(g.x)$.

The relation $(g^*)^{-1}(W) = \bigcup_{x \in \Lambda([\text{supp}\mu])} (g^*)^{-1}(W(x)) = \bigcup_{x \in \Lambda([\text{supp}\mu])} W(g.x)$ shows that W is $[\text{supp}\mu^*]$ -

invariant. Then condition i-p implies $W = \mathbb{P}^{d-1}$, $r = 1, p = d - 1, d(x) = d - 1, m(x) = 1$ for any $x \in \mathbb{P}^{d-1}$, hence the Lemma. \square

Proof of the theorem

We use the Markov kernel π_x^s defined by $\pi_x^s = \frac{\nu_x^s}{\nu_x^s(1)}$ with ν_x^s given in Lemma 3.6.

Then we have the harmonicity equation :

$$\pi_x^s = \int q^s(x, g)g^*.\pi_{g.x}^s d\mu(g)$$

and the continuity in variation of π_x^s . The above equation implies that the sequence $S_n^*(\omega).\pi_{S_n(\omega).x}^s$ is a \mathbb{Q}_x^s -martingale with respect to the natural filtration on Ω . Since $\pi_x^s \in M^1(\mathbb{P}^{d-1})$ we can apply the martingale convergence theorem.

Since, by Lemma 3.9, π_x^s is proper and, by definition, $x \rightarrow \pi_x^s$ is continuous in variation, we can use the same method as in ([3], [6], [28]) : because of i-p condition, the martingale

$S_n^*(\omega) \cdot \pi_{S_n(\omega).x}^s$ converges $\mathbb{Q}_x^s - a.e$ to a Dirac measure. Then, using Lemma 3.4, $S_n^*.m$ converges $\mathbb{Q}_x^s - a.e$ to a Dirac measure $\delta_{z^*(\omega)}$. Then, from above, for any $x \in \mathbb{P}^{d-1}$:

$$\mathbb{Q}_x^s(\Omega') = 1 \quad , \quad z^*(\mathbb{Q}_x^s) = \pi_x^s.$$

Hence the law of $z^*(\omega)$ under \mathbb{Q}_x^s is π_x^s . It follows, by integration :

$$\mathbb{Q}^s(\Omega') = 1 \quad , \quad z^*(\mathbb{Q}^s) = \int z^*(\mathbb{Q}_x^s) d\pi^s(x) = \int \pi_x^s d\pi^s(x).$$

In view of the formulas for ν_x^s , π_x^s and the relation $\pi^s = \frac{e^s \nu^s}{\nu^s(e^s)}$, we get $z^*(\mathbb{Q}^s) = {}^*\pi^s$. Then Lemma 3.9 and the definition of π_x^s give that ${}^*\pi^s$ is proper. The relations :

$\lim_{n \rightarrow \infty} \delta(S_n(\omega).x, S_n(\omega).y) = 0$, $\lim_{n \rightarrow \infty} \frac{|S_n(\omega)x|}{|S_n(\omega)|} = |\langle z^*(\omega), x \rangle|$, $\lim_{n \rightarrow \infty} \sigma(S_n, \xi) = -\infty$, follow from the geometrical Lemma 3.3, since $S_n^*.m$ converges to $\delta_{z^*(\omega)}$.

Using Lemma 3.5 we know that \mathbb{Q}_x^s is absolutely continuous with respect to \mathbb{Q}^s . We calculate $\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega)$ as follows.

By definition of \mathbb{Q}_x^s and \mathbb{Q}^s :

$$\mathbb{E}_x^s \left(\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega) | g_1, \dots, g_n \right) = \frac{q_n^s(x, S_n(\omega))}{\int q_n^s(y, S_n(\omega)) d\pi^s(y)}$$

Furthermore :

$$\frac{q_n^s(x, S_n(\omega))}{q_n^s(y, S_n(\omega))} = \frac{|S_n(\omega)x|^s}{|S_n(\omega)y|^s} \frac{e^s(S_n(\omega).x)}{e^s(S_n(\omega).y)} \frac{e^s(y)}{e^s(x)}.$$

The martingale convergence theorem gives :

$$\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega) = \lim_{n \rightarrow \infty} \frac{q_n^s(x, S_n(\omega))}{\int q_n^s(y, S_n(\omega)) d\pi^s(y)}.$$

Using the relation $\lim_{n \rightarrow \infty} \frac{|S_n(\omega)x|}{|S_n(\omega)|} = |\langle z(\omega), x \rangle|$, if $\omega \in \Omega'$, we get :

$$\lim_{n \rightarrow \infty} \frac{q_n^s(x, S_n(\omega))}{q_n^s(y, S_n(\omega))} = \left| \frac{\langle z^*(\omega), x \rangle}{\langle z^*(\omega), y \rangle} \right|^s \frac{e^s(y)}{e^s(x)}.$$

Hence $\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega) = \frac{|\langle z^*(\omega), x \rangle|^s}{e^s(x)} \left[\int \frac{|\langle z^*(\omega), y \rangle|^s}{e^s(y)} d\pi^s(y) \right]^{-1}$. Since, from above π^s is proper and the \mathbb{Q}^s -law of $z^*(\omega)$ is π^s , we have for any $x \in \mathbb{P}^{d-1}$: $|\langle z^*(\omega), x \rangle| > 0$, $\mathbb{Q}^s - a.e$.

Hence $\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}^s}(\omega) > 0$ $\mathbb{Q}^s - a.e$, i.e \mathbb{Q}_x^s is equivalent to \mathbb{Q}^s ; Also, using the formulas above :

$$\frac{d\mathbb{Q}_x^s}{d\mathbb{Q}_y^s}(\omega) = \left| \frac{\langle z^*(\omega), x \rangle}{\langle z^*(\omega), y \rangle} \right|^s \frac{e^s(y)}{e^s(x)} \quad \square$$

3) The law of large numbers for $\text{Log}|S_n(\omega)x|$ with respect to \mathbb{Q}^s

Here, by derivatives of a function φ at the boundaries of an interval $[a, b]$ we will mean finite half derivatives *i.e* we write :

$$\varphi'(a) = \varphi'(a_+) \in \mathbb{R} \quad , \quad \varphi'(b) = \varphi'(b_-) \in \mathbb{R}$$

Theorem 3.10

Let $\mu \in M^1(G)$, $s \in I_\mu$. Assume $[\text{supp}\mu]$ satisfies i-p, and $\text{Log}\gamma(g)$ is μ -integrable. Assume also that $|g|^s \text{Log}\gamma(g)$ is μ -integrable and write

$$L_\mu(s) = \int \text{Log}|gx|q^s(x, g)d\pi^s(x) d\mu(g).$$

Then, for any $x \in \mathbb{P}^{d-1}$, we have $\mathbb{Q}^s - a.e$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)x| = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)| = L_\mu(s).$$

This convergence is valid in $\mathbb{L}^1(\mathbb{Q}^s)$ and in $\mathbb{L}^1(\mathbb{Q}_x^s)$ for any $x \in \mathbb{P}^{d-1}$. Furthermore $k(t)$ has a continuous derivative on $[0, s]$ and if $t \in [0, s]$, $x \in \mathbb{P}^{d-1}$:

$$L_\mu(t) = \frac{k'(t)}{k(t)} = \lim_{n \rightarrow \infty} \frac{1}{nk^n(t)} \int |gx|^t \text{Log}|gx|d\mu^n(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \frac{\int |g|^t \text{Log}|g|d\mu^n(g)}{\int |g|^t d\mu^n(g)}.$$

In particular if $\alpha > 0$ satisfies $k(\alpha) = 1$, then $k'(\alpha) > 0$.

Proof

We consider the function $f(x, \omega)$ on ${}^a\Omega$ defined by $f(x, \omega) = \text{Log}|g_1(\omega)x|$. If $|x| = 1$, we have $-\text{Log}|g^{-1}| \leq \text{Log}|gx| \leq \text{Log}|g|$, hence $f(x, \omega)$ is ${}^a\mathbb{Q}^s$ -integrable.

Moreover $\int f(x, \omega)d^a\mathbb{Q}(x, \omega) = \int q^s(x, g) \text{Log}|gx|d\nu^s(x)d\mu(g) = L_\mu(s)$, and

$$\sum_0^{n-1} (f \circ {}^a\theta^k)(x, \omega) = \text{Log}|S_n(\omega)x|.$$

As mentioned above ${}^a\mathbb{Q}^s$ is ${}^a\theta$ -ergodic, hence we get using Birkhoff's theorem :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)x| = L_\mu(s), \quad {}^a\mathbb{Q}^s - a.e.$$

On the other hand, we can apply the subadditive ergodic theorem to the sequence $\text{Log}|S_n(\omega)|$ and to the ergodic system $(\Omega, \theta, \mathbb{Q}^s)$.

This gives that there exists $L(s) \in \mathbb{R}$ such that, $\mathbb{Q}^s - a.e$ and in $\mathbb{L}^1(\mathbb{Q}^s)$, the sequence $\frac{1}{n} \text{Log}|S_n(\omega)|$ converges to $L(s)$. We know, using Theorem 3.2 that, for fixed x and $\mathbb{Q}^s - a.e$,

$$\lim_{n \rightarrow \infty} \frac{|S_n(\omega)x|}{|S_n(\omega)|} = | \langle z^*(\omega), x \rangle |,$$

and furthermore the law of $z^*(\omega)$ under \mathbb{Q}^s is proper. Hence, for fixed x we have :

$$| \langle z^*(\omega), x \rangle | > 0, \quad \mathbb{Q}^s - a.e.$$

Then for fixed $x \in \mathbb{P}^{d-1}$ and $\mathbb{Q}^s - a.e$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)x| = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n(\omega)| = L_\mu(s).$$

Using Lemma 3.6, since $\mathbb{Q}_x^s \leq c(s) \mathbb{Q}^s$ this convergence is also valid $\mathbb{Q}_x^s - a.e.$ Hence by definition of ${}^a\mathbb{Q}^s$, we have $L(s) = L_\mu(s)$. The first assertion follows.

In order to get the \mathbb{L}^1 -convergences, we observe that Fatou's Lemma gives :

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)x| d\mathbb{Q}^s(\omega) \geq L_\mu(s).$$

On the other hand, the subadditive ergodic theorem gives :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)| d\mathbb{Q}^s(\omega) = L(s) = L_\mu(s).$$

Since $|S_n(\omega)x| \leq |S_n(\omega)|$ if $|x| = 1$, these two relations imply, for every $x \in \mathbb{P}^{d-1}$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)x| d\mathbb{Q}^s(\omega) = L_\mu(s).$$

Now we write :

$$\frac{1}{n} |\text{Log}|S_n(\omega)x| - L(s)| \leq \frac{1}{n} (\text{Log}|S_n(\omega)| - \text{Log}|S_n(\omega)x|) + \frac{1}{n} |\text{Log}|S_n(\omega)| - L(s)|.$$

From the above calculation, the integral of the first term converge to zero. The subadditive ergodic theorem implies the same for the second term.

Hence $\lim_{n \rightarrow +\infty} \int |\text{Log}|S_n(\omega)x| - L(s)| d\mathbb{Q}^s(\omega) = 0$. Since $\mathbb{Q}_x^s \leq c(s)\mathbb{Q}^s$, this convergence is also valid in $\mathbb{L}^1(\mathbb{Q}_x^s)$ for any fixed x . This gives the second assertion, in particular :

$$L_\mu(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)x| d\mathbb{Q}_x^s(\omega) = \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}_x^s(\text{Log}|S_n(\omega)x|).$$

The above limit can be expressed as follows.

Let φ be a continuous function on \mathbb{P}^{d-1} . Then Theorem 2.6 implies :

$$\lim_{n \rightarrow \infty} \mathbb{E}_x^s(\varphi(S_n(\omega).x)) = \lim_{n \rightarrow \infty} (Q^s)^n \varphi(x) = \pi^s(\varphi),$$

uniformly in $x \in \mathbb{P}^{d-1}$.

Hence $L_\mu(s)\pi_s(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x^s(\varphi(S_n(\omega).x) \text{Log}|S_n(\omega)x|)$.

In particular with $\varphi = \frac{1}{e^s}$ and any x :

$$L_\mu(s) = \lim_{n \rightarrow \infty} \frac{1}{nk^n(s)} \int |gx|^s \text{Log}|gx| d\mu^n(g).$$

We denote $v_n(s) = \int |gx|^s d\mu^n(g)$ and we observe that $v'_n(s) = \int |gx|^s \text{Log}|gx| d\mu^n(g)$.

Using Theorem 2.6, we get : $\lim_{n \rightarrow \infty} \frac{v_n(s)}{k^n(s)} = \pi^s \left(\frac{1}{e^s} \right) = 1$.

Then the above formula for $L_\mu(s)$ reduces to $L_\mu(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \frac{v'_n(s)}{v_n(s)}$.

On the other-hand : $\frac{1}{n} \text{Log} v_n(s) = \frac{1}{n} \int_0^s \frac{v'_n(t)}{v_n(t)} dt$, $\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} v_n(s) = \text{Log} k(s)$.

The convexity of $\text{Log} v_n$ on $[0, s]$ gives :

$$\frac{v'_n(0)}{v_n(0)} \leq \frac{v'_n(t)}{v_n(t)} \leq \frac{v'_n(s)}{v_n(s)}.$$

Then the convergence of the sequences $\frac{1}{n} \frac{v'_n(0)}{v_n(0)}$ and $\frac{1}{n} \frac{v'_n(s)}{v_n(s)}$ implies that the sequence $\frac{1}{n} \frac{v'_n(t)}{v_n(t)}$ is uniformly bounded on $[0, s]$. On the other hand, Hölder inequality implies the μ -integrability of $|g|^t \text{Log}|g|$ if $t \in [0, s]$, and the bound :

$$\int |g|^t |\text{Log}|g|| d\mu(g) \leq \left(\int |\text{Log}|g|| d\mu(g) \right)^{\frac{s-t}{s}} \left(\int |g|^s |\text{Log}|g|| d\mu(g) \right)^{t/s}.$$

Hence, as above, we have the convergence of $\frac{1}{n} \frac{v'_n(t)}{v_n(t)}$, to $L(t)$. Then dominated convergence gives the convergence of $\frac{1}{n} \text{Log} v_n(s)$ to $\text{Log} k(s) = \int_0^s L(t) dt$.

We have $L(t) = \int \text{Log}|gx| q^t(x, g) d\mu(g) d\pi^t(x)$, and the continuity of q^s, π^s given by Theorem 2.6. Then the bound of $\int |g|^t |\text{Log}|g|| d\mu(g)$ given above imply the continuity of $L(t)$ on $[0, s]$. The integral expression of $\text{Log} k(s)$ in terms of $L(t)$ implies that k has a derivative and $\frac{k'(t)}{k(t)} = L(t)$ if $t \in [0, s]$. This gives the first part of the last relation in the theorem. In order to get the rest, we consider $u_n(t) = \int |g|^t d\mu^n(g)$ and write for $t \in [0, s]$:

$$\frac{u'_n(t)}{u_n(t)} = \frac{\int |g|^t \text{Log}|g| d\mu^n(g)}{\int |g|^t d\mu^n(g)}$$

The convergence of $\frac{1}{n} \text{Log} u_n(t)$ to $\text{Log} k(t)$ and the convexity of the functions $\text{Log} u_n(t)$, $\text{Log} k(t)$ give for $t \in [0, s]$:

$$\frac{k'(t_-)}{k(t)} \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \frac{u'_n(t)}{u_n(t)} \leq \limsup_{n \rightarrow \infty} \frac{u'_n(t)}{u_n(t)} \leq \frac{k'(t_+)}{k(t)}.$$

Since if $t \in]0, s[$, we have $k'(t_-) = k'(t_+) = k'(t)$, we get $\lim_{n \rightarrow \infty} \frac{1}{n} \frac{u'_n(t)}{u_n(t)} = \frac{k'(t)}{k(t)}$ if $t < s$.

Furthermore, by continuity we have $\lim_{n \rightarrow \infty} \frac{1}{n} \frac{u'_n(s)}{u_n(s)} = \frac{k'(s_-)}{k(s)}$. Now the rest of the formula follows from the expression of $\frac{u'_n(t)}{u_n(t)}$ given above. The relation $L_\mu(\alpha) > 0$ follows from the formula $L_\mu(t) = \frac{k'(t)}{k(t)}$ and the strict convexity of $\text{Log} k(t)$. \square

4) Lyapunov exponents and spectral gaps.

We begin with a more general situation than above. As special cases, it contains the Markov chains on \mathbb{P}^{d-1} considered in section 2. In particular simplicity of the dominant Lyapunov exponent, given by Theorem 3.17 below, will be a simple consequence of their special properties and of the general Proposition 3.11 below. For $s = 0$, this result was

shown in [28] under condition i-p. For the use of the Zariski closure as a tool to show condition i-p see [22], [41]. We give corresponding notations.

Let X be a compact metric space, $\mathcal{C}(X, X)$ the semigroup of continuous maps of X into itself which is endowed with by the topology of uniform convergence. We denote by $g.x$ the action of $g \in \mathcal{C}(X, X)$ on $x \in X$ and we consider a closed subsemigroup Σ of $\mathcal{C}(X, X)$. Let μ be a probability measure on Σ and let $q(x, g)$ be a continuous non negative function on $X \times \text{supp}\mu$ such that $\int q(x, g)d\mu(g) = 1$. We will denote by $(X, q \otimes \mu, \Sigma)$ this set of datas and we will say that $(X, q \otimes \mu, \Sigma)$ is a Markov system on (X, Σ) . We write $\Omega = \Sigma^{\mathbb{N}}$, we denote by Ω_n the set of finite sequences of length n on Σ and for $\omega = (g_1, g_2, \dots, g_n)$ in $\Omega_n, x \in X$, we write $q_n(x, \omega) = \prod_1^n q(S_{k-1}.x, g_k)$ where $S_n = g_n \cdots g_1, S_0 = Id$.

We define a probability measure \mathbb{Q}_x^n on Ω_n by $\mathbb{Q}_x^n = q_n(x, \cdot)\mu^{\otimes n}$ and we denote by \mathbb{Q}_x the probability measure on Ω which is the projective limit of this system. If ν is a probability measure on X we set $\mathbb{Q}_\nu = \int \mathbb{Q}_x d\nu(x)$. We will consider the extended shift ${}^a\theta$ on ${}^a\Omega = X \times \Omega$ which is defined by : ${}^a\theta(x, \omega) = (g_1.x, \theta\omega)$, and also the Markov chain on X with kernel Q defined by $Q\varphi(x) = \int \varphi(g.x)q(x, g)d\mu(g)$. Clearly, when endowed with the corresponding shift, the spaces of paths of this Markov chain is a factor system of $(X \times \Omega, {}^a\theta)$. If π is a Q -stationary measure on X , the measure \mathbb{Q}_π on Ω is shift-invariant and ${}^a\mathbb{Q}_\pi = \int \delta_x \otimes \mathbb{Q}_x d\pi(x)$ is ${}^a\theta$ -invariant. In this situation we will say that $(X, q \otimes \mu, \Sigma, \pi)$ is a stationary Markov system. If π is Q -ergodic, then ${}^a\mathbb{Q}_\pi$ is ${}^a\theta$ -ergodic and \mathbb{Q}_π is θ -ergodic. We will denote by $\mathbb{E}_x, \mathbb{E}_\pi$ the corresponding expectations symbols.

In particular we will consider below Markov systems of the form $(X, q \otimes \mu, T)$ where $\Sigma = T \subset GL(d, \mathbb{R})$ ($d > 1$), and also flag-extensions of them. We can extend the action of $g \in T$ to $X \times \mathbb{P}^{d-1}$ by $g(x, v) = (g.x, g.v)$ and define a new Markov chain with kernel Q_1 by $Q_1\varphi(x, v) = \int \varphi(g.x, g.v)q(x, g)d\mu(g)$. Given a Q -stationary probability measure π , we will denote by C_1 the compact convex set of probability measures on $X \times \mathbb{P}^{d-1}$ which have projection π on X . The same considerations apply if \mathbb{P}^{d-1} is replaced by \mathbb{P}_2^{d-1} , the manifold of 2-planes or $\mathbb{P}_{1,2}^{d-1}$, the manifold of contact elements in \mathbb{P}^{d-1} . Then we define similarly the kernels $Q_2, Q_{1,2}$ and the convex sets $C_2, C_{1,2}$.

Since $S_n = g_n \cdots g_1$ and the g_k are \mathbb{Q}_π -stationary random variables where \mathbb{Q}_π is θ -invariant and ergodic, the Lyapunov exponents of the product S_n are well defined as soon as :

$$\int \text{Log}|g_1(\omega)|d\mathbb{Q}_\pi(\omega) \text{ and } \int \text{Log}|g_1^{-1}(\omega)|d\mathbb{Q}_\pi(\omega)$$

are finite (see [42]). In particular the two largest ones γ_1 and γ_2 are given by :

$$\gamma_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|S_n(\omega)|d\mathbb{Q}_\pi(\omega), \quad \gamma_1 + \gamma_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|\wedge^2 S_n(\omega)|d\mathbb{Q}_\pi(\omega)$$

In order to study the values of γ_1, γ_2 we need to consider the above Markov operators $Q_1, Q_2, Q_{1,2}$ and the convex sets $C_1, C_2, C_{1,2}$ of corresponding stationary measures. We denote $q(g) = |q(\cdot, g)|$ and we assume $\int \text{Log}q(g)d\mu(g) < \infty$. For $\eta_1 \in C_1$, we will write $I_1(\eta_1) = \int \sigma_1(g, v)d\eta_1(x, v)d\mu(g)$. Similarly with $\eta_2 \in C_2, \eta_{1,2} \in C_{1,2}$ we define $I_2(\eta_2), I_{1,2}(\eta_{1,2})$. The following result will be a basic tool in this subsection.

Proposition 3.11

With the above notations, let T be a closed subsemigroup of $GL(d, \mathbb{R})$, $(X, q \otimes \mu, T, \pi)$ a stationary and uniquely ergodic Markov system. Assume that $S_n^*.m$ converges $\mathbb{Q}_\pi - a.e$ to a Dirac measure $\delta_{z^*(\omega)}$ such that for any $v \in \mathbb{P}^{d-1}$, $\langle v, z^*(\omega) \rangle \neq 0$ $\mathbb{Q}_\pi - a.e$. Assume that $\int \text{Log} \gamma(g) q(g) d\mu(g)$ is finite. Then we have $\gamma_2 - \gamma_1 = \sup\{I_{1,2}(\eta); \eta \in C_{1,2}\} < 0$, and the sequence $\frac{1}{n} \sup_{x,v,v'} \mathbb{E}_x(\text{Log} \frac{\delta(S_n(\omega).v, S_n(\omega).v')}{\delta(v, v')})$ converges to $\gamma_2 - \gamma_1 < 0$.

The proof uses the same arguments as in [3] and [6], [29]. The condition $\lim_{n \rightarrow \infty} S_n^*.m = \delta_{z^*(\omega)}$ is satisfied in the examples of subsection 3.2 (see Theorem 3.2).

Lemma 3.12

Let m_p be the natural $SO(d)$ -invariant probability measure on the submanifold of p -decomposable unit multivectors $x = v_1 \wedge v_2 \wedge \dots \wedge v_p$. Then there exists $c > 0$ such that for any $u \in \text{End}V : 0 < \text{Log} |\wedge^p u| - \int \text{Log} |(\wedge^p u)(x)| dm_p(x) \leq c$.

Proof

We write u in polar form $u = kak'$ with $k, k' \in SO(d)$, $a = \text{diag}(a_1, a_2, \dots, a_d)$ and $a_1 \geq a_2 \geq \dots \geq a_d > 0$. We write also $x = k''\varepsilon^p$ with $k'' \in SO(d)$, $\varepsilon^p = e_1 \wedge e_2 \wedge \dots \wedge e_p$, hence :

$$\int \text{Log} |\wedge^p ux| dm_p(x) = \int \text{Log} |\wedge^p ak'k''\varepsilon^p| d\tilde{m}(k'') = \int \text{Log} |\wedge^p ak\varepsilon^p| d\tilde{m}(k),$$

where \tilde{m} is the normalized Haar measure on $SO(d)$. Furthermore :

$$\begin{aligned} |\wedge^p ak\varepsilon^p| &\geq |\langle \wedge^p ak\varepsilon^p, \varepsilon^p \rangle| = |\wedge^p a| \langle k\varepsilon^p, \varepsilon^p \rangle|, \\ \int \text{Log} |\wedge^p ux| dm_p(x) &\geq \text{Log} |\wedge^p u| + \int \text{Log} \langle k\varepsilon^p, \varepsilon^p \rangle |d\tilde{m}(k)| \\ &= \text{Log} |\wedge^p u| + \int \text{Log} \langle x, \varepsilon^p \rangle |dm_p(x)|. \end{aligned}$$

Hence it suffices to show the finiteness of the integral in the right hand side. But the set of unit decomposable p -vectors is an algebraic submanifold of the unit sphere of $\wedge^p V$ and m_p is its natural Riemannian measure. Since the map $x \rightarrow \langle x, \varepsilon^p \rangle^2$ is polynomial, there exists $q \in \mathbb{N}$, $c > 0$ such that : $ct^q \leq m_p\{x; \langle x, \varepsilon^p \rangle^2 \leq t\} \leq 1$.

The push forward σ of m_p by this map is an absolutely continuous probability measure on $[0,1]$ which satisfies $\sigma(0, t) \geq ct^{q/2}$. Then :

$$\int \text{Log} \langle x, \varepsilon^p \rangle |dm_p(x) = \int_0^1 \text{Log} t d\sigma(t) > -\infty,$$

since the integral $\int_0^1 t^{q/2-1} dt$ is finite for $q > 0$. \square

We recall from [35] the following :

Lemma 3.13

Let (E, θ, ν) be a dynamical system where ν is a θ -invariant probability measure, f a ν -integrable function. If $\sum_1^n f \circ \theta^k$ converges $\nu - a.e$ to $-\infty$, then one has $\nu(f) < 0$.

Lemma 3.14

Let E be a compact metric space, P a Markov kernel on E which preserves the space of continuous functions on E , $C(P)$ the compact convex set of P -stationary measures. Then, for every continuous function f on E , the sequence $\sup_{x \in E} \frac{1}{n} \sum_{k=0}^{n-1} P^k f(x)$ converges to $\sup\{\eta(f); \eta \in C(P)\}$. In particular, if for all $\eta, \eta' \in C(P)$ we have $\eta(f) = \eta'(f)$, then we have the uniform convergence : $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k f(x) = \eta(f)$.

Proof

Let $J \subset \mathbb{R}$ be the set of cluster values of the sequences $\frac{1}{n} \sum_{k=0}^{n-1} (P^k f)(x_n)$ with $x_n \in E$. We will show that the convex envelope of J is equal to $\{\eta(f) ; \eta \in C(P)\}$. If the sequence $\frac{1}{n_k} \sum_{k=0}^{n_k-1} (P^k f)(x_{n_k})$ converges to $c \in \mathbb{R}$, we can consider the sequence of probability measures $\eta_k = \frac{1}{n_k} \sum_{k=0}^{n_k-1} P^k \delta_{x_{n_k}}$ and extract a convergent subsequence with limit $\eta \in C(P)$. Then, since f is continuous :

$$\eta(f) = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{k=0}^{n_k-1} (P^k f)(x_{n_k}) = c.$$

Conversely, if $\eta \in C(P)$ is ergodic, Birkhoff's theorem applied to the sequence $\frac{1}{n} \sum_{k=0}^{n-1} (P^k f)(x)$ gives $\eta - a.e$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (P^k f)(x) = \eta(f),$$

hence there exists $x \in E$ such that $\eta(f)$ is the limit of $\frac{1}{n} \sum_{k=0}^{n-1} (P^k f)(x)$. If η is not ergodic, η is a barycenter of ergodic measures, hence $\eta(f)$ belongs to the convex envelope of J . In view of the above, this shows the claim. Since J is closed, the convex envelope of J is a closed interval $[a, b]$, hence $b = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in E} \sum_{k=0}^{n-1} (P^k f)(x) = \sup_{\eta \in C(P)} \eta(f)$. \square

Lemma 3.15

We have the formulas

$$\gamma_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x,v} \int \text{Log}|S_n(\omega)v| d\mathbb{Q}_x(\omega) = \sup_{\eta \in C_1} I_1(\eta),$$

$$\gamma_1 + \gamma_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x,v,v'} \int \text{Log}|S_n(\omega)v \wedge S_n(\omega)v'| d\mathbb{Q}_x(\omega) = \sup_{\eta \in C_2} I_2(\eta).$$

Proof

We consider the Markov chain on $X \times \mathbb{P}^{d-1}$ with kernel Q_1 defined by :

$$Q_1 \varphi(x, v) = \int \varphi(g.x, g.v) q(x, g) d\mu(g),$$

and the function $\psi(x, v) = \int \sigma_1(g, v) q(x, g) d\mu(g)$.

