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Experimental identification of stochastic processes using an uncertain computational non-linear dynamical model

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Abstract. The problem presented deals with tubes bundles in Pressurized Water Reactors. The final objective is to identify a model of the external loads applied to theses tubes bundles through the knowledge of dynamical responses. In complex dynamical systems, such an identification is difficult due to the size of the computational model and due to the high number of parameters to be identified. As a consequence, a simplified computational model is constructed. The introduction of such a simplified model introduces model uncertainties. We are first interested in the implementation (modelling and identification) of a probabilistic approach of uncertainties in the mean computational model using the non-parametric probabilistic approach for parameter uncertainties and model uncertainties. In addition, a probabilistic model for the stochastic loads is constructed to take into account model uncertainties in the probabilistic model of the stochastic loads. Finally, the non-linear stochastic dynamical system submited to the uncertain stochastic loads is used to identify the probability model of its uncertainties. In a first part, the theory is presented. The second part is devoted to the validation of the theory in presenting an application.

1. Introduction

The present research has been developed in the context of the dynamical analysis of the tubes bundles in Pressurized Water Reactors. The tubes are excited by a turbulent flow which induces a non-linear dynamical response of the dynamical system made up of a structure coupled with the fluid. The final objective is to identify a mathematical model of the stochastic loads applied to the tubes bundles and induced by the turbulent flow using both experimental responses of the real non-linear dynamical system and a simplified computational non-linear dynamical model. For this identification, the use of a simplified computational model induces model uncertainties. A non-parametric probabilistic approach is then used to take into account both parameter uncertainties and model uncertainties.

The real dynamical system (real tubes bundles) is replaced by a reference non-linear dynamical system made up of five tubes and three grids which allows the reference responses to be simulated. Therefore, the experimental responses of the tubes related to one configuration of

the real system are replaced by the numerical responses of the reference system calculated with this reference computational model.

The simplified computational model is derived from the reference computational model .

In a first step, the probability model of uncertainties is identified in the simplified computational model using the reference computational model and the maximum likelihood method. We then deduce a stochastic simplified computational model which allows a robust identification of stochastic loads to be carried out with respect to uncertainties in the dynamical system.

The second step is devoted to the stochastic inverse problem consisting in identifying the stochastic loads. The stochastic loads used in the simplified computational model are represented by a vector-valued centred stationary Gaussian stochastic process. Such a stochastic process is then completely defined by a matrix-valued spectral density function. The use of a rough spatial discretization of the random field in the simplified computational model introduces uncertainties in the stochastic process which models the stochastic loads. These uncertainties are then taken into account in introducing a probabilistic model for the matrix-valued spectral density function which becomes a random quantity which has to be constructed and identified.

2. Stochastic non-linear simplified computational model, including system uncertainties, and identification

2.1. Reference computational model

Let Ω be the domain of a three dimensional damped structure having a non-linear behaviour (the non-linearities are not distributed but are localized). The structure is fixed on the part Γ_0 of the boundary Γ of Ω. Let $\mathbf{u}^{ref}(t)$ be the vector of the n degrees of freedom at time t. Let $[\mathbb{M}^{ref}]$, $[\mathbb{D}^{ref}]$ and $[\mathbb{K}^{ref}]$ be respectively the mass, damping and stiffness matrices of the linear part of the finite element model. Since there are no rigid body displacements, these three matrices are positive definite. The domain Ω is decomposed in two subdomains, the subdomain Ω^A_{ref} which corresponds to a non-linear subsystem made up of one part of the structure containing the localized non-linearities and the subdomain Ω^B_{ref} which corresponds to a linear subsystem made up of the second part of the structure and which has a linear behaviour. These two subsytems are coupled on the coupling interface Γ_C . The finite element model of the linear subsytem Ω^B_{ref} is analyzed in the frequency band of analysis $\mathcal{B} = [-\omega_{max}, \omega_{max}]$. Let $[\mathbb{A}^{B,ref}(\omega)] = -\omega^2[\mathbb{M}^{B,ref}] + i\omega[\mathbb{D}^{B,ref}] + [\mathbb{K}^{B,ref}]$ be the dynamic stiffness matrix of this linear subsystem with free coupling interface, where $[\mathbb{M}^{B,ref}]$, $[\mathbb{D}^{B,ref}]$ and $[\mathbb{K}^{B,ref}]$ are the mass, damping and stiffness matrices which are positive definite. Introducing the vector $\mathbf{u}_p^{B,ref}(\omega)$ of the n_p internal DOF and the vector $\mathbf{u}_c^{B,ref}(\omega)$ of the n_c coupling DOF on the interface. A reduction of the linear subsystem is performed using the Craig Bampton method [4]. The block decomposition of the reduced dynamical stiffness matrix related to the generalized coordinates $\mathbf{y}^B(\omega)$ and the coupling DOF $\mathbf{u}_c^B(\omega)$ is written as

