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# A general regularization of the hypersingular integrals in the symmetric Galerkin boundary element method

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## SUMMARY

The symmetric Galerkin boundary element method is used to solve boundary value problems by keeping the symmetric nature of the matrix obtained after discretization. The matrix elements are obtained from a double integral involving the double derivative of Green's operator, which is highly singular. The paper presents a regularization of the hypersingular integrals which depend only on the properties of Green's tensor. The method is presented in the case of Laplace's operator, with an example of application. The case of elasticity is finally addressed theoretically, showing an easy extension to any case of anisotropy.

KEY WORDS: finite element; integral equation; symmetric Galerkin boundary element method; Green's function

## 1. INTRODUCTION

It is well known that finite element methods (FEM) and boundary element methods (BEM) contain features that are complementary, FEM method deals easily with non-linear problems and BEM allows to account for unbounded domains or singularities. The solutions allowing to perform the coupling between FEM and BEM were therefore the subject of numerous works. The major part (but not all) of these methods lead to the formation of a 'stiffness matrix' of the infinite domain which is next added to the FEM stiffness matrix. The computation of a stiffness matrix from the classical 'collocation BEM' [1] has been early recognized as leading to complications : the stiffness matrix built from this method is non-symmetrical [2], which is contrary to the Maxwell–Betti theorem, and the use of the symmetrical part of that non-symmetrical matrix can lead to erroneous results [3, 4]. The introduction of the symmetric Galerkin boundary element method

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(SGBEM) [5–11] leads to a symmetric system of equations when used alone and to a symmetric stiffness matrix if the coupling with finite elements is in view [9].

The kernel of the operators appearing in the SGBEM is however strongly singular, because it involves the second derivative of Green's function related to the differential operator, whose singularity is the same as the one of the fundamental singular solution of the operator, typically  $\ln(r)$  for 2D problems and  $1/r$  for 3D problems. The singularity in SGBEM is one degree higher than the one obtained in the classical 'collocation' formulation of the BEM. The stiffness matrix itself, obtained by an integration involving the singular kernel, is therefore affected by this singularity.

The computation of the hypersingular integrals which appears in the context of the SGBEM has been the subject of a significant number of works which include closed-form evaluations of the integrals involved in the computation of matrix elements or regularization methods [8–10, 12–15]. These methods are often limited in their field of application (use of properties of the elasticity tensor or restriction to the plane case), except for the generalization of SGBEM by the Fourier method [13] which transforms the singularity by Fourier transform into a problem of convergence at infinite of the Fourier transform and seems quite general. By comparison, the method described thereafter presents the same appealing simplicity and generality as the classical regularization method used for the 'collocation' method [8, 16] and reduces completely the hypersingularity in the 2D case.

The purpose of the paper is to present this method of regularization of the hypersingular integrals used in SGBEM. After presentation of the method in the case of Laplace equation, a numerical application is shown. Finally, the case of elastostatics is also treated from a theoretical point of view.

## 2. REGULARIZATION OF THE HYPERSINGULAR INTEGRALS RELATED TO THE LAPLACE OPERATOR

### 2.1. Formulation of SGBEM and computation of the equivalent stiffness matrix

The purpose of the following is related to a solution  $u$  of the Laplace equation, which is therefore solution within a given domain  $D$  of

$$\Delta(u) = \sum_i \frac{\partial^2 u}{\partial x_i^2} = 0 \quad (1)$$

The formulation of SGBEM starts from the Maxwell–Betti reciprocity theorem, which reads (without sources):

$$\int_{\Gamma} u^*(\mathbf{x}) p(\mathbf{x}) d\Gamma(\mathbf{x}) = \int_{\Gamma} u(\mathbf{x}) p^*(\mathbf{x}) d\Gamma(\mathbf{x}) \quad (2)$$

where  $p$  is the normal flux induced by the potential  $u$  along the boundary and  $u^*$  is the potential induced along the boundary by a repartition of sources along an outside boundary  $\Gamma^*$ ,  $p^*$  being the normal flux induced by  $u^*$ .

$u^*$  is chosen as induced by a repartition of sources  $F^*$  (flux discontinuities) and  $D^*$  (potential discontinuities) along the boundary at current point  $\mathbf{x}'$ , so that:

$$u^*(\mathbf{x}) = \int_{\Gamma^*} G_{uu}(\mathbf{x}, \mathbf{x}') F^*(\mathbf{x}') d\Gamma^*(\mathbf{x}') + \int_{\Gamma^*} G_{up}(\mathbf{x}, \mathbf{x}', \mathbf{n}, \mathbf{n}') D^*(\mathbf{x}') d\Gamma^*(\mathbf{x}') \quad (3)$$

$$p^*(\mathbf{x}) = \int_{\Gamma^*} G_{pu}(\mathbf{x}, \mathbf{x}', \mathbf{n}, \mathbf{n}') F^*(\mathbf{x}') d\Gamma^*(\mathbf{x}') + \int_{\Gamma^*} G_{pp}(\mathbf{x}, \mathbf{x}', \mathbf{n}, \mathbf{n}') \cdot D^*(\mathbf{x}') d\Gamma^*(\mathbf{x}') \quad (4)$$

The different kernels are functions of the normal  $\mathbf{n}$  to  $\Gamma$  and  $\mathbf{n}'$  to  $\Gamma^*$  and of the position of current points. The kernel  $G_{uu}$  is computed from the fundamental solution of the Laplace operator. Kernels  $G_{up}$  and  $G_{pu}$  are computed from the normal derivatives of  $G_{uu}$  and kernel  $G_{pp}$  is computed from the double normal derivative of  $G_{uu}$ .