We observe that : $\int \sigma_1(S_n(\omega), v) d\mathbb{Q}_x(\omega) = \sum_{k=0}^{n-1} Q_1^k \psi(x, v)$,

and ψ is continuous since $\int \text{Log} \gamma(g) q(g) d\mu(g) < \infty$. Also, since π is the unique Q -stationary measure, any Q_1 -stationary measure has projection π on X . Then, using Lemma 3.14 :

$$\sup_{\eta \in C_1} I_1(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x,v} \int \sigma_1(S_n(\omega), v) d\mathbb{Q}_x(\omega),$$

which gives the second part of the first formula. In order to show the first part we consider $\eta \in C_1$ which is Q_1 -ergodic, the extended shift $\tilde{\theta}$ on $X \times \mathbb{P}^{d-1} \times \Omega$ and the function $f(x, v, \omega) = \sigma_1(g_1(\omega), v)$. Then :

$$\tilde{\theta}(x, v, \omega) = (g_1.x, g_1.v, \theta\omega) \text{ and } \sigma_1(S_n(\omega), v) = \sum_0^{n-1} f \circ \tilde{\theta}^k(x, v, \omega).$$

Also, $f(x, v, \omega)$ is $\widehat{\mathbb{Q}}_\eta$ -integrable where $\widehat{\mathbb{Q}}_\eta = \int \delta_{(x,v)} \otimes \mathbb{Q}_x d\eta(x, v)$.

Using the subadditive ergodic theorem :

$$I_1(\eta) = \frac{1}{n} \int \sigma_1(S_n(\omega), v) d\mathbb{Q}_x(\omega) d\eta(x, v) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log} |S_n(\omega)| d\mathbb{Q}_\pi(\omega) = \gamma_1.$$

We show now that for some $\eta \in C_1$ we have $\gamma_1 = I_1(\eta)$.

Using Lemma 3.12, we know that :

$$0 \leq \text{Log} |S_n(\omega)| - \int \text{Log} |S_n(\omega)v| dm(v) \leq c,$$

hence, integrating with respect to \mathbb{Q}_π :

$$0 \leq \int dm(v) \int (\text{Log} |S_n(\omega)| - \text{Log} |S_n(\omega)v|) d\mathbb{Q}_\pi(\omega) \leq c.$$

Then the sequence of non negative functions $h_n(v)$ on \mathbb{P}^{d-1} given by :

$$h_n(v) = \frac{1}{n} \int (\text{Log} |S_n(\omega)| - \text{Log} |S_n(\omega)v|) d\mathbb{Q}_\pi(\omega)$$

satisfies $0 \leq h_n(v) \leq \frac{c}{n}$ with c given by Lemma 3.11, $\lim_{n \rightarrow \infty} \int h_n(v) dm(v) = 0$. It follows that we can find a subsequence h_{n_j} such that $h_{n_j}(v)$ converges to zero $m - a.e$, hence :

$$\gamma_1 = \lim_{j \rightarrow \infty} \frac{1}{n_j} \int \sigma_1(S_{n_j}(\omega), v) d\mathbb{Q}_\pi(\omega) \quad m - a.e,$$

in particular the convergence is valid for some $v \in \mathbb{P}^{d-1}$. The sequence of probability measures $\frac{1}{n_j} \sum_1^{n_j} Q_1^k(\pi \otimes \delta_v)$ has a weakly convergent subsequence η_j with a Q_1 -invariant limit η . Furthermore, the function ψ considered above is continuous, hence :

$$\eta(\psi) = \lim_{j \rightarrow \infty} \frac{1}{n_j} \int \sigma_1(S_{n_j}(\omega)v) d\mathbb{Q}_\pi(\omega) = \gamma_1, \quad \gamma_1 = I_1(\eta) = \sup_{\eta_1 \in C_1} I_1(\eta_1).$$

The same argument is valid for $\text{Log} |S_n(\omega)(v \wedge v')|$ with m replaced by m_2 , hence the second formula. \square

Lemma 3.16

For any $\eta \in C_1$, we have $\gamma_1 = I_1(\eta)$

Proof

As in the proof of Lemma 3.15 we have $I_1(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \sigma_1(S_n(\omega), v) \quad \mathbb{Q}_\eta - a.e$, hence the

existence of $v \in \mathbb{P}^{d-1}$ such that $\mathbb{Q}_\pi - a.e$:

$$I_1(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} |S_n(\omega)v|$$

Then, using Theorem 3.2 and Lemma 3.15 we get, $\mathbb{Q}_\pi - a.e$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} \frac{|S_n(\omega)v|}{|S_n(\omega)|} = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} | \langle z^*(\omega), v \rangle | = 0,$$

since $\langle z^*(\omega), v \rangle \neq 0$, $\mathbb{Q}_\pi - a.e$.

Hence $I_1(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log} |S_n(\omega)| d\mathbb{Q}_\pi(\omega) = \gamma_1 \quad \square$

Proof of the Proposition

We have $\gamma_2 - \gamma_1 = (\gamma_1 + \gamma_2) - 2\gamma_1$, $\gamma_1 = I_1(\eta_1)$ for any $\eta_1 \in C_1$ and $\gamma_1 + \gamma_2 = \sup_{\eta_2 \in C_2} I_2(\eta_2)$.

Using the theorem of Markov-Kakutani, for the inverse image of $\eta_2 \in C_2$ in $C_{1,2}$ we know that any $\eta_2 \in C_2$ is the projection of some $\eta_{1,2} \in C_{1,2}$, hence $\gamma_1 + \gamma_2 = \sup_{\eta_{1,2} \in C_{1,2}} I_2(\eta_{1,2})$. If

η'_1 is the projection of $\eta_{1,2}$ on \mathbb{P}^{d-1} , we have $I_{1,2}(\eta_{1,2}) = I_2(\eta_{1,2}) - 2I_1(\eta'_1)$ and from Lemma 3.16, $\gamma_1 = I_1(\eta'_1)$. It follows $\gamma_2 - \gamma_1 = \sup_{\eta_{1,2} \in C_{1,2}} I_{1,2}(\eta_{1,2})$.

Since $I_{1,2}(\eta_{1,2})$ depends continuously of $\eta_{1,2}$ and $C_{1,2}$ is compact, in order to show that $\gamma_2 - \gamma_1$ is negative it suffices to prove $I_{1,2}(\eta) < 0$, for any $\eta \in C_{1,2}$. We consider the extended shift $\tilde{\theta}$ on $X \times \mathbb{P}_{1,2}^{d-1} \times \Omega$ defined by $\tilde{\theta}(x, \xi, \omega) = (g_1.x, g_1.\xi, \theta\omega)$, the function $f(\xi, \omega) = \sigma(g_1, \xi)$ and the $\tilde{\theta}$ -invariant measure $\tilde{\mathbb{Q}}_\eta = \int \delta_{(x, \xi)} \otimes \mathbb{Q}_x d\eta(x, \xi)$. Since $S_n^*.m$ converges $\mathbb{Q}_\pi - a.e$ to $\delta_{z^*(\omega)}$, Lemma 3.4 implies $\lim_{n \rightarrow \infty} \sigma(S_n(\omega), \xi) = -\infty$, $\mathbb{Q}_\pi - a.e$ if the origin v of ξ satisfies $\langle v, z^*(\omega) \rangle \neq 0$. By hypothesis, this condition is satisfied for any ξ and $\mathbb{Q}_\pi - a.e$, hence we have $\lim_{n \rightarrow \infty} \sum_1^n f \circ \theta^k = -\infty$ $\mathbb{Q}_\pi - a.e$ for any ξ . It follows that this convergence is valid $\tilde{\mathbb{Q}}_\eta - a.e$, hence Lemma 3.13 gives : $\eta(f) = I_{1,2}(\eta) < 0$.

We consider $I_n = \frac{1}{n} \mathbb{E}_x \left(\text{Log} \frac{\delta(S_n(\omega).v, S_n(\omega).v')}{\delta(v, v')} \right)$. With $|v| = |v'| = 1$, $\delta_1(v, v') = |v \wedge v'|$:

$$\text{Log} \frac{\delta_1(S_n(\omega).v, S_n(\omega).v')}{\delta_1(v, v')} = \text{Log} \frac{|S_n(\omega)v \wedge S_n(\omega)v'|}{|v \wedge v'|} - \text{Log} |S_n(\omega)v| - \text{Log} |S_n(\omega)v'|.$$

By Lemma 3.15, we have :

$$\gamma_1 + \gamma_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x, v, v'} \mathbb{E}_x \left(\text{Log} \frac{|(S_n(\omega)v \wedge S_n(\omega)v')|}{|v \wedge v'|} \right).$$

Also, by Lemmas 3.14 and 3.16, we have the convergence of $\sup_{x, v} \frac{1}{n} \mathbb{E}_x(\text{Log} |S_n(\omega)v|)$ and

$\inf_{x,v} \frac{1}{n} \mathbb{E}_x(\text{Log}|S_n(\omega)v|)$ to $I_1(\eta) = \gamma_1$. The uniform convergence of $\frac{1}{n} \mathbb{E}_x(\text{Log}|S_n(\omega)v|)$ to γ_1

follows. Then the equivalence of δ_1, δ implies that $\sup_{x,v,v'} \frac{1}{n} I_n$ converges to $\gamma_2 - \gamma_1$. \square

With the notations of paragraph 3 above we have the following corollaries, for products of random matrices.

Theorem 3.17

Assume $d > 1$, the closed subsemigroup $T \subset GL(d, \mathbb{R})$ satisfies condition i-p, $s \in I_\mu$ and $\int |g|^s \text{Log} \gamma(g) d\mu(g)$ is finite. Then the dominant Lyapunov exponent of $S_n(\omega)$ is simple and :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x,v,v'} \mathbb{E}_x^s \left(\text{Log} \frac{\delta(S_n(\omega) \cdot v, S_n(\omega) \cdot v')}{\delta(v, v')} \right) = L_{\mu,2}(s) - L_{\mu,1}(s) < 0$$

where $L_{\mu,1}(s), L_{\mu,2}(s)$ are the two highest Lyapunov exponents of $S_n(\omega)$ with respect to \mathbb{Q}^s .

In particular : $\lim_{n \rightarrow \infty} \frac{1}{n} \sup_{v,v'} \mathbb{E}^s \left(\text{Log} \frac{\delta(S_n(\omega)v, S_n(\omega)v')}{\delta(v, v')} \right) \leq L_{\mu,2}(s) - L_{\mu,1}(s) < 0$.

Proof

In view of Theorems 3.2, 2.6, the conditions of Proposition 3.11 are satisfied by :

$X = \mathbb{P}^{d-1}, q \otimes \mu = q^s \otimes \mu, \pi = \pi^s$. On the other hand we have $\mathbb{Q}^s = \int \mathbb{Q}_x^s d\pi(x)$, hence the second formula. \square

We will use Theorem 3.17 to establish certain functional inequalities for the operators Q^s, \tilde{Q}^s on $\mathbb{P}^{d-1}, \mathbb{S}^{d-1}$ defined in section 2 and acting on $H_\varepsilon(\mathbb{P}^{d-1})$ or $H_\varepsilon(\mathbb{S}^{d-1})$. Using [32], spectral gap properties will follow. We will say that Q satisfies a "Doeblin-Fortet inequality" on $H_\varepsilon(X)$, where X is a compact metric space if we have for any $\varphi \in H_\varepsilon(X)$: $[Q^{n_0} \varphi]_\varepsilon \leq \rho[\varphi]_\varepsilon + D|\varphi|$ for some $n_0 \in \mathbb{N}$ where $\rho < 1, D \geq 0$.

Corollary 3.18

For ε sufficiently small and $0 \leq s < s_\infty$, if $\int |g|^s \gamma^\delta(g) d\mu(g) < \infty$ for some $\delta > 0$,

$$\lim_{n \rightarrow \infty} \left(\sup_{x,y} \mathbb{E}^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)} \right) \right)^{1/n} = \rho(\varepsilon) < 1.$$

If $k'(s) > 0$ then $\lim_{n \rightarrow \infty} \left(\sup_x \mathbb{E}^s \left(\frac{1}{|S_n x|^\varepsilon} \right) \right)^{1/n} < 1$.

Proof

The proof of the first formula is based on the theorem and is given below. The proof of the second formula follows from a similar argument (see also [38], Theorem 1, for $s = 0$).

We denote $\alpha_n(x, y, \omega) = \text{Log} \frac{\delta(S_n(\omega) \cdot x, S_n(\omega) \cdot y)}{\delta(x, y)}$ and we observe that :

$$e^{\varepsilon \alpha_n} \leq 1 + \varepsilon \alpha_n + \varepsilon^2 \alpha_n^2 e^{\varepsilon |\alpha_n|}, \quad |\alpha_n| \leq 2 \text{Log} \gamma(S_n)$$

Since $t^2 e^{|t|} \leq e^{3|t|}$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$:

$$\alpha_n^2 e^{\varepsilon|\alpha_n|} \leq \frac{1}{\varepsilon_0^2} e^{3\varepsilon_0|\alpha_n|} \leq \frac{1}{\varepsilon_0^2} (\gamma(S_n))^{\varepsilon_0}.$$

We observe that $I_n = \frac{1}{\varepsilon_0^2} \mathbb{E}^s(\gamma^{6\varepsilon_0}(S_n))$ is finite for $s < s_\infty$ and ε_0 sufficiently small (see the proof of Corollary 3.20. below).

It follows :

$$\begin{aligned} \mathbb{E}^s(e^{\varepsilon\alpha_n(x,y,\omega)}) &\leq 1 + \varepsilon \mathbb{E}^s(\alpha_n(x,y,\omega)) + \varepsilon^2 I_n, \\ \sup_{x,y} \mathbb{E}^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x,y)} \right) &\leq 1 + \varepsilon \sup_{x,y} \mathbb{E}^s \left(\text{Log} \frac{\delta(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x,y)} \right) + \varepsilon^2 I_n. \end{aligned}$$

Also the quantity $\rho_n(\varepsilon) = \sup_{x,y} \mathbb{E}^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x,y)} \right)$ satisfies $\rho_{m+n}(\varepsilon) \leq \rho_m(\varepsilon)\rho_n(\varepsilon)$, hence we have $\rho(\varepsilon) = \lim_{n \rightarrow \infty} \rho_n(\varepsilon)^{1/n} = \text{Inf}_{n \in \mathbb{N}} (\rho_n(\varepsilon))^{1/n}$. It follows that, in order to show $\rho(\varepsilon) < 1$, for ε small it suffices to show $\rho_{n_0}(\varepsilon) < 1$ for some n_0 . To do that, we choose n_0 such that $\sup_{x,y} \mathbb{E}^s \left(\text{Log} \frac{\delta(S_{n_0} \cdot x, S_{n_0} \cdot y)}{\delta(x,y)} \right) = c < 0$ which is possible using the theorem, and we take ε sufficiently small so that $\varepsilon^2 I_{n_0} + \varepsilon c < 0$. Then we get :

$$\rho_{n_0}(\varepsilon) \leq 1 + \varepsilon^2 I_{n_0} + \varepsilon c < 1. \quad \square$$

Corollary 3.19

Let $H_\varepsilon(\mathbb{P}^{d-1})$ be the Banach space of ε -Hölder functions on \mathbb{P}^{d-1} with the norm $\varphi \rightarrow [\varphi]_\varepsilon + |\varphi|$ and assume $0 \leq s < s_\infty$, $\int |g|^s \gamma^\delta(g) d\mu(g)$ for some $\delta > 0$. Then for ε sufficiently small the operator Q^s (defined in Theorem 2.6) on $H_\varepsilon(\mathbb{P}^{d-1})$ satisfies the following Doeblin-Fortet inequality where $B \geq 0$ $n_0 \in \mathbb{N}$, $\rho^{n_0}(\varepsilon) \leq \rho'(\varepsilon) < 1$: $[(Q^s)^{n_0} \varphi]_\varepsilon \leq \rho'(\varepsilon)[\varphi]_\varepsilon + B|\varphi|$. In particular the operator P^s admits the following spectral decomposition in $H_\varepsilon(\mathbb{P}^{d-1})$:

$$P^s = k(s)(\nu^s \otimes e^s + U_s)$$

where U_s commutes with the projection $\nu^s \otimes e^s$ and has spectral radius less than 1.

Proof

From Lemma 3.6, we know that $\mathbb{Q}_x^s \leq c(s)\mathbb{Q}^s$, hence, using Corollary 3.18, for $n \geq n_0$ sufficiently large and with $\rho'(\varepsilon) \in]\rho^{n_0}(\varepsilon), 1[$: $\sup_{x,y} \mathbb{E}_x^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x,y)} \right) \leq \rho'(\varepsilon)$.

We can write :

$$\begin{aligned} (Q^s)^n \varphi(x) - (Q^s)^n \varphi(y) &= \mathbb{E}_x^s(\varphi(S_n \cdot x)) - \mathbb{E}_y^s(\varphi(S_n \cdot y)) = \mathbb{E}_x^s(\varphi(S_n \cdot x) - \varphi(S_n \cdot y)) + \\ &(\mathbb{E}_x^s - \mathbb{E}_y^s)(\varphi(S_n \cdot y)). \end{aligned}$$

The first term in the right hand side is bounded by $[\varphi]_\varepsilon \delta^\varepsilon(x,y) \sup_{x,y} \mathbb{E}_x^s \left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x,y)} \right)$ i.e by $[\varphi]_\varepsilon \delta^\varepsilon(x,y) \rho'(\varepsilon)$.

Using Lemma 3.5, we know that the second term is bounded by $B|\varphi| \delta^{\bar{s}}(x,y)$. Hence with $\varepsilon < \bar{s}$ we get the required inequality.

From Theorem 2.6 we know that Q^s has a unique stationary measure π^s and 1 is the unique eigenvalue with modulus one. Then the above Doeblin-Fortet inequality implies (see [32]) the relation $Q^s = \pi^s \otimes 1 + V_s$ where V_s commutes with the projection $\pi^s \otimes 1$ and has spectral radius less than one, hence the required formula for P^s . \square

Corollary 3.20

With the notations and hypothesis of Corollary 3.19, the following Doeblin-Fortet inequality is valid, if $z = s + it$, $0 \leq s < s_\infty$:

$$[(Q^z)^{n_0} \varphi]_\varepsilon \leq \rho'(\varepsilon) [\varphi]_\varepsilon + (B + A_{n_0}(\varepsilon) |t|^\varepsilon) |\varphi|,$$

where $0 \leq A_{n_0}(\varepsilon) < \infty$. For $t \neq 0$, the spectral radius of Q^z is less than 1.

Furthermore $k(s)$ is analytic on $]0, s_\infty[$, and 1 is a simple eigenvalue of Q^s .

Proof

By definition of $Q^z = Q^{s+it}$ we have $(Q^{s+it})^n \varphi(x) = \mathbb{E}_x^s(|S_n x|^{it} \varphi(S_n \cdot x))$, hence :

$$|(Q^z)^n \varphi(x) - (Q^z)^n \varphi(y)| \leq |(\mathbb{E}_x^s - \mathbb{E}_y^s)(|S_n x|^{it} \varphi(S_n \cdot x))| + |\mathbb{E}_y^s(|S_n x|^{it} \varphi(S_n \cdot x) - |S_n y|^{it} \varphi(S_n \cdot y))|.$$

Using Corollary 3.18 the first term is bounded by $B\delta^\varepsilon(x, y)\mathbb{E}^s(|\varphi|)$ i.e by $B|\varphi|\delta^\varepsilon(x, y)$. The second term is dominated by

$$\mathbb{E}_y^s(|S_n x|^{it} - |S_n y|^{it})|\varphi| + \mathbb{E}_y^s(|\varphi(S_n \cdot x) - \varphi(S_n \cdot y)|).$$

As in the proof of Corollary 3.5, for $n \geq n_0$:

$$\begin{aligned} \mathbb{E}_y^s(|\varphi(S_n \cdot x) - \varphi(S_n \cdot y)|) &\leq [\varphi]_\varepsilon \mathbb{E}_y^s(\delta^\varepsilon(S_n \cdot x, S_n \cdot y)) \\ &\leq [\varphi]_\varepsilon \delta^\varepsilon(x, y) \sup_{x, y} \mathbb{E}_y^s\left(\frac{\delta^\varepsilon(S_n \cdot x, S_n \cdot y)}{\delta^\varepsilon(x, y)}\right) \leq [\varphi]_\varepsilon \delta^\varepsilon(x, y) \rho'(\varepsilon). \end{aligned}$$

On the other hand, using the relation :

$$\begin{aligned} ||u|^{it} - |v|^{it}| &\leq 2|t|^\varepsilon |Log|u| - Log|v||^\varepsilon \leq 2|t|^\varepsilon \sup\left(\frac{1}{|u|}, \frac{1}{|v|}\right)^\varepsilon ||u| - |v||^\varepsilon \text{ we get :} \\ |(|S_n x| - |S_n y|)^{it}| &\leq 2|t|^\varepsilon \sup_{|v|=1} \frac{1}{|S_n v|^\varepsilon} |S_n(x - y)|^\varepsilon \leq 2|t|^\varepsilon \sup_{|v|=1} \frac{|S_n|^\varepsilon}{|S_n v|^\varepsilon} \delta^\varepsilon(x, y). \end{aligned}$$

Since $|S_n v| \geq |S_n^{-1}|^{-1}$ we get :

$$\mathbb{E}_y^s(|S_n x|^{it} - |S_n y|^{it}) \leq 2c(s) |t|^\varepsilon \delta^\varepsilon(x, y) \mathbb{E}^s(\gamma^{2\varepsilon}(S_n)).$$

Since $\gamma(S_{m+n}) \leq \gamma(S_m)\gamma(S_n \circ \theta^m)$ and Q^s is shift-invariant :

$$\mathbb{E}^s(\gamma^{2\varepsilon}(S_n)) \leq (\mathbb{E}^s(\gamma^{2\varepsilon}(S_1)))^n < \infty.$$

Then for n fixed and ε sufficiently small the hypothesis implies that $|\mathbb{E}_y^s(|S_n x|^{it} - |S_n y|^{it})|$ is bounded by $A_n(\varepsilon) |t|^\varepsilon \delta^\varepsilon(x, y)$. Finally for $n = n_0$:

$$[(Q^z)^{n_0} \varphi]_\varepsilon \leq \rho'(\varepsilon) |\varphi]_\varepsilon + (B + A_{n_0}(\varepsilon) |t|^\varepsilon) |\varphi|.$$

Then, using [32], one gets that the possible unimodular spectral values of Q^z are eigenvalues. Using Theorem 2.6, if $t \neq 0$, one get that no such eigenvalue exists, hence the spectral radius of Q^z is less than 1.

In order to show the analyticity of $k(s)$ on $]0, s_\infty[$, we consider the operator P^z for $z \in \mathbb{C}$ close to s . We begin by showing the holomorphy of P^z for $Re z \in]0, s_\infty[$. Let γ be a loop contained in the strip $Re z \in I_\mu$ and $\varphi \in H_\varepsilon(\mathbb{P}^{d-1})$. Then, since $z \rightarrow |gx|^z$ is holomorphic:

$$\int_\gamma P^z \varphi(x) dz = \int_{G \times \gamma} \varphi(g \cdot x) |gx|^z d\mu(g) dz = \int_G \varphi(g \cdot x) d\mu(g) \int_\gamma |gx|^z dz = 0,$$

On the other hand, if γ is a small cercle of center $k(s) \in \mathbb{C}$, the spectral gap property of the operator P^s implies that $k(s)$ is a simple pole of the function $\zeta \rightarrow (\zeta I - P^s)^{-1}$, hence by functional calculus :

$$k(s)\nu^s \otimes e^s = \frac{1}{2i\pi} \int_{\gamma} (\zeta I - P^s)^{-1} d\zeta.$$

Since P^z depends continuously of z , the function $(\zeta I - P^z)^{-1}$ has a pole inside the small disk defined by γ , if z is close to s . Then by perturbation theory P^z has an isolated spectral value $k(z)$ close to $k(s)$. The corresponding projection $\nu^z \otimes e^z$ satisfies :

$$k(z)\nu^z \otimes e^z = \frac{1}{2i\pi} \int_{\gamma} (\zeta I - P^z)^{-1} d\zeta.$$

This formula and the holomorphy of P^z shows that $k(z)$ is holomorphic in a neighbourhood of s . The analyticity of $k(s)$ follows. The fact that 1 is a simple eigenvalue of Q^s follows from Theorem 2.6. \square

Corollary 3.21

Assume $\int |g|^s \gamma^\delta(g) d\mu(g) < \infty$ for some $\delta > 0$. Then given $\varepsilon > 0$ sufficiently small, for any $\varepsilon_0 > 0$ there exists $\delta_0 = \delta_0(\varepsilon_0)$, $n_0 = n_0(\varepsilon_0)$ such that if $x, y \in \mathbb{S}^{d-1}$ satisfy $\tilde{\delta}(x, y) \leq \delta_0$, then $\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0}.x, S_{n_0}.y)) \leq \varepsilon_0 \delta^\varepsilon(x, y)$.

In particular, one has the following Doeblin-Fortet inequality with $D \geq 0$, $\rho_0 = \varepsilon_0 c(s) < 1$: $[(\tilde{Q}^s)^{n_0} \varphi]_\varepsilon \leq \rho_0 [\varphi]_\varepsilon + D |\varphi|$, where $c(s)$ satisfies $\mathbb{Q}_x^s \leq c(s) \mathbb{Q}^s$.

In case I, the \tilde{Q}^s -invariant functions are constant. In case II, the space of \tilde{Q}^s -invariant functions is generated by p_+^s and p_-^s .

If $t \neq 0$ the spectral radius of \tilde{Q}^z is less than 1.

Furthermore, 1 is the unique unimodular eigenvalue of \tilde{Q}^s except in case I, where -1 is the unique non trivial possibility.

Proof

Assume ε is as in Corollary 3.18. We will use for any $n \in \mathbb{N}$, $t > 0$ the relation :

$$\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_n.x, S_n.y)) = \mathbb{E}^s(\tilde{\delta}^\varepsilon(S_n.x, S_n.y) 1_{\{\gamma(S_n) > t\}}) + \mathbb{E}^s(\tilde{\delta}^\varepsilon(S_n.x, S_n.y) 1_{\{\gamma(S_n) \leq t\}})$$

In view of Corollary 3.18 we have for some n_0 , any $\bar{x}, \bar{y} \in \mathbb{P}^{d-1}$, given $\varepsilon_0 > 0$:

$$\mathbb{E}^s(\delta^\varepsilon(S_{n_0}.\bar{x}, S_{n_0}.\bar{y})) \leq \frac{\varepsilon_0}{2} \delta^\varepsilon(\bar{x}, \bar{y}).$$

Using Lemma 2.11 we have for $x, y \in \mathbb{S}^{d-1}$: $\tilde{\delta}(S_{n_0}.x, S_{n_0}.y) \leq 2\gamma^2(S_{n_0})\tilde{\delta}(x, y)$, hence :

$$\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0}.x, S_{n_0}.y) 1_{\{\gamma(S_{n_0}) > t\}}) \leq 2\mathbb{E}^s(\gamma^{2\varepsilon}(S_{n_0}) 1_{\{\gamma(S_{n_0}) > t\}}).$$

Since, as in the proof of Corollary 3.20 we have if ε is sufficiently small, $\mathbb{E}^s(\gamma^{2\varepsilon}(S_{n_0})) < \infty$, we can choose $t_0 > 0$ so that $\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0}.x, S_{n_0}.y) 1_{\{\gamma(S_{n_0}) > t_0\}}) \leq \frac{\varepsilon_0}{2} \tilde{\delta}^\varepsilon(x, y)$. Then, on the set $\{\gamma(S_{n_0}) \leq t_0\}$ we have :

$$\tilde{\delta}(S_{n_0}.x, S_{n_0}.y) \leq 2\gamma^2(S_{n_0})\tilde{\delta}(x, y) \leq 2t_0^2 \tilde{\delta}(x, y).$$

We observe that, if $\tilde{\delta}(u, v) \leq \sqrt{2}$ with $u, v \in \mathbb{S}^{d-1}$, then $\delta(\bar{u}, \bar{v}) = \tilde{\delta}(u, v)$. Hence, if $2t_0^2 \tilde{\delta}(u, v) \leq \sqrt{2}$, we get : $\tilde{\delta}(S_{n_0}.x, S_{n_0}.y) = \delta(S_{n_0}.\bar{x}, S_{n_0}.\bar{y})$ on the set $\{\gamma(S_{n_0}) \leq t_0\}$.

It follows, if $\tilde{\delta}(x, y) \leq \frac{\sqrt{2}}{2t_0} = \delta_0$:

$$\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y) 1_{\{\gamma(S_{n_0}) \leq t_0\}}) \leq \frac{\varepsilon_0}{2} \tilde{\delta}^\varepsilon(x, y)$$

Hence we get, if $\tilde{\delta}(x, y) \leq \delta_0$: $\mathbb{E}^s(\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y)) \leq \varepsilon_0 \tilde{\delta}^\varepsilon(x, y)$.

Using $\mathbb{Q}_x^s < c(s)\mathbb{Q}^s$ we obtain : $\sup_{\tilde{\delta}(x,y) \leq \delta_0} \mathbb{E}_x^s(\frac{\tilde{\delta}^\varepsilon(S_{n_0} \cdot x, S_{n_0} \cdot y)}{\tilde{\delta}^\varepsilon(x, y)}) \leq c(s)\varepsilon_0$.