$$[A^{B,ref}(\omega)] = \begin{bmatrix} A_{yy}^{B,ref}(\omega) & A_{yc}^{B,ref}(\omega) \\ A_{cy}^{B,ref}(\omega) & A_{cc}^{B,ref}(\omega) \end{bmatrix}. \tag{1}$$

In order to perform the identification of stochastic simplified computational model, for the reference computational model, we introduce the finite positive real number $J^{ref} = \int_{\mathcal{B}} \|[Z^{B,ref}(\omega)]^{-1}\|_F^2 \, d\omega$ where $[Z^{B,ref}(\omega)] = [A_{cc}^{B,ref}(\omega)] - [A_{cy}^{B}(\omega)][A_{yy}^{B,ref}(\omega)]^{-1}[A_{yc}^{B,ref}(\omega)]$ is the condensed dynamical stiffness matrix of the linear subsystem on the coupling interface of the reference computational model. Consequently J^{ref} gives a measure over $\mathcal{B}$ of the dynamical effects of subsystem Ω^B_{ref} on the subsystem Ω^A at the coupling interface.

2.2. Mean reduced linear subsystem of the simplified computational model
The simplified computational model is constructed from the reference computational model. The non-linear subsystems of the two models are the same (see Fig. 2.2). And consequently, the degrees of freedom on the coupling interface are the same.

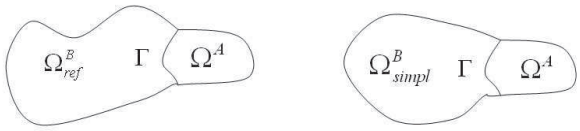

Figure 1. Reference model (left) and simplified model (right).

2.3. System uncertainties modeling using the non-parametric probabilistic approach
The non-parametric probabilistic approach is used to take into account both model uncertainties and parameter uncertainties in the dynamical system. This approach has recently been introduced (see [1], [2]) and consists in replacing the matrices of reduced mean model by random matrices for which the probability distributions are constructed by using the maximum entropy principle with constraints defined by the available information. Such an approach has been validated for different cases. Therefore, the mean reduced dynamical stiffness matrix $[\underline{A}^B(\omega)]$ is replaced by the complex random matrix $[\mathbf{A}^B(\omega)] = -\omega^2[\mathbf{M}^B] + i\omega[\mathbf{D}^B] + [\mathbf{K}^B]$ in which the matrices $[\underline{M}^B]$, $[\underline{D}^B]$ and $[\underline{K}^B]$ of the reduced mean system are replaced by the random matrices $[\mathbf{M}^B]$, $[\mathbf{D}^B]$ and $[\mathbf{K}^B]$ defined on a probability space $(\Theta, \mathcal{T}, \mathcal{P})$ and whose probability distributions depend respectively on the dispersion parameters δ^B_M, δ^B_D and δ^B_K. The random condensed dynamical stiffness matrix $[\mathbf{Z}^B(\omega)]$ of the linear subsystem on the coupling interface is then given by $[\mathbf{Z}^B(\omega)] = [\mathbf{A}^B_{cc}(\omega)] - [\mathbf{A}^B_{cy}(\omega)][\mathbf{A}^B_{yy}(\omega)]^{-1}[\mathbf{A}^B_{yc}(\omega)]$.