The discrete form of the integral equation is obtained by using the discretized form of the potential (and its gradient) and of the sources

$$u(\mathbf{x}) = \Psi_u(\mathbf{x})[\mathbf{U}] \quad (5)$$

$$p(\mathbf{x}) = \Psi_p(\mathbf{x})[\mathbf{P}] \quad (6)$$

$$F^*(\mathbf{x}') = \Psi_p^*(\mathbf{x}')[\mathbf{F}^*] \quad (7)$$

$$D^*(\mathbf{x}') = \Psi_u^*(\mathbf{x}')[\mathbf{D}^*] \quad (8)$$

where  $\Psi_u$ ,  $\Psi_p$ ,  $\Psi_p^*$ ,  $\Psi_u^*$  are the row vectors containing interpolation functions for each of the boundary variables.

It is shown in [17] that the SGBEM formulation is restricted to interpolation functions which comply with:

$$\Psi_u^* = \Psi_u \quad (9)$$

$$\Psi_p^* = \Psi_p \quad (10)$$

Different uses can be made of the SGBEM, but the main objective is to couple the SGBEM with the FEM, by obtaining the 'stiffness matrix' equivalent to the boundary element formulation.

This can be obtained by looking for the energy induced by a compatible pair of potentials and fluxes along the boundary of the domain (the trace on the boundary of a compatible state  $(u, p)$ , where  $p = \text{grad}(u) \cdot \mathbf{n}$ , within the considered domain  $D$ ). The potential energy is equal to:

$$W = \frac{1}{2} \int_{\Gamma} p(\mathbf{x}) \cdot u(\mathbf{x}) d\Gamma(\mathbf{x}) \quad (11)$$

It is shown in [9] that the SGBEM formulation, obtained at the limit when  $\Gamma^* = \Gamma$  leads to the discretized form of the elastic energy given by

$$W = \frac{1}{2} [\mathbf{U}]^T [\mathbf{K}] [\mathbf{U}] \quad (12)$$

where the stiffness matrix  $\mathbf{K}$  is equal to  $\mathbf{K} = \mathbf{K}_u + \mathbf{K}_p$ , with:

$$[\mathbf{K}_u] = ([\mathbf{E}_{up}] + [\mathbf{L}_{up}])^T [\tilde{\mathbf{G}}_{uu}]^{-1} ([\mathbf{E}_{up}] + [\mathbf{L}_{up}]) \quad (13)$$

$$[\mathbf{K}_p] = -[\tilde{\mathbf{G}}_{pp}] \quad (14)$$

The matrices appearing in  $[\mathbf{K}_u]$  and  $[\mathbf{K}_p]$  are defined below:

$$[\tilde{\mathbf{G}}_{ij}] = \int_{\Gamma} \int_{\Gamma} \Psi_{i'}^T(\mathbf{x}') G_{ij}(\mathbf{x}, \mathbf{x}') \cdot \Psi_{j'}(\mathbf{x}) d\Gamma(\mathbf{x}') d\Gamma(\mathbf{x}) \quad (15)$$

In this expression  $i, i'$  are  $u, p$  or  $p, u$ , similarly for  $j, j'$

The integral on  $\Gamma$  is weakly singular for  $i = j = u$ , is taken as the Cauchy principal value for  $i \neq j$ , and is the Hadamard finite part integral for  $i = j = p$

$$[\mathbf{E}_{up}] = \int_{\Gamma} \kappa(\mathbf{x}') \Psi_p^T(\mathbf{x}') \Psi_u(\mathbf{x}') d\Gamma(\mathbf{x}') \quad (16)$$

and:

$$[\mathbf{L}_{up}] = \int_{\Gamma} \int_{\Gamma} \Psi_p^T(\mathbf{x}') G_{up}(\mathbf{x}, \mathbf{x}') [\Psi_u(\mathbf{x}') - \Psi_u(\mathbf{x})] d\Gamma d\Gamma \quad (17)$$

The term  $\kappa(\mathbf{x}')$  is equal to 1 or 0, depending on whether the unit normal  $\mathbf{n}$  is interior or exterior to the surface  $\Gamma$ .

The hypersingular integral is related to the term  $\mathbf{K}_p$  of the stiffness matrix which reads:

$$\begin{aligned} [\mathbf{K}_p] &= -[\tilde{\mathbf{G}}_{pp}] = - \int_{\Gamma} \int_{\Gamma} \Psi_u(\mathbf{x}') \frac{\partial^2 G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}) \partial \mathbf{n}'(\mathbf{x}')} \Psi_u(\mathbf{x}) d\Gamma(\mathbf{x}') d\Gamma(\mathbf{x}) \\ &= - \lim_{\Gamma^* \rightarrow \Gamma} \int_{\Gamma^*} \int_{\Gamma} \Psi_u(\mathbf{x}') \frac{\partial^2 G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}) \partial \mathbf{n}'(\mathbf{x}')} \Psi_u(\mathbf{x}) d\Gamma^*(\mathbf{x}') d\Gamma(\mathbf{x}) \end{aligned} \quad (18)$$

The integral in the first line of (18) must be taken as the 'Hadamard finite part integral'. It is shown in [9, 17] that this finite part integral can be computed in the following way: in a first step, the integrals are taken over two close but distinct curves  $\Gamma^*$  and  $\Gamma$ , where  $\Gamma^*$  surrounds  $\Gamma$  and is in one to one correspondence with  $\Gamma$ . Next the interpolation functions, the unknown boundary and tests functions, are introduced on  $\Gamma$  and  $\Gamma^*$ . Then, the limit when  $\Gamma^*$  tends to  $\Gamma$  is taken as in the second line of (18). In the case of linear interpolation functions, the limit can be taken analytically [17].