On the other hand, if $\varphi \in H_\varepsilon(\mathbb{S}^{d-1})$:

$$(\tilde{Q}^s)^n \varphi(x) - (\tilde{Q}^s)^n \varphi(y) = \mathbb{E}_x^s(\varphi(S_n \cdot x) - \varphi(S_n \cdot y)) + (\mathbb{E}_x - E_y)(\varphi(S_n \cdot y))$$

In view of Lemma 3.5, the second term is bounded by $B \tilde{\delta}^{\bar{s}}(x, y)|\varphi|$. Then, for $\tilde{\delta}(x, y) \leq \delta_0$ we obtain, since $\varepsilon \leq \bar{s}$:

$$|(\tilde{Q}^s)^{n_0} \varphi(x) - (\tilde{Q}^s)^{n_0} \varphi(y)| \leq c(s)\varepsilon_0[\varphi]_\varepsilon \tilde{\delta}^\varepsilon(x, y) + B|\varphi| \tilde{\delta}^\varepsilon(x, y)$$

If $\tilde{\delta}(x, y) \geq \delta_0$ we have trivially : $\tilde{\mathbb{E}}_x(|\varphi(S_{n_0} \cdot x) - \varphi(S_{n_0} \cdot y)|) \leq 2 c(s) \frac{\tilde{\delta}^\varepsilon(x, y)}{\delta_0^\varepsilon} |\varphi|$

Finally on \mathbb{S}^{d-1} :

$$[(\tilde{Q}^s)^{n_0} \varphi]_\varepsilon \leq c(s)\varepsilon_0[\varphi]_\varepsilon + (B + 2\frac{c(s)}{\delta_0^\varepsilon})|\varphi|$$

hence the result with $D = B + 2\frac{c(s)}{\delta_0^\varepsilon}$. The structure of the space of \tilde{Q}^s -invariant functions is given by Theorem 2.17. Doeblin-Fortet inequality implies that the possible unimodular spectral values of \tilde{Q}^z are eigenvalues. Then, as in the end of proof of Theorem 2.7, one would have for some $\varphi \in H_\varepsilon(\mathbb{S}^{d-1})$ $e^{i\theta} \in \mathbb{C}$, and any $g \in \text{supp}\mu : |g x|^{it} \varphi(g \cdot x) = e^{i\theta} \varphi(x)$. This would imply $|\lambda_g|^{it} = e^{i\theta}$ for any $g \in \text{supp}\mu$, wich contradicts Proposition 2.5 if $t \neq 0$. The last assertion is a direct consequence of Corollary 2.19. \square

Proof of Theorem A

The spectral decomposition $P^s = k(s)(\nu^s \otimes e^s + U^s)$ is part of Corollary 3.19. The analyticity of $k(s)$ on $]0, s_\infty[$ is stated in Corollary 3.20. The strict convexity of $\text{Log}k(s)$ is stated in Theorem 2.6. The fact that the spectral radius of P^z is less than $k(s)$ follows from the corresponding assertion for Q^z in Corollary 3.20. \square

IV Renewal theorems for products of random matrices

In this section using the results of subsections 3.2, 3.3, we show that the renewal theorem of [34] can be applied to our situation and we give the corresponding statements for products of random matrices. In particular, under condition i-p, if the largest Lyapunov exponent is negative and there exists $\alpha > 0$ with $k(\alpha) = 1$, we get the matricial analogue of Cramer's estimate of ruin for random walk on \mathbb{R} , ([15]) *i.e* there exists $A > 0$ such that for any $x \in \mathbb{S}^{d-1}$:

$$\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}\{\sup_{n > 0} |S_n(\omega)x| > t\} = A e^\alpha(x) \text{ where } A > 0.$$

Hence, under i-p condition, we extend the results of [33], [37] to the general case. Theorems 4.7, 4.7' below will play an essential role in section 5.

1) The renewal theorem for fibered Markov chains

We begin by summarizing, with a few changes and comments, the basic notations of [34]. Let (S, δ) be a complete separable metric space, P (resp \overline{P}) a Markov kernel on $S \times \mathbb{R}$ (resp S) which preserves $C_b(S \times \mathbb{R})$ (resp $C_b(S)$). We assume that P commutes with the translations $(x, t) \rightarrow (x, t + a)$ on $S \times \mathbb{R}$, \overline{P} is the factor kernel of P on S , and π is a fixed \overline{P} -stationary probability measure on S . In our applications, we will have S compact and $S = \mathbb{P}^{d-1}$ or $S \subset \mathbb{S}^{d-1}$. Here we consider paths on $S \times \mathbb{R}$ starting from $(x, 0) \in S \times \{0\}$. Such a path can be written as $(X_n, V_n)_{n \in \mathbb{N}}$ with $V_0 = 0$, $X_0 = x$, $V_n - V_{n-1} = U_n$ ($n \geq 1$). The corresponding space of paths will be denoted ${}^a\Omega = S \times \prod_1^\infty(S \times \mathbb{R})$, the Markov measures on ${}^a\Omega$ will be denoted by ${}^a\mathbb{P}_x$, and the expectation symbol will be written ${}^a\mathbb{E}_x$. The space of bounded measurable functions on a measurable space Y will be denoted $\mathcal{B}(Y)$. We observe that the Markov kernel P on $S \times \mathbb{R}$ is completely defined by the family of measures $F(du|x, y)$ ($x, y \in S$) where $F(du|x, y)$ is the conditional law of V_1 given $X_0 = x$, $X_1 = y$. The number $\int uF(du|x, y)\overline{P}(x, dy)d\pi(x)$ will be called the mean of P , if the corresponding integral $\int |u|F(du|x, y)\overline{P}(x, dy)d\pi(x)$ is finite. In that case, we say that P has a 1-moment. If $t \in \mathbb{R}_+$ we define the hitting time $N(t)$ of the interval $]t, \infty[$ by :

$N(t) = \text{Min}\{n \geq 1 ; V_n > t\}$ ($= +\infty$ if no such n exists).

On the event $N(t) < +\infty$ we take $W(t) = V_{N(t)} - t$, $Z(t) = X_{N(t)}$. Then $W(t)$ is the residual waiting time of the interval $]t, \infty[$ (see [15]). The law of $(Z(t), W(t))$ under \mathbb{P}_x is the hitting measure of $S \times]t, \infty[$ starting from $(x, 0)$.

One needs some technical definitions concerning direct Riemann integrability, non arithmeticity of the Markov chain defined by P on $S \times \mathbb{R}$, and the possibility of comparing ${}^a\mathbb{P}_x, {}^a\mathbb{P}_y$ in a weak sense for different points x, y in S . We add some comments as follows. Given a fibered Markov chain on $S \times \mathbb{R}$, we denote :

$$C_k = \{x \in S ; {}^a\mathbb{P}_x\{\frac{V_m}{m} > \frac{1}{k} \quad \forall m \geq k\} \geq \frac{1}{2}\}, C_0 = \phi.$$

For any $f \in \mathcal{B}({}^a\Omega)$ and $\varepsilon > 0$ we write :

$$f^\varepsilon(x_0, x_1, \dots, v_1 \dots) = \text{sup}\{f(y_0, y_1, \dots, w_1, \dots) ; \delta(x_i, y_i) + |v_i - w_i| < \varepsilon \text{ if } i \in \mathbb{N}\}.$$

Definition 4.1

A Borel function $\varphi \in \mathcal{B}(S \times \mathbb{R})$ is said to be *d.R.i* (for directly Riemann integrable) if :

$$\sum_0^\infty \sum_{\ell=-\infty}^{+\infty} (k+1) \text{sup}\{|\varphi(x, t)| ; x \in C_{k+1} \setminus C_k, \ell \leq t \leq \ell + 1\} < +\infty$$

and for every fixed $x \in S$ and any $\beta > 0$, the function $t \rightarrow \varphi(x, t)$ is Riemann integrable on $[-\beta, \beta]$.

In our setting below we will have $C_k = S$ for some $k > 0$ and for some $\varepsilon > 0$ any $x \in S$ and m sufficiently large ${}^a\mathbb{P}_x(\frac{V_m}{m} \geq \varepsilon) \geq \frac{1}{2}$. Then the following stronger form of the above definition will be used

Definition 4.1'

$\varphi \in \mathcal{B}(S \times \mathbb{R})$ is said to be boundedly Riemann integrable (*b.R.i*) if

$$\sum_{\ell=-\infty}^{\ell=\infty} \sup\{|\varphi(x, t)|; x \in S, t \in [\ell, \ell + 1]\} < \infty$$

and for any fixed $x \in S$, any $\beta > 0$, the function $t \rightarrow \varphi(x, t)$ is Riemann integrable on $[-\beta, \beta]$.

Remark

Definition 4.1' corresponds to $\sup\{|\varphi(x, t)|; x \in S\}$ directly Riemann integrable in the sense of [15]. If $C_k = S$ for some $k \in \mathbb{N}$, then condition *b.R.i* implies condition *d.R.i*

The following will help us to express the appropriate aperiodicity condition for (P, π) .

Definition 4.2

The kernel P , the space (S, δ) and the measure $\pi \in M^1(S)$ being as above we consider a point $(\zeta, \lambda, y) \in \mathbb{R} \times [0, 1] \times S$ and we say that (P, π) satisfies distortion (ζ, λ) at y if for any $\varepsilon > 0$, there exists $A \in \mathcal{B}(S)$ with $\pi(A) > 0$ and $m_1, m_2 \in \mathbb{N}$, $\tau \in \mathbb{R}$ such that for any $x \in A$:

$${}^a\mathbb{P}_x\{\delta(X_{m_1}, y) < \varepsilon, |V_{m_1} - \tau| \leq \lambda\} > 0, \quad {}^a\mathbb{P}_x\{\delta(X_{m_2}, y) < \varepsilon, |V_{m_2} - \tau - \zeta| \leq \lambda\} > 0.$$

Definition 4.3

We will say that the kernel P on $S \times \mathbb{R}$ is non-expanding if for each fixed $x \in S$, $\varepsilon > 0$, there exists $r_0 = r_0(x, \varepsilon)$ such that for all real valued $f \in \mathcal{B}({}^a\Omega)$ and for all y with $\delta(x, y) < r_0$ one has :

$${}^a\mathbb{P}_y(f) \leq {}^a\mathbb{P}_x(f^\varepsilon) + \varepsilon|f|, \quad {}^a\mathbb{P}_x(f) \leq {}^a\mathbb{P}_y(f^\varepsilon) + \varepsilon|f|.$$

We denote by I.1 -I.4 the following conditions

I.1 For every open set O with $\pi(O) > 0$, and ${}^a\mathbb{P}_x - a.e.$, for each $x \in S$, we have :

$${}^a\mathbb{P}_x\{X_n \in O \text{ for some } n\} = 1.$$

I.2 P has a 1-moment and for all $x \in S$, ${}^a\mathbb{P}_x - a.e.$:

$$\lim_{n \rightarrow \infty} \frac{V_n}{n} = L = \int uF(du|x, y)\bar{P}(x, dy)d\pi(x) > 0.$$

I.3 There exists a sequence $(\zeta_i)_{i \geq 1} \subset \mathbb{R}$ such that the group generated by ζ_i is dense in \mathbb{R} and such that for any $i \geq 1$ and $\lambda \in [0, 1]$, there exists $y = y(i, \lambda) \in S$ such that (P, π) satisfies distortion (ζ_i, λ) at y .

I.4 The kernel P on $S \times \mathbb{R}$ is non-expanding.

Then, the following extension of the classical renewal theorem (see [15]) is proved in [34].

Theorem 4.4

Assume conditions I.1-I.4 are satisfied for P . Then there exists a positive measure ψ on S such that for any $x \in S$ and $\varphi \in C_b(S \times]0, \infty[)$:

$$\lim_{t \rightarrow \infty} {}^a\mathbb{E}_x \varphi(Z(t), W(t)) = \frac{1}{L} \int \varphi(z, s) 1_{[0, t]}(s) {}^a\mathbb{P}_y \{X_{N(0)} \in dz, V_{N(0)} \in dt\} d\psi(y) ds = H(\varphi)$$

Moreover, if $\varphi \in C_b(S \times \mathbb{R})$ is *d.R.i.*, then, for any $x \in S$:

$$\lim_{t \rightarrow \infty} {}^a\mathbb{E}_x \left(\sum_0^\infty \varphi(X_n, V_n - t) \right) = \frac{1}{L} \int \varphi(y, s) d\pi(y) ds$$

Furthermore ψ is an invariant measure for the Markov chain on S with kernel ${}^aP(y, dz) = {}^a\mathbb{P}_y(X_{N(0)} \in dz)$ and $\int \mathbb{E}_y(N(0)) d\psi(y) = 1$, $\int \mathbb{E}_y(V_{N(0)}) d\psi(y) = L$.

Remark

If S is compact, condition I.1 is a consequence of uniqueness of the \bar{P} -stationary measure π . This follows from the law of large numbers for Markov chains with a unique stationary measure [7] : for any continuous function with $0 \leq f \leq 1$, $\pi(f) > 0$ we have ${}^a\mathbb{P}_x - a.e$ for

all $x \in S$: $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_0^{n-1} f(X_k) = \pi(f) > 0$. This implies condition I.1.

The construction of ψ in [34] is based on Kac's recurrence theorem and implies the absolute continuity of ψ with respect to π , hence the probability H is independant of x and absolutely continuous with respect to $\pi \otimes \ell$.

2) Under condition i-p and a moment hypothesis conditions I1-I4 are valid

We verify conditions I1-I4 in four related situations. Here \mathbb{R} is identified with \mathbb{R}_+^* through the map $t \rightarrow e^t$. If $d > 1$ we use condition i-p. If $d = 1$ we use non arithmeticity of μ .

The first and simpler one, corresponds to $S = \mathbb{P}^{d-1}$, $S \times \mathbb{R} = \dot{V}$, $P(v, \cdot) = \mu * \delta_v$ where P is the operator denoted \dot{P} in section 2. Also we write on \mathbb{P}^{d-1} : $\bar{P}(x, \cdot) = \mu * \delta_x$ if $x \in \mathbb{P}^{d-1}$. We will begin the verifications by this case and show how to modify the arguments in the other cases.

In the second case, $S \times \mathbb{R} \subset V \setminus \{0\}$, S is a compact subset of \mathbb{S}^{d-1} and P (resp \tilde{P}) will be the restriction to $S \times \mathbb{R}$ (resp S) of the kernel already denoted P (resp \tilde{P}) in section 2. Since, for any $t \in \mathbb{R}_+^*$ and $g \in G$, we have $g(tv) = tg(v)$, the kernels P and \tilde{P} will satisfy the commutation condition required in the above paragraph. As shown at the end of section 2, we need to consider two cases for \tilde{P} , depending of the fact that P preserves a proper convex cone (case II) or not (case I). In case I (resp II) we will have $S = \mathbb{S}^{d-1}$ (resp $S = Co(\Lambda_+([supp\mu])$). With these choices, there exists a unique \tilde{P} -stationary measure on S , as follows from Theorem 2.17. We denote by $\alpha \in I_\mu$ the positive number (if it exists)

such that $k(\alpha) = 1$, where $k(s) = \lim_{n \rightarrow \infty} \left(\int |g|^s d\mu^n(g) \right)^{1/n}$.

We know from section 3, that for any $s \in I_\mu$, there are two Markov kernels Q^s on \mathbb{P}^{d-1} and \tilde{Q}^s on \mathbb{S}^{d-1} , naturally associated with the operator P^s considered in section 2. We are

here mainly interested in the cases $s = \alpha$ and $s = 0$, with $Q^0 = \bar{P}$ and $\tilde{Q}^0 = \tilde{P}$, but we observe that our considerations are valid for any $s \in I_\mu$. In what follows the notation S will be used in all the cases above.

We denote by ${}^a\mathbb{Q}_x^s$ the natural Markov measures on the path space ${}^a\Omega$. If $s = 0$ we use the notation ${}^a\mathbb{P}_x$. We write $V_k = \text{Log}|S_k x|$, $X_k = S_k \cdot x$ and we denote by Δ_x the map from $\Omega = G^{\mathbb{N}}$ to ${}^a\Omega$ given by $(g_1, g_2, \dots) \rightarrow (x, X_1, V_1, X_2, V_2, \dots)$. Clearly ${}^a\mathbb{Q}_x^\alpha$ (resp ${}^a\mathbb{P}_x$) is the push-forward of \mathbb{Q}_x^α (resp \mathbb{P}) by Δ_x , hence we can translate the results of section 3 in the new setting.

The validity of condition I.1, in all cases, is a direct consequence of the remark following the theorem, since by Theorem 2.6 (see also Theorem 2.17) the kernels \bar{P} , Q^α , \tilde{P} , \tilde{Q}^α have unique stationary measures on S .

In order to verify I.2 we begin by $S = \mathbb{P}^{d-1}$, $S \times \mathbb{R} = \dot{V}$, $\bar{P}(x, \cdot) = \mu * \delta_x$, $\dot{P}(v, \cdot) = \mu * \delta_v$. Then $F(A|x, y) = \mu\{g \in G, \text{Log}|gx| \in A, g \cdot x = y\}$ for any Borel set $A \subset \mathbb{R}$, and $\pi = \nu$ with $\mu * \nu = \nu$. We observe that $|\text{Log}|gx|| \leq \text{Log}\gamma(g)$ if $|x| = 1$ and $\gamma(g) = \sup(|g|, |g^{-1}|)$. The finiteness of $\int |u|F(du|x, y)\bar{P}(x, dy)d\pi(x)$, follows since $\text{Log}\gamma(g)$ is μ -integrable. Also $\int uF(du|x, y)\tilde{P}(x, dy)d\pi(x) = \int \text{Log}|gx|d\mu(g)d\nu(x) = L_\mu$. Then the relation ${}^a\mathbb{P}_x = \Delta_x(\mathbb{P})$ and Theorem 3.10 imply, for every $x \in \mathbb{P}^{d-1}$ (case $s = 0$), and \mathbb{P} -a.e :

$$L_\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S_n x| = \int \text{Log}|gx|d\mu(g)d\nu(x).$$

This is condition I.2 in the first case. For $S \subset \mathbb{S}^{d-1}$, the result is the same, since the involved quantities depend only of $|gx|$ with $x \in \mathbb{P}^{d-1}$, and \tilde{P} has a unique stationary measure on S .

In the cases of Q^α and \tilde{Q}^α it suffices also to consider the case $S = \mathbb{P}^{d-1}$.

The 1-moment condition of I.2 follows from $\int |g|^\alpha |\text{Log}\gamma(g)|d\mu(g) < +\infty$. The convergence part follows from Theorem 3.10 with $L_\mu(\alpha) = \int q^\alpha(x, g)\text{Log}|gx|d\pi^\alpha(x)d\mu(g) = \frac{k'(\alpha)}{k(\alpha)} > 0$.

We show I.3 as follows.

If $d > 1$, since the semigroup $T = \bigcup_{n \geq 0} (\text{supp}\mu)^n$ satisfies (i-p), we know, using Proposition 2.5 that the set $\Delta = \{\text{Log}|\lambda_h| ; h \in T^{\text{prox}}\}$ is dense in \mathbb{R} .

The same is true of $2\Delta = \{\text{Log}\lambda_{h^2} ; h \in T^{\text{prox}}\}$.

If $d = 1$, the same properties follow from the non arithmeticity of μ .

We take for ζ_i ($i \in \mathbb{N}$) a dense countable subset of 2Δ . Let $\zeta_i = \text{Log}\lambda_g \in 2\Delta$, with $\lambda_g > 0$, $g = h^2$, $h = u_1 \cdots u_n$, $u_i \in \text{supp}\mu$ ($1 \leq i \leq n$) and $y = y(\zeta_i, \lambda) = \bar{v}_g \in \mathbb{P}^{d-1} = S$. We observe that, if ε is sufficiently small and $B_\varepsilon = \{x \in \mathbb{P}^{d-1}; \delta(x, \bar{v}_g) \leq \varepsilon\}$, then $g \cdot B_\varepsilon \subset B_{\varepsilon'}$, with $\varepsilon' < \varepsilon$ and g as above.

Also, $\lambda > 0$ being fixed, and ε sufficiently small, we have $|\text{Log}\lambda_g - \text{Log}|gx|| < \lambda$ if $x \in B_\varepsilon$. These relations remain valid for g' instead of g if g' is sufficiently close to g .

Then we have for $x \in B_\varepsilon$, and $S_{2n} = g \in (\text{supp}\mu)^{2n}$ as above :

$$\mathbb{P}\{\delta(S_{2n} \cdot x, \bar{v}_g) < \varepsilon, |\text{Log}|S_{2n} x| - \text{Log}\lambda_g| < \lambda\} > 0.$$

With $\zeta_i = \text{Log}\lambda_g$, $y = \bar{v}_g$, $A = B_\varepsilon$, $\tau = 0$, $m_1 = 0$, $m_2 = 2n$, this implies condition I.3 for

the probability ${}^a\mathbb{P}_x = \Delta_x(\mathbb{P})$.

The definition of \mathbb{Q}_x^α shows its equivalence to \mathbb{P} on the σ -algebra of the sets depending of the first n coordinates. Then the relation ${}^a\mathbb{Q}_x^\alpha = \Delta_x(\mathbb{Q}_x^\alpha)$ implies with g as above :

$${}^a\mathbb{Q}_x^\alpha \{ \delta(S_{2n} \cdot x, \tilde{v}_g) < \varepsilon, \quad |Log|S_{2n}x| - Log\lambda_g| < \lambda \} > 0.$$

Hence condition I.3 is valid for ${}^a\mathbb{Q}_x^\alpha$ also.

If we consider \mathbb{S}^{d-1} instead of \mathbb{P}^{d-1} , i.e $S = \mathbb{S}^{d-1}$ or $S = Co(\Lambda_+([supp\mu]))$, and the metric $\tilde{\delta}$ on S , the above geometrical argument remains valid with $g = h^2$, $y = \tilde{v}_g \in \Lambda_+([supp\mu])$ in the second case, $\lambda_g > 0$ and ε sufficient small. This shows I.3 in this setting.

Condition I.4 follows from the proof of Proposition 1 of [33]. The proof of the corresponding part of this proposition is a consequence of the condition :

$${}^a\mathbb{P}_x \{ \exists C > 0 \text{ with } |S_n x| \geq C|S_n| \text{ for all } n \} = 1$$

for all $x \in S$, which implies that $|S_n x|$ and $|S_n y|$ are comparable if x and y are close.

For the proof of the above condition, we observe that if $s \in I_\mu$, in particular if $s = 0$ or α ,

this condition has been proved in the stronger form $\lim_{n \rightarrow \infty} \frac{|S_n x|}{|S_n|} = | \langle z^*(\omega), x \rangle |$, $\mathbb{Q}_x^s - a.e$,

since as shown in Theorem 3.2 $| \langle z^*(\omega), x \rangle | > 0$, $\mathbb{Q}_x^s - a.e$ for any fixed $x \in \mathbb{P}^{d-1}$. Hence condition I.4 is valid in all the cases under consideration.

3) Direct Riemann integrability

In case of the spaces $S = \mathbb{P}^{d-1}$ or $S \subset \mathbb{S}^{d-1}$ considered above, under condition i-p for $[supp\mu]$, the *d.R.i* condition takes the simple form given by Lemma 4.5 below, in multiplicative notation.

We assume now that the hypothesis of Theorem 3.10 is satisfied, use the corresponding notations, and C_k is as in Definition 4.1.

Lemma 4.5

Assume $\varphi \in C_b(\mathbb{S}^{d-1} \times \mathbb{R}_+^*)$ is *b.R.i*, i.e φ is locally Riemann integrable and satisfies :

$$\sum_{\ell=-\infty}^{\ell=+\infty} \sup\{ |\varphi(x, t)| ; x \in \mathbb{S}^{d-1}, t \in [2^\ell, 2^{\ell+1}] \} < +\infty.$$

Then $C_k = \mathbb{S}^{d-1}$ for k large, hence φ is *d.R.i* with respect to \mathbb{P} and \mathbb{Q}^α .

Proof

We consider first ${}^a\mathbb{P}_x$. Using Theorem 3.10 for $s = 0$, we get for any fixed x :

$$\lim_{n \rightarrow +\infty} \frac{Log|S_n(\omega)x|}{n} = L_\mu, \quad \mathbb{P} - a.e.$$

We observe that for any $x, y \in \mathbb{S}^{d-1}$, $\|S_n y| - |S_n x|\| \leq |S_n| \tilde{\delta}(x, y) \leq 2|S_n|$.

It follows :

$$\left| \frac{|S_n y|}{|S_n x|} - 1 \right| \leq 2 \frac{|S_n|}{|S_n x|}, \quad |Log|S_n y| - Log|S_n x|| \leq 2 \frac{|S_n|}{|S_n x|},$$

Using Theorem 3.2, we get that the sequence $\frac{|S_n|}{|S_{nx}|}$ converges $\mathbb{P} - a.e$ to $\frac{1}{|\langle z(\omega), x \rangle|} < \infty$, hence the sequence $\frac{1}{n} \frac{|S_n|}{|S_{nx}|}$ converges $\mathbb{P} - a.e$ to zero.

It follows that $\frac{1}{n} \text{Log}|S_{nx}| - \frac{2}{n} \frac{|S_n|}{|S_{nx}|}$ converges $\mathbb{P} - a.e$ to L_μ .

Hence there exists $m_0 > 0$ such that $\mathbb{P}\{\frac{1}{n} \text{Log}|S_{nx}| - \frac{2}{n} \frac{|S_n|}{|S_{nx}|} > \frac{1}{2}L_\mu \text{ for all } n \geq m_0\} \geq 1/2$.

In view of the inequality $\frac{1}{n} \text{Log}|S_{ny}| \geq \frac{1}{n} \text{Log}|S_{nx}| - \frac{2}{n} \frac{|S_n|}{|S_{nx}|}$, we have for any $y \in \mathbb{S}^{d-1}$, $\mathbb{P}\{\frac{1}{n} \text{Log}|S_{ny}| > L_\mu/2 \text{ for all } n \geq m_0\} > \frac{1}{2}$.

This implies $C_k = \mathbb{S}^{d-1}, C_{k+1} \setminus C_k = \phi$ if $\frac{1}{k} \leq \text{Inf}\left(\frac{1}{m_0}, \frac{L_\mu}{2}\right)$. Then, φ is *d.R.i.*

If $s = \alpha$, the argument is the same with \mathbb{P} replaced by \mathbb{Q}^α and the relation $\mathbb{Q}_x^\alpha \leq c(\alpha)\mathbb{Q}^\alpha$ is used as follows.

$$\mathbb{Q}^\alpha\left\{\frac{1}{n} \text{Log}|S_{ny}| > L_\mu(\alpha)/2 \text{ for all } n \geq m_0\right\} > 1 - \frac{1}{2c(\alpha)}.$$

Since $\mathbb{Q}_y^\alpha \leq c(\alpha)\mathbb{Q}^\alpha$, this gives for any $y \in \mathbb{P}^{d-1}$:

$$\mathbb{Q}_y^\alpha\left\{\frac{1}{n} \text{Log}|S_{ny}| > L_\mu(\alpha)/2 \text{ for all } n \geq m_0\right\} > \frac{1}{2}.$$

Then also $C_k = \mathbb{S}^{d-1}$ for $\frac{1}{k} \leq \text{Inf}\left(\frac{1}{m_0}, \frac{L_\mu(\alpha)}{2}\right)$. Hence we conclude as above

4) The renewal theorems for products of random matrices.

We consider $V \setminus \{0\} = \mathbb{S}^{d-1} \times \mathbb{R}_+^*$, $\dot{V} = \mathbb{P}^{d-1} \times \mathbb{R}_+^*$ and we study the asymptotics of the potential kernels of the corresponding random walks defined by μ . We denote :

$$\dot{V}_1 = \{v \in \dot{V} ; |v| > 1\}, \quad V_1 = \{v \in V ; |v| > 1\}$$

and we consider also the entrance measures $H_1(v, \cdot)$ or $\dot{H}_1(v, \cdot)$ of $S_n v$ in V_1 or \dot{V}_1 , starting from $v \neq 0$. Since conditions *I* are valid, their behaviours for v small are given by Theorem 4.4, and we will state them below. We denote by $\dot{\Lambda}([\text{supp}\mu])$ the inverse image of $\Lambda([\text{supp}\mu])$ in \dot{V} . Also we denote :

$$\dot{\Lambda}_1([\text{supp}\mu]) = \{v \in \dot{V} ; \bar{v} \in \Lambda([\text{supp}\mu]), |v| \geq 1\}$$

$$\Lambda_1([\text{supp}\mu]) = \{v \in V ; \bar{v} \in \Lambda([\text{supp}\mu]), |v| \geq 1\}.$$

As shown below these closed sets support the limits of $\dot{H}_1(v, \cdot)$ and $H_1(v, \cdot)$.

The results will take two forms according as $L_\mu > 0$ or $L_\mu < 0$. Also, each of these situations leads to two results depending of the geometrical case \dot{V} or V , under consideration.

Theorem 4.6

Assume $\mu \in M^1(G)$ is such that the semigroup $[\text{supp}\mu]$ satisfies condition i-p, if $d > 1$ or μ is non arithmetic if $d = 1$, $\text{Log}\gamma(g)$ is μ -integrable and $L_\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \int \text{Log}|g| d\mu^n(g) > 0$.

Then if $v \in \dot{V}$, $\sum_0^\infty \mu^k * \delta_v$ is a Radon measure on \dot{V} such that on $C_c(\dot{V})$ we have the vague convergence :

$$\lim_{v \rightarrow 0} \sum_0^\infty \mu^k * \delta_v = \frac{1}{L_\mu} \nu \otimes \ell$$

where $\nu \in M^1(\Lambda([\text{supp}\mu]))$ is the unique \bar{P} -invariant measure on \mathbb{P}^{d-1} .