2.4. Identification of the dispersion parameters
The vector-valued dispertion paramter $\boldsymbol{\delta} = (\delta^B_M, \delta^B_D, \delta^B_K)$ is identified using the maximum likelihood method ([3]) applied to the random variable $J(\boldsymbol{\delta}) = \int_{\mathcal{B}} \|[\mathbf{Z}^B(\omega)]^{-1}\|^2_F \, d\omega$. This random variable corresponds to the real number J^{ref} introduced in Section 2.1. We then have to construct the probability density function $x \mapsto p_J(x, \boldsymbol{\delta})$ of the random variable $J(\boldsymbol{\delta})$. For any x fixed in $[0, +\infty[$ and for any value of the vector $\boldsymbol{\delta}$ in the admissible space $\mathcal{C}_{ad}$ of the dispersion parameters, the real number $p_J(x, \boldsymbol{\delta})$ is estimated by using the stochastic non-linear simplified computational model and the Monte Carlo simulation. We then have to solve the following optimisation problem

$$\boldsymbol{\delta}^{opt} = \arg\max_{\boldsymbol{\delta} \in \mathcal{C}_{ad}} (p_J(J^{ref}; \boldsymbol{\delta})) \quad , \tag{2}$$

where $\boldsymbol{\delta}^{opt}$ is the identified value of vector $\boldsymbol{\delta}$.

2.5. Random dynamical transient response of the stochastic non-linear simplified computational model
For the stochastic system, the displacement vector of the stochastic linear subsystem Ω^B_{simpl} is denoted by $(\mathbf{U}^B_p(t), \mathbf{U}^B_c)(t)$. The displacement vector of the stochastic non-linear subsystem Ω^A is denoted by $(\mathbf{U}^A_p(t), \mathbf{U}^A_c(t))$. Then, the random variable $\mathbf{Q}(t) = (\mathbf{Y}^A(t), \mathbf{Y}^B(t), \mathbf{U}^c(t))$ which is composed of the random generalized coordinates $\mathbf{Y}^A(t)$ of the random non-linear subsystem, of the random generalized coordinates $\mathbf{Y}^B(t)$ of the linear subsystem and of the random coupling DOF $\mathbf{U}_c(t)$, is solution of the random non-linear dynamical system

$$[\tilde{\mathbf{M}}]\ddot{\mathbf{Q}}(t) + [\tilde{\mathbf{D}}]\dot{\mathbf{Q}}(t) + [\tilde{\mathbf{K}}]\mathbf{Q}(t) + \mathbf{f}^{NL}(\mathbf{Q}(t)) = \tilde{\mathbf{f}}(t) \quad , \quad t \in \mathbb{R} \tag{3}$$

where $\tilde{\mathbf{f}}(t)$ is the generalized forces vector and $\mathbf{f}^{NL}(\mathbf{Q}(t))$ is the vector of the non-linear forces.

3. Probabilistic model of the stochastic load with model uncertainties

The vector of the non zero components of the transient load is denoted by $\mathbf{f}(t)$ and is modelled by a second order stationary Gaussian stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$ defined on a probability space $(\Theta', \mathcal{T}', \mathcal{P}')$, indexed by $\mathbb{R}$, centred, mean square continuous on $\mathbb{R}$ and physically realizable. This stochastic process is then completely defined by its matrix-valued spectral density function $[\underline{S}_{\mathbf{F}}(\omega)]$. In addition, for all ω in $\mathcal{B} \subset \mathbb{R}$, the matrix valued spectral density function being positive definite, the Hermitian matrix $[\underline{S}_{\mathbf{F}}(\omega)]$ is invertible and its Cholesky decomposition yields $[\underline{S}_{\mathbf{F}}(\omega)] = [\underline{L}_{\mathbf{F}}(\omega)]^*[\underline{L}_{\mathbf{F}}(\omega)]$. Since the probabilistic model developed below will only be a simple representation of the real stochastic load applied to the structure, model uncertainties must be introduced in order to improve the efficiency of the representation which will be used for the identification of this stochastic load. We then take into account these uncertainties in introducing an additional probabilistic model of uncertainties for this stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$ which is then rewritten as $\{\mathbf{F}(t, [\underline{S}_{\mathbf{F}}]), t \in \mathbb{R}\}$ in which $[\underline{S}_{\mathbf{F}}] = \{[\underline{S}_{\mathbf{F}}(\omega)], \omega \in \mathbb{R}\}$. Since a stationary centred Gaussian process is only defined by its matrix valued spectral density function, the uncertainties are introduced by a probabilistic model of this matrix. Consequently, the deterministic funtion $[\underline{S}_{\mathbf{F}}] = \{[\underline{S}_{\mathbf{F}}(\omega)], \omega \in \mathbb{R}\}$, is then replaced by a random function with values in $\mathbb{M}_m^+(\mathbb{C})$ and denoted by $[\mathbf{S}_{\mathbf{F}}] = \{[\mathbf{S}_{\mathbf{F}}(\omega)], \omega \in \mathbb{R}\}$, defined on a probability space $(\Theta", \mathcal{T}", \mathcal{P}")$. The uncertain stochastic process $\{\mathbf{F}(t, [\underline{S}_{\mathbf{F}}]), t \in \mathbb{R}\}$ is then rewritten as $\{\mathbf{F}^{\text{unc}}(t), t \in \mathbb{R}\}$ and is such that $\mathbf{F}^{\text{unc}}(t, \theta', \theta'') = \mathbf{F}(t, \theta'; [\mathbf{S}_{\mathbf{F}}](\theta''))$ for all $\theta' \in \Theta'$ and $\theta" \in \Theta"$.