A regularized form is given in [9] for this matrix in the case of isotropic linear elasticity and for higher-order interpolation functions, but it uses the Galerkin representation of the displacement field in isotropic elasticity, which cannot be used for example in the case of anisotropic elasticity. The purpose of the following is to obtain a regularized form of the formulation which leads to an easy extension to all operators by taking the integration on two neighbouring surfaces  $\Gamma$  and  $\Gamma^*$  and by showing that the double integral on these neighbouring surfaces is equal to a double integral which does not contain anymore singularity or hypersingularity at the limit.

## 2.2. Regularization of the hypersingular integrals

As for the classical regularization, the use is made of the fundamental property of Green's function, which reads:

$$\Delta_x G(\mathbf{x}, \mathbf{x}') + \delta(\mathbf{x} - \mathbf{x}') = 0 \quad (19)$$

where derivatives are computed by using the components of  $\mathbf{x}$ .

This equation leads, for any finite domain  $D$ , bounded by  $\Gamma$ , to:

$$\int_{\Gamma} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})} d\Gamma(\mathbf{x}) + c(\mathbf{x}') = 0 \quad (20)$$

where  $c(\mathbf{x}') = 0$  if  $\mathbf{x}'$  is outside  $D$  and  $c(\mathbf{x}') = 1$  if  $\mathbf{x}'$  is within  $D$ ,  $\Gamma$  being the boundary of  $D$ .

Let us consider the function

$$F(\mathbf{x}') = \int_{\Gamma} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x})} d\Gamma(\mathbf{x})$$

This function is constant for any value of  $\mathbf{x}'$  which is not on  $\Gamma$  in each subdomain  $D_i =$ interior of  $D$  and  $D_0 =$ exterior of  $D$ . In addition, the integrand is derivable at any order. Then, the integral of the derivative of the integrand with respect to  $\mathbf{x}'$  is equal to the derivative of  $F(\mathbf{x}')$  for any point  $F(\mathbf{x}')$  which is not on  $\Gamma$  (in fact the integrand needs only to have a continuous first derivative [18]). For any point which is not on  $\Gamma$ , the partial derivatives of  $c(\mathbf{x}')$  being null within  $D_i$  and  $D_o$ , it leads to:

$$\int_{\Gamma} \frac{\partial^2 G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{x}' \partial \mathbf{n}(\mathbf{x})} d\Gamma(\mathbf{x}) = 0 \quad (21)$$

Conversely, taking into account the symmetry properties of  $G$  leads to:

$$\int_{\Gamma^*} \frac{\partial^2 G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{x} \partial \mathbf{n}'(\mathbf{x}')} d\Gamma^*(\mathbf{x}') = 0 \quad (22)$$

This produces the following double integrals on  $\Gamma$ :

$$\int_{\Gamma^*} \int_{\Gamma} f(\mathbf{x}) \frac{\partial^2 G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}) \partial \mathbf{n}'(\mathbf{x}')} d\Gamma^*(\mathbf{x}') d\Gamma(\mathbf{x}) = 0 \quad (23)$$

for any function  $f(\mathbf{x})$ , and

$$\int_{\Gamma^*} \int_{\Gamma} g(\mathbf{x}') \frac{\partial^2 G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}) \partial \mathbf{n}'(\mathbf{x}')} d\Gamma^*(\mathbf{x}') d\Gamma(\mathbf{x}) = 0 \quad (24)$$

for any function  $g(\mathbf{x}')$

Let us now consider the part  $W_p$  of the discretized potential energy, related to the part  $[\mathbf{K}_p]$  of the stiffness matrix, given by:

$$W_p = \frac{1}{2} [\mathbf{U}]^T [\mathbf{K}_p] [\mathbf{U}] = -\frac{1}{2} [\mathbf{U}]^T [\tilde{\mathbf{G}}_{pp}] [\mathbf{U}] \quad (25)$$

This quadratic form is equal to

$$\begin{aligned} W_p &= -\frac{1}{2}[\mathbf{U}]^T[\tilde{\mathbf{G}}_{pp}][\mathbf{U}] = -\frac{1}{2}\int_{\Gamma}\int_{\Gamma}u(\mathbf{x}')\frac{\partial^2 G(\mathbf{x},\mathbf{x}')}{\partial\mathbf{n}(\mathbf{x})\partial\mathbf{n}'(\mathbf{x}')}u(\mathbf{x})d\Gamma(\mathbf{x}')d\Gamma(\mathbf{x}) \\ &= -\frac{1}{2}\lim_{\Gamma^*\rightarrow\Gamma}\int_{\Gamma^*}\int_{\Gamma}u(\mathbf{x}')\frac{\partial^2 G(\mathbf{x},\mathbf{x}')}{\partial\mathbf{n}(\mathbf{x})\partial\mathbf{n}'(\mathbf{x}')}u(\mathbf{x})d\Gamma^*(\mathbf{x}')d\Gamma(\mathbf{x}) \end{aligned} \quad (26)$$

where  $u$  is the discretized form of the boundary potential, computed from the discretized displacement  $[\mathbf{U}]$ .