This convergence is valid on any bounded continuous function f which satisfy on \dot{V} :

$$\sum_{-\infty}^{+\infty} \sup\{|f(v)| ; 2^\ell \leq |v| \leq 2^{\ell+1}\} < \infty.$$

Furthermore we have the weak convergence :

$$\lim_{v \rightarrow 0} \dot{H}_1(v, \cdot) = \dot{H}_1 \in M^1(\dot{\Lambda}_1([\text{supp}\mu])).$$

Proof

In view of the verifications of conditions I in subsections 2, 3 and of Lemma 4.5, this is a direct consequence of Theorem 4.4 applied in the case $S = \mathbb{P}^{d-1}$, $S \times \mathbb{R} = \dot{V} \setminus \{0\}$.

The fact that \dot{H}_1 is supported on $\dot{\Lambda}_1([\text{supp}\mu])$ follows from the $\text{supp}\mu$ -invariance of $\dot{\Lambda}([\text{supp}\mu])$, the existence of the limit and the fact that, if $\bar{v} \in \dot{\Lambda}([\text{supp}\mu])$ then $\dot{H}_1(v, \cdot) \in M^1(\dot{\Lambda}_1([\text{supp}\mu]))$.

□

Theorem 4.6'

Assume μ is as in Theorem 4.6. Then there are 2 cases as in Theorem 2.17 :

Case I : No proper convex cone in V is $\text{supp}\mu$ -invariant.

Then, in vague topology :

$$\lim_{v \rightarrow 0} \sum_0^\infty \mu^k * \delta_v = \frac{1}{L_\mu} \tilde{\nu} \otimes \ell$$

where $\tilde{\nu}$ is the unique μ -stationary measure on \mathbb{S}^{d-1}

Case II : Some proper convex cone in V is $\text{supp}\mu$ -invariant.

Then, for any $u \in \mathbb{S}^{d-1}$, in vague topology,

$$\lim_{t \rightarrow 0_+} \sum_0^\infty \mu^k * \delta_{tu} = \frac{1}{L_\mu} (p_+(u)\nu_+ \otimes \ell + p_-(u)\nu_- \otimes \ell)$$

where ν_+ is the unique μ -stationary measure on $\Lambda_+([\text{supp}\mu])$, ν_- is symmetric of ν_+ , $p_+(u)$ is the entrance probability of $S_n.u$ in $Co(\Lambda_+([\text{supp}\mu]))$, $p_-(u) = 1 - p_+(u)$.

In the two cases these convergences are also valid on any bounded continuous functions f

on $V \setminus \{0\}$ such that $\sum_{-\infty}^{+\infty} \sup\{|f(v)| ; 2^\ell \leq |v| < 2^{\ell+1}\} < \infty$.

In addition, for any $u \in \mathbb{S}^{d-1}$, in weak topology :

$$\lim_{t \rightarrow 0_+} H_1(tu, \cdot) = H_{1,u} \in M^1(\Lambda_1([\text{supp}\mu])).$$

In case I, $H_{1,u}$ is independent of u .

In case II, with $\Lambda_{1,+}([\text{supp}\mu]) = \{v \in V ; \tilde{v} \in \Lambda_+([\text{supp}\mu]), |v| \geq 1\}$ we have :

$$H_{1,u} = p_+(u)H_{1,+} + p_-(u)H_{1,-},$$

where $H_{1,+} \in M^1(\Lambda_{1,+}(\text{supp}\mu))$, and $H_{1,-}$ is symmetric of $H_{1,+}$

Proof

In case I, the proof is the same as for Theorem 4.6 with $S = \mathbb{S}^{d-1}$ instead of \mathbb{P}^{d-1} .

In case II, we take $S = Co(\Lambda_+([\text{supp}\mu])$ and we observe that $S \times \mathbb{R}_+^*$ is a $\text{supp}\mu$ -invariant convex cone with non zero interior to which we can apply Theorem 4.4.

If $u \in S$ (resp $u \in -S$) we have :

$$\lim_{t \rightarrow 0_+} \sum_0^\infty \mu^k * \delta_{tu} = \frac{1}{L_\mu}(\nu_+ \otimes \ell) \quad (\text{resp } \lim_{t \rightarrow 0_+} \sum_0^\infty \mu^k * \delta_{tu} = \frac{1}{L_\mu}(\nu_- \otimes \ell))$$

If $u \in V \setminus \{0\}$, we denote by $p_+(u, dv)$ (resp $p_-(u, dv)$) the entrance measure of $S_n.u$ in the cone $\Phi = S \times \mathbb{R}_+^*$ (resp $-\Phi$). Clearly the mass of $p_+(u, dv)$ is $p_+(u)$, and $p_+(u, dv)$ is supported on Φ .

We denote also by $U(v, \cdot) = \sum_0^\infty \mu^k * \delta_v$ the potential kernel of the linear random walk $S_n(\omega)v$

starting from v in $V \setminus \{0\}$. Then, for any $\varphi \in C_c(\Phi \cup -\Phi)$, $U(tu, \varphi) = \int U(v, \varphi)(p_+(tu, dv) + p_-(tu, dv))$.

Clearly the kernel $p_+(x, dv)$ commutes with the scaling $x \rightarrow tx$ ($t > 0$). Then it follows from above that, on $C_c(\Phi \cup -\Phi)$:

$$\lim_{t \rightarrow 0_+} U(tu, \cdot) = \frac{1}{L_\mu}(p_+(u)\nu_+ \otimes \ell + p_-(u)\nu_- \otimes \ell).$$

If $\varphi \in C_c(V \setminus \{0\})$ vanishes on $\Phi \cup -\Phi$, Theorem 4.6 implies $\lim_{t \rightarrow 0_+} U(tu, \varphi) = 0$.

Finally we have $\lim_{t \rightarrow 0_+} \sum_0^\infty \mu^k * \delta_{tu} = \frac{1}{L_\mu}(p_+(u)\nu_+ \otimes \ell + p_-(u)\nu_- \otimes \ell)$. The existence of

$H_{1,u}$ follow from the first formula in Theorem 4.4. In particular the right hand side of this formula is independant of $x \in S$. Hence, in case I, $H_{1,u}$ is independent of u . In case II, we use $S = C_0(\Lambda_+([\text{supp}\mu])$ and we argue as above in order to obtain the formula $H_{1,u} = p_+(u)H_{1,+} + p_-(u)H_{1,-}$ where $H_{1,+} = H_{1,u}$ for $u \in C_0(\Lambda_+([\text{supp}\mu])$ and $H_{1,-} = H_{1,u}$ for $u \in C_0(\Lambda_-([\text{supp}\mu])$. \square

Theorem 4.7

Assume that $\mu \in M^1(G)$ is such that $[\text{supp}\mu]$ satisfies i-p, if $d > 1$ or μ is non arithmetic if $d = 1$. Assume $L_\mu < 0, \alpha > 0$ exists with $k(\alpha) = 1, \int |g|^\alpha \text{Log}\gamma(g)d\mu(g) < \infty$ and write :

$$L_\mu(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \int |g|^\alpha \text{Log}|g| d\mu^n(g) = \frac{k'(\alpha)}{k(\alpha)}.$$

Then $L_\mu(\alpha) > 0$ and for any $u \in \mathbb{P}^{d-1}$, we have the vague convergence in \dot{V} :

$$\lim_{t \rightarrow 0} t^{-\alpha} \sum_0^\infty \mu^k * \delta_{tu} = \frac{e^\alpha(u)}{L_\mu(\alpha)} \nu^\alpha \otimes \ell^\alpha$$

where $\nu^\alpha \in M^1(\mathbb{P}^{d-1})$ (resp $e^\alpha \in C(\mathbb{P}^{d-1})$, $\nu^\alpha(e^\alpha) = 1$) is the unique solution of $P^\alpha \nu^\alpha = \nu^\alpha$ (resp $P^\alpha e^\alpha = e^\alpha$) and ν^α has support $\Lambda([\text{supp}\mu])$.

Furthermore, on $C_b(\dot{V}_1)$ and for any $u \in \mathbb{P}^{d-1} \subset \dot{V}$ we have the vague convergence :

$$\lim_{t \rightarrow 0} t^{-\alpha} \dot{H}_1(tu, \cdot) = e^\alpha(u) \dot{H}_1^\alpha,$$

where \dot{H}_1^α is a positive measure supported on $\dot{\Lambda}_1([\text{supp}\mu])$.

In particular, for some $A > 0$ and any $u \in \mathbb{P}^{d-1}$: $\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}\{\sup_{n \geq 1} |S_n u| > t\} = Ae^\alpha(u)$.

The first convergence is valid on any continuous function f such that $f_\alpha(v) = |v|^{-\alpha} f(v)$ satisfies $\sum_{-\infty}^{+\infty} \text{sup}\{|f_\alpha(v)| ; 2^\ell \leq |v| \leq 2^{\ell+1}\} < \infty$.

Proof

We observe that the function $e^\alpha \otimes h^\alpha$ on \dot{V} satisfies $\dot{P}(e^\alpha \otimes h^\alpha) = e^\alpha \otimes h^\alpha$, hence we can consider the associated Markov operator \dot{Q}_α on \dot{V} defined by

$$\dot{Q}_\alpha(f) = \frac{1}{e^\alpha \otimes h^\alpha} \dot{P}(fe^\alpha \otimes h^\alpha).$$

Then the potential kernel of \dot{Q}_α is given by

$$\sum_0^\infty (\dot{Q}_\alpha)^k(f) = \frac{1}{e^\alpha \otimes h^\alpha} \sum_0^\infty \dot{P}^k(fe^\alpha \otimes h^\alpha).$$

Clearly \dot{Q}_α commutes with dilations, hence defines a fibered Markov kernel on \dot{V} .

Also the mean of \dot{Q}_α is $L_\mu(\alpha) > 0$. Then, taking $f = \frac{\varphi}{e^\alpha \otimes h^\alpha}$, since conditions I are valid, the result follows from Theorem 4.4. Cramer estimation for $\mathbb{P}\{|S_n u| > t\}$ follows with $A = \dot{H}_1^\alpha(\dot{V}_1) > 0$. \square

Theorem 4.7'

Assume μ and α are as in Theorem 4.7. Then for any $u \in \mathbb{S}^{d-1}$ we have the vague convergence : $\lim_{t \rightarrow 0_+} t^{-\alpha} \sum_0^\infty \mu^k * \delta_{tu} = \frac{e^\alpha(u)}{L_\mu(\alpha)} \tilde{\nu}_u^\alpha \otimes \ell^\alpha$, where $\tilde{\nu}_u^\alpha \in M^1(\tilde{\Lambda}(T))$ is \tilde{P}^α -invariant.

These are 2 cases like in Theorem 4.6'.

Case I : $\tilde{\nu}_u^\alpha = \tilde{\nu}^\alpha$ has support $\tilde{\Lambda}(T)$

Case II : $\tilde{\nu}_u^\alpha = p_+^\alpha(u)\nu_+^\alpha + p_-^\alpha(u)\nu_-^\alpha$ where $p_+^\alpha(u)$ (resp $p_-^\alpha(u)$) denotes the entrance measure under \mathbb{Q}_u^α of $S_n \cdot u$ in the convex envelope of $\Lambda_+(T)$ (resp $\Lambda_-(T)$).

The above convergences are valid on any continuous function f such that $f_\alpha(v) = |v|^{-\alpha} f(v)$, satisfies :

$$\sum_{\ell=-\infty}^{\ell=+\infty} \text{sup}\{|f_\alpha(v)| ; 2^\ell \leq |v| \leq 2^{\ell+1}\} < \infty.$$

Furthermore, on $C_b(V_1)$ for any $u \in \mathbb{S}^{d-1}$, we have the vague convergence :

$$\lim_{t \rightarrow 0_+} t^{-\alpha} H_1(tu, \cdot) = e^\alpha(u) (p_+^\alpha(u) H_{1,+}^\alpha + p_-^\alpha(u) H_{1,-}^\alpha).$$

In case I, $H_{1,+}^\alpha = H_{1,-}^\alpha = H_1^\alpha$ is a positive measure supported on $\Lambda_1([\text{supp}\mu])$. In case II, $H_{1,+}^\alpha$ is a positive measure supported on $\Lambda_{1,+}([\text{supp}\mu])$ and $H_{1,-}^\alpha$ is symmetric of $H_{1,+}^\alpha$.

Proof

The proof is a simple combination of the proofs of Theorems 4.7, 4.6'. \square

For the existence of $\alpha > 0$ such that $k(\alpha) = 1$, we have the following sufficient condition where we denote by $r(g)$ the spectral radius of $g \in G$.

Proposition 4.8

Let $\mu \in M^1(G)$ and assume that $k(s) = \lim_{n \rightarrow \infty} (\int |g|^s d\mu^n(g))^{1/n}$ is finite for any $s > 0$. For any $p \in \mathbb{N}$ and $g \in (\text{supp}\mu)^p$ we have :

$$\lim_{s \rightarrow \infty} \frac{\text{Log} k(s)}{s} \geq p \text{Log} r(g).$$

In particular if some $g \in [\text{supp}\mu]$ satisfies $r(g) > 1$, then $k(s) > 1$ for s sufficiently large.

The proof is based on the following elementary lemma which we state without proof.

Lemma 4.9

Let $g \in G$. Then for any $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ and a neighbourhood $V(\varepsilon)$ of g such for any sequence $g_k \in V(\varepsilon)$ one has $|g_n \cdots g_1| \geq c(\varepsilon) r^n(g)(1 - \varepsilon)^n$

Proof of the proposition

The convexity of $\text{Log} k(s)$ implies that $\lim_{s \rightarrow \infty} \frac{\text{Log} k(s)}{s}$ exists. Let $g \in \text{supp}\mu$, hence given $\varepsilon > 0$ there exists a neighbourhood $V(\varepsilon)$ of g such that $\mu(V(\varepsilon)) = C(\varepsilon) > 0$. From the

lemma we have, with $k(s) = \lim_{n \rightarrow \infty} (\int |g_n \cdots g_1|^s d\mathbb{P}(\omega))^{1/n}$:

$$k(s) \geq \lim_{n \rightarrow \infty} (c^s(\varepsilon) r^{ns}(g)(1 - \varepsilon)^{ns} C^n(\varepsilon))^{1/n} = r^s(g)(1 - \varepsilon)^s C(\varepsilon)$$

$$\frac{\text{Log} k(s)}{s} \geq \text{Log}(1 - \varepsilon) + \text{Log} r(g) + \frac{\text{Log} C(\varepsilon)}{s}, \quad \lim_{s \rightarrow \infty} \frac{\text{Log} k(s)}{s} \geq \text{Log} r(g)$$

We observe that if μ is replaced by μ^p , then $k(s)$ is replaced by $k^p(s)$. Hence for $g \in (\text{supp}\mu)^p$ we have from above the required inequality.

If $g \in [\text{supp}\mu]$ then we can assume $g \in (\text{supp}\mu)^p$ for some $p \in \mathbb{N}$; since $r(g) > 1$, we have $\text{Log} r(g) > 0$, hence $\lim_{s \rightarrow \infty} \frac{\text{Log} k(s)}{s} > 0$. \square

Proofs of Theorems B, B' and Corollary

Theorem B (resp B') is a direct consequence of Theorem 4.6 (resp 4.7). The assertion of the corollary is part of Theorem 4.7. \square

V The tails of an affine stochastic recursion.

1) Notations and main result

Let H be the affine group of the d -dimensional Euclidean space V , i.e the set of maps f of V into itself of the form $f(x) = gx + b$ where $g \in GL(V) = G, b \in V$. Let λ be a probability measure on H , μ its projection on G . We consider the affine random walk on $V = \mathbb{R}^d$ defined by λ , i.e the Markov chain on V described by the stochastic recursion :

$$X_{n+1}^x = A_{n+1}X_n^x + B_{n+1}, \quad X_0^x = x \in V,$$

where (A_n, B_n) are H -valued i.i.d random variables with law λ . We denote $\widehat{\Omega} = H^{\mathbb{N}}$ and we endow $\widehat{\Omega}$ with the shift $\widehat{\theta}$ and the product measure $\widehat{\mathbb{P}} = \lambda^{\otimes \mathbb{N}}$; by abuse of notation the expectation symbol with respect to $\widehat{\mathbb{P}}$ will be denoted by \mathbb{E} . We have :

$$X_n^x = A_n \cdots A_1 x + \sum_1^n A_n \cdots A_{k+1} B_k.$$

We are interested in the case where $R_n = \sum_1^n A_1 \cdots A_{k-1} B_k$ converges $\widehat{\mathbb{P}} - a.e$ to a random variable R and X_n^x converges in law to R . We observe that $X_n^x - A_n \cdots A_1 x$ and R_n have the same law. In that case we have :

$$R = \sum_0^\infty A_1 \cdots A_k B_{k+1},$$

hence R satisfies the following equation : (S) $R = ARo\widehat{\theta} + B$, and the law ρ of R satisfies the convolution equation $\rho = \lambda * \rho = \int h \rho d\lambda(h)$. Also, if R is unbounded, we will be interested in the tail of R in direction u , *i.e* the asymptotics ($t \rightarrow \infty$) of $\widehat{\mathbb{P}}\{\langle R, u \rangle > t\}$ (resp $\widehat{\mathbb{P}}\{|\langle R, u \rangle| > t\}$ where $u \in \mathbb{S}^{d-1}$ (resp $u \in \mathbb{P}^{d-1}$). We are mainly interested in the "shape at infinity" of ρ *i.e* the asymptotics ($t \rightarrow 0_+$) of the measure $t.\rho$ where $t.\rho$ is the push-forward of ρ by the dilation $v \rightarrow tv$ in $V(t > 0)$. It turns out that this "shape at infinity" depends only of the semigroups T and Σ ; the main burden occurs when T preserves a convex cone but Σ does not (case II' below). A basic role will be played by the top Lyapunov exponent L_μ of the product of random matrices $S_n = A_n \cdots A_1$, and μ will be assumed to satisfy $\int \text{Log} \gamma(g) d\mu(g) < \infty$ where $\gamma(g) = \sup(|g|, |g^{-1}|)$. The main hypothesis will be on μ , which is always assumed to satisfy $L_\mu < 0$ and condition i-p of section 2 if $d > 1$, or μ non arithmetic if $d = 1$. We recall that the function $k(s)$ is defined on the interval $I_\mu \subset [0, \infty[$ by $k(s) = \lim_{n \rightarrow \infty} (\int |g|^s d\mu^n(g))^{1/n}$ and $\text{Log} k(s)$ is strictly convex. Also it will be assumed that $\text{supp} \lambda$ has no fixed point in V . Let Σ (resp T) be the closed subsemigroup of H (resp G) generated by $\text{supp} \lambda$ (resp $\text{supp} \mu$). We denote by $\Delta_a(\Sigma)$ the set of fixed attractive points of the elements of Σ , *i.e* fixed points $h^+ \in V$ of elements $h = (g, b) \in \Sigma$ such that $\lim_{n \rightarrow \infty} |g^n|^{1/n} < 1$. For $v \in V \setminus \{0\}$ we denote $H_v^+ = \{x \in V; \langle v, x \rangle > 1\}$ and, for a bounded measure ξ on V we consider its Radon transform $\widehat{\xi}$, *i.e* the function on $V \setminus \{0\}$ defined by $\widehat{\xi}(v) = \xi(H_v^+) = \widehat{\mathbb{P}}\{\langle R, u \rangle > t\}$. We also write $u = tv$ with $u \in \mathbb{S}^{d-1}$, $t > 0$ and $\widehat{\xi}(u, t) = \widehat{\xi}(\frac{u}{t})$. In particular, the directional tails of ξ are described by the function $\widehat{\xi}(v)$ ($v \rightarrow 0$). We start with the basic :

Proposition 5.1

Assume $L_\mu < 0$ and $\mathbb{E}(\text{Log}|B|) < \infty$. Then R_n converges $\widehat{\mathbb{P}} - a.e$ to $R = \sum_1^\infty A_1 \cdots A_{k-1} B_k$, and for any $x \in V$, X_n^x converges in law to R . If $\beta \in I_\mu$ satisfies $k(\beta) < 1$ and $\mathbb{E}(|B|^\beta) < \infty$, then $\mathbb{E}(|R|^\beta) < \infty$.

The law ρ of R is the unique λ -stationary measure on V . The closure $\overline{\Delta_a(\Sigma)} = \Lambda_a(\Sigma)$ is the unique Σ -minimal subset in V and is equal to $\text{supp}\rho$. If T contains an element g with $\lim_{n \rightarrow \infty} |g^n|^{1/n} > 1$ and T satisfies condition i-p then $\Lambda_a(\Sigma)$ is unbounded.

If furthermore $\text{supp}\lambda$ has no fixed point in V , then $\rho(W) = 0$ for any affine subspace W .

Proof

Under the conditions $L_\mu < 0$ and $\mathbb{E}(\text{Log}|B|) < \infty$ the $\widehat{\mathbb{P}}$ - a.e convergence of R_n is well known as well as the moment condition $\mathbb{E}(|R|^\beta) < \infty$ if $k(\beta) < 1$ (see for example [5]). We complete the argument by observing that, since $L_\mu < 0$, we have $\lim_{n \rightarrow \infty} |A_n \cdots A_1 x| = 0$ hence, since $X_n^x - A_n \cdots A_1 x$ has the same law as R_n , the convergence in law of X_n^x to R for any x follows. In particular, if $x \in V$ is distributed according to $\xi \in M^1(V)$, the law of X_n^x is $\lambda^n * \xi = \int \lambda^n * \delta_x d\xi(x)$, hence has limit ρ at $n = \infty$. If ξ is λ -stationary, we have $\lambda^n * \xi = \xi$, hence $\xi = \rho$.

Since $L_\mu < 0$, there exists $h = (g, b) \in \Sigma$, such that $|g| < 1$, hence $\lim_{n \rightarrow \infty} |g^n|^{1/n} < 1$.

If $h = (g, b) \in \Sigma$ satisfies $\lim_{n \rightarrow \infty} |g^n|^{1/n} < 1$, then $I - g$ is invertible, hence the unique fixed point h^+ of h satisfies $(I - g)h^+ = b$, and for any $x \in V$, $h^n x - h^+ = g^n(x - h^+)$, hence $\lim_{n \rightarrow \infty} h^n x = h^+$. Taking x in $\text{supp}\rho$ we get $h^+ \in \text{supp}\rho$, since $\text{supp}\rho$ is h -invariant.

Furthermore, for any $x \in V$ and $h' \in \Sigma$ we have $\lim_{n \rightarrow \infty} h' h^n x = h'(h^+)$ and $h' h^n \in \Sigma$ satisfies $\lim_{n \rightarrow \infty} |g' g^n| = 0$, hence the unique fixed point x_n of $h' h^n$ satisfies $\lim_{t \rightarrow \infty} x_n = h'(h^+)$. Then

$\overline{\Delta_a(\Sigma)} = \Lambda_a(\Sigma)$ is a Σ -invariant non trivial, closed subset of $\text{supp}\rho$.

On the other hand, for $x \in \Delta_a(\Sigma)$ we have : $\lim_{n \rightarrow \infty} \lambda^n * \delta_x = \rho$, $(\lambda^n * \delta_x)(\Lambda_a(\Sigma)) = 1$ for all n hence $\rho(\Lambda_a(\Sigma)) = 1$, i.e $\Lambda_a(\Sigma) = \text{supp}\rho$. The Σ -minimality of $\Lambda_a(\Sigma)$ follows from the fact that, for any $x \in V$ and $h = (g, b)$ with $|g| < 1$, one has $\lim_{t \rightarrow \infty} h^t x = h^+ \in \Lambda_a(\Sigma)$ hence

$\overline{\Sigma x} \supset \Lambda_a(\Sigma)$. This implies also the uniqueness of the Σ -minimal set.

We observe that, if $\text{supp}\rho$ is bounded, then $Co(\text{supp}\rho)$ is compact. Also any $h \in \Sigma$ preserves $\text{supp}\rho$ and $Co(\text{supp}\rho)$. Then Markov-Kakutani theorem implies that the affine map h has a fixed point h^0 in $Co(\text{supp}\rho)$. Using condition i-p we know that the vector space generated by $h^0 - \text{supp}\rho$ is equal to V . If $h = (g, b) \in \Sigma$ satisfies $\lim_{n \rightarrow \infty} |g^n|^{1/n} > 1$, then

$$(I - g)h^0 = b \quad \text{and} \quad h^n x - h^0 = g^n(x - h^0).$$

From above we can find $x \in \Lambda_a(\Sigma)$ with $x - h^0 \notin V_g^<$. Then $\Lambda_a(\Sigma)$ is unbounded since $\lim_{n \rightarrow \infty} |h^n x| = \infty$ and $h^n x \in \Lambda_a(\Sigma)$.

Let $\mathcal{W} = \{W_i ; i \in I\}$ be the set of affine subspaces of minimal dimension with $\rho(W_i) > 0$. Since $\dim(W_i \cap W_j) < \dim W_i$ if $i \neq j$, we have $\rho(W_i \cap W_j) = 0$, hence $\sum_{i \in I} \rho(W_i) \leq 1$. It

follows that, for any $\varepsilon > 0$, the set $\{W_j ; j \in I \text{ and } \rho(W_j) \geq \varepsilon\}$ has cardinality at most $\frac{1}{\varepsilon}$, hence $\rho(W_i)$ reaches its maximum on a finite set $\{W_j ; j \in J \subset I\}$ of affine subspaces. Then the stationarity equation $\lambda * \rho = \rho$ gives on such a subspace W_j :

$$\rho(W_j) = \int \rho(h^{-1}W_j)d\lambda(h).$$

Since $\rho(h^{-1}W_j) \leq \rho(W_j)$ we get, for any $h \in \text{supp}\lambda$, $j \in J$:

$$\rho(h^{-1}W_j) = \rho(W_j), \text{ i.e } h^{-1}W_j = W_i, \text{ for some } i \in J.$$

In other words the set $\{W_j ; j \in J\}$ is $\text{supp}\lambda$ -invariant. If $\dim W_j > 0$, one gets that the set of directions $\overline{W_j}(j \in J)$ is a $\text{supp}\mu$ -invariant finite set of subspaces of V , which contradicts condition i-p. Hence each $W_j(j \in J)$ is reduced to a point w_j . Then the barycenter of the finite set $\{w_j ; j \in J\}$ is invariant under $\text{supp}\lambda$, which contradicts the hypothesis. Hence $\rho(W) = 0$ for any affine subspace W . \square

In order to state the main result of this section we consider the compactification $V \cup \mathbb{S}_\infty^{d-1}$, the sets $\tilde{\Lambda}^\infty(T)$, $\Lambda_+^\infty(T)$, $\Lambda_-^\infty(T)$ defined in section 1. The closure $\overline{\Lambda_a(\Sigma)}$ of $\Lambda_a(\Sigma)$ in the compact space $V \cup \mathbb{S}_\infty^{d-1}$ is T -invariant hence $\overline{\Lambda_a(\Sigma)} \cap \mathbb{S}_\infty^{d-1}$, which is non void if $\text{supp}\rho = \Lambda_a(\Sigma)$ is unbounded, is a closed T -invariant subset of \mathbb{S}_∞^{d-1} .

Then Proposition 2.15 applied to $\overline{\Lambda_a(\Sigma)} \cap \mathbb{S}_\infty^{d-1} \subset \mathbb{S}_\infty^{d-1}$ gives the following trichotomy, since condition i-p is satisfied by T :

case I : T has no invariant proper convex cone and $\overline{\Lambda_a(\Sigma)} \supset \tilde{\Lambda}^\infty(T)$

case II' : T has an invariant proper convex cone and $\overline{\Lambda_a(\Sigma)} \supset \tilde{\Lambda}^\infty(T)$

case II'' : T has an invariant proper convex cone and $\overline{\Lambda_a(\Sigma)}$ contains only one of the sets $\Lambda_+^\infty(T)$, $\Lambda_-^\infty(T)$, say $\Lambda_+^\infty(T)$, hence $\overline{\Lambda_a(\Sigma)} \cap \Lambda_-^\infty(T) = \phi$.

As in Theorem 4.7', we consider the \tilde{P}^α -invariant measures $\tilde{\nu}^\alpha, \nu_+^\alpha, \nu_-^\alpha$.

In case $\alpha \in]0, s_\infty[$ exists with $k(\alpha) = 1$ (see Proposition 4.8 for a sufficient condition). The following corresponds to Theorem C of the introduction and describes the asymptotics of the probability measure $t.\rho(t \rightarrow 0_+)$:

Theorem 5.2

With the above notations assume $L_\mu < 0$, Σ has no fixed point in V , T satisfies condition i-p, there exists $\alpha \in]0, s_\infty[$ such that $k(\alpha) = 1$ and $\mathbb{E}(|B|^{\alpha+\delta}) < \infty$, $\mathbb{E}(|A|^\alpha \gamma^\delta(A)) < \infty$ for some $\delta > 0$. If $d = 1$ assume also that μ is non arithmetic.

Then $\text{supp}\rho$ is unbounded and we have the following vague convergence on $V \setminus \{0\}$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho) = \Lambda = C(\sigma^\alpha \otimes \ell^\alpha),$$

where $C > 0$, $\sigma^\alpha \in M^1(\tilde{\Lambda}(T))$ and the measure $\Lambda = C(\sigma^\alpha \otimes \ell^\alpha)$ satisfies $\mu * \Lambda = \Lambda$.

In case I, we have $\sigma^\alpha = \tilde{\nu}^\alpha$.