3.1. Construction of the random function $[\boldsymbol{S}_{\boldsymbol{F}}]$

The random function $[\mathbf{S}_{\mathbf{F}}]$ is constructed using the information theory using the maximum entropy principle (Shannon 48). The available information concerning the random function $[\mathbf{S}_{\mathbf{F}}(\omega)], \omega \in \mathbb{R}$ is the following. For all ω in $\mathbb{R}$, $[\mathbf{S}_{\mathbf{F}}(\omega)]$ is a random matrix with values in $\mathbb{M}_m^+(\mathbb{C})$ and by construction $E\{[\mathbf{S}_{\mathbf{F}}(\omega)]\} = [\underline{S}_{\mathbf{F}}(\omega)]$. Consequently, for all ω in $\mathbb{R}$, the random matrix $[\mathbf{S}_{\mathbf{F}}(\omega)]$ is invertible almost surely, which means that for $\mathcal{P}"$-almost $\theta"$ in $\Theta"$, the matrix $[\mathbf{S}_{\mathbf{F}}(\omega, \theta")]^{-1}$ exists. By construction we will impose that the random matrix $[\mathbf{S}_{\mathbf{F}}(\omega)]^{-1}$ is a second order random variable which means that $E\{\|[\mathbf{S}_{\mathbf{F}}(\omega)]^{-1}\|^2\} < +\infty$. Then, for all ω in $\mathcal{B}$, the random matrix $[\mathbf{S}_{\mathbf{F}}(\omega)]$ is normalized as $[\mathbf{S}_{\mathbf{F}}(\omega)] = [\underline{L}_{\mathbf{F}}(\omega)]^*[\mathbf{G}_m][\underline{L}_{\mathbf{F}}(\omega)]$, in which the random matrix $[\mathbf{G}_m]$ belongs to the *normalized positive-definite ensemble* denoted by SG^+ (see [2]). This random matrix which is independent of ω is such that $[\mathbf{G}_m] \in \mathbb{M}_m^+(\mathbb{R})$, $E\{[\mathbf{G}_m]\} = [I_m]$ and $E\{\|[\mathbf{G}_m]^{-1}\|_F^2\} < +\infty$. The dispersion of $[\mathbf{S}_{\mathbf{F}}]$ is independant of ω and is controlled by the dispersion parameter δ_F which is such that $\delta_F = \{(1/m)E\{\|[\mathbf{G}_m] - [I_m]\|\}_F^2\}^{1/2}$.

With such a construction, it can be proven that the stochastic process $\mathbf{F}^{\text{unc}}(t)$ is a second order stationary stochastic process, centred, mean square continous on $\mathbb{R}$, physically realizable whose matrix-valued spectral density function $[S_{\mathbf{F}^{\text{unc}}}(\omega)]$ is such that $[S_{\mathbf{F}^{\text{unc}}}(\omega)] = [\underline{S}_{\mathbf{F}}(\omega)]$ for all $\omega \in \mathbb{R}$. Nevertheless, it can be proved that the stochastic process $\mathbf{F}^{\text{unc}}(t)$ is not Gaussian.