Replacing  $f$  and  $g$  by  $u^2/4$  in (23) and (24) leads to

$$\frac{1}{4}\int_{\Gamma^*}\int_{\Gamma}u^2(\mathbf{x})\frac{\partial^2 G(\mathbf{x},\mathbf{x}')}{\partial\mathbf{n}(\mathbf{x})\partial\mathbf{n}'(\mathbf{x}')}d\Gamma^*(\mathbf{x}')d\Gamma(\mathbf{x})=0 \quad (27)$$

and to a similar expression obtained by replacing  $u^2(\mathbf{x})$  by  $u^2(\mathbf{x}')$

Adding these expressions to (26) leads to:

$$\begin{aligned} W_p &= \frac{1}{2}[\mathbf{U}]^T[\mathbf{K}_p][\mathbf{U}] = \frac{1}{4}\lim_{\Gamma^*\rightarrow\Gamma}\int_{\Gamma^*}\int_{\Gamma}(u(\mathbf{x}')-u(\mathbf{x}))^2\frac{\partial^2 G(\mathbf{x},\mathbf{x}')}{\partial\mathbf{n}(\mathbf{x})\partial\mathbf{n}'(\mathbf{x}')}d\Gamma^*(\mathbf{x}')d\Gamma(\mathbf{x}) \\ &= \frac{1}{4}\int_{\Gamma}\int_{\Gamma}(u(\mathbf{x}')-u(\mathbf{x}))^2\frac{\partial^2 G(\mathbf{x},\mathbf{x}')}{\partial\mathbf{n}(\mathbf{x})\partial\mathbf{n}'(\mathbf{x}')}d\Gamma(\mathbf{x}')d\Gamma(\mathbf{x}) \end{aligned} \quad (28)$$

Contrary to the original integral (26), the integral in (28) is now regular or weakly singular. Indeed, under the condition that  $u \in C^{0,\alpha}$  (or  $u$  is Hölder continuous, meaning  $\exists(\alpha, C) > 0, |u(\mathbf{x}) - u(\mathbf{x}')| < C|\mathbf{x} - \mathbf{x}'|^\alpha$ ), and assuming that  $\alpha$  is greater than  $\frac{1}{2}$ , the quantity  $(u(\mathbf{x}') - u(\mathbf{x}))^2$  is inferior to  $r^{2\alpha}$ , where  $r$  is the distance between  $\mathbf{x}'$  and  $\mathbf{x}$ , while the kernel behaves as  $1/r^d$ , where  $d$  is the space dimension, as reported in [9]. As a result, the integrand has the order  $2\alpha - 2 > -1$  and is therefore at least weakly singular. In fact, the practice is to use interpolation functions that are piecewise differentiable at any order, leading to  $\alpha = 1$ . As a consequence, the integral in (28) is regular for 2D problems and weakly singular (order  $1/r$ ) for 3D problems.

This method is quite similar to the classical regularization for singular integrals of the collocation BEM which involves, when  $\mathbf{x}$  is on  $\Gamma$ , the weakly singular integral (for 3D problems) [16]:

$$\int_{\Gamma}(u(\mathbf{x}')-u(\mathbf{x}))\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial\mathbf{n}'(\mathbf{x}')}d\Gamma(\mathbf{x}') \quad (29)$$

Up to now, the process has produced a regularized form of the potential energy. It is easy to show that this process leads to a regularized expression of the initially hypersingular part of the stiffness matrix and finally of the SGBEM formulation.

Indeed, let us call by  $[\mathbf{K}_p^*]$  the matrix obtained after discretization of the right part of (28). It is obvious that

$$W_p = -\frac{1}{2}[\mathbf{U}]^T[\tilde{\mathbf{G}}_{pp}][\mathbf{U}] = \frac{1}{2}[\mathbf{U}]^T[\mathbf{K}_p][\mathbf{U}] = \frac{1}{2}[\mathbf{U}]^T[\mathbf{K}_p^*][\mathbf{U}] \quad (30)$$

for any discretized boundary potential field  $[\mathbf{U}]$ . Taking into account that the three matrices are symmetric leads to  $[\tilde{\mathbf{G}}_{pp}] = [\mathbf{K}_p] = [\mathbf{K}_p^*]$ .

Finally, after the discretization of (28), the matrix  $[\tilde{\mathbf{G}}_{pp}] = [\mathbf{K}_p] = [\mathbf{K}_p^*]$  can be obtained under a regularized form and used either within the computation of the stiffness matrix, or within the classical formulation of the SGBEM (without, in this case, computation of the stiffness matrix).

### 3. EXAMPLE OF A NUMERICAL APPLICATION IN THE CASE OF THE LAPLACE OPERATOR WITHIN A CIRCULAR DOMAIN

Let us consider the application to a circular domain. The construction of the stiffness matrix related to this domain as shown in Figure 1 is performed now.

The boundary of the circular domain is split into contour elements  $e_n$  related to the  $N$  points  $\mathbf{x}_i$  on the contour as shown in Figure 1, with  $e_n = (\mathbf{x}_n, \mathbf{x}_{n+1})$  for  $1 < n < N - 1$  and  $e_N = (\mathbf{x}_N, \mathbf{x}_1)$ .

The stiffness matrix depends obviously on the choice of interpolation functions. In the following, linear interpolation functions are chosen, but other choices are obviously possible, because the regularization method does not use any special form of the interpolation functions.

Introducing the decomposition of the boundary integral into partial integrals along the elements within (28) leads to:

$$W_p = \frac{1}{2} [\mathbf{U}]^T [\mathbf{K}_p] [\mathbf{U}] = \frac{1}{4} \sum_{p,q} \int_{e_p} \int_{e_q} (u(\mathbf{x}) - u(\mathbf{x}'))^2 H(\mathbf{x}, \mathbf{x}') ds(\mathbf{x}') ds(\mathbf{x}) \quad (31)$$