In case II', there exists $C_+, C_- > 0$ with $C\sigma^\alpha = C_+\nu_+^\alpha + C_-\nu_-^\alpha$

In case II'', $\sigma^\alpha = \nu_+^\alpha$

In case I the Radon measure $\tilde{\nu}^\alpha \otimes \ell^\alpha$ on $V \setminus \{0\}$ is a minimal μ -harmonic measure.

In case II, $\nu_+^\alpha \otimes \ell^\alpha$ and $\nu_-^\alpha \otimes \ell^\alpha$ are minimal μ -harmonic measures on $V \setminus \{0\}$.

Remarks

In general $\overline{\text{supp}(\sigma^\alpha \otimes \ell^\alpha)} \cap \mathbb{S}_\infty^{d-1}$ is smaller than $\overline{\text{supp}\rho} \cap \mathbb{S}_\infty^{d-1}$ and $\text{supp}\sigma^\alpha$ has a fractal structure.

As the proof below shows, if the moment condition on $|A|$ is replaced by $\mathbb{E}(|A|^\alpha \text{Log}\gamma(A)) < \infty$, the convergence remains valid on the sets H_v^+ .

2) Asymptotics of directional tails

We apply Theorem 4.7' to μ^* -potentials of suitable functions ; we pass, using the map $\eta \rightarrow \widehat{\eta}$, from the convolution equation $\lambda * \rho = \rho$ to a Poisson type equation on $V \setminus \{0\}$ which involves μ^* and $\widehat{\rho}$. The corresponding convergences will play an essential role in the proof of Theorem 5.2. We denote by ρ_1 the law of $R - B$ and we consider the signed measure $\rho_0 = \rho - \rho_1$, hence $\rho_0(V) = 0$. Also we show that ρ_0 is "small at infinity", we define $C, C_+, C_-, \sigma^\alpha$.

Proposition 5.3

With the hypothesis of Theorem 5.2, we denote by ${}^*\widetilde{\nu}_u^\alpha$ the positive kernel on \mathbb{S}^{d-1} given by Theorem 4.7' and associated with μ^* . Then one has the equations on $V \setminus \{0\}$:

$$\rho = \sum_0^\infty \mu^k * (\rho - \rho_1) \quad , \quad \widehat{\rho}(v) = \sum_0^\infty ((\mu^*)^k * \delta_v)(\widehat{\rho} - \widehat{\rho}_1).$$

For $u \in \mathbb{S}^{d-1}$, the function $t \rightarrow t^{\alpha-1} \widehat{\rho}_0(u, t)$ is Riemann-integrable in generalised sense on $]0, \infty[$ and, one has with $r_\alpha(u) = \int_0^\infty t^{\alpha-1} \widehat{\rho}_0(u, t) dt$:

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\rho}(u, t) = \frac{{}^*e^\alpha(u)}{L_\mu(\alpha)} {}^*\widetilde{\nu}_u^\alpha(r_\alpha) = C(\sigma^\alpha \otimes \ell^\alpha)(H_u^+)$$

where $C = 2 \frac{{}^*\widetilde{\nu}^\alpha(r_\alpha)\alpha}{L_\mu(\alpha)p(\alpha)} \geq 0$ and $\sigma^\alpha \in M^1(\widetilde{\Lambda}(T))$ satisfies $\mu * (\sigma^\alpha \otimes \ell^\alpha) = \sigma^\alpha \otimes \ell^\alpha$.

Furthermore $\text{supp}\rho$ is unbounded and,

In case I : $\sigma^\alpha = \widetilde{\nu}^\alpha$

In case II : ${}^*\widetilde{\nu}^\alpha(r_\alpha)\sigma^\alpha = \frac{1}{2}({}^*\nu_+^\alpha(r_\alpha)\nu_+^\alpha + {}^*\nu_-^\alpha(r_\alpha)\nu_-^\alpha)$ where ${}^*\nu_+^\alpha(r_\alpha) \geq 0, {}^*\nu_-^\alpha(r_\alpha) \geq 0$.

The proof will follow from a series of Lemmas.

We start with the following simple Tauberian Lemma (see [19]).

Lemma 5.4

For a non negative and non increasing function f on \mathbb{R}_+^* and $s \geq 0$, we denote : $f^s(t) = \frac{1}{t} \int_0^t x^s f(x) dx$. Then the condition $\lim_{t \rightarrow \infty} f^s(t) = c$ implies $\lim_{t \rightarrow \infty} t^s f(t) = c$.

Proof

Let b be a positive real number with $b > 1$ and let us observe that, since f is non increasing :

$$\frac{1}{t} \int_t^{bt} x^s f(x) dx \leq f(t) \frac{1}{t} \int_t^{bt} x^s dx = \frac{t^s}{s+1} (b^{s+1} - 1) f(t).$$

It follows :

$$\frac{b^{s+1}-1}{s+1} t^s f(t) \geq b f^s(bt) - f^s(t).$$

Then the hypothesis gives :

$$\liminf_{t \rightarrow \infty} f \frac{b^{s+1} - 1}{s + 1} t^s f(t) \geq (b - 1)c$$

Using the relation $\lim_{b \rightarrow 1} \frac{b^{s+1} - 1}{(s+1)(b-1)} = 1$ we get $\lim_{t \rightarrow \infty} \inf(t^s f(t)) \geq c$. An analogous argument gives $\lim_{t \rightarrow \infty} \sup(t^s f(t)) \leq c$. It follows $\lim_{t \rightarrow \infty} t^s f(t) = c$. \square

We will use the multiplicative structure of the group $\mathbb{R}_+^* =]0, \infty[$, and we recall that Lebesgue measure ℓ on the multiplicative group \mathbb{R}_+^* is given by $\frac{dt}{t}$.

Lemma 5.5

Assume that the V -valued random variable R satisfies equation (S), and $\mathbb{E}(|B|^{\alpha+\delta}) < \infty$, with $\delta > 0$. For $u \in \mathbb{S}^{d-1}$ and $t, x > 0$ we write : $r^\alpha(u, t) = \frac{1}{t} \int_0^t x^\alpha \widehat{\rho}_0(u, x) dx$. Then $|r^\alpha(u, t)| \leq \frac{t^\alpha}{\alpha+1}$. For any $\delta' \in]0, \frac{\delta}{\delta+\alpha+1}[$ there exists $C(\delta') > 0$ such that if $t \geq 1$, $|r^\alpha(u, t)| \leq C(\delta') t^{-\delta'}$. In particular the function $r^\alpha(u, t)$ is *b.R.i* on $\mathbb{S}^{d-1} \times \mathbb{R}_+^*$.

Proof

The inequality $|r^\alpha(u, t)| \leq \frac{t^\alpha}{\alpha+1}$ follows from $|\widehat{\rho}_0(u, t)| \leq 1$.

We write $\widehat{\rho}_0(u, x) = r_1(u, x) - r_2(u, x)$ with :

$$r_1(u, x) = \widehat{\mathbb{P}}\{x - \langle B, u \rangle < \langle R - B, u \rangle \leq x\},$$

$$r_2(u, x) = \widehat{\mathbb{P}}\{x < \langle R - B, u \rangle \leq x - \langle B, u \rangle\}$$

and $r_1^\alpha(u, t) = \frac{1}{t} \int_0^t x^\alpha r_1(u, x) dx$, $r_2^\alpha(u, t) = \frac{1}{t} \int_0^t x^\alpha r_2(u, x) dx$.

In order to estimate r_1^α , we choose $\varepsilon \in]0, 1[$ with $\varepsilon > \frac{\alpha}{\alpha+\delta}$ and write for $t \geq 2$:

$$r_1^\alpha(u, t) \leq \frac{1}{t} \int_2^t x^\alpha \widehat{\mathbb{P}}\{\langle B, u \rangle \geq x^\varepsilon\} dx + \frac{1}{t} \int_2^t x^\alpha \widehat{\mathbb{P}}\{x - x^\varepsilon < \langle R - B, u \rangle \leq x\} dx + \frac{2^\alpha}{t}.$$

Then Tchebitchev's inequality gives : $\widehat{\mathbb{P}}\{\langle B, u \rangle \geq x^\varepsilon\} \leq x^{-(\alpha+\delta)\varepsilon} \mathbb{E}(|B|^{\alpha+\delta})$.

Hence the first term $I_1^\varepsilon(t)$ in the above equality satisfies :

$$I_1^\varepsilon(t) \leq \mathbb{E}(|B|^{\alpha+\delta}) \frac{1}{t} \int_2^t x^{\alpha-\varepsilon(\alpha+\delta)} dx \leq \mathbb{E}(|B|^{\alpha+\delta}) t^{\alpha-\varepsilon(\alpha+\delta)}.$$

For $t - t^\varepsilon \geq 2$, the second term $I_2^\varepsilon(t)$ satisfies :

$$I_2^\varepsilon(t) = \frac{1}{t} \int_2^t x^\alpha \widehat{\mathbb{P}}\{\langle R - B, u \rangle > x - x^\varepsilon\} dx - \frac{1}{t} \int_2^{t-t^\varepsilon} x^\alpha \widehat{\mathbb{P}}\{\langle R - B, u \rangle > x\} dx.$$

In the second integral above we use the change of variables $x \rightarrow x - x^\varepsilon$ and we get :

$$I_2^\varepsilon(t) \leq \frac{1}{t} \int_2^t [x^\alpha - (x - x^\varepsilon)^\alpha (1 - \varepsilon x^{\varepsilon-1})] \widehat{\mathbb{P}}\{\langle R - B, u \rangle > x - x^\varepsilon\} dx + \frac{k_0(\varepsilon)}{t}$$

with $0 < k_0(\varepsilon) < \infty$.

We observe that there exists $k_1(\varepsilon) < \infty$ such that for any $x \geq 2$:

$$x^\alpha - (x - x^\varepsilon)^\alpha (1 - \varepsilon x^{\varepsilon-1}) \leq k_1(\varepsilon) x^{\alpha+\varepsilon-1}$$

For any $\beta \in]0, \alpha[$, Proposition 5.1 implies that $\mathbb{E}(|R|^\beta) < \infty$. Also R satisfies equation (S) and $A, R\theta$ are independant. Hence Tchebitchev's inequality gives :

$$\widehat{\mathbb{P}}\{\langle R - B, u \rangle > x\} \leq x^{-\beta} \mathbb{E}(|A|^\beta) \mathbb{E}(|R|^\beta) \leq k_2(\beta) x^{-\beta} \text{ with } k_2(\beta) < \infty.$$

It follows that for any t with $t - t^\varepsilon > 2$:

$$I_2^\varepsilon(t) \leq \frac{k_0(\varepsilon)}{t} + k_1(\varepsilon) k_2(\beta) \frac{1}{t} \int_2^t \frac{x^{\alpha+\varepsilon-1}}{(x-x^\varepsilon)^\beta} dx \leq k_3(\varepsilon, \beta) t^{\alpha-\beta+\varepsilon-1}$$

We can choose β very close to $\frac{\alpha+1}{\alpha+1+\delta} > \frac{\alpha}{\alpha+\delta-1}$ such that for $\delta' \in]0, \frac{\delta}{\delta+\alpha+1}[$ we have $\sup\{t^{\delta'} |r_1^\alpha(u, t)|; t \geq 1\} < \infty$, as follows from the above estimations of $I_1^\varepsilon(t)$ and $I_2^\varepsilon(t)$. Hence, there exists $k_3 < \infty$ such that for $t \geq 1$: $r_1^\alpha(u, t) \leq k_3 t^{-\delta'}$.

The same argument is valid for r_2^α , hence for $\delta' < \frac{\delta}{\delta+\alpha+1}$ and $t \geq 1$, we have : $r^\alpha(u, t) \leq C(\delta')t^{-\delta'}$, with $C(\delta') < \infty$. Furthermore, for $t \in]0, 1[$ we have $|r^\alpha(u, t)| \leq \frac{t^\alpha}{\alpha+1}$, hence the function $r^\alpha(u, t)$ is *b.R.i.* on $\mathbb{S}^{d-1} \times \mathbb{R}_+^*$ \square

Lemma 5.6

We denote by r the finite measure on \mathbb{R}_+^* defined by $r(dx) = 1_{]0,1[}(x)x^\alpha dx$ and we write $\rho_0 = \rho - \rho_1$. Then the function f_α on $V \setminus \{0\}$ defined by

$f_\alpha(v) = |v|^{-\alpha}(r * \widehat{\rho}_0)(v) = \frac{1}{t} \int_0^t x^\alpha \widehat{\rho}_0(u, x) dx$, $u = tv (t > 0, |u| = 1, v \in V \setminus \{0\})$ is *b.R.i.* and one has $(\delta_u \otimes \ell^\alpha)(r * \widehat{\rho}_0) = \int_0^\infty t^{\alpha-1} \widehat{\rho}_0(u, t) dt = r_\alpha(u)$ where $t \rightarrow t^{\alpha-1} \widehat{\rho}_0(u, t)$ is Riemann-integrable on $]0, \infty[$ in generalised sense.

Proof

By definition :

$$f_\alpha(v) = |v|^{-\alpha}(r * \widehat{\rho}_0)(v) = t^\alpha (r * \widehat{\rho}_0)\left(\frac{u}{t}\right) = \frac{1}{t} \int_0^t y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy = r^\alpha(u, t),$$

$$(\delta_u \otimes \ell^\alpha)(r * \widehat{\rho}_0) = \int_0^\infty t^{\alpha-1} \frac{dt}{t^{\alpha+1}} \int_0^t y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy = \lim_{T \rightarrow \infty} \int_0^T \frac{dt}{t^2} \int_0^t y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy.$$

Lemma 5.5 implies that $f_\alpha(v)$ is b.R.i and has limit 0 at $|v| = \infty$. Integration by parts in the above formula gives :

$$\int_0^T \frac{dt}{t^2} \int_0^t y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy = -\frac{1}{T} \int_0^T y^\alpha \widehat{\rho}_0\left(\frac{u}{y}\right) dy + \int_0^T t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right) dt = -r^\alpha(u, T) + \int_0^T t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right) dt.$$

Since, using Lemma 5.5 $\lim_{T \rightarrow \infty} r^\alpha(u, T) = 0$, it follows that $\int_0^T t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right) dt$ has a finite limit at $T = \infty$, hence $t \rightarrow t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right)$ is Riemann-integrable on \mathbb{R}_+^* in generalised sense and, for $u \in \mathbb{S}^{d-1}$:

$$(\delta_u \otimes \ell^\alpha)(r * \widehat{\rho}_0) = \int_0^\infty t^{\alpha-1} \widehat{\rho}_0\left(\frac{u}{t}\right) dt = r_\alpha(u). \quad \square$$

Proof of Proposition 5.3

If $u = tv$ with $u \in \mathbb{S}^{d-1}$, $t > 0$ the function $\widehat{\rho}(v) = \widehat{\mathbb{P}}\{\langle R, u \rangle > t\}$ is bounded, right continuous and non increasing. Since equation (S) can be written as $R - B = ARo\widehat{\theta}$ and $A, Ro\widehat{\theta}$ are independant we have : $\rho_1 = \mu * \rho$, $\rho - \mu * \rho = \rho - \rho_1 = \rho_0$. Furthermore :

$$\rho = \sum_0^n \mu^k * \rho_0 + \mu^{n+1} * \rho, \quad \widehat{\rho}(v) = \sum_0^n ((\mu^*)^k * \delta_v)(\widehat{\rho}_0) + ((\mu^*)^{n+1} * \delta_v)(\widehat{\rho}).$$

Also if r^- denote the push-forward of r by $x \rightarrow x^{-1}$:

$$r^- * \rho = \sum_0^n \mu^k * (r^- * \rho_0) + \mu^{n+1} * (r^- * \rho).$$

Since $L_\mu < 0$ the subadditive ergodic theorem applied to $\text{Log}|S_n(\omega)|$ gives the convergence of $S_n(\omega)v$ to 0. In particular, for $\xi \in M^1(V)$, the sequence $\mu^n * \xi$ converges in law to δ_0 hence : $\lim_{n \rightarrow \infty} \widehat{\mu^n * \xi}(v) = (\mu^*)^n * \delta_v(\widehat{\xi}) = 0$.

From the above convergence on V : $\rho - \delta_0 = \sum_0^\infty \mu^k * \rho_0$, $\widehat{\rho}(v) = \sum_0^\infty ((\mu^*)^k * \delta_v)(\widehat{\rho}_0)$.

But, by Proposition 5.1, $\rho(\{0\}) = 0$, hence the stated vague convergence of $\sum_0^\infty \mu^k * \rho_0$.

Since the sequence $(\mu^{n+1} * (r^- * \rho))(\psi)$ converges to zero for any bounded Borel function

ψ on V such that $\lim_{v \rightarrow 0} \psi(v) = 0$, we have on such functions : $r^- * \rho = \sum_0^\infty (\mu^*)^k * (r^- * \rho_0)$.

We observe that, for any bounded measure ξ , we have $(\mu^k * \widehat{\xi}(v) = ((\mu^*)^k * \delta_v)(\widehat{\xi})$ and $\widehat{r^- * \xi} = r * \widehat{\xi}$. It follows from the above equality that the potential $\sum_0^\infty ((\mu^*)^k * \delta_v)(r * \widehat{\rho}_0)$ is finite and equal to $(r * \widehat{\rho})(v)$.

We have observed in Lemma 5.6 that the function $v \rightarrow |v|^{-\alpha}(r * \widehat{\rho}_0)(v)$ is b.R.i, hence the renewal Theorem 4.7' applied to μ^* and to the function $r * \widehat{\rho}_0$ gives for $u \in \mathbb{S}^{d-1}$:

$$\lim_{t \rightarrow \infty} t^\alpha (r * \widehat{\rho})(u, t) = \frac{{}^*e^\alpha(u)}{L_\mu(\alpha)} ({}^*\widetilde{\nu}_u^\alpha \otimes \ell^\alpha)(r * \widehat{\rho}_0).$$

Since for fixed u , $\widehat{\rho}(u, x) = \mathbb{P}\{\langle R, u \rangle > x\}$ is non increasing, Lemma 5.4 gives :

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\rho}(u, t) = \frac{{}^*e^\alpha(u)}{L_\mu(\alpha)} {}^*\widetilde{\nu}_u^\alpha(r_\alpha).$$

In particular, we have ${}^*\widetilde{\nu}_u^\alpha(r_\alpha) \geq 0$. In case I this gives ${}^*\widetilde{\nu}^\alpha(r_\alpha) \geq 0$ since ${}^*\widetilde{\nu}_u^\alpha = {}^*\widetilde{\nu}^\alpha$. Also, in case II, taking $u \in \Lambda_+(T^*)$ and using ${}^*p_+^\alpha(u) = 1$, this gives ${}^*\nu_+^\alpha(r_\alpha) \geq 0$. Also, in the same way ${}^*\nu_-^\alpha(r_\alpha) \geq 0$. Furthermore, in case II, using Theorem 4.7' :

$${}^*\widetilde{\nu}_u^\alpha(r_\alpha) = {}^*p_+^\alpha(u) {}^*\nu_+^\alpha(r_\alpha) + {}^*p_-^\alpha(u) {}^*\nu_-^\alpha(r_\alpha).$$

If ${}^*\widetilde{\nu}^\alpha(r_\alpha) > 0$, in case II we can define a probability measure σ^α on $\widetilde{\Lambda}(T)$ by :

$${}^*\widetilde{\nu}^\alpha(r_\alpha) \sigma^\alpha = \frac{1}{2} ({}^*\nu_+^\alpha(r_\alpha) \nu_+^\alpha + {}^*\nu_-^\alpha(r_\alpha) \nu_-^\alpha)$$

while in case I : $\sigma^\alpha = \widetilde{\nu}^\alpha$. If ${}^*\widetilde{\nu}^\alpha(r_\alpha) = 0$, we have also ${}^*\nu_+^\alpha(r_\alpha) = {}^*\nu_-^\alpha(r_\alpha) = 0$, hence we can leave σ^α with projection ν^α on \mathbb{P}^{d-1} undefined in the above formulas. In any case $\sigma^\alpha \otimes \ell^\alpha$ is μ -harmonic.

We get another expression for the above limit, by using the formulas for ${}^*e^\alpha(u)$, ${}^*p_+^\alpha(u)$, ${}^*p_-^\alpha(u)$ of section 2, for case II as follows (see Theorem 2.17) :

$${}^*e^\alpha(u) {}^*p_+^\alpha(u) p(\alpha) = \int \langle u, u' \rangle_+^\alpha d\nu_+^\alpha(u') = \alpha (\nu_+^\alpha \otimes \ell^\alpha)(H_u^+).$$

From above, we get :

$${}^*e^\alpha(u) {}^*\widetilde{\nu}_u^\alpha(r_\alpha) = \frac{\alpha}{p(\alpha)} (({}^*\nu_+^\alpha(r_\alpha) \nu_+^\alpha + {}^*\nu_-^\alpha(r_\alpha) \nu_-^\alpha) \otimes \ell^\alpha)(H_u^+).$$

Hence, with $C = 2 \frac{{}^*\widetilde{\nu}^\alpha(r_\alpha) \alpha}{L_\mu(\alpha) p(\alpha)}$, σ^α as above and $C_+ = \frac{{}^*\widetilde{\nu}_+^\alpha(r_\alpha) \alpha}{L_\mu(\alpha) p(\alpha)}$, $C_- = \frac{{}^*\widetilde{\nu}_-^\alpha(r_\alpha) \alpha}{L_\mu(\alpha) p(\alpha)}$

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\rho}(u, t) = C (\sigma^\alpha \otimes \ell^\alpha)(H_u^+), \quad C \sigma^\alpha = C_+ \nu_+^\alpha + C_- \nu_-^\alpha.$$

In case I we get the corresponding formula.

In order to show that $supp \rho$ is unbounded, in view of Proposition 5.1, it suffices to show that there exists $g \in T$ with $\lim_{n \rightarrow \infty} |g^n|^{1/n} > 1$. If not, then the trace $Tr g$ of g is bounded by d on T . On the other hand, condition i-p implies the irreducibility of the action of T on $V \otimes \mathbb{C}$, as shown now. Let $W \subset V \otimes \mathbb{C}$ be a proper T -invariant subspace of $V \otimes \mathbb{C}$ and \overline{W} its complex conjugate. Then $W \cap \overline{W}$ and $W + \overline{W}$ are complexified subspaces of subspaces of V which are also T -invariant. Using irreducibility of T we get $W \cap \overline{W} = \{0\}$, $W + \overline{W} = V \otimes \mathbb{C}$ hence $V \otimes \mathbb{C} = W \oplus \overline{W}$. Let $g \in T^{prox}$ and $v \in V \setminus \{0\}$ with $gv = \lambda_g v$ and write $v = w + \bar{w}$ with $w \in W$, hence since W, \overline{W} are g -invariant $gw = \lambda_g w$, $g\bar{w} = \lambda_g \bar{w}$. Since λ_g is a simple eigenvalue we get $w = \bar{w}$ i.e $W \cap \overline{W} \neq \{0\}$ which gives a contradiction with condition i-p, hence T acts irreducibly on $V \otimes \mathbb{C}$. Then Burnside's density theorem

implies that $End(V \otimes \mathbb{C})$ is generated as an algebra by T , i.e there exists a base $g_i \in T$ ($i = 1, \dots, d^2$) of $EndV$. Then the linear forms $u \rightarrow Tr(ug_i)$ ($i = 1, \dots, d^2$) form a basis of the dual space of $EndV$. In particular, for some constant $c > 0$ we have for any $g \in T$, $|g| \leq c \sum_{i=1}^{d^2} |Tr(gg_i)| \leq cd^3$. Then for any $n \in \mathbb{N}$, and $s > 0$: $\int |g|^s d\mu^n(g) \leq cd^{3s}$, $k(s) \leq 1$. Since $k'(\alpha) > 0$ with $\alpha \in]0, s_\infty[$, this contradicts the convexity of $Logk(s)$. \square

Corollary 5.7

For any $v \in V \setminus \{0\}$, we have :

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{| \langle R, v \rangle | > t\} = C \frac{p(\alpha)}{\alpha} (*e^\alpha \otimes h^\alpha)(v)$$

In particular, there exists $b > 0$ such that $\widehat{\mathbb{P}}\{|R| > t\} \leq bt^{-\alpha}$.

Proof

By definition of $*\tilde{\nu}_u^\alpha$ and since $*p_+^\alpha(u) = *p_-^\alpha(-u)$ we have : $\frac{1}{2}(*\nu_u^\alpha + *\nu_{-u}^\alpha) = *\tilde{\nu}^\alpha$. Hence, using the proposition :

$$\lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{| \langle R, u \rangle | > t\} = 2 \frac{*e^\alpha(u)}{L_\mu(\alpha)} *\tilde{\nu}^\alpha(r_\alpha) = C \frac{p(\alpha)}{\alpha} *e^\alpha(u).$$

The formula in the corollary follows by α -homogeneity.

We take a base $u_i \in V$ ($1 \leq i \leq d$) and write $|R| \leq b' \sum_{i=1}^d | \langle R, u_i \rangle |$ with $b' > 0$. For t large : $\widehat{\mathbb{P}}\{|R| > t\} \leq \sum_{i=1}^d \widehat{\mathbb{P}}\{| \langle R, u_i \rangle | > \frac{t}{b'}\} \leq (C' + \varepsilon)t^{-\alpha} b'^\alpha \sum_{i=1}^d *e^\alpha(u_i)$, with $\varepsilon > 0$, $C' = C \frac{p(\alpha)}{\alpha}$, hence the result. \square

3) A dual Markov walk and the positivity of directional tails

Here we go over from the process $\langle R_n, v \rangle$ to its extension $(S'_n v, r + \langle R_n, v \rangle)$ and we study the corresponding generalized ladder process (see([15]), p 391).

Let M be a T^* -minimal subset of \mathbb{S}^{d-1} and $X = M \times \mathbb{R}$, hence $M = \tilde{\Lambda}(T^*)$ in case I and $M = \Lambda_+(T^*)$ (or $\Lambda_-(T^*)$) in case II. We denote by $\Lambda_a^*(\Sigma)$ the set of $u \in \mathbb{S}^{d-1}$ such that the projection of ρ on the line $\mathbb{R}u$ has unbounded support in direction u . The following will play an essential role in the discussion of positivity for C, C_+, C_- .

Proposition 5.8

With the hypothesis of Theorem 5.2 if $M \subset \mathbb{S}^{d-1}$ is T^* -minimal and $\Lambda_a^*(\Sigma) \supset M$, then for any $u \in M$:

$$C_M(u) = \lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{\langle R, u \rangle > t\} > 0.$$

We begin by introducing notations for the proof. We observe that $R_n = \sum_0^{n-1} A_1 \cdots A_k B_{k+1}$ satisfies the relation $\langle R_{n+1}, v \rangle = \langle R_n, v \rangle + \langle B_{n+1}, S'_n v \rangle$ where $S'_n = (A_1 \cdots A_n)^*$. Also $h \in H$ acts on $E = (V \setminus \{0\}) \times \mathbb{R}$ according to the formula $h(v, r) = (g^*v, r + \langle b, v \rangle)$, hence the pair $(S'_n v, r + \langle R_n, v \rangle)$ is a λ -random walk on the right homogeneous H -space E . If D is the orientation preserving affine group of the line, we note that the above

action of H commutes with the D -action on E defined by $d(v, r) = (tv, tr + x)$ where $d = (t, x) \in D$, hence $(S'_n v, r + \langle R_n, v \rangle)$ is also a D -fibered Markov chain over \mathbb{S}^{d-1} . The study of this Markov chain will play an essential role below.

The random walk $(S'_n v, r + \langle R_n, v \rangle)$ has projection $S'_n v$ on V , hence ${}^* \widehat{P}({}^* e^\alpha \otimes h^\alpha) = {}^* e^\alpha \otimes h^\alpha$, and we can consider the new relativized fibered Markov kernel ${}^* \widehat{P}_\alpha$ on $E = (M \times \mathbb{R}) \times \mathbb{R}^*$. The projection x_n of $(S'_n v, r + \langle R_n, v \rangle)$ on $X = M \times \mathbb{R}$ has a corresponding kernel ${}^* \widehat{Q}^\alpha$ given by ${}^* \widehat{Q}^\alpha \varphi(u, p) = \int \varphi(g^* \cdot u, h^u p) {}^* q^\alpha(u, g) d\lambda(h)$ where $h^u p = \frac{1}{|g^* u|} (p + \langle b, u \rangle)$ and ${}^* q^\alpha$ corresponds to q^α as in section 3. It will turn out that ${}^* \widehat{Q}^\alpha$ and its extension ${}^* \widehat{Q}^\alpha$ to $M \times \mathbb{R}$ have similar stochastic properties.