4. Identification of the stochastic load

The identification of the stochastic load $\{\mathbf{F}^{\text{unc}}(t), t \in \mathbb{R})\}$ consists in identifying the mean value $[\underline{S}_{\mathbf{F}}(\omega)]$ of the matrix-valued spectral density function and the parameter δ_F that controls the level of uncertainties. In practice, the parameter $[\underline{S}_{\mathbf{F}}]$ which has to be identified is in fact a function $\omega \mapsto [\underline{S}_{\mathbf{F}}(\omega)]$. This identification will be performed in introducing a parametric representation of this function which is written as $[\underline{S}_{\mathbf{F}}(\omega)] = [S(\omega, \mathbf{r})]$ for all $\omega \in \mathbb{R}$. Let $\mathcal{C}_r \subset \mathbb{R}^{\nu_r}$ be the admissible set of the parameter $\mathbf{r}$. The vector $\mathbf{r}$ and the dispersion parameter δ_F are identified separetly using two different cost functions.

4.1. Identification of the parameter $\boldsymbol{r}$

Such an identification is performed using the stochastic equation deduced from Eqs. (3) in which the deterministic load $\mathbf{f}(t)$ is replaced by the stochastic load $\mathbf{F}(t;[\underline{S}_{\mathbf{F}}])$. We then extract the $\mathbb{R}^{\mu}$-valued random variable $\mathbf{Z}_s(t) = (Z_{s,1}(t), ..., Z_{s,\mu}(t))$ which represents the observations of the simplified stochastic model. Therefore, for all $\theta \in \Theta$, the matrix-valued spectral density function $\{[\mathbf{S}_{\mathbf{Z}_s}(\omega,\theta)], \omega \in \mathbb{R}\}$ of the stationary stochastic process $\{\mathbf{Z}_s(t,\theta), t \in \mathbb{R}\}$ can be estimated. Generating ν_θ independent realizations of the random matrices $[\tilde{\mathbf{M}}]$, $[\tilde{\mathbf{D}}]$ and $[\tilde{\mathbf{K}}]$, the matrix-valued spectral density function $[S_{\mathbf{Z}_s}]$ is estimated by the Monte Carlo simulation method, i.e., for all $\omega \in \mathbb{R}$, $[S_{\mathbf{Z}_s}(\omega)] = (1/\nu_\theta)\sum_{i=1}^{\nu_\theta}[\mathbf{S}_{\mathbf{Z}_s}(\omega,\theta_i)]$. Let $\{\mathbf{Z}_s^{exp}(t) = (Z_{s,1}^{exp}(t), ..., Z_{s,\mu}^{exp}(t)), t \in \mathbb{R}\}$ be the $\mathbb{R}^{\mu}$-valued stationary stochastic process which is measured for the manufactured real system and corresponding to the observation stochastic process $\{\mathbf{Z}_s(t), t \in \mathbb{R}\}$. The matrix-valued spectral density function $\{[S_{\mathbf{Z}_s^{exp}}(\omega)], \omega \in \mathbb{R}\}$ of this stochastic process is estimated using the periodogram method (see [5]). The parameter $\mathbf{r}$ is then estimated minimizing the distance $D(\mathbf{r}) = \int_{\mathcal{B}} \|[S_{\mathbf{Z}_s}(\omega,\mathbf{r})] - [S_{\mathbf{Z}_s^{exp}}(\omega)]\|_F^2 \, d\omega$ relative to the usual second order description of the two stochastic processes. We then have to solve the following optimization problem

$$\mathbf{r}_{opt} = \underset{\mathbf{r}\in\mathcal{C}_r}{\operatorname{argmin}} \, D(\mathbf{r}) \quad , \tag{4}$$

in which $\mathbf{r}_{opt}$ is the identified value of the vector $\mathbf{r}$.

4.2. Identification of the dispersion parameter δ_F

The dispersion parameter δ_F is estimated using the maximum likelihood method for the random variable J_s such that, for all $(\theta,\theta'') \in \Theta \times \Theta''$, one has $J_s(\theta,\theta'') = \int_{\mathcal{B}} \|[\mathbf{S}_{\mathbf{Z}'_s}(\omega,\theta,\theta'')]\|_F^2 d\omega$, in which, $\{[\mathbf{S}_{\mathbf{Z}'_s}(\omega,\theta,\theta'')], \omega \in \mathbb{R}\}$ is the matrix-valued spectral density function of the stationary stochastic process $\{\mathbf{Z}'_s(t,\theta,\theta''), t \in \mathbb{R}\}$. The probabilty density function $x \mapsto p_{J_s}(x)$ of the random variable J_s is then estimated by the Monte Carlo method to solve the uncertain stochastic reduced computational model. We then have to solve the following optimization problem