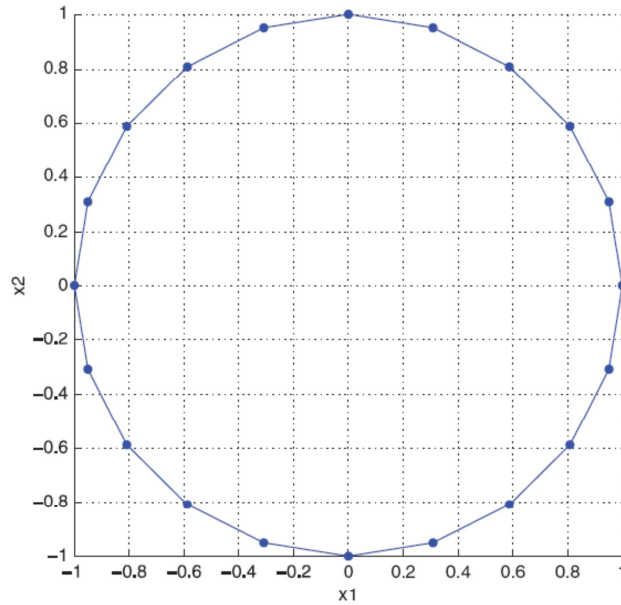


Figure 1. Circular domain bounded by 20 boundary elements.



where

$$H(\mathbf{x}, \mathbf{x}') = \frac{\partial^2 G(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}) \partial \mathbf{n}(\mathbf{x}')} \quad (32)$$

replacing now the function  $u$  by its interpolation along each element leads to:

$$W_p = \frac{1}{4} \sum_{p,q} \int_{-1}^1 \int_{-1}^1 (\Psi_1(\zeta)u_p + \Psi_2(\zeta)u_{p+1} - \Psi_1(\eta)u_q + \Psi_2(\eta)u_{q+1})^2 \\ \times H(\mathbf{x}, \mathbf{x}') J(\zeta) J(\eta) d\zeta d\eta \quad (33)$$

where  $\zeta$  is the local coordinate on  $e_p$  and  $\eta$  is the local coordinate on  $e_q$  and the Jacobian  $J(\zeta)$  is the length of the element  $e_p$ . The position of the current point  $\mathbf{x}(\zeta)$  on  $e_p$  is defined from those of the nodes  $\mathbf{x}_p$  and  $\mathbf{x}_{p+1}$  defining the element  $e_p$  by

$$\mathbf{x}(\zeta) = \Psi_1(\zeta)\mathbf{x}_p + \Psi_2(\zeta)\mathbf{x}_{p+1} \quad (34)$$

with a similar expression for  $\mathbf{x}'(\eta)$

The contribution of the couple of elements  $(e_p, e_q)$  leads therefore to partial matrix elements of the matrix  $K_{ij}$ . The integrations on  $e_p$  and  $e_q$  lead to the contributions related to the matrix elements given in formulas (A1)–(A11) of the Appendix.

The computation of these elements requires Green's function  $G$ . In order to check the capacity of the regularization, the choice has been made to use the first Green's function related to the circular domain instead of the fundamental solution for an infinite domain. As shown in [9], the interest of this choice is to avoid the inversion of  $[\tilde{\mathbf{G}}_{uu}]$  in Equation (13). In fact, the part  $[\mathbf{K}_u]$  of the stiffness matrix is null in this case, taking into account the properties of the first Green's function, leading to  $[\mathbf{K}] = [\mathbf{K}_p]$ . It is worthwhile to notice that, in this case, the use of  $G$  expanded along a series of orthogonal functions would lead to the 'Dirichlet-to-Neumann' formulation [19, 20].

Green's function for a circular domain is given in polar coordinates by [21]

$$G(r, \phi; \rho, \psi) = \frac{1}{4\pi} \frac{R^2 - 2r\rho \cos(\phi - \psi) + (R^{-1}r\rho)^2}{r^2 - 2r\rho \cos(\phi - \psi) + \rho^2} \quad (35)$$

where  $(r, \phi)$  are the polar coordinates of  $\mathbf{x}'$  and  $(\rho, \psi)$  the coordinates of  $\mathbf{x}$

The kernel  $H$  within Equation (31) is obtained by differentiation and given by:

$$H = \frac{1}{4\pi R^2 \sin^2((\phi - \psi)/2)} \quad (36)$$

The positions of  $\mathbf{x}$  and  $\mathbf{x}'$  on the contour of the macroelement are defined by the angles  $\phi$  and  $\psi$ .

The discretization of the circle is now performed by  $N$  elements characterized by the same angular increment  $\Delta\phi$ , such that the angular position of the point  $\mathbf{x}_n$  is given by  $\phi_n = (n-1)\Delta\phi$ .

For such a discretization, the elementary matrices related to  $e_p, e_q$  needed to compute the matrix elements from relations (A1) to (A11) of the Appendix are finally given by relations given in Equations (A12)–(A23) of the Appendix.

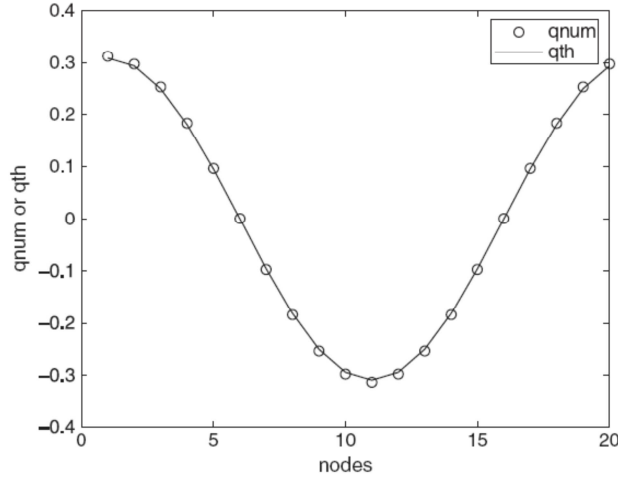


Figure 2. Comparison between the nodal fluxes computed numerically (qnum) and from the explicit formula (37) (qth).