We observe that, if H_u^p is the affine hyperplane defined by $\langle y, u \rangle + p = 0$, and $h \in H$, the hyperplane $h^{-1}(H_u^p)$ is defined by $\langle y, g^* \cdot u \rangle + h^u p = 0$, hence the formula $h(u, p) = (g^* \cdot u, h^u p)$ corresponds to the action of h^{-1} on the space W of affine oriented hyperplanes in V . We will consider on $\widehat{\Omega}$ the projective limit ${}^* \widehat{Q}_u^\alpha$ of the system ${}^* q_n^\alpha(u, \cdot) \lambda^{\otimes n}$ and, by abuse of notation, the corresponding expectation will be written \mathbb{E}_u^α . If a ${}^* \widehat{Q}^\alpha$ -stationary measure $\tilde{\pi}_M^\alpha$ is given on M , we write ${}^* \widehat{Q}^\alpha = \int {}^* \widehat{Q}_u^\alpha d\tilde{\pi}_M^\alpha(u)$ and we denote by \mathbb{E}^α the corresponding expectation symbol. If η is a probability measure on X , the associated Markov measure on ${}^a \widehat{\Omega} = X \times \widehat{\Omega}$, is denoted by ${}^* \widehat{Q}_\eta^\alpha$, and the extended shift by ${}^a \widehat{\theta}$ where ${}^a \widehat{\theta}(x, \widehat{\omega}) = (h_1 x, \widehat{\theta} \widehat{\omega})$. It will be convenient to use the functions $a(g, u), b(h, u)$ defined by $h^u p = a(g, u)p + b(h, u)$, and the random variables a_k, b_k defined by $a_k(\widehat{\omega}, u) = a(g_k, S'_{k-1} \cdot u)$, $b_k(\widehat{\omega}, u) = b(h_k, S'_{k-1} \cdot u)$. Then we can express the action of $h_n \cdots h_1 \in H$ on X as : $u_n = S'_n \cdot u$, $y_n^p(u) = (h_n \cdots h_1)^u p$, where :

$$y_n^p(u) = a(S'_n, u)p + y_n^\circ(u) \text{ and } y_n^\circ(u) = \sum_1^n a_{k+1}^n(u) b_k(u)$$

with $a_k^n(u) = a(g_n \cdots g_k, S'_{k-1} \cdot u)$. The random variables a_k, b_k are ${}^* \widehat{Q}^\alpha$ -stationary and y_n° has the same law as $p_n^\circ(\widehat{\omega}, u) = \sum_0^{n-1} a_1 \cdots a_k b_{k+1}$.

Also if η is ${}^* \widehat{Q}^\alpha$ -stationary we will consider the bilateral associated system $(\Omega^\#, {}^a \widehat{\theta}, \eta^\#)$ where $\Omega^\# = X \times H^\mathbb{Z}$, ${}^a \widehat{\theta}$ is the bilateral shift and $\eta^\#$ is the unique ${}^a \widehat{\theta}$ -invariant measure with projection ${}^* \widehat{Q}_\eta^\alpha$ on $X \times \widehat{\Omega}$. The Birkhoff sum $\text{Log}(|S'_n u| |\frac{p_n}{p}|)$ occurs below and can be extended as an \mathbb{R} -valued \mathbb{Z} -cocycle on X , again denoted by the same formula. The following lemma is a fibered version of a well known fact in the context of generalized autoregressive processes (see [5]).

Lemma 5.9

Let M be a T^* -minimal subset of \mathbb{S}^{d-1} , π_M^α the unique ${}^* \widehat{Q}^\alpha$ -stationary measure on M . With the above notations, we consider the Markov chain $x_n = (u_n, p_n)$, on $M \times \mathbb{R}$ given by :

$$u_{n+1} = g_{n+1}^* \cdot u_n, \quad p_{n+1} = \frac{p_n + \langle b_{n+1}, u_n \rangle}{|g_{n+1}^* u_n|}, \quad p_0 = p, \quad u_0 = u,$$

where (g_n, b_n) are distributed according to ${}^* \widehat{Q}_u^\alpha$. Then x_n converges in ${}^* \widehat{Q}^\alpha$ -law to the unique ${}^* \widehat{Q}^\alpha$ -stationary measure κ , the projection of κ on M is π_M^α , $\kappa(M \times \{p\}) = 0$ for

any $p \in \mathbb{R}$, and $\int |p|^\varepsilon d\kappa(u, p) < \infty$ for ε small. We have $\kappa^\# - a.e$:

$$\limsup_{n \rightarrow \infty} |S'_n u| |p_n| = \infty, \quad \lim_{n \rightarrow \infty} |S'_{-n} u| |p_{-n}| = 0.$$

If $\Lambda_a^*(\Sigma) \supset M$, then $\kappa(M \times]0, \infty[) > 0$ and $\limsup_{n \rightarrow \infty} |S'_n u| |p_n| = \infty$, ${}^* \widehat{\mathcal{Q}}_\kappa^\alpha - a.e$.

Proof

We estimate $\mathbb{E}^\alpha(|p_n^\circ|^\varepsilon)$ for $0 < \varepsilon < \delta$ and ε small, where $p_n^\circ(\widehat{\omega}, u) = \sum_0^{n-1} a_1 \cdots a_k b_{k+1}$. Since $(a_1 \cdots a_k)(\omega, u) = a(S'_k(\omega), u)$ we get $\mathbb{E}^\alpha(|a_1 \cdots a_k|^\varepsilon) = \mathbb{E}^\alpha(|S'_k u|^{-\varepsilon})$. Hence Corollary 3.18 gives $\lim_{k \rightarrow \infty} (\mathbb{E}^\alpha(|a_1 \cdots a_k|^\varepsilon))^{1/k} < 1$. Also for ε small,

$$\mathbb{E}^\alpha(|b_k|^\varepsilon) = \mathbb{E}^\alpha(|< \frac{b_{1,u}}{|g^* u}| >|^\varepsilon) \leq \mathbb{E}(|B_1|^\varepsilon) \gamma^\varepsilon(A) < \infty,$$

using ${}^* \widehat{\mathcal{Q}}^\alpha$ -stationarity, Hölder inequality and the condition $\mathbb{E}(|B_1|^{\alpha+\delta}) + \mathbb{E}(|A_1|^{\alpha+\delta}) < \infty$.

Since $|p_n^\circ(\widehat{\omega}, u)|^\varepsilon \leq \sum_0^{n-1} |a_1 \cdots a_k|^\varepsilon |b_{k+1}|^\varepsilon$, we get that $\mathbb{E}^\alpha(|p_n^\circ|^{\varepsilon/2})$ is bounded. The ${}^* \widehat{\mathcal{Q}}^\alpha - a.e$

convergence of the partial sum $p_n^\circ(\widehat{\omega}, u)$ to $p(\widehat{\omega}, u) = \sum_0^\infty a_1 \cdots a_k b_{k+1}$ and the finiteness of

$\mathbb{E}^\alpha(|p|^\varepsilon)$ follows. By definition, $p(\widehat{\omega}, u)$ satisfies the functional equation : $p = a_1 p \circ {}^a \widehat{\theta} + b_1$. It follows that the probability measure κ on $M \times \mathbb{R}$ given by $\kappa = \int \delta_u \otimes \delta_{p(\widehat{\omega}, u)} d {}^* \widehat{\mathcal{Q}}^\alpha(\widehat{\omega}, u)$ is ${}^* \widehat{\mathcal{Q}}^\alpha$ -invariant.

As observed above, the ${}^* \widehat{\mathcal{Q}}^\alpha$ -laws of y_n° and p_n° are the same. Since the product of $\widetilde{\pi}_M^\alpha$ with the law of y_n° is $({}^* \widehat{\mathcal{Q}}^\alpha)^n (\pi_M^\alpha \otimes \delta_\circ)$ we have in weak topology : $\lim_{n \rightarrow \infty} ({}^* \widehat{\mathcal{Q}}^\alpha)^n (\pi_M^\alpha \otimes \delta_\circ) = \kappa$.

Since $|y_n^p(\widehat{\omega}, u) - y_n^{p'}(\widehat{\omega}, u)| = a(S'_n(\omega), u) |p - p'|$ and $a(S'_n(\omega), u) = |S'_n u|^{-1}$ converges ${}^* \widehat{\mathcal{Q}}_u^\alpha - a.e$ to zero, we get the convergence of $({}^* \widehat{\mathcal{Q}}^\alpha)^n (\pi_M^\alpha \otimes \delta_p)$ to κ , for any p . On the other hand, if η' is a ${}^* \widehat{\mathcal{Q}}^\alpha$ -stationary measure on $M \times \mathbb{R}$, its projection on M is ${}^* \widehat{\mathcal{Q}}^\alpha$ -stationary, hence equal to π_M^α , since M is T^* -minimal. Then, from above $({}^* \widehat{\mathcal{Q}}^\alpha)^n \eta'$ converges to κ , hence $\eta' = \kappa$. The ${}^* \widehat{\mathcal{Q}}^\alpha$ -ergodicity of κ implies the ${}^a \widehat{\theta}$ -ergodicity of ${}^* \widehat{\mathcal{Q}}_\kappa^\alpha$ and $\kappa^\#$.

Assume $\kappa(M \times]0, \infty[) = 0$, i.e the ${}^* \widehat{\mathcal{Q}}^\alpha$ -invariant set $supp \kappa$ is contained in $M \times]-\infty, 0]$. Then, for any $(u, p) \in supp \kappa$: $p+ < R_n, u > \leq 0$, ${}^* \widehat{\mathcal{Q}}_u^\alpha - a.e$, i.e $< R_n, u > \leq -p$ ${}^* \widehat{\mathcal{Q}}_u^\alpha - a.e$ for any $n \in \mathbb{N}$, for some $p \leq 0$, $u \in M$. It follows $< R_n, u > \leq -p$, $\lambda^{\otimes n} - a.e$, and $< R, u > \leq -p$, $\widehat{\mathbb{P}} - a.e$. This implies that the support of the projection of ρ on $\mathbb{R}u$ is bounded in direction u , hence contradicts the condition $\Lambda_a^*(\Sigma) \supset M$. Furthermore, arguments as in the proof of Proposition 5.1, using that $supp \lambda$ has no fixed point in V , show $\kappa(M \times \{p\}) = 0$

for any $p \in \mathbb{R}$. From Theorem 3.10 we know that $\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} |S'_n u| = L_\mu(\alpha) > 0$, ${}^* \widehat{\mathcal{Q}}_u^\alpha - a.e$. Furthermore since κ is ${}^* \widehat{\mathcal{Q}}^\alpha$ -ergodic and $\kappa(M \times \{0\}) = 0$ we have $\limsup_{n \rightarrow \infty} |p_n| > 0$ ${}^* \widehat{\mathcal{Q}}_\kappa^\alpha - a.e$.

Then we get $\limsup_{n \rightarrow \infty} |S'_n u| |p_n| = \infty$ ${}^* \widehat{\mathcal{Q}}_\kappa^\alpha - a.e$. If $\Lambda_a^*(\Sigma) \supset M$, we have $\kappa(M \times]0, \infty[) > 0$,

and again using ergodicity : $\limsup_{n \rightarrow \infty} p_n > 0$. Since $\lim_{n \rightarrow \infty} |S'_n u| = \infty$ ${}^* \widehat{\mathcal{Q}}_u^\alpha - a.e$, it follows

$\limsup_{n \rightarrow \infty} |S'_n u| |p_n| = \infty$ ${}^* \widehat{\mathcal{Q}}_\kappa^\alpha - a.e$. Using Theorem 3.2, we have also , for any $u \in M$ and

${}^*\widehat{\mathbb{Q}}_u^\alpha$ -a.e : $\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}|S'_{-n}u| = -L_\mu(\alpha) < 0$. The condition $\int |p|^\varepsilon d\kappa(u, p) < \infty$ implies :
 $\limsup_{n \rightarrow \infty} \frac{\text{Log}|p_n|}{|n|} \leq 0$ $\kappa^\#$ -a.e. Then we get : $\lim_{n \rightarrow \infty} |S'_{-n}u| |p_{-n}| = 0$ $\kappa^\#$ -a.e. \square

Remarks

The moment condition on A can be replaced by the finiteness of the expectation of $|A|^\alpha \text{Log}\gamma(A)$, without changing the conclusion. The proof uses Theorem 3.10 instead of Corollary 3.18

We consider the $\mathbb{N} \cup \{\infty\}$ -valued optional time τ on $X \times \widehat{\Omega}$ given for $p \neq 0$ by :

$$\tau = \text{Inf}\{n > 0 ; p^{-1} < R_n, u \gg > 0\}.$$

We observe that τ is independant of p as long as $p > 0$ or $p < 0$.

By definition of $p_n : \tau = \text{Inf}\{n > 0 ; p^{-1}p_n|S'_n u| > 1\}$, in particular $p^{-1}p_\tau > 0$. Also we define $\tau_n = \tau \circ ({}^a\widehat{\theta})^{\tau_{n-1}}$, so that τ can be interpreted as the first ladder epoch and $p^{-1}p_\tau|S'_\tau u|$ as the first ladder height of the \mathbb{Z} -cocycle $p^{-1}p_n|S'_n u|$. These two descriptions of τ will be used below. On the other hand, by Poincaré recurrence theorem we have, $\kappa^\#$ -a.e $\limsup_{n \rightarrow \infty} p^{-1}p_n > 0$; since $\lim_{n \rightarrow \infty} |S'_n u| = \infty$, τ, τ_n are finite ${}^*\widehat{\mathbb{Q}}_\kappa^\alpha$ -a.e and $p^{-1}p_{\tau_n} > 0$.

It will be convenient to use (u, p, t') coordinates with $t' = r^{-1} \in \mathbb{R}^*$, $p|v| = r \in \mathbb{R}$ so that, the λ -random walk on E can be written as :

$$u_{n+1} = g_{n+1}^* \cdot u_n, \quad p_{n+1} = \frac{p_n + \langle \widehat{b}_{n+1}, u_n \rangle}{|g_{n+1}^* u_n|}, \quad t'_{n+1} = t'_n |g_{n+1}^* u_n|^{-1} p_n p_{n+1}^{-1}$$

hence $r + \langle R_n, v \rangle = p_n |S'_n v|$, $t'_n = t' (|S'_n u| \frac{p_n}{p})^{-1}$, $p_n = \frac{p + \langle R_n, u \rangle}{|S'_n u|}$.

We consider also the stopped operator ${}^*\widehat{P}^\tau$ on $V \setminus \{0\} \times \mathbb{R}$. In (u, p, t') coordinates with $x = (u, p)$ the associated process at time n starting from $(u, p, t') \in X \times \mathbb{R}_+^*$ is $(x_{\tau_n}, t'|S'_{\tau_n} u|^{-1} p_{\tau_n}^{-1} p)$, hence ${}^*\widehat{P}^\tau$ is a Markov fibered operator on $X \times \mathbb{R}_+^*$. Since ${}^*\widehat{P}({}^*e^\alpha \otimes h^\alpha) = {}^*e^\alpha \otimes h^\alpha$ and τ is finite ${}^*\mathbb{Q}_\kappa^\alpha$ -a.e, the positive kernel ${}^*\widehat{P}_\alpha^\tau$ given by ${}^*\widehat{P}_\alpha^\tau \varphi = ({}^*e^\alpha \otimes h^\alpha)^{-1} \widehat{P}^\tau ({}^*e^\alpha \otimes h^\alpha \varphi)$ is a fibered kernel on $X \times \mathbb{R}_+^*$ which satisfies ${}^*\widehat{P}_\alpha^\tau 1 = 1, \kappa \otimes \ell$ a.e.

The following lemma expresses the function $\widehat{\rho}(p^{-1}v) = \widehat{\mathbb{P}}\{p^{-1} < R, u \gg > t\} = \psi(v, p)$ as a ${}^*\widehat{P}^\tau$ -potential of a non negative function. Its asymptotics will give the positivity of $C_M(u)$ in Proposition 5.8.

Lemma 5.10

With $tv = u \in \mathbb{S}^{d-1}$, $t > 0$ $p \neq 0$, we write : $\tau = \text{Inf}\{n > 0 ; p^{-1} < R_n, u \gg > 0\}$

$\psi(v, p) = \widehat{\mathbb{P}}\{p^{-1} < R, u \gg > t\}$, $\psi_\tau(v, p) = \widehat{\mathbb{P}}\{t < p^{-1} < R, u \gg \leq t + p^{-1} < B^\tau, u \gg ; \tau < \infty\}$

where $B^\tau = \sum_0^{\tau-1} A_1 \cdots A_k B_{k+1}$, $\psi^\alpha = ({}^*e^\alpha \otimes h^\alpha)^{-1} \psi$, $\psi_\tau^\alpha = ({}^*e^\alpha \otimes h^\alpha)^{-1} \psi_\tau$. Then :

$$\psi = \sum_0^\infty ({}^*\widehat{P}^\tau)^k \psi_\tau, \quad \psi^\alpha = \sum_0^\infty ({}^*\widehat{P}_\alpha^\tau)^k \psi_\tau^\alpha.$$

Proof

We write $\langle R - R_n, v \rangle = \langle R^n, S'_n v \rangle$, hence if $\tau < \infty : \langle R - B^\tau, v \rangle = \langle R^\tau, S'_\tau v \rangle$. By

definition of τ , since $t > 0$:

$$\psi_\tau(v, p) = \psi(v, p) - \widehat{\mathbb{P}}\{\langle R - B^\tau, u \rangle p^{-1} > t ; \tau < \infty\}$$

On the other hand, from above, since $p^{-1}p_\tau > 0$:

$$\widehat{\mathbb{P}}\{\langle R - B^\tau, u \rangle p^{-1} > t ; \tau < \infty\} = \widehat{\mathbb{P}}\{\langle R^\tau, u_\tau \rangle p^{-1} > \frac{t}{|S'_\tau u|} ; \tau < \infty\} = {}^*\widehat{P}^\tau \psi(v, p),$$

since $p^{-1}p_\tau > 0$. It follows $\psi_\tau = \psi - {}^*\widehat{P}^\tau \psi$, $\psi = \sum_0^{n-1} ({}^*\widehat{P}^\tau)^k \psi_\tau + {}^*\widehat{P}^{\tau_n} \psi$ with :

$${}^*\widehat{P}^{\tau_n} \psi(v, p) = \widehat{\mathbb{P}}\{t |S'_{\tau_n} u|^{-1} < p^{-1} < R^{\tau_n}, u_{\tau_n} \rangle ; \tau_n < \infty\}.$$

On the set $\{\tau_n = \infty\}$, we have ${}^*\widehat{P}^{\tau_n} \psi = 0$, while on the set $\lim_{n \rightarrow \infty} \tau_n = \infty$, we have $\widehat{\mathbb{P}} - a.e$

$\lim_{n \rightarrow \infty} |S'_{\tau_n} u|^{-1} = \infty$. Since $|R| < \infty$, $\widehat{\mathbb{P}} - a.e$ and R^{τ_n} is $\widehat{\mathbb{P}}$ -independant of S'_{τ_n} we get

$$\lim_{n \rightarrow \infty} {}^*\widehat{P}^{\tau_n} \psi = 0, \psi = \sum_0^\infty ({}^*\widehat{P}^\tau)^k \psi_\tau.$$

The last relation follows from the definitions of ψ_τ^α and ${}^*\widehat{P}_\alpha^\tau$. \square

Let τ be as above, $\Lambda_a^*(\Sigma) \supset M$ and let ${}^*\widehat{Q}^{\alpha, \tau}(x, \cdot)$ be the law of x_τ under ${}^*\widehat{Q}_x^\alpha$. Since $\tau < \infty$, ${}^*\widehat{Q}^\alpha - a.e$ and $p^{-1}p_\tau > 0$ we have $p_\tau > 0$ if $p > 0$, hence the operator ${}^*\widehat{Q}^{\alpha, \tau}$ preserves $X_+ \equiv M \times]0, \infty[$ and satisfies ${}^*\widehat{Q}^{\alpha, \tau} 1 = 1, \kappa - a.e$. We will also consider the first return time to X_+ and the corresponding Markov operator ${}^*\widehat{Q}_+^\alpha$ on X_+ . Following closely ([34], Lemma 2) we can construct a stationary measure for ${}^*\widehat{Q}^{\alpha, \tau}$ on X_+ , and show the finiteness of the corresponding expectation of τ

Lemma 5.11

Assume $\Lambda_a^*(\Sigma) \supset M$ and write :

$$\tau = \text{Inf}\{n > 0 ; p^{-1}p_n |S'_n u| > 1\} = \text{Inf}\{n > 0 ; p^{-1} < R_n, u \rangle > 0\}.$$

Then the stopped operator ${}^*\widehat{Q}^{\alpha, \tau}$ preserves X_+ and admits a stationary ergodic measure κ^τ on X_+ which is absolutely continuous with respect to κ .

The integral $\mathbb{E}_0^\alpha(\tau) = \int \mathbb{E}_u^\alpha(\tau) d\kappa^\tau(u, p)$ is finite.

If $\gamma_\tau^\alpha = L_\mu(\alpha) \mathbb{E}_0^\alpha(\tau)$ one has $\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}(|S'_{\tau_n} u| \frac{p_{\tau_n}}{p}) = \gamma_\tau^\alpha \in]0, \infty[, {}^*\widehat{Q}_{\kappa^\tau}^\alpha - a.e$.

Also $V_\tau = p_\tau p^{-1} \text{Log}|S'_\tau u|$ has finite expectation with respect to ${}^*\widehat{Q}_{\kappa^\tau}^\alpha$.

Proof

Since $\Lambda_a^*(\Sigma) \supset M$, lemma 5.9 gives $\kappa(X_+) > 0$. In order to deal only with positive values of the \mathbb{Z} -cocycle $|S'_n u| p_n p^{-1}$ it is convenient to consider the two sided Markov chain $x_{n_k} (k \in \mathbb{Z}, x_\circ \in X_+)$, induced on X_+ by x_n hence n_1 is the first return time of x_n to X_+ and $n_1 \leq \tau$ since $p^{-1}p_\tau > 0$. We note that the normalized restriction κ_+ of κ is a stationary ergodic measure for x_{n_k} . Also the Markov kernel ${}^*\widehat{P}_\alpha$ on $X \times \mathbb{R}^*$ induces a fibered Markov kernel ${}^*\widehat{P}_{\alpha,+}$ on $X_+ \times \mathbb{R}^*$. The corresponding bilateral Markov chain is (x_{n_k}, t'_{n_k}) with $t'_{n_k} = t' p p_{n_k}^{-1} |S'_{n_k} u|^{-1}$. We observe that, if $t' > 0, p > 0$, then $t' p_{n_k} p^{-1} |S'_{n_k} u| > 0$, hence ${}^*\widehat{P}_{\alpha,+}$ preserves $X_+ \times \mathbb{R}_+^*$ and ${}^*\widehat{P}_{\alpha,+}(\kappa_+ \otimes \ell) = \kappa_+ \otimes \ell$. We denote by $\Omega_+^\#$ the subset of $\Omega^\#$

defined by the conditions $x \in X_+$, $x_n \in X_+$ infinitely often for $n > 0$ and $n < 0$, by $\kappa_+^\#$ the normalized restriction of $\kappa^\#$ to $\Omega_+^\#$ and by ${}^a\hat{\theta}_+$ the induced shift. Also let $\Omega_0^\#$ be the subset of $\Omega_+^\#$ defined by the conditions $x \in X_+$, $\sup_{k>0} (p_{n-k} p^{-1} |S'_{n-k} u|) < 1$. From Lemma

5.9, we know that $\kappa_+^\# - a.e$:

$$\lim_{n \rightarrow \infty} |S'_{-n} u| p_{-n} p^{-1} = 0, \text{ hence } \lim_{k \rightarrow \infty} |S'_{n-k} u| p_{n-k} p^{-1} = 0, |S'_{n-k} u| p_{n-k} p^{-1} > 0.$$

It follows that the index $-\nu_0 < 0$ of the strict last maximum of the sequence $p_{n-k} p^{-1} |S'_{n-k} u|$ ($k > 0$) is finite $\kappa_+^\# - a.e$. Furthermore if $\omega^\# \in \Omega_0^\#$ we see that $\tau(\omega^\#)$ is the first return time of $({}^a\hat{\theta})^n(\omega^\#)$ to $\Omega_0^\#$, hence ${}^a\hat{\theta}_+^\tau$ is the transformation on $\Omega_0^\#$ induced by ${}^a\hat{\theta}_+$. This allows to proceed as in ([34] Lemma 2) with the \mathbb{R}_+^* -valued \mathbb{Z} -cocycle $|S'_{n_k} u| p_{n_k} p^{-1}$. Since $\kappa_+^\#$ is ${}^a\hat{\theta}_+$ -invariant we see that $\kappa_+^\#(\Omega_0^\#) > 0$ and we can apply Kac's recurrence theorem to $\Omega_0^\#$, ${}^a\hat{\theta}_+^\tau$ and $\Omega_+^\#$ (see [43]), hence the normalized restriction $\kappa_0^\#$ of $\kappa_+^\#$ to $\Omega_0^\#$ is ${}^a\hat{\theta}_+^\tau$ -ergodic and stationary, the return time τ has finite expectation $\mathbb{E}_0^\alpha(\tau)$ and $\lim_{n \rightarrow \infty} \frac{\tau_n}{n} = \mathbb{E}_0^\alpha(\tau)$, $\kappa_0^\# - a.e$.

Since $\kappa_0^\#$ is absolutely continuous with respect to $\kappa^\#$, Theorem 3.10 gives :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} |S'_{\tau_n} u| = \left(\lim_{n \rightarrow \infty} \frac{1}{\tau_n} \text{Log} |S'_{\tau_n} u| \right) \left(\lim_{n \rightarrow \infty} \frac{\tau_n}{n} \right) = \mathbb{E}_0^\alpha(\tau) L_\mu(\alpha) \quad , \quad \kappa_0^\# - a.e.$$

Using Birkhoff's theorem for the non-negative increments of $V_{\tau_n} = \text{Log} \left(\frac{p_{\tau_n}}{p} |S'_{\tau_n} u| \right)$, we get the $\kappa_0^\#$ -a.e convergence of $\frac{1}{n} V_{\tau_n}$. Since $\kappa_0^\#$ is ${}^a\hat{\theta}_+^\tau$ -invariant $\frac{1}{n} \text{Log} \frac{p_{\tau_n}}{p}$ converges to zero in $\kappa_0^\#$ -measure, hence using the $\kappa_0^\# - a.e$ convergence of $\frac{1}{n} \text{Log} |S'_{\tau_n} u|$, we get the $\kappa_0^\# - a.e$ convergence of $\frac{1}{n} V_{\tau_n}$ to γ_τ^α . In particular V_τ has finite expectation.

In order to relate $\kappa_0^\#$ and the kernel $*\hat{Q}^{\alpha, \tau}$ we consider the Markov kernel adjoint to $*\hat{Q}_+^\alpha(x, \cdot)$ with respect to κ_+ , and we denote by $*\hat{Q}_+^\alpha \otimes \delta_x$ the corresponding Markov measure on $H^{\mathbb{Z}_-} \times X$ with $\mathbb{Z}_- = \mathbb{N} \cup \{0\}$. Also we write $*\hat{Q}_x^\alpha = \delta_x \otimes *\bar{Q}_x^\alpha$ where $*\bar{Q}_x^\alpha$ is supported on $H^{\mathbb{N}}$ and we observe that $\kappa_+^\# = \int *\hat{Q}_+^\alpha \otimes \delta_x \otimes *\hat{Q}_x^\alpha d\kappa_+(x)$, hence $\kappa_0^\# = \int 1_{\Omega_0^\#} (*\hat{Q}_+^\alpha \otimes \delta_x \otimes *\hat{Q}_x^\alpha) d\kappa_+(x)$ with $\Omega_0^\# = \bar{\Omega}_0^\# \times H^{\mathbb{N}}$ and $\bar{\Omega}_0^\# \subset H^{\mathbb{Z}_-} \times X$. We denote by κ^τ the projection of $\kappa_0^\#$ on X_+ , hence κ^τ has density $u(x) = (*\hat{Q}_+^\alpha \otimes \delta_x)(\bar{\Omega}_0^\#)$ with respect to κ_+ . It follows that the projection of $\kappa_0^\#$ on $X \times \hat{\Omega}$ can be expressed as :

$$\int u(x) \delta_x \otimes *\bar{Q}_x^\alpha d\kappa_+(x) = \int \delta_x \otimes *\bar{Q}_x^\alpha d\kappa^\tau(x) = *\hat{Q}_{\kappa^\tau}^\alpha.$$

Since $\kappa_0^\#$ is invariant and ergodic with respect to the bilateral shift ${}^a\hat{\theta}_+^\tau$, the same is valid for $*\hat{Q}_{\kappa^\tau}^\alpha$ with respect to the unilateral shift ${}^a\hat{\theta}_+^\tau$. Since the kernel $x \rightarrow *\hat{Q}_x^\alpha$ commutes with ${}^a\hat{\theta}_+^\tau$ and $*\hat{Q}^{\alpha, \tau}$ the $*\hat{Q}^{\alpha, \tau}$ -invariance and ergodicity of κ^τ follows. Also we have $\mathbb{E}_0^\alpha(\tau) = \int \mathbb{E}_u^\alpha(\tau) d\kappa^\tau(u, p)$ and the above convergences are valid $*\hat{Q}_{\kappa^\tau}^\alpha - a.e$ \square

Remark

If $S = X_+$, $P = *P_{\alpha, +}$, $\pi = \kappa_+$ the measure κ^τ is closely connected with the measure ψ of Theorem 4.4. However, in our context, the function $\text{Log}|p|$ is not known to be κ -integrable.

Proof of Proposition 5.8

Assume that for some $u \in M$, $C_M(u) = \lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{\langle R, u \rangle > t\} = 0$.

For $p > 0$, with the notations of Lemma 5.10, this means $\lim_{t \rightarrow \infty} \psi^\alpha(v, p) = 0$. Using Proposition 5.3 we know that this implies $\lim_{t \rightarrow \infty} \psi^\alpha(v, p) = 0$ for any $u = tv \in M$ ($t > 0$). Also, from Lemma 5.11, we have, since $\Lambda_a^*(\Sigma) \supset M$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Log}(|S'_{\tau_n} u| \frac{p^{\tau_n}}{p}) = \gamma_\tau^\alpha > 0, \quad * \widehat{\mathbb{Q}}_{\kappa^\tau}^\alpha - a.e.$$

Since the canonical Markov measure associated with κ^τ and $* \widehat{\mathbb{Q}}^{\alpha, \tau}$ is a push-forward of $* \widehat{\mathbb{Q}}_{\kappa^\tau}^\alpha$, this convergence is also valid with respect to the canonical measure.