$$\delta_F^{opt} = \arg \max_{\delta_F \in \mathcal{C}_{\delta_F}} \left(p_{J_s}(J_s^{\exp}, \delta_F)\right) \quad , \tag{5}$$

in which δ_F^{opt} is the identified value of the variable δ_F, $\mathcal{C}_{\delta_F}$ is the admissible set for the dispersion parameter δ_F and where $J_s^{\exp} = \int_{\mathcal{B}} \|[S_{\mathbf{Z}_s^{exp}}(\omega)]\|_F^2 d\omega$.

5. Application

5.1. Data for the reference computational model

The reference computational model is made up of one linear subsystem and one non-linear subsystem. The linear subsystem is made up of four parallel Euler beams fixed at their ends. The non-linear subsytem is made up of a beam fixed at its ends, parallel to the other beams and with one transverse symmetric elastic stop (two identical tranverse stops). The five beams are linked by three transversal grids, each grid being modelled by four transverval springs (see Fig. 2). Therefore, the coupling interface between the two subsytems is composed of three points located in the neutral fiber of the beam of the non-linear subsystem. Each beam has a constant circular section with radius 0.5 m, thickness 0.2 m, length 16 m, mass density 250 kg/m^3, Young's modulus 450 N/mm^2 and dampding rate 0.02. The Young's modulus of the beam of the non-linear subsystem is 750 N/mm^2.The elastic stops are localized at 6 m from the left fixed end, the gap of each stop is $1.5 \cdot 10^{-6}$ m and the choc stiffness is 10^8 N/m. The stiffness of each spring of the tranversal grid is 4.10^7 N/m. Each beam is modelled by eight beam Euler finite elements of equal lengths and nine nodes. The DOF of the two nodes at the ends of the beam are locked.

The twelve springs in the three tranversal grids are modelled by twelve spring elements. We are only interested in the transversal displacements in direction y for the plane xy of the beam of the non-linear subsystem (see Fig. 2). Consequently, each beam has 14 DOF of y translation and z rotation. The beam of the nonlinear subsystem is exited by 7 transversal forces applied following

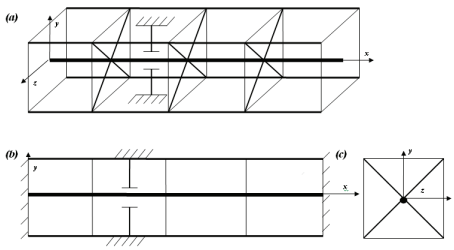

Figure 2. (a) Reference model:3D view.(b) Transversal view.(c) Tranversal view in the plane of one grid:the 6 diagonal lines represent the 12 springs.

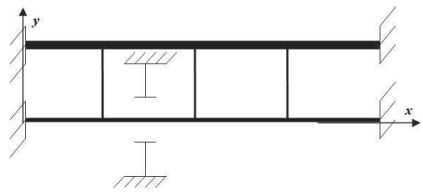

Figure 3. Simplified model.

the y direction. The vector of these 7 forces are denoted by $\mathbf{f}^{\text{ref}}$. Then $\{\mathbf{f}^{\text{ref}}(t), t \in \mathbb{R}\}$ is modeled by a second-order centred stationary Gaussian stochastic process for which its matrix-valued spectral density function $[S_{\mathbf{f}^{\text{ref}}}(\omega)]$ is such that (1) for all i in $\{1, \ldots, 7\}$, $[S_{\mathbf{f}^{\text{ref}}}(\omega)]_{ii}$ is a constant equal to $1.3N^2/Hz$ on the frequency band of analysis $\mathcal{B} = [-100, 100]$ Hz and (2) for all i and j in $\{1, \ldots, 7\}$, $|[S_{\mathbf{f}^{\text{ref}}}(\omega)]_{ij}|^2 = \gamma_{ij}(\omega)[S_{\mathbf{f}^{\text{ref}}}(\omega)]_{ii}[S_{\mathbf{f}^{\text{ref}}}(\omega)]_{jj}$ where $\gamma_{ij}(\omega) = exp(-|x_i - x_j|/\lambda)$ in which $|x_i - x_j|$ is the distance between the two excited points and $\lambda = 4\,m$ is a reference lenght related to the correlation lengh.