It is easy to check that all integrands are bounded functions, because the denominator of the integrand can be null only when the argument  $\alpha$  within  $\sin(\alpha)$  inside the denominator of the integrand is null (with  $\alpha = \Delta\phi(\zeta - \eta)/4$  or  $\alpha = \Delta\phi(\zeta - \eta - 2)/4$ ). It is easy to check that the integrand is then equal, up to a multiplicative constant, to  $\alpha^2/\sin^2(\alpha) = 1/\text{sinc}^2(x)$  ( $\text{sinc} = \text{cardinal sine}$ ) which is obviously finite even when  $\alpha=0$ . It is also clear that these integrals are regular only because the numerator tends to zero simultaneously with the denominator. Without regularization, the computation of  $[\mathbf{K}_p]$  would lead to singular integrals which would be more difficult to evaluate numerically.

A numerical application has been performed on a circular domain having a unit radius whose boundary is split into 20 elements as shown in Figure 1. A linear potential given by  $u = x_1$  is applied on the boundary. Let us define the column vector  $[\mathbf{U}]$  made of the values  $U_l = u(\mathbf{x}_l)$  of the potential at all nodes  $\mathbf{x}_l$  of the boundary. The stiffness matrix  $[\mathbf{K}]$  of the macroelement is computed as explained in the previous section. The result  $[\mathbf{q}]$  of the matrix product  $[\mathbf{q}] = [\mathbf{K}][\mathbf{U}]$  is made of the 'nodal' fluxes. Each nodal flux  $q_l$  may be considered as the integral of  $\partial u / \partial \mathbf{n}$  along the part of the circle defined by  $\phi$  included between  $\phi_l - \Delta\phi/2$  and  $\phi_l + \Delta\phi/2$ . It is therefore easy to compute all nodal fluxes  $q_l$  related to the given potential, which are equal, for a unit radius to  $q_{th} = q_l$ , where:

$$q_l = \sin(\phi_l + \Delta\phi/2) - \sin(\phi_l - \Delta\phi/2) \quad (37)$$

Figure 2 shows the comparison of the variation of the nodal flux at the nodes of the boundary between the values given by the numerical method (qnum) and those given by the explicit formula (37) above for the theoretical value  $q_{th}$ . The results show a good agreement between both sets of nodal fluxes along the boundary.

#### 4. REGULARIZATION OF THE OPERATOR OF ELASTOSTATICS

Let us now consider the case of the elasticity. Assuming that body forces are not present, the components  $u_i$  of the displacement field are solutions within a domain  $D$  of

$$[C_{mnij}u_{i,j}]_{,n} = 0 \quad (38)$$

where  $C_{mnij}$  are the components of the elasticity tensor.

'Green's tensor'  $U_i^k(\mathbf{x}, \mathbf{y})$  for elasticity is solution of

$$[C_{mnij}U_{i,j}^k]_{,n} + \delta_{mk}\delta(\mathbf{y} - \mathbf{x}) = 0 \quad (39)$$

where  $k$  denotes the direction of the unit force inside the domain  $D$ .

Let us consider the boundary  $\Gamma$  of the domain  $D$ . The traction vector  $T_i^k$  induced by  $U_i^k$  along  $\Gamma$  is given by

$$T_i^k = P_{imj}U_m^k n_j(\mathbf{y}) \quad (40)$$

which is obtained from the differential operator  $P_{imj}$  given by:

$$P_{imj}(\partial\mathbf{y}) = C_{ijmn} \frac{\partial}{\partial y_n} \quad (41)$$

From the previous relations and using Gauss's theorem, the following property of  $T_i^k$  is obtained:

$$\int_{\Gamma} T_i^k(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) + \delta_{ik} \cdot p(\mathbf{x}') = 0 \quad (42)$$

where  $p(\mathbf{x}') = 1$  if  $\mathbf{x}'$  is within  $\Gamma$  and  $p(\mathbf{x}') = 0$  if  $\mathbf{x}'$  is outside  $\Gamma$ .

As for the case of Laplace equation, the potential energy  $W$  related to the displacement field  $u_p$  and to the compatible traction  $T_p$  along the boundary  $\Gamma$  is given by:

$$W = \frac{1}{2} \int_{\Gamma} u_p(\mathbf{x}) \cdot T_p(\mathbf{x}) d\Gamma(\mathbf{x}) \quad (43)$$

The discretized potential energy leads to the stiffness matrix by

$$W = \frac{1}{2} [\mathbf{U}]^T [\mathbf{K}] [\mathbf{U}] \quad (44)$$

where  $[\mathbf{U}]$  contains now the nodal components of displacement and where the stiffness matrix  $\mathbf{K}$  is still equal to  $\mathbf{K} = \mathbf{K}_u + \mathbf{K}_p$ , like in (13) and (14).

The part  $\mathbf{K}_p$  of the stiffness matrix related to the hypersingular integral is obtained by discretization of the integral:

$$\begin{aligned} W_p &= -\frac{1}{2} \int_{\Gamma} \int_{\Gamma} u_p(\mathbf{x}) \cdot u_i(\mathbf{x}') H_{pi}(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) d\Gamma(\mathbf{x}') \\ &= -\frac{1}{2} \lim_{\Gamma^* \rightarrow \Gamma} \int_{\Gamma^*} \int_{\Gamma} u_p(\mathbf{x}) \cdot u_i(\mathbf{x}') H_{pi}(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) d\Gamma^*(\mathbf{x}') \end{aligned} \quad (45)$$

where the components  $H_{pi}$  are given by:

$$H_{pi}(\mathbf{x}, \mathbf{x}') = P_{pkq}(\partial\mathbf{x}) \cdot P_{imj}(\partial\mathbf{x}') U_m^k(\mathbf{x}, \mathbf{x}') n_j(\mathbf{x}') n_q(\mathbf{x}) \quad (46)$$

Green's tensor  $U_i^k$  complies to the symmetry property  $U_i^k(\mathbf{x}, \mathbf{x}') = U_k^i(\mathbf{x}', \mathbf{x})$ .