Then, using Lemma 5.10 and $\psi_\tau^\alpha \geq 0$, we can apply Lemma A.1 to the Markov kernel $* \widehat{P}_\alpha^\tau$ on $X_+ \times \mathbb{R}_+^*$, the potential $\sum_0^\infty (* \widehat{P}_\alpha^\tau)^k \psi_\tau^\alpha$ of the non negative function $\psi_\tau^\alpha \leq (*e^\alpha \otimes h^\alpha)^{-1}$ and

the $* \widehat{\mathbb{Q}}^{\alpha, \tau}$ stationary measure κ^τ : we get $\psi_\tau^\alpha = 0$, $\kappa^\tau \otimes \ell - a.e.$, i.e

$$\widehat{\mathbb{P}}\{t < p^{-1} < R, u \rangle \leq t + p^{-1} < B^\tau, u \rangle, \tau < \infty\} = 0.$$

Since $p^{-1} < B^\tau, u \rangle > 0$, this gives $p^{-1} < R, u \rangle \leq 0$ $\kappa^\tau \otimes \widehat{\mathbb{P}} - a.e$ on $\{\tau < \infty\}$, in particular for some $(u, p) \in X_+$, $p^{-1} < R, u \rangle \leq 0$ i.e $\langle R, u \rangle \leq 0$ $\widehat{\mathbb{P}} - a.e$ on $\{\tau < \infty\}$. Since $\Lambda^*(\Sigma) \supset M$, Lemma 5.9 implies that, for any $u \in M$ the set $\{\langle R, u \rangle > 0, \tau < \infty\} = \{\langle R, u \rangle > 0\}$ is not $\widehat{\mathbb{P}}$ -negligible, hence the required contradiction. \square

The following will be useful when applying Proposition 5.8.

Lemma 5.12

In cases I or II' : $\Lambda_a^*(\Sigma) = \mathbb{S}^{d-1}$. In case II'' : $\Lambda_a^*(\Sigma) \supset \Lambda_+(T^*)$

Proof

Let $\Lambda_a^\infty(\Sigma) = \overline{\Lambda_a(\Sigma)} \cap \mathbb{S}^{d-1}$ and $u \in \mathbb{S}^{d-1}$, $u'_\infty \in \Lambda_a^\infty(\Sigma)$ corresponds to $u' \in \mathbb{S}^{d-1}$. If $\langle u', u \rangle > 0$, then $u \in \Lambda_a^*(\Sigma)$. Hence the complement of $\Lambda_a^*(\Sigma)$ in \mathbb{S}^{d-1} is contained in the set $\{u \in \mathbb{S}^{d-1} ; \langle u, u' \rangle \leq 0 \forall u'_\infty \in \Lambda_a^\infty(\Sigma)\}$. From the discussion at the beginning of this section we know that $\Lambda_a^\infty(\Sigma) \neq \phi$ is T -invariant and closed, hence contains $\widetilde{\Lambda}^\infty(T)$ in cases I, II' or only $\Lambda_+^\infty(T)$ in case II'' with $\Lambda_+^\infty(T) \cap \Lambda_a^\infty(\Sigma) = \phi$.

Since $\widetilde{\Lambda}^\infty(T)$ is symmetric and condition i-p is valid it follows $\Lambda_a^*(\Sigma) = \mathbb{S}^{d-1}$ in cases I, II'. In case II'', we know from the end of proof of Theorem 2.17 that the complement of $\Lambda_a^*(\Sigma)$ is contained in the set denoted $-\widehat{\Lambda}_+(T^*)$. Since $\Lambda_+(T^*) \cap -\widehat{\Lambda}_+(T^*) = \phi$ we get $\Lambda_+(T^*) \subset \Lambda_a^*(\Sigma)$. \square

The following improves Corollary 5.7.

Corollary 5.13

For any $u \in \mathbb{S}^{d-1}$:

$$\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}\{|\langle R, u \rangle| > t\} = C \frac{p(\alpha)}{\alpha} *e^\alpha(u) > 0$$

In cases I, for any $u \in \mathbb{S}^{d-1}$:

$$\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}\{\langle R, u \rangle > t\} = \frac{1}{2} C \frac{p(\alpha)}{\alpha} {}^*e^\alpha(u) > 0$$

In case II, for any $u \in \Lambda_+(T^*)$, if $\Lambda_a(\Sigma) \supset \Lambda_+^\infty(T)$:

$$\lim_{t \rightarrow \infty} t^\alpha \mathbb{P}\{\langle R, u \rangle > t\} = \frac{p(\alpha)}{\alpha} C_+ {}^*e^\alpha(u) > 0.$$

Proof

This a trivial consequence of Proposition 5.3, Corollary 5.7, Proposition 5.8 and Lemma 5.12. \square

Corollary 5.14

With the above notations we write :

$\gamma_\tau^\alpha = L_\mu(\alpha) \mathbb{E}_0^\alpha(\tau)$, $\psi_\tau^\alpha(v, p) = \widehat{\mathbb{P}}\{t < p^{-1} < R, u \rangle \leq t + p^{-1} < B^\tau, u \rangle, \tau < \infty\} {}^*e^\alpha(u)^{-1} t^\alpha$ and we denote by κ^τ the ${}^*Q^{\alpha, \tau}$ -stationary measure on X_+ , given by Lemma 5.11. Then, if $\Lambda_a(\Sigma) \supset \Lambda_+^\infty(T)$:

$$C_+ \geq \frac{\alpha}{p(\alpha) \gamma_\tau^\alpha} \int_{]0, \infty[\times X_+} \psi_\tau^\alpha(v, p) t^{-1} d\kappa^\tau(u) dt > 0.$$

Proof

With the notations of Lemma 5.10, we have $\psi^\alpha = \sum_0^\infty ({}^*\widehat{P}_\alpha^\tau)^k \psi_\tau^\alpha$ where ${}^*\widehat{P}_\alpha^\tau$ is a fibered Markov kernel on $(X_+ \cup \{\infty\}) \times \mathbb{R}_+^*$ which satisfies the conditions of Lemma A.1 and ψ_τ^α is bounded by $({}^*e^\alpha \otimes h^\alpha)^{-1}$. Hence, as in the proof of the proposition, if $\overline{\psi}_\tau^\alpha$ is a Borel function with compact support, bounded by ψ_τ^α :

$$\lim_{t \rightarrow \infty} \psi^\alpha(v, p) \geq \limsup_{t \rightarrow \infty} \sum_0^\infty {}^*\widehat{P}_\alpha^\tau \overline{\psi}_\tau^\alpha(v, p),$$

and, using Corollary 5.13, since $\lim_{t \rightarrow \infty} \psi^\alpha(v, p)$ is constant on X_+ :

$$\lim_{t \rightarrow \infty} \psi^\alpha(v, p) \geq \frac{1}{\gamma_\tau^\alpha} \int_{]0, \infty[\times X_+} \overline{\psi}_\tau^\alpha(v', p) d\kappa^\tau(u', p) \frac{dt}{t}.$$

Hence, approximating from below ψ_τ^α by $\overline{\psi}_\tau^\alpha$:

$$\lim_{t \rightarrow \infty} \psi^\alpha(v, p) \geq \frac{1}{\gamma_\tau^\alpha} \int_{]0, \infty[\times X_+} \psi_\tau^\alpha(v', p) \frac{dt}{t} d\kappa^\tau(u', p),$$

$$C_M(u) = \lim_{t \rightarrow \infty} t^\alpha \widehat{\mathbb{P}}\{\langle R, u \rangle > t\} = {}^*e^\alpha(u) \lim_{t \rightarrow \infty} \psi^\alpha(v, p),$$

$$C_M(u) \geq \frac{{}^*e^\alpha(u)}{\gamma_\tau^\alpha} \int_{]0, \infty[\times X_+} \psi_\tau^\alpha(v', p) t^{-1} dt d\kappa^\tau(u', p).$$

The final formula follows from Corollary 5.13. \square

Remark

We observe that, if $d = 1$, and A, B are positive, a formula of this type for $C = C_+$, with equality, is given in [14]. We don't know if such an equality is valid in our setting. However if λ is non singular, then theorem 4.4 can be used instead of Lemma A.1, since from the corollary ψ_τ^α is $\kappa^\tau \otimes \ell$ -integrable, hence equality in the above formula is valid.

4) Homogeneity at infinity of the stationary measure ρ and proof of the main result

For the proof of Theorem 5.2 we prepare the following lemmas. If $\alpha \notin \mathbb{N}$, it follows from [2] that Theorem 5.2 is a consequence of Propositions 5.3 and 5.8. If $\alpha \in \mathbb{N}$, as follows from [45] the situation is different. Here, in general we will need to use the Choquet-Deny type results of Appendix 2.

Lemma 5.15

For any compact subset K of $V \setminus \{0\}$, there exists a constant $C(K) > 0$ such that $\sup_{t>0} t^{-\alpha}(t.\rho)(K) \leq C(K)$. In particular the family $\rho_t = t^{-\alpha}(t.\rho)$ is relatively compact for the topology of vague convergence and any cluster value η of the family ρ_t satisfies $\sup_{t>0} t^{-\alpha}(t.\eta)(K) \leq C(K)$, hence $\sup_{t>0} t((e^\alpha \otimes h^\alpha)\eta)(K) \leq C'(K)$ with $C'(K) > 0$.

Proof

For some $\delta > 0$ we have $K \subset \{x \in V ; |x| > \delta\}$, hence using Corollary 5.7, $t^{-\alpha}\mathbb{P}\{|R| > \frac{\delta}{t}\} \leq \frac{b}{\delta^\alpha} = C(K)$. The relative compactness of the family ρ_t follows. Also :

$$(tt_n)^{-\alpha}(tt_n.\rho)(K) \leq C(K), \quad t^{-\alpha}(t.\eta)(K) = \lim_{n \rightarrow \infty} (tt_n)^{-\alpha}(tt_n.\rho)(K) \leq C(K),$$

Hence $\sup_{t>0} t^{-\alpha}(t.\eta)(K) \leq C(K)$.

Since $e^\alpha \otimes h^\alpha$ is α -homogeneous : $t((e^\alpha \otimes h^\alpha)\eta) = (e^\alpha \otimes h^\alpha)(t^{-\alpha}(t.\eta))$.

With $C_K = \sup_{v \in K} (e^\alpha \otimes h^\alpha)(v)$, we get :

$$t((e^\alpha \otimes h^\alpha)\eta)(K) \leq C_K t^{-\alpha}(t.\eta)(K) \leq C_K C(K) = C'(K). \quad \square$$

Lemma 5.16

Assume η is the vague limit of $t_n^{-\alpha}(t_n.\rho)$ ($t_n \rightarrow 0_+$). Then η is μ -harmonic, i.e $\mu * \eta = \eta$.

Proof

Let φ be ε -Hölder continuous on V with compact support contained in the set $\{x \in V ; |x| \geq \delta\}$ with $\delta > 0$, and let us show $\lim_{t \rightarrow 0_+} t^{-\alpha} I_t(\varphi) = 0$ where $I_t(\varphi) = (t.\rho - t.(\mu * \rho))(\varphi)$.

By definition :

$$I_t(\varphi) = \mathbb{E}(\varphi(tR) - \varphi(tA_1 R o \hat{\theta})) \text{ with } \varphi(tR) = 0 \text{ if } |tR| < \delta \text{ and } \varphi(tA_1 R o \hat{\theta}) = 0 \text{ if } |tA_1 R o \hat{\theta}| < \delta. \text{ Hence : } I_t(\varphi) \leq [\varphi]_\varepsilon t^\varepsilon (\mathbb{E}(|B_1|^\varepsilon 1_{\{|tR| > \delta\}}) + \mathbb{E}(|B_1|^\varepsilon 1_{\{|tA_1 R o \hat{\theta}| > \delta\}})).$$

We write :

$$I_t^1 = t^{\varepsilon - \alpha} \mathbb{E}(|B_1|^\varepsilon 1_{\{|tR| > \delta\}}), \quad I_t^2 = t^{\varepsilon - \alpha} \mathbb{E}(|B_1|^\varepsilon 1_{\{|tA_1 R o \hat{\theta}| > \delta\}})$$

and we estimate I_t^1, I_t^2 as follows. We have $I_t^1 \leq \delta^{\varepsilon - \alpha} \mathbb{E}(|B_1|^\varepsilon |R|^{\alpha - \varepsilon} 1_{\{|tR| > \delta\}})$. Since $|R|^{\alpha - \varepsilon} \leq c(\alpha)(|A_1 R o \hat{\theta}|^{\alpha - \varepsilon} + |B_1|^{\alpha - \varepsilon})$, using independance of $R o \hat{\theta}$ and $|B_1|^\varepsilon |A_1|^{\alpha - \varepsilon}$:

$$\mathbb{E}(|B_1|^\varepsilon |R|^{\alpha - \varepsilon}) \leq c \mathbb{E}(|B_1|^\alpha) + c \mathbb{E}(|A_1|^{\alpha - \varepsilon} |B_1|^\varepsilon) \mathbb{E}(|R|^{\alpha - \varepsilon}).$$

Using Hölder inequality we get $\mathbb{E}(|A_1|^{\alpha - \varepsilon} |B_1|^\varepsilon) < \infty$. Also using Proposition 5.1, we get $\mathbb{E}(|R|^{\alpha - \varepsilon}) < \infty$. It follows that $|B_1|^\varepsilon |R|^{\alpha - \varepsilon} 1_{\{|tR| > \delta\}}$ is bounded by the integrable function $|B_1|^\varepsilon |R|^{\alpha - \varepsilon}$. Then by dominated convergence : $\lim_{t \rightarrow 0} I_t^1 = 0$.

In the same way we have :

$$I_t^2 \leq \delta^{\varepsilon-\alpha} \mathbb{E}(|B_1|^\varepsilon |A_1 R \widehat{\theta}|^{\alpha-\varepsilon} 1_{\{t|A_1 R \widehat{\theta}| > \delta\}})$$

Also, using independence and Hölder inequality :

$$\mathbb{E}(|B_1|^\varepsilon |A_1 R \widehat{\theta}|^{\alpha-\varepsilon}) \leq \mathbb{E}(|B_1|^\varepsilon |A_1|^{\alpha-\varepsilon}) \mathbb{E}(|R|^{\alpha-\varepsilon}) < \infty.$$

Then by dominated convergence $\lim_{t \rightarrow 0_+} I_t^2 = 0$. Hence $\lim_{t \rightarrow 0_+} t^{-\alpha} I_t(\varphi) = 0$.

By definition of η we have, for any $g \in G$: $\lim_{t_n \rightarrow 0_+} t_n^{-\alpha} (t_n \cdot g \rho)(\varphi) = (g\eta)(\varphi)$.

Furthermore we have $|\varphi(x)| \leq |\varphi| 1_{\{x \in V; |x| \geq \delta\}}$ and,

$$|(g\eta)(\varphi)| \leq |\varphi| \eta(\{x \in V; |gx| \geq \delta\}) \leq |\varphi| \lim_{n \rightarrow \infty} t_n^{-\alpha} \mathbb{P}\{|R| > \frac{\delta}{|g|t_n}\}.$$

Using Corollary 5.7, we get : $|(g\eta)(\varphi)| \leq \frac{\delta}{\delta^\alpha} |\varphi| |g|^\alpha$.

Since $\int |g|^\alpha d\mu(g) < \infty$ and for any $g \in G$ $\lim_{t_n \rightarrow 0_+} t_n^{-\alpha} (t_n \cdot g \rho)(\varphi) = g\eta(\varphi)$, we have by domi-

nated convergence : $\lim_{t_n \rightarrow 0_+} t_n^{-\alpha} (t_n \cdot (\mu * \rho))(\varphi) = (\mu * \eta)(\varphi)$. Then the property $\lim_{t \rightarrow 0_+} t^{-\alpha} I_t = 0$

implies $(\mu * \eta)(\varphi) = \eta(\varphi)$, hence $\mu * \eta = \eta$. \square

Lemma 5.17

Assume η and $\sigma \otimes \ell^\alpha$ are μ -harmonic Radon measures on $V \setminus \{0\}$ with $\sigma \in M^1(\mathbb{S}^{d-1})$. Assume also that for any $v \in V \setminus \{0\}$, $\eta(H_v^+) = (\sigma \otimes \ell^\alpha)(H_v^+)$. Then we have $\eta = \sigma \otimes \ell^\alpha$

Proof

As in the proof of Corollary 5.7, we observe that the condition $\eta(H_v^+) = (\sigma \otimes \ell^\alpha)(H_v^+)$ implies for any $\delta > 0$:

$$\sup_{t > 0} t^{-\alpha} (t \cdot \eta)(\{x \in V; |x| > \delta\}) < \infty$$

Hence, as in Lemma 5.14, for any compact $K \subset V \setminus \{0\}$, with $\eta^\alpha = (e^\alpha \otimes h^\alpha)\eta$:

$$\sup_{t > 0} t^{-\alpha} (t \cdot \eta)(K) \leq C(K), \quad \sup_{t > 0} (t \cdot \eta^\alpha)(K) \leq C'(K).$$

It follows that η^α is dilation-bounded.

On the other hand, the projection $\bar{\sigma} \otimes \ell^\alpha$ (resp $\dot{\eta}$) of $\sigma \otimes \ell^\alpha$ (resp η) on \dot{V} satisfy :

$$\mu * (\bar{\sigma} \otimes \ell^\alpha) = \bar{\sigma} \otimes \ell^\alpha \quad (\text{resp } \mu * \dot{\eta} = \dot{\eta}), \quad \text{hence } Q_\alpha(e^\alpha \bar{\sigma} \otimes \ell) = e^\alpha \bar{\sigma} \otimes \ell, \quad \dot{Q}_\alpha(\dot{\eta}^\alpha) = \dot{\eta}^\alpha.$$

We observe that the fibered Markov operator \dot{Q}_α satisfies condition D of the appendix, in view of Corollary 3.20 and of the moment condition on A .

Then Theorem 2.6 implies $\bar{\sigma} = \nu^\alpha$. Also, in view of Corollary 3.20 and the above observations, we can apply the second part of Theorem A.4 to $\dot{\eta}^\alpha$ with $P = \dot{Q}_\alpha$, hence $\dot{\eta}^\alpha$ is proportional to $\pi^\alpha \otimes \ell$, i.e $\dot{\eta}$ is proportional to $\nu^\alpha \otimes \ell^\alpha$. Since $\bar{\sigma} = \nu^\alpha$ and $\eta(H_v^+) = (\sigma \otimes \ell^\alpha)(H_v^+)$ we get $\dot{\eta} = \nu^\alpha \otimes \ell^\alpha$.

We denote for $v \in \mathbb{S}^{d-1}$: $\lambda_v = |\langle v, \cdot \rangle|^\alpha \sigma \otimes \ell^\alpha$, $\eta_v = |\langle v, \cdot \rangle|^\alpha \eta$. Since $\sigma \otimes \ell^\alpha$ and η are μ -harmonic, we have $\int g \lambda_{g^* v} d\mu(g) = \lambda_v$, $\int g \eta_{g^* v} d\mu(g) = \eta_v$. The projections λ_v and η_v on \dot{V} satisfy the same equation hence are equal. As in section 3, we get that

the sequences of Radon measures $g_1 \cdots g_n \lambda_{g_n^* \cdots g_1^* v}$ and $g_1 \cdots g_n \eta_{g_n^* \cdots g_1^* v}$ are vaguely bounded ${}^* \mathbb{Q}_v^\alpha$ -martingales. On \dot{V} we get, using Theorem 3.2, for some $z(\omega) \in \mathbb{P}^{d-1}$ and ${}^* \mathbb{Q}_v^\alpha - a.e$:

$$\lim_{n \rightarrow \infty} g_1 \cdots g_n \lambda_{g_n^* \cdots g_1^* v} = \lim_{n \rightarrow \infty} g_1 \cdots g_n \eta_{g_n^* \cdots g_1^* v} = \delta_{z(\omega)} \otimes \ell.$$

Let $z_+(\omega)$ and $z_-(\omega)$ be opposite points on \mathbb{S}^{d-1} with projection $z(\omega)$ on \mathbb{P}^{d-1} . The martingale convergence on $V \setminus \{0\}$ gives that $g_1 \cdots g_n \lambda_{g_n^* \cdots g_1^* v}$ (resp $g_1 \cdots g_n \eta_{g_n^* \cdots g_1^* v}$) converges vaguely to $p(\omega) \delta_{z_+(\omega)} + q(\omega) \delta_{z_-(\omega)}$ (resp $p'(\omega) \delta_{z_+(\omega)} + q'(\omega) \delta_{z_-(\omega)}$) with $p(\omega) + q(\omega) = p'(\omega) + q'(\omega) = 1$. The condition $\eta(H_v^+) = (\sigma \otimes \ell^\alpha)(H_v^+)$ implies in the limit :

$$p(\omega) \delta_{z_+(\omega)} + q(\omega) \delta_{z_-(\omega)} = p'(\omega) \delta_{z_+(\omega)} + q'(\omega) \delta_{z_-(\omega)}.$$

Hence, taking expectations we get $\eta = \sigma \otimes \ell^\alpha$. \square

Proof of Theorem 5.2

The convergence of $\rho_t = t^{-\alpha}(t.\rho)$ to $C(\sigma^\alpha \otimes \ell^\alpha)$ on the sets H_v^+ and the positivity properties of C, C_+, C_- follows from Corollary 5.13. For the vague convergence of ρ_t we observe that Lemma 5.15 gives the vague compactness of ρ_t . If $\eta = \lim_{t_n \rightarrow 0_+} t_n^{-\alpha}(t_n.\rho)$, Lemma

5.16 gives the μ -harmonicity of η . Since $\eta(H_v^+) = C(\sigma^\alpha \otimes \ell^\alpha)(H_v^+)$, Lemma 5.17 gives $\Lambda = C(\sigma^\alpha \otimes \ell^\alpha) = \eta$, hence the vague convergence of ρ_t to Λ . The detailed form of Λ follows from Proposition 5.3.

For the final minimality assertions one uses the second part of Theorem A.4 and we replace η by $(e^\alpha \otimes h^\alpha)\eta$. We verify condition D as follows. Condition D_1 follows directly from Corollary 3.21. For D_2 we note that the replacement of μ by $\mu' = \frac{1}{2}(\delta_e + \mu)$ don't affect the condition $k(\alpha) = 1$, the harmonicity of $\tilde{\nu}^\alpha, \nu_+^\alpha, \nu_-^\alpha$, but avoid unimodular eigenvalues other than 1 for \tilde{Q}^α . In case I, 1 is a simple eigenvalue of \tilde{Q}^α and, if θ is positive Radon measure with $\mu * \theta = \theta$, $\theta \leq \tilde{\nu}^\alpha \otimes \ell^\alpha$, we have $\mu' * \theta = \theta$, hence θ is proportional to $\tilde{\nu}^\alpha \otimes \ell^\alpha$. In case II and for ν_+^α , one restricts \tilde{Q}^α to the convex cone generated to $\Lambda_+(T)$, so as to achieve the simplicity of 1 as an eigenvalue of \tilde{Q}^α . Then condition D_2 is satisfied for the corresponding operator and the above argument is also valid for $\nu_+^\alpha \otimes \ell^\alpha$. Condition D_3 follows from the moment conditions assumed on λ . \square

Corollary 5.18

Let $\mathcal{B}_{\varepsilon, \alpha}$ be the set of locally bounded Borel functions on $V \setminus \{0\}$ such that the set of discontinuities of f is Λ -negligible and for $\varepsilon > 0$:

$$K_f(\varepsilon) = \sup\{|v|^{-\alpha} |\text{Log} v|^{1+\varepsilon} |f(v)| ; v \neq 0\} < \infty$$

Then for any $f \in \mathcal{B}_{\varepsilon, \alpha}$: $\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho)(f) = \Lambda(f)$.

The proof depends of two lemmas in which we will use the norm $\|v\| = \sup_{1 \leq i \leq d} | \langle v, e_i \rangle |$ instead of $|v|$ where $e_i (1 \leq i \leq d)$ is a basis of V . Also for $\delta > 0$ and $0 < \delta_1 < \delta_2$ we write

$$B_\delta = \{v \in V ; \|v\| \leq \delta\}, B_{\delta_1, \delta_2} = B_{\delta_2} \setminus B_{\delta_1}, B'_\delta = V \setminus B_\delta.$$

Lemma 5.19

For any $f \in \mathcal{B}_{\varepsilon, \alpha}$, $0 < \delta_1 < \delta_2$:

$$\lim_{t \rightarrow 0_+} t^{-\alpha}(t.\rho)(f1_{B_{\delta_1, \delta_2}}) = \Lambda(f1_{B_{\delta_1, \delta_2}}).$$

Proof

From the fact that ν^α gives measure zero to any projective subspace and the homogeneity of $\Lambda = \sigma^\alpha \otimes \ell^\alpha$, we know that Λ gives measure zero to any affine hyperplane, hence the boundary of B_{η_1, η_2} is Λ -negligible. Then the proof follows from the vague convergence of $t^{-\alpha}(t.\rho)$ to Λ and the hypothesis of Λ -negligibility of the discontinuity set of f . \square

Lemma 5.20

a) There exists $C > 0$ such that for any $f \in \mathcal{B}_{\varepsilon, \alpha}$, $t > 0$ and $\delta_2 > e$:

$$|t^{-\alpha}(t.\rho)(f1_{B_{\delta_2}})| \leq CK_f(\varepsilon)|\text{Log}\delta_2|^{-\varepsilon}$$

b) Then exists $C(\varepsilon) > 0$ such that for any $f \in \mathcal{B}_{\varepsilon, \alpha}$, $t > 0$, $\delta_1 < e^{-1}$:

$$|t^{-\alpha}(t.\rho)(f1_{B_{\delta_1}})| \leq C(\varepsilon)K_f(\varepsilon)|\text{Log}\delta_1|^{-\varepsilon}$$

Proof

a) Let $\varphi_\varepsilon(x)$ be the function on $\mathbb{R}_+ \setminus \{1\}$ given by $\varphi_\varepsilon(x) = x^\alpha |\text{Log}x|^{-1-\varepsilon}$. For $x \geq e$ we have $\varphi'_\varepsilon(x) \leq \alpha x^{\alpha-1} |\text{Log}x|^{-1-\varepsilon}$. We denote $F_t(x) = \mathbb{P}(\|tR\| > x)$ and we observe that, using Proposition 5.1, the non increasing function F_t is continuous. We have :

$$|t^{-\alpha}(t.\rho)(f1_{B'_{\delta_2}})| \leq t^{-\alpha}K_f(\varepsilon) \int_{\delta_2}^\infty \varphi_\varepsilon(x) dF_t(x)$$

Integrating by parts, we get :

$$|t^{-\alpha}(t.\rho)(f1_{B'_{\delta_2}})| \leq t^{-\alpha}K_f(\varepsilon)[\varphi_\varepsilon(x)F_t(x)]_{\delta_2}^\infty + t^{-\alpha}K_f(\varepsilon) \int_{\delta_2}^\infty \varphi'_\varepsilon(x)F_t(x)dx.$$

From Corollary 5.7 we know that, for some $C > 0$, $F_t(x) \leq C t^\alpha x^{-\alpha}$. Then, using the above estimation of $\varphi'_\varepsilon(x)$, we get :

$$|t^{-\alpha}(t.\rho)(f1_{B'_{\delta_2}})| \leq CK_\varepsilon(f) \int_{\delta_2}^\infty \frac{\alpha}{|\text{Log}x|^{1+\varepsilon}} \frac{dx}{x} \leq CK_\varepsilon(f)|\text{Log}\delta_2|^{-\varepsilon}$$

b) The proof follows the same lines and uses the estimation of $|\varphi'_\varepsilon(x)|$ by

$$(\alpha + 1 + \varepsilon)x^{\alpha-1} |\text{Log}x|^{-1-\varepsilon} \text{ for } x \leq e^{-1}. \quad \square$$

Proof of the Corollary

For $\delta \geq e$ and with $D > 0$ we have $\int_{B'_\delta} \varphi_\varepsilon(\|v\|)d\Lambda(v) \leq D \int_\delta^\infty \frac{x^\alpha}{(\text{Log}x)^{1+\varepsilon}} \frac{dx}{x^{\alpha+1}} = \frac{D}{|\text{Log}\delta|^\varepsilon}$ hence

$\lim_{\delta \rightarrow \infty} \int_{B'_\delta} \varphi_\varepsilon(\|v\|)d\Lambda(v) = 0$. Also for $\delta < 1$:

$$\int_{B'_\delta} \varphi_\varepsilon(\|v\|)d\Lambda(v) \leq D \int_0^\delta \frac{x^\alpha}{|\text{Log}x|^{1+\varepsilon}} \frac{dx}{x^{\alpha+1}} = \frac{D}{|\text{Log}\delta|^\varepsilon},$$

hence $\lim_{\delta \rightarrow \infty} \int_{B'_\delta} \varphi_\varepsilon(\|v\|)d\Lambda(v) = 0$.