5.2. Data for the simplified computational model

The simplified computational model consists in replacing the linear substem composed of 4 beams by a linear subsystem composed of an equivalent Euler beam (see Fig. 3). The non-linear subsystem is the same for the 2 models. The section of the equivalent beam is arbitrarily defined and is chosen as a constant circular section with radius 0.5 m, thickness 0.2 m, length 16 m. Its Young's modulus and its the mass density are identified so that the three first eigenfrequency of the simplified computational model are the same as the three first eigenfrequency of the reference computational model. It should be noted that this choice of simplified model as an equivalent beam does not allow several eigenfrequencies to be correctly fitted. After identification, the equivalent beam has a mass density $4 * 250 = 1000$ kg/m^3 and a Young's modulus $4 * 450 = 1800$ N/mm^2.

5.3. Comparison between the dynamical response of the reference computational model and the dynamical response of the mean simplified computational model.

For the two models, the stationary stochastic response is calculated in the time interval $[0, 220]$ s using an explicit Euler integration scheme for which the time step is 3 ms. Let P_{obs} be the impact point of the non-linear subsystem. The power spectral density function of the stochastic transversal displacement responses in point P_{obs} (see Fig. 4) is estimated using the periodogram method. It can be seen that there are significant differences in the frequency band $[35, 100]$ Hz induced by model uncertainties. This is the reason why the model uncertainties are taken into account in order to extend the domain of validity of the mean simplified computational model in the frequency band $[35, 100]$ Hz.

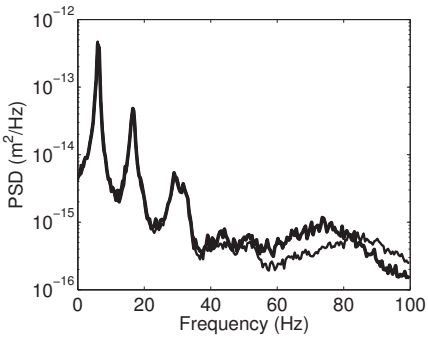

Figure 4. Power spectral density function (PSD) for the transversal displacement responses. Comparison between the reference computational model in point *Pobs* (thin line) and the response given by the mean simplified computational model (thick line).

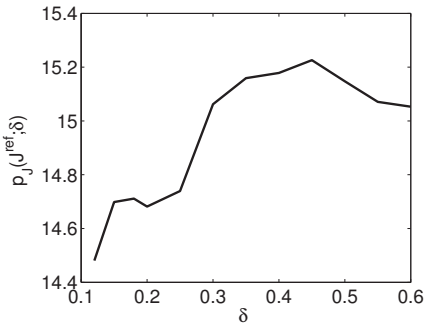

Figure 5. Graph of function $\delta \mapsto p_J(J^{ref}; \boldsymbol{\delta})$.

5.4. System uncertainties modelling and dispersion parameter identification.
The non-parametric probabilistic model of model uncertainites introduced in Section 2.3 is used for stiffness part of the linear subsystem of the mean simplified computational model. We then have to identify the dispersion parameter $\boldsymbol{\delta} = (\delta_K^B)$. The estimation of each probability density function in Eq. (2) is carried out with 200 realizations for the Monte Carlo simulation in order to solve stochastic simplified computational model. Fig. 5 shows the likelihood function calculated using Eq. (2) with $\mathcal{C}_{ad} = [0, \sqrt{23/27}]$. The maximum is reached for $\boldsymbol{\delta}^{opt} = 0.45$. Using Eq. (3) the confidence region associated with a probability level $P_c = 0.95$ of the reponse of the stochastic simplified computational model can then be estimated. The calculations are carried out with 100 simulations. The comparison between the reference solution with the response constructed with the stochastic simplified computational model is given in Fig. 6.

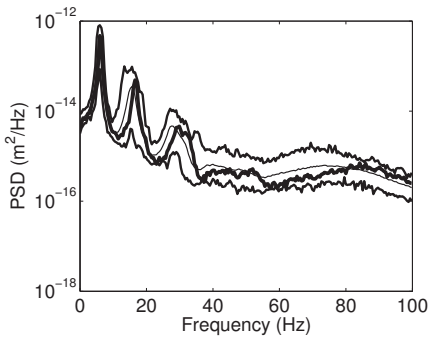

Figure 6. Power spectral density function (PSD) for the stochastic transversal displacement in point *Pobs*: confidence region calculated with the simplified model (thin line) and reference model (thick line).