Introducing this relation in the definition of  $H_{pi}$  leads to:

$$H_{pi}(\mathbf{x}, \mathbf{x}') = H_{ip}(\mathbf{x}', \mathbf{x}) \quad (47)$$

Taking into account this symmetry,  $W_p$  reads now:

$$W_p = -\frac{1}{4} \lim_{\Gamma^* \rightarrow \Gamma} \int_{\Gamma^*} \int_{\Gamma} (u_p(\mathbf{x}) \cdot u_i(\mathbf{x}') + u_i(\mathbf{x}) \cdot u_p(\mathbf{x}')) H_{pi}(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) d\Gamma^*(\mathbf{x}') \quad (48)$$

From (39) it is easy to show, as in Section 2 that the following relations are obtained:

$$\lim_{\Gamma^* \rightarrow \Gamma} \int_{\Gamma^*} \int_{\Gamma} f(\mathbf{x}) \cdot H_{pi}(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) d\Gamma^*(\mathbf{x}') = 0 \quad (49)$$

and:

$$\lim_{\Gamma^* \rightarrow \Gamma} \int_{\Gamma^*} \int_{\Gamma} g(\mathbf{x}') \cdot H_{pi}(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) d\Gamma^*(\mathbf{x}') = 0 \quad (50)$$

Taking  $f(\mathbf{x}) = \frac{1}{4} u_i(\mathbf{x}) \cdot u_p(\mathbf{x})$  and  $g(\mathbf{x}') = \frac{1}{4} u_i(\mathbf{x}') \cdot u_p(\mathbf{x}')$  leads to:

$$\frac{1}{4} \lim_{\Gamma^* \rightarrow \Gamma} \int_{\Gamma^*} \int_{\Gamma} (u_p(\mathbf{x}) \cdot u_i(\mathbf{x}) + u_i(\mathbf{x}') \cdot u_p(\mathbf{x}')) H_{pi}(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) d\Gamma^*(\mathbf{x}') = 0 \quad (51)$$

Adding the integrals in Equations (48) and (51) leads to:

$$\begin{aligned} W_p &= \frac{1}{4} \lim_{\Gamma^* \rightarrow \Gamma} \int_{\Gamma^*} \int_{\Gamma} (u_i(\mathbf{x}') - u_i(\mathbf{x}))(u_p(\mathbf{x}') - u_p(\mathbf{x})) H_{pi}(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) d\Gamma^*(\mathbf{x}') \\ &= \frac{1}{4} \int_{\Gamma} \int_{\Gamma} (u_i(\mathbf{x}') - u_i(\mathbf{x}))(u_p(\mathbf{x}') - u_p(\mathbf{x})) H_{pi}(\mathbf{x}, \mathbf{x}') d\Gamma(\mathbf{x}) d\Gamma(\mathbf{x}') \end{aligned} \quad (52)$$

This integral is now a regular or weakly singular integral, for the same reasons as those given in Section 2 for the Laplace operator. This one can be discretized as in the previous section to obtain the stiffness matrix related to the domain  $D$ , all singularities being now removed in the case of plane elasticity or reduced to weak (integrable) singularities for 3D computations.

## 5. CONCLUSION

The SGBEM leads to the computation of the 'boundary stiffness matrix' which is symmetric, compared with the classical 'collocation' method, which leads to a non-symmetric stiffness matrix. One main difficulty in the use of SGBEM is the presence of hypersingular integrals and of singular integrals. The regularization of the singular integrals can be performed as in [9]. A new regularization of the hypersingular integrals has been obtained in this paper.

The method presented in this paper is to use a regularization process which rests only on the fundamental properties of Green's function, like for the classical regularization method of [16]. A numerical example has been treated in the case of Laplace equation. Finally, the regularization method has been extended to the case of elasticity. The regularization method allows to replace the hypersingular integrals by regular integrals in the case of 2D problems and by weakly singular

integrals in the case of 3D problems. The method can easily be extended to others operators, and can be used in the case of anisotropic elasticity, because it rests only on the use of the properties of Green's function (or Green's tensor) instead of properties related to specific operators or specific interpolation functions.

#### APPENDIX A: DETAIL OF THE COMPUTATION OF THE MATRIX ELEMENTS

The computation of matrix elements from interpolation functions are given as:

For  $q \neq p$ ,  $q \neq p+1$ ,  $q \neq p-1$  the contribution of  $e_p, e_q$  is a  $4 \times 4$  matrix related to lines and columns  $p, p+1, q, q+1$

$$K(e_p, e_q) = \begin{pmatrix} M_1(1, 1) & M_1(1, 2) & -M_2(1, 1) & -M_2(1, 2) \\ M_1(1, 2) & M_1(2, 2) & -M_2(2, 1) & M_2(2, 2) \\ -M_2(1, 1) & -M_2(2, 1) & M_3(1, 1) & M_3(1, 2) \\ -M_2(1, 2) & -M_2(2, 2) & M_3(1, 2) & M_3(2, 2) \end{pmatrix} \quad (A1)$$

For  $q = p+1$ , the contribution of  $e_p, e_{p+1}$  is a  $3 \times 3$  matrix related to lines and columns  $p, p+1, p+2$  given by:

$$K(e_p, e_{p+1}) = \begin{pmatrix} M_1(1, 1) & M_4(1, 2, 1) & -M_2(1, 2) \\ M_4(1, 2, 1) & M_5(2, 1) & -M_6(2, 2, 1) \\ -M_2(1, 2) & -M_6(2, 2, 1) & M_3(2, 2) \end{pmatrix} \quad (A2)$$