Then the corollary follows of the lemmas. \square

Proof of Theorem C

Except for the last assertion, Theorem C is a reformulation of Theorem 5.2. The last assertion is the content of Corollary 5.18. \square

A) Appendix

Here we give the proofs of two general results used in section 5.

1) A weak renewal theorem for Markov walks

Let (X, ν) be a complete separable metric space, where ν is a probability measure. As in section 4 we consider a general Markov chain on $X \times \mathbb{R}$ with kernel P , we assume that P commutes with the \mathbb{R} -translations and we denote by ℓ Lebesgue measure on \mathbb{R} . We assume that the measure $\nu \otimes \ell$ is P -invariant. We write a path of this Markov chain as (x_n, V_n) where $x_n \in X$ and $V_n \in \mathbb{R}$, we denote by ${}^a\mathbb{P}_x$ the Markov measure on the paths starting from $x \in X$ and we write ${}^a\mathbb{P}_\nu = \int {}^a\mathbb{P}_x d\nu(x)$, ${}^a\mathbb{E}_x$ for the corresponding expectation symbol. In this context the following weak analogue of the renewal theorem holds.

Lemma A.1

With the above notations, assume that ψ is a bounded non negative Borel function on $X \times \mathbb{R}$, supported on $X \times [-a, a]$, the potential $U\psi = \sum_0^\infty P^k \psi$ is bounded on $X \times [-c, c]$ ($c > 0$) for any $c > 0$ and we have for any $\varepsilon > 0$:

$$\lim_{n \rightarrow \infty} {}^a\mathbb{P}_\nu \left\{ \left| \frac{V_n}{n} - \gamma \right| > \varepsilon \right\} = 0, \text{ with } \gamma > 0.$$

$$\text{Then : } \lim_{t \rightarrow \infty} \frac{1}{t} \int_{-t}^0 ds \int_X U\psi(x, s) d\nu(x) = \frac{1}{\gamma} (\nu \otimes \ell)(\psi).$$

Furthermore if ψ is a non negative Borel function on $X \times \mathbb{R}$ and $\lim_{t \rightarrow -\infty} U\psi(x, t) = 0$, $\nu - a.e$ then $\psi = 0$, $\nu \otimes \ell - a.e$.

If ψ is a Borel function on $X \times \mathbb{R}$ which satisfies

$$|\psi|_b = \sum_{\ell=-\infty}^{\ell=\infty} \sup \{ |\psi(x, s)| ; x \in X, s \in [\ell, \ell + 1] \} < \infty,$$

then the above convergence is valid.

Proof

We observe that the maximum principle implies $|U\psi| = \sup_{x,t} |U\psi|(x, t) < \infty$, since $U\psi$ is locally bounded. For $\varepsilon > 0$, $t > 0$ we denote $n_1 = n_1(t) = [\frac{1}{\gamma}\varepsilon t]$, $n_2 = n_2(t) = [\frac{1}{\gamma}(1 + \varepsilon)t]$ where $[t]$ denotes integer part of $t > 0$.

$$\text{We write : } \sum_0^\infty P^k \psi = U\psi, \sum_0^{n-1} P^k \psi = U_n \psi, \sum_{n+1}^\infty P^k \psi = U^{n+1} \psi, \sum_n^m P^k \psi = U_n^m \psi$$

and $I(t) = \frac{1}{t} \int_{-t}^0 ds \int_X (U\psi)(x, s) d\nu(x) = \sum_1^3 I_k(t) - I_4(t)$ where :

$$I_1(t) = \frac{1}{t} \int_X d\nu(x) \int_{-\infty}^\infty U_{n_1}^{n_2} \psi(x, s) ds, \quad I_2(t) = \frac{1}{t} \int_X d\nu(x) \int_{-t}^0 U_{n_1} \psi(x, s) ds$$

$$I_3(t) = \frac{1}{t} \int_X d\nu(x) \int_{-t}^0 U^{n_2} \psi(x, s) ds \quad I_4(t) = \frac{1}{t} \int_X d\nu(x) \int_{\mathbb{R} \setminus [-t, 0]} U_{n_1}^{n_2} \psi(x, s) ds.$$

We estimate each term $I_k(t)$.

We have since the measure $\nu \otimes \ell$ is P -invariant :

$$I_1(t) = \frac{n_2 - n_1 + 1}{t} (\nu \otimes \ell)(\psi), \text{ hence } \lim_{t \rightarrow \infty} I_1(t) = \frac{1}{\gamma} (\nu \otimes \ell)(\psi).$$

Furthermore :

$$|I_4(t)| \leq \frac{|\psi|}{t} (n_2 - n_1 + 1) \sup_{n_1 \leq n \leq n_2} \int_X ({}^a\mathbb{P}_x\{V_n \leq a\} + {}^a\mathbb{P}_x\{V_n \geq t - a\}) d\nu(x).$$

Since $n^{-1}V_n$ converges to $\gamma > 0$ in probability, the above integral has limit zero, hence

$$\lim_{t \rightarrow \infty} I_4(t) = 0.$$

We have also : $|I_2(t)| \leq \frac{\varepsilon}{\gamma} \nu \otimes \ell(\psi)$.

In order to estimate $I_3(t)$ we denote for $n \in \mathbb{N}$, $s > 0$: $\rho_n^s = \text{Inf}\{k \geq n ; -a \leq V_n - s \leq a\}$ and we use the interpretation of $U^n\psi$ as the expected number of visits to ψ after time n :

$$U^n\psi(x, s) \leq |U\psi|^a \mathbb{P}_x\{\rho_n^s < \infty\}$$

Taking $n = \lceil \frac{(1+\varepsilon)t}{\gamma} \rceil = n_2$ we get

$$I_3(t) \leq |U\psi| \int_X {}^a\mathbb{P}_x\{V_k - t \leq a \text{ for some } k \geq \lceil \frac{(1+\varepsilon)t}{\gamma} \rceil\} d\nu(x).$$

Since $\frac{V_n}{n}$ converges to $\gamma > 0$ in probability, we get $\lim_{t \rightarrow \infty} I_3(t) = 0$.

Since ε is arbitrary we get finally : $\lim_{t \rightarrow \infty} I(t) = \frac{1}{\gamma} (\nu \otimes \ell)(\psi)$.

The second conclusion follows by restriction and truncation of ψ on $X \times [-a, a]$.

For the proof of the last assertion we observe that for any $\ell \in \mathbb{Z}$

$$\Delta = |U1_{X \times [0,1]}| = |U1_{X \times [\ell, \ell+1]}| < \infty. \text{ Writing } \psi = \sum_{\ell=-\infty}^{\ell=\infty} \psi 1_{X \times [\ell, \ell+1]} \text{ we get } |U\psi| \leq \Delta |\psi|_b.$$

The quantity $\psi \rightarrow |\psi|_b$ is a norm on the space \mathcal{H} of Borel functions on $X \times \mathbb{R}$ such that $|\psi|_b < \infty$. Since the set of bounded functions supported on $X \times [-c, c]$ for some $c > 0$ is dense in \mathcal{H} and $\psi \rightarrow (\nu \otimes \ell)(\psi)$ is a continuous functional on \mathcal{H} , the above relation extends by density to any $\psi \in \mathcal{H}$. \square

2) A Choquet-Deny type property

Here, as in section 4, we consider a fibered Markov chain on $S \times \mathbb{R}$ but we reinforce the hypothesis on the Markov kernel P . Hence S is a compact metric space, P commutes with \mathbb{R} -translations and acts continuously on $C_b(S \times \mathbb{R})$. We define for $t \in \mathbb{R}$, the Fourier operator P^{it} on $C(S)$ by :

$$P^{it}\varphi(x) = P(\varphi \otimes e^{it\cdot})(x, 0)$$

For $t = 0$ $P^{it} = P^0$ is equal to \bar{P} , the factor operator on S defined by P . We assume that for each $t \in \mathbb{R}$, P^{it} preserves the space $H_\varepsilon(S)$ of ε -Hölder functions and is a bounded operator therein. Moreover we assume that $P^{it}(t \in \mathbb{R})$, and P satisfies the following condition D (compare [30]) :

1) For any $t \in \mathbb{R}$, one can find $n_0 \in \mathbb{N}$, $\rho(t) \in [0, 1[$ and $C(t) > 0$ for which

$$[(P^{it})^{n_0}\varphi]_\varepsilon \leq \rho(t)[\varphi]_\varepsilon + C(t)|\varphi|.$$

2) For any $t \in \mathbb{R}$, the equation $P^{it}\varphi = e^{i\theta}\varphi$, $\varphi \in H_\varepsilon(S)$, $\varphi \neq 0$ has only the trivial solution $e^{i\theta} = 1$, $t = 0$, $\varphi = \text{cte}$.

3) For some $\delta > 1$: $M_\delta = \sup_{x \in S} \int |a|^\delta P((x, 0), d(y, a)) < \infty$.

Conditions 1,2 above imply that \overline{P} has a unique stationary measure π and the spectrum of \overline{P} in $H_\varepsilon(S)$ is of the form $\{1\} \cup \Delta$ where Δ is a compact subset of the open unit disk (see [32]). They imply also that, for any $t \neq 0$, the spectral radius of P^{it} is less than one. With the notations of section 4, condition 3 above will allow to estimate ${}^a\mathbb{E}(|V_n|^p)$ for $p \leq \delta$ and to show the continuity of $t \rightarrow |P^{it}|$.

The following is a simple consequence of conditions 1,2,3 above.

Lemma A.2

With the above notations, let $I \subset \mathbb{R}$ be a compact subset of $\mathbb{R} \setminus \{0\}$. Then there exists $D > 0$ and $\sigma \in [0, 1[$ such that for any $n \in \mathbb{N} : \sup_{t \in I} |(P^{it})^n| \leq D\sigma^n$

Proof

Conditions 1 and 2 for P^{it} ($t \neq 0$) imply that the spectral radius r_t of P^{it} satisfies $r_t < 1$ (see [30]). Hence there exists $C_t > 0$ such that for any $n \in \mathbb{N} : |(P^{it})^n| \leq C_t(\frac{1}{2} + \frac{r_t}{2})^n$. On the other hand $t \rightarrow |P^{it}|$ is continuous as the following calculation shows. For $a, t, t' \in \mathbb{R}, \delta' \in [0, 1]$, we have $|e^{iat} - e^{iat'}| \leq 2|a|^{\delta'} |t - t'|^{\delta'}$ hence :

$$|P^{it}\varphi(x) - P^{it'}\varphi(x)| \leq 2|\varphi| |t - t'|^{\delta'} \int |a|^{\delta'} P((x, 0), d(y, a)), \quad |P^{it} - P^{it'}| \leq 2M_{\delta'} |t - t'|^{\delta'}$$

For each $t \in I$ we fix $n_t \in \mathbb{N}$ such that $|(P^{it})^{n_t}| \leq \frac{1}{3}$. Then the above continuity of P^{it} , hence of $(P^{it})^{n_t}$, gives for t' sufficiently close to $t : |(P^{it'})| \leq \frac{1}{2}$; Then, using compactness of I we find n_1, \dots, n_k such that one of the inequalities $|(P^{it})^{n_j}| \leq \frac{1}{2}$ ($1 \leq j \leq k$) is valid at any given point of I . Then, since $|P^{it}| \leq 1$ with $n_0 = n_1 \dots n_k$ we get $|P^{it})^{n_0}| \leq \frac{1}{2}$. Using Euclidean division of n by n_0 , we get the required inequality. \square

We are interested in the action of P^n on functions on $S \times \mathbb{R}$ which are of the form $u \otimes f$ where $u \in C(S), f \in \mathbb{L}^1(\mathbb{R})$ and also in P -harmonic Radon measures which satisfy boundedness conditions. In some proofs, since $H_\varepsilon(S)$ is dense in $C(S)$ it will be convenient to assume $u \in H_\varepsilon(S)$.

Definition A.3

We say that the Radon measure θ on $S \times \mathbb{R}$ is translation-bounded if for any compact subset K of $S \times \mathbb{R}$, any $a \in \mathbb{R}$, there exists $C(K) > 0$ such that $|\theta(a + K)| \leq C(K)$ where $a + K$ is the compact subset of $S \times \mathbb{R}$ obtained from K using translation by $a \in \mathbb{R}$.

We are led to consider a positive function ω on \mathbb{R}^d which satisfies $\omega(x + y) \leq \omega(x)\omega(y)$.

For example, if $p \geq 0$, such a function ω_p is defined by $\omega_p(a) = (1 + |a|)^p$.

We denote $\mathbb{L}_\omega^1(\mathbb{R}) = \{f \in \mathbb{L}^1(\mathbb{R}) ; \omega f \in \mathbb{L}^1(\mathbb{R})\}$ and we observe that $f \rightarrow \|\omega f\|_1 = \|f\|_{1,\omega}$ is a norm under which $\mathbb{L}_\omega^1(\mathbb{R})$ is a Banach algebra. The dual space of $\mathbb{L}_\omega^1(\mathbb{R})$ is the space $\mathbb{L}_\omega^\infty(\mathbb{R})$ of measurable functions g such that $g\omega^{-1} \in \mathbb{L}^\infty(\mathbb{R})$ and the duality is given by $\langle g, f \rangle = \int g(a)f(a)da$. The Fourier transform \widehat{f} of $f \in \mathbb{L}_\omega^1(\mathbb{R})$ is well defined by $\widehat{f}(t) = \int f(a)e^{ita}da$. We denote by J^c the ideal of $\mathbb{L}^1(\mathbb{R})$ which consists of functions $f \in \mathbb{L}^1(\mathbb{R})$ such that \widehat{f} has a compact support not containing 0 and we write $J_\omega^c = \mathbb{L}_\omega^1(\mathbb{R}) \cap J^c$. Also we denote by $\mathbb{L}_0^1(\mathbb{R})$ the ideal of $\mathbb{L}^1(\mathbb{R})$ defined by the condition $\widehat{f}(0) = 0$. It is well known

that J^c is dense in $\mathbb{L}_0^1(\mathbb{R})$, hence for $\omega = \omega_p$ $J_{\omega,2}^c$ is dense in $\mathbb{L}_\omega^1(\mathbb{R})$ (see [31] p 187).

Theorem A.4

With the above notations, assume that the family $P^{it}(t \in \mathbb{R})$ satisfies conditions D and let $\omega = \omega_p$ with $p < \delta$. Then for any $f \in \mathbb{L}_\omega^1(\mathbb{R}) \cap J^c$, $u \in C(S)$, we have the convergence :

$$\lim_{n \rightarrow \infty} \sup_{x \in S} \|P^n(u \otimes f)(x, \cdot)\|_{1,\omega} = 0.$$

If θ is a P -harmonic Radon measure which is translation-bounded, then θ is proportional to $\pi \otimes \ell$. In particular $\pi \otimes \ell$ is a minimal P -harmonic Radon measure.

The proof follows from the above considerations and the following lemmas.

Lemma A.5

Assume π_n is a sequence of bounded measures on \mathbb{R} , ω is a positive Borel function on \mathbb{R} such that for any $x, y \in \mathbb{R}$, $\omega(x+y) \leq \omega(x)\omega(y)$ and assume that the total variation measures $|\pi_n|$ of π_n satisfy $\sup\{|\pi_n|(\omega) ; n \in \mathbb{N}\} < \infty$. Let $f \in \mathbb{L}_\omega^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$ and assume A_n, B_n are sequences of Borel subsets of \mathbb{R} such that, with $A'_n = \mathbb{R} \setminus A_n$, $B'_n = \mathbb{R} \setminus B_n$:

- a) $\lim_{n \rightarrow \infty} |\pi_n|(\omega) \|f 1_{B'_n}\|_{1,\omega} = 0$
- b) $\lim_{n \rightarrow \infty} |\pi_n|(\omega 1_{A'_n}) = 0$
- c) $\lim_{n \rightarrow \infty} \|\pi_n * f\|_2 \|\omega^2 1_{A_n+B_n}\|_1^{1/2} = 0.$

Then we have $\lim_{n \rightarrow \infty} \|\pi_n * f\|_{1,\omega} = 0$. Furthermore, if the measures π_n depend of a parameter λ and if the convergences in a), b), c) are uniform in λ , then the convergence of $\|\pi_n * f\|_{1,\omega}$ is also uniform in λ .

Proof

Let η, η' be two bounded measures on \mathbb{R} and let A, B be Borel subsets with complements A', B' in \mathbb{R} . Observe that, since $0 \leq \omega(x+y) \leq \omega(x)\omega(y)$:

$$\omega(x+y) \leq \omega(x+y)1_{A+B}(x+y) + (\omega 1_A)(x)(\omega 1_{B'})(y) + (\omega 1_{A'})(x)\omega(y).$$

It follows

$$|(\eta * \eta')(\omega) \leq |\eta * \eta'|(\omega 1_{A+B}) + |\eta|(\omega) |\eta'|(\omega 1_{B'}) + |\eta|(\omega 1_{A'})|\eta'|(\omega).$$

Then we take $\eta = \pi_n$, $\eta' = f(a)da$, $A = A_n$, $B = B_n$ and we get :

$$\|\pi_n * f\|_{1,\omega} \leq \int |\pi_n * f|(a) |\omega 1_{A_n+B_n}|(a) da + |\pi_n|(\omega 1_{A_n}) \|f 1_{B'_n}\|_{1,\omega} + |\pi_n|(\omega 1_{A'_n}) \|f\|_{1,\omega}.$$

Conditions a, b imply that the two last terms in the above inequality have limits zero. Using condition c and Schwarz inequality we see that the first term has also limit zero. If π_n depends of a parameter λ , the uniformity of the convergence of $\|\pi_n * f\|_{1,\omega}$ follows directly from the bound for $\|\pi_n * f\|_{1,\omega}$ given above. \square

The following lemma is an easy consequence of condition 3 on the Markov kernel P and of Hölder inequality

Lemma A.6

For any $p \in [1, \delta]$, there exists $C_p > 0$ such that : $\sup_x^a \mathbb{E}_x((|V_n|)^p) \leq C_p n^p$

In particular, for any $L > 0$: $\sup_{x,n}^a \mathbb{P}_x\{|V_n| > nL\} \leq \frac{C_p}{L^p}$.

We leave to the reader the proof of the first inequality. The second one follows from Markov inequality.

For a Radon measure θ on $S \times \mathbb{R}$ and $b \in \mathbb{R}$, we denote by $\theta * \delta_b$ the Radon measure defined by $(\theta * \delta_b)(\varphi) = \int \varphi(x, a + b) d\theta(x, a)$ and for $\varphi \in C_c(S \times \mathbb{R})$, θ translation-bounded we write $|\theta|_\varphi = \sup\{|\theta * \delta_b(\varphi)| ; b \in \mathbb{R}\}$. For such measures and any bounded measure r on \mathbb{R} , $\theta * r$ is well defined by $(\theta * r)(\varphi) = \int (\theta * \delta_b)(\varphi) dr(b)$ and we have $|\theta * r|_\varphi \leq \|r\| \theta_\varphi$ where $\|r\|$ is the total variation of r . In particular, $f \in \mathbb{L}^1(\mathbb{R})$ can be identified with the measure $r_f = f(a) da$, we can define $\theta * f = \theta * r_f$ and if $f_n \in \mathbb{L}^1(\mathbb{R})$ converges in \mathbb{L}^1 -norm to $f \in \mathbb{L}^1(\mathbb{R})$, then $\theta * f_n$ converges to $\theta * f$ in the vague topology. On the other hand, if r has compact support and θ is a Radon measure on \mathbb{R} , $\theta * r$ is well defined as a Radon measure.

Lemma A.7

With the above notations, assume that θ is a translation-bounded non negative Radon measure on $S \times \mathbb{R}$. Let r be a non negative continuous function on \mathbb{R} with compact support containing 0. Then for $p > 1$, there exists a non negative bounded measure $\bar{\theta}$ on S such that $\theta * r \leq (1_S \otimes \omega_p)(\bar{\theta} \otimes \ell)$.

Proof

For simplicity of notations, assume $r > 0$ on $[0, 1]$. We denote by $\theta_k * \delta_k$ the restriction of θ to $S \times [k, k + 1[$ ($k \in \mathbb{Z}$) and we write $\theta = \sum_{k \in \mathbb{Z}} \theta_k * \delta_k$ with $\text{supp } \theta_k \subset S \times [0, 1]$. We observe that, since θ is translation-bounded, the mass of θ_k is bounded for $k \in \mathbb{Z}$, hence $\dot{\theta} = \sum_{k \in \mathbb{Z}} (1 + |k|)^{-p} \theta_k$ is a bounded measure supported on $S \times [0, 1]$. We have clearly

$$\theta_k \leq (1 + |k|)^p \dot{\theta}, \quad \theta \leq \dot{\theta} * \sum_{k \in \mathbb{Z}} (1 + |k|)^p \delta_k, \quad \theta * r \leq \dot{\theta} * (r * \sum_{k \in \mathbb{Z}} (1 + |k|)^p \delta_k)$$

But, by definition of ω_p and since $\text{supp}(r)$ is compact, we have $r * \sum_{k \in \mathbb{Z}} (1 + |k|)^p \delta_k \leq c \omega_p$ for

some $c > 0$ and it follows, $\theta * r \leq c \dot{\theta} * \omega_p$. We desintegrate the bounded measure $\dot{\theta}$ as $\dot{\theta} =$

$\int \delta_x \otimes \theta^x d\tilde{\theta}(x)$ where $\tilde{\theta}$ is the projection of $\dot{\theta}$ on S and θ^x is a probability measure. Hence

$\dot{\theta} * \omega_p = \int \delta_x \otimes (\theta^x * \omega_p) d\tilde{\theta}(x)$. But, since θ^x is supported on $[0, 1]$: $\theta^x * \omega_p \leq 2^p \theta^x([0, 1]) \omega_p$.

Hence $\dot{\theta} \leq 2^p (1_S \otimes \omega_p)(\tilde{\theta} \otimes \ell)$ and finally $\theta * r \leq 2^p c (1_S \otimes \omega_p)(\tilde{\theta} \otimes \ell) = (1_S \otimes \omega_p)(\bar{\theta} \otimes \ell)$ with $\bar{\theta} = 2^p c \tilde{\theta}$. \square

Proof of the theorem

We fix $p \in [1, \delta]$, $\omega = \omega_p$, $u \in H_\varepsilon(S)$, $u \geq 0$ and for $x \in S$, we define the positive measure π_n^x on \mathbb{R} by $\pi_n^x(\varphi) = P^n(u \otimes \varphi)(x, 0)$ where φ is a non negative Borel function on \mathbb{R} . We

observe that $\pi_n^x(1) \leq |u|$, and for $f \in C_b(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R})$:

$$P^n(u \otimes f)(x, a) = \pi_n^x(f * \delta_a) = (\pi_n^x * f^*)(a)$$

where $f^*(a) = f(-a)$. It follows $\|P^n(u \otimes f)(x, \cdot)\|_1 \leq |u| \|f\|_1$ and also for $f \in \mathbb{L}_\omega^1(\mathbb{R})$, $\|P^n(u \otimes f)(x, \cdot)\|_{1,\omega} \leq \|f\|_{1,\omega} \sup\{\pi_n^x(\omega) ; x \in S\}$. From condition 3, since $p < \delta$:

$$\pi_n^x(\omega) \leq |u|^a \mathbb{E}_x(1 + |V_n|)^p \leq 2^p C_p n^p$$

where we have used Lemma A.6 in order to bound ${}^a\mathbb{E}_x(1 + |V_n|)^p$.

We fix $\delta > 1$, we denote $B_n = \{a \in \mathbb{R} ; |a| \leq n^{1+\delta}\}$, $A_n = \{a \in \mathbb{R} ; |a| \leq n^{c+1}\}$ with $c > 0$ to be defined later and we verify the conditions a, b, c of Lemma A.5 for $\pi_n = \pi_n^x$, uniformly in $x \in S$ for $f \in \mathbb{L}_{\omega^2}^1(\mathbb{R})$.

Since $f\omega^2 \in \mathbb{L}^1(\mathbb{R})$, Markov inequality gives $\|f1_{B_n'}\|_{1,\omega} \leq \|f\|_{1,\omega^2} n^{-p(1+\delta)}$. Then, using the bound of $\pi_n^x(\omega)$ given above :

$$\pi_n^x(\omega) \|f1_{B_n'}\|_{1,\omega} \leq 2^p C_p n^{-p\delta} \|f\|_{1,\omega^2}.$$

Hence condition a) is satisfied.

We write $\pi_n^x(\omega 1_{A_n'}) \leq |u|^a \mathbb{E}_x(\omega(V_n) 1_{A_n'}(V_n))$ and use Hölder inequality for $p' > 1$, $\frac{1}{q'} = 1 - \frac{1}{p'}$ and $p'p < \delta$:

$$\pi_n^x(\omega 1_{A_n'}) \leq C' 2^{2p} n^p \mathbb{P}\{|V_n| \geq n^{c+1}\}^{1/q'} \leq C'' n^{p-cp'/q'}$$

where we have used the fact that $\sup_x {}^a\mathbb{E}_x(|U_1|^{pp'}) < \infty$ for $pp' < \delta$ and Lemma A.6. If

we take $c > q' = \frac{p'}{p'-1}$, we see that $\lim_{n \rightarrow \infty} \pi_n^x(\omega 1_{A_n'}) = 0$ uniformly, hence condition b) is satisfied.

In order to verify condition c) we observe that $\widehat{\pi}_n^x(t) = (P^{it})^n u(x)$. For $f \in J^c \subset \mathbb{L}^2(\mathbb{R})$ we denote $Y = \text{supp } \widehat{f} \subset \mathbb{R} \setminus \{0\}$ and we know from Lemma A.2, that there exists $D > 0$ and $\sigma \in [0, 1[$ such that for any $t \in Y$, $n \in \mathbb{N}$: $|(P^{it})^n| \leq D\sigma^n$.

From Plancherel formula, we get,

$$\|\pi_n^x * f\|_2 = \left(\int |\widehat{\pi}_n^x(t)|^2 |\widehat{f}(t)|^2 dt \right)^{1/2} \leq \|f\|_2 |u| \sup_{t \in Y} |(P^{it})^n| \leq D |u| \sigma^n \|f\|_2.$$

On the other hand, $\|\omega^2 1_{A_n+B_n}\|_1$ is bounded by a polynomial in n . Since $\sigma < 1$, condition c) is satisfied. Hence, Lemma A.5 gives : $\lim_{n \rightarrow \infty} \|\pi_n^x * f\|_{1,\omega} = 0$ uniformly. By density the same relation is valid for all $f \in \mathbb{L}_\omega^1(\mathbb{R})$

Now, let us choose $p \in]1, \delta[$ and $\omega = \omega_p$. Since θ is translation-bounded we can assume θ to be non negative and translation-bounded. Then Lemma A.7 gives for any r as in the lemma : $\theta * r \leq (1_S \otimes \omega)(\bar{\theta} \otimes \ell)$. Taking $r = r_n$ as an approximate identity we have $\theta = \lim_{n \rightarrow \infty} \theta * r_n$ in the weak sense. Hence we can assume $\theta \leq (1_S \otimes \omega)(\bar{\theta} \otimes \ell)$ where $\bar{\theta}$ is a bounded measure on S . Let $f \in \mathbb{L}_\omega^1(\mathbb{R}) \cap J^c(\mathbb{R})$, $u \in H_\varepsilon(S)$ be as above, hence f, u satisfy for every $n \in \mathbb{N}$:

$$\begin{aligned} \theta(u \otimes f) &= (P^n \theta)(u \otimes f) = \theta(P^n(u \otimes f)) = \int P^n(u \otimes f)(x, a) d\theta(x, a), \\ |\theta(u \otimes f)| &\leq \int d\bar{\theta}(x) \int |P^n(u \otimes f)(x, a)| \omega(a) da \leq \|\bar{\theta}\| \sup_x \|P^n(u \otimes f)(x, \cdot)\|_{1,\omega}. \end{aligned}$$

From the first part of the proof we get $\theta(u \otimes f) = 0$ for any $u \in H_\varepsilon(S)$, $f \in J_\omega^c = \mathbb{L}_\omega^1(\mathbb{R}) \cap J^c$. This relation remains valid for f in the ideal I_ω^c of $\mathbb{L}_\omega^1(\mathbb{R})$ generated by J_ω^c . Using regularisation on Fourier transforms we see that the closure in $\mathbb{L}^1(\mathbb{R})$ of J_ω^c contains

J^c , hence the unique Fourier exponential which vanish on I_ω^c is 1. Then using classical Fourier Analysis (see [31] p.187) we get that I_ω^c is dense in $\mathbb{L}_0^1(\mathbb{R})$. As observed above, since θ is translation-bounded, this implies $\theta(u \otimes f) = 0$ for any $f \in \mathbb{L}_0^1(\mathbb{R})$. Since $H_\varepsilon(S)$ is dense in $C(S)$, we get that θ is invariant by \mathbb{R} -translation. Then we have $\theta = \bar{\theta} \otimes \ell$ where $\bar{\theta}$ is a positive measure on S which satisfies $\overline{P} \bar{\theta} = \bar{\theta}$. Using conditions D.1, D.2, this implies that $\bar{\theta}$ is proportional to π , hence θ is proportional to $\pi \otimes \ell$.

For the final assertion we observe that, if θ is a P -harmonic positive Radon measure with $\theta \leq C \pi \otimes \ell$, for some $C > 0$, then θ is translation bounded. Hence as above, θ is proportional to $\pi \otimes \ell$. \square

Remark

The above result extends naturally to the multidimensional situation. This improves the corresponding result of [30] mentioned above.

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