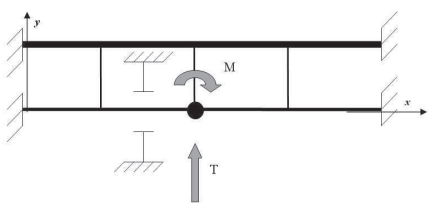

Figure 7. Definition of the stochastic load.

5.5. Probabilistic model of the stochastic load $\boldsymbol{F}(t)$.
We recall that the real model of the stochastic load used to construct the experimental responses in Section 5.4 is now assumed unknown and has to be identified using the stochastic simplified computational model. Consequently, we then have to define a model as simple as possible for the stochastic load $\mathbf{F}(t)$. We have then chosen to model $\mathbf{F}(t)$ as $\{\mathbf{F}(t) = (T(t), M(t)), t \in \mathbb{R}\}$ in which $T(t)$ is a tranversal force and $M(t)$ a moment applied to the middle of the nonlinear beam (see Fig. 7). This force and this moment are second order centred stationary Gaussian independent stochastic processes. So, they are both completely defined by their power spectral density functions $[\underline{S}_{\mathbf{T}}(\omega)]$ and $[\underline{S}_{\mathbf{M}}(\omega)]$. The matrix-valued spectral density function of the stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$ is then defined by

$$[\underline{S}_{\mathbf{F}}(\omega)] = \begin{bmatrix} \underline{S}_{\mathbf{T}}(\omega) & 0 \\ 0 & \underline{S}_{\mathbf{M}}(\omega) \end{bmatrix} \quad , \quad \omega \in \mathbb{R} \quad . \tag{6}$$

It is assumed that the function $\omega \mapsto [\underline{S}_{\mathbf{F}}(\omega)]$ is constant in the frequency band of analysis $\mathcal{B}$. As explained in Section 3, the stochastic process $\{\mathbf{F}^{\text{unc}}(t), t \in \mathbb{R}\}$ including the probabilistic model of uncertainties is contructed from the stochastic process $\{\mathbf{F}(t), t \in \mathbb{R}\}$.

5.6. Identification of the stochastic load $\boldsymbol{F}^{\text{unc}}(t)$.
The function $\omega \mapsto [\underline{S}_{\mathbf{F}}(\omega)]$ which is a constant diagonal hermitian matrix over the frequency band of analysis $\mathcal{B}$ can then be rewritten as

$$[\underline{S}_{\mathbf{F}}(\omega)] = [S(\omega, \mathbf{r})] = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \quad , \quad \omega \in \mathcal{B} \quad , \quad \mathbf{r} \in \mathcal{C}_r \quad , \tag{7}$$

in which the admissible space $\mathcal{C}_r = \{\mathbf{r} = (r_1, r_2); r_1 > 0, r_2 > 0\}$. This vector $\mathbf{r}$ is identified using the trial method, consisting in calculating the cost function $D(\mathbf{r})$ for 100 values of the vector $\mathbf{r}$. Then, the optimal value $\mathbf{r}_{opt}$ defined by Eq. (4) is such that $\mathbf{r}_{opt} = (18\,N^2/Hz, 20\,(N/m)^2/Hz)$. The dispersion paramater δ_F is identified using Eq. (5) with $\mathcal{C}_{\delta_F} =]0, \sqrt{3/7}[$. The maximum of the likelihood function $\delta_F \mapsto p_{J_s}(J_s^{\text{exp}}; \delta_F)$ is reached for $\delta_F^{opt} = 0.04$.

6. Conclusions

We have then presented a complete methodology to identify the stochastic load taking into account uncertainites in the computational model and in the stochastic loads representation. With respect to the state of the art, this work proposes a new way to perform the experimental identification of a stochastic load with a robust method. The robustness is introduced in taking into account (1) uncertainties in the simplified computational non-linear dynamical model used to carry out this identification and (2) uncertainties in the mathematical representation of the stochastic process which models the loads to be identified.

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