For  $q = p-1$ , the contribution of  $e_{p-1}, e_p$  is a  $3 \times 3$  matrix related to lines and columns  $p-1, p, p+1$  given by:

$$K(e_{p-1}, e_p) = \begin{pmatrix} M_3(1, 1) & -M_6(1, 1, 2) & -M_2(1, 2) \\ -M_6(1, 1, 2) & M_5(1, 2) & M_4(2, 1, 2) \\ -M_2(1, 2) & M_4(2, 1, 2) & M_1(2, 2) \end{pmatrix} \quad (A3)$$

And for  $q = p$ :

$$K(e_p, e_p) = \begin{pmatrix} M_7(1, 1) & M_7(1, 2) \\ M_7(1, 2) & M_7(2, 2) \end{pmatrix} \quad (A4)$$

Where the functions  $M_1$  to  $M_7$  are defined by:

$$M_1(m, n) = \int_{-1}^1 \int_{-1}^1 \Psi_m(\zeta) \cdot \Psi_n(\zeta) \cdot H(\mathbf{x}(\zeta), \mathbf{x}'(\eta)) J(\zeta) \cdot J(\eta) d\zeta d\eta \quad (A5)$$

$$M_2(m, n) = \int_{-1}^1 \int_{-1}^1 \Psi_m(\zeta) \cdot \Psi_n(\eta) \cdot H(\mathbf{x}(\zeta), \mathbf{x}'(\eta)) J(\zeta) \cdot J(\eta) d\zeta d\eta \quad (A6)$$

$$M_3(m, n) = \int_{-1}^1 \int_{-1}^1 \Psi_m(\eta) \cdot \Psi_n(\eta) \cdot H(\mathbf{x}(\zeta), \mathbf{x}'(\eta)) J(\zeta) \cdot J(\eta) d\zeta d\eta \quad (A7)$$

$$M_4(m, n, r) = \int_{-1}^1 \int_{-1}^1 \Psi_m(\zeta) \cdot (\Psi_n(\zeta) - \Psi_r(\eta)) \\ \times H(\mathbf{x}(\zeta), \mathbf{x}'(\eta)) \cdot J(\zeta) \cdot J(\eta) d\zeta d\eta \quad (\text{A8})$$

$$M_5(m, n) = \int_{-1}^1 \int_{-1}^1 (\Psi_m(\zeta) - \Psi_n(\eta))^2 \cdot H(\mathbf{x}(\zeta), \mathbf{x}'(\eta)) J(\zeta) \cdot J(\eta) d\zeta d\eta \quad (\text{A9})$$

$$M_6(m, n, r) = \int_{-1}^1 \int_{-1}^1 \Psi_m(\eta) (\Psi_n(\zeta) - \Psi_r(\eta)) \\ \times H(\mathbf{x}(\zeta), \mathbf{x}'(\eta)) J(\zeta) \cdot J(\eta) d\zeta d\eta \quad (\text{A10})$$

$$M_7(m, n) = \int_{-1}^1 \int_{-1}^1 (\Psi_m(\zeta) - \Psi_m(\eta)) \cdot (\Psi_n(\zeta) - \Psi_n(\eta)) \\ \times H(\mathbf{x}(\zeta), \mathbf{x}'(\eta)) J(\zeta) \cdot J(\eta) d\zeta d\eta \quad (\text{A11})$$

For the case of the first Green's function related to a circle, these matrix elements become: for  $i = 1, 3$

$$M_i(m, n) = C \iint \frac{P_i(m, n, \zeta, \eta)}{\sin^2(\Delta\phi(2p - 2q + \zeta - \eta)/4)} d\zeta d\eta \quad (\text{A12})$$

where

$$P_3(1, 1, \zeta, \eta) = P_1(1, 1, \zeta, \eta) = (1 - \zeta)^2 \quad (\text{A13})$$

$$P_3(2, 2, \zeta, \eta) = P_1(2, 2, \zeta, \eta) = (1 + \zeta)^2 \quad (\text{A14})$$

$$P_3(1, 2, \zeta, \eta) = P_1(1, 2, \zeta, \eta) = (1 - \zeta)(1 + \zeta) \quad (\text{A15})$$

$$P_2(1, 1, \zeta, \eta) = P_2(2, 2, \zeta, \eta) = (1 - \zeta)(1 - \eta) \quad (\text{A16})$$

$$P_2(1, 2, \zeta, \eta) = P_2(2, 1, \zeta, \eta) = (1 - \zeta)(1 + \eta) \quad (\text{A17})$$

for  $i = 4, 5, 6$

$$M_4(1, 2, 1) = C \iint \frac{(1 - \zeta)(\zeta + \eta)}{\sin^2(\Delta\phi(\zeta - \eta - 2)/4)} d\zeta d\eta \quad (\text{A18})$$

$$M_5(2, 1) = M_5(1, 2) = C \iint \frac{(\zeta + \eta)^2}{\sin(\Delta\phi(\zeta - \eta - 2)/4)} d\zeta d\eta \quad (\text{A19})$$

$$M_4(2, 1, 2) = -M_6(2, 2, 1) = -M_6(1, 1, 2) = M_4(1, 2, 1) \quad (\text{A20})$$

for  $i = 7$ :

$$M_7(1, 1) = C \iint \frac{(\zeta - \eta)^2}{\sin^2(\Delta\phi(\zeta - \eta)/4)} d\zeta d\eta \quad (\text{A21})$$

$$M_7(2, 2) = -M_7(1, 2) = -M_7(2, 1) = M_7(1, 1) \quad (\text{A22})$$

$$C = \frac{\Delta\phi^2}{128\pi} \quad (\text{A23})$$

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