Probabilistic and asymptotic methods with the Perron Frobenius's operator

Guy Cirier

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Résumé

1- Écart résolvant . Soit f *une application polynomiale d'un compact* C *de* R^{d} *dans lui-même. Une mesure* P *est dite invariante sous* f *ssi* $P = P_f$ *où* $P_f = P \circ f^{-1}$. P_f *est appelé l'opérateur de Perron Frobenius (PF). Soient* $e_f^n(y) = \partial^n \left[\mathrm{e}^{ya} - \mathrm{e}^{yf(a)} \right] / \partial a^n \big|_{a=0}$ *et* $H_n(y) = \partial^n \mathrm{e}^{yf(a)} / \partial a^n \big|_{a=0}$ *où* 0 *est point fixe de* f. *Soit* E_n *l'idéal engendré par* $d+1$ *polynômes* $e_f^n(y) = 0$ *et* $e_f^{n+1_\ell}(y) = 0$. *Sous cette hypothèse et via la transformation de Laplace-Steiltjes, la solution est obtenue à partir de la distribution asymptotique des zéros réels de* E_n *quand* $n \to \infty$. *En effet, l'idéal* E_n *définit des variétés algébriques affines réelles. Si ces variétés sont* 0 *-dimensionnelles et ont une distribution asymptotique de densité dérivable* q d *-dimensionnelle, la densité invariante sera de la forme sous réserve de convergence:*

$$\boxed{p(a) = \prod_{\ell=1}^{\ell=d} (-a_\ell) \partial^d q(a) / \partial a_1 .. \partial a_\ell .. \partial a_d}$$

2-Approximation. Sous les hypothèses de la méthode du col, la distribution asymptotique des zéros des $H_n(y)$ *(variétés algébriques affines) est image réciproque de distributions uniformes* κ_ℓ *sur (0,1) :*

$$\boxed{\pi \kappa_\ell = s_\ell \operatorname{Im} f_\ell(a) - \vartheta_\ell} \quad \ell = 1,2,..,p$$

pour toutes les $p \le d$ *coordonnées complexes du point critique* a *de* $\gamma(a) = s\zeta f(a) - \zeta \log a$. ϑ_ℓ *est l'argument de* a_ℓ , $s_\ell = y_\ell / n$). *Si la hessienne de* $\gamma(a)$ *est dégénérée, comme dans le cas de linéarités partielles, nous obtenons des familles de variétés aléatoires (exemple Henon).*

3- Applications. À toute EDO ou EDP, de la forme $\partial a / \partial t = F(a)$, *on peut associer l'itération infinitésimale* $f(a,\delta) = a + \delta F(a)$ *et lui appliquer la densité de Perron Frobenius. Si* $F(a)$ *est partiellement linéaire, alors une distribution aléatoire peut être solution asymptotique de cette équation. Les applications les plus remarquables de l'étude sont les solutions asymptotiques des équations de Lorenz, de Navier Stokes ou d'Hamilton.*

4- Si $\lambda^k = 1$, *il y a résonance. Sous les hypothèses précédentes, si* $n = kq \to \infty$, *au point critique* a *de* $\gamma(a)$, *on a la condition of résonance*

$$\boxed{1 = a^{k-1}}$$

Abstract

1- Solving deviation. Let f *be a polynomial application of a compact* C *of* R^{d} *in* C. *A measure* P *is invariant under* f *iff* $P = P_f$ *where* $P_f = P \circ f^{-1}$. P_f *is said Perron Frobenius operator (PF). Denote* $e_f^n(y) = \partial^n \left[\mathrm{e}^{ya} - \mathrm{e}^{yf(a)} \right] / \partial a^n \big|_{a=0}$ *and* $H_n(y) = \partial^n \mathrm{e}^{yf(a)} / \partial a^n \big|_{a=0}$ *where* 0 *is a fixed point of* f. *Let* E_n *be the ideal spanned by the* $d+1$ *polynomials* $e_f^n(y) = 0$ *and* $e_f^{n+1_\ell}(y) = 0$. *Under this hypothesis on* f , *via the Laplace-Steiltjes transform, we prove that the solution is given by the asymptotic distribution of the zeros of* E_n *when* $n \to \infty$. *The ideal* E_n *defines real algebraic affine manifolds. If these manifolds are 0-dimensional and have an asymptotic distribution with a derivable* d *-dimensional density* q, *the invariant density is:*

$$\boxed{p(a) = \prod_{\ell=1}^{\ell=d} (-a_\ell) \partial^d q(a) / \partial a_1 .. \partial a_\ell .. \partial a_d}$$

2-Approximation. Under assumptions of the steepest descent, the asymptotic distribution of the zeros of the $H_n(y)$ *(algebraic affine manifolds) getting the solution of (PF), is a function of iid* κ_ℓ *on (0,1)*

$$\boxed{\pi\kappa_\ell = s_\ell \operatorname{Im} f_\ell(a) - \vartheta_\ell} \quad \ell = 1,2,..,p$$

for all the p *complex coordinates of the critical point* a *of* $\gamma(a) = s\zeta f(a) - \zeta \log a$. ϑ_ℓ *is the argument of* a_ℓ , $s_\ell = \lim_{N\to\infty} y_\ell / n$. *If the Hessian is degenerated, which is the case of partial linearity, we get a family of random manifolds as in the Henson's case.*

3 – Applications to EDP, EDO. Asymptotic behaviours of ODE or PDE, as $\partial a/\partial t = F(a)$, *via the Perron Frobenius's density, are most interesting. Here, the infinitesimal iteration is* $f(a,\delta) = a + \delta F(a)$. *But, if the Hessian is degenerated, the asymptotic design can be probabilistic. Among applications, are asymptotic solutions of Lorenz, Navier-Stokes or Hamilton's equations.*

4- If $\lambda^k = 1$, *there is resonance. Under the previous hypothesis when* $n = kq \to \infty$, *at the critical point* a *of* $\gamma(a)$, *we have the condition of resonance*

$$\boxed{1 = a^{k-1}}$$

Introduction

The Perron-Frobenius's equation (PF) is a very difficult functional equation. Ulam and Von Neumann gave the only well-known first solution [23] for the logistic $x1 = 4(x - x^2)$. Therefore, two important directions are used in physics: quantum chaos [11] and thermodynamics [22]. This paper brings many improvements and corrections to the first preprints [3] [4] [5].

The subject of this paper is to describe the structures of invariant probabilistic distributions. They can exist, even they are masked by others cycles or distributions. The convergence to some point or some distribution depends on zones of domination, as we match with the topographic zones of attraction of certain cycles or fixed points. This problem is very difficult but not completely studied here. The reader is warned that we put forward here a very static and asymptotic conception of the physics phenomena. No transitions, no dynamics, no orbits, except perhaps for the cycles, only a probability of presence somewhere after infinite iterations in a compact set.

Chapter 1: The resolving deviation

I- Hypothesis, notations and definitions

1--H0- Hypothesis on the polynomial application f

Let f *be a polynomial application of a compact set* C *of* R^{d} *in* C *with at least one fixed point* $f(0)=0$ *well isolated in* C. *Diameter of* C *is* D. f *is locally invertible at* 0. $f=(f_1, f_2, .., f_\ell ., f_d)$.

Then, f *is* C^∞ and have a finite number v of reciprocal images.

2- Collection of iterations

Let $C(f)$ *be the collection of iterations* $C(f)=(f, f^{(2)}, ..., f^{(k)}...)$ *where* $f^{(k)}=f\circ f^{(k-1)}$.

We can index generically an $f^{(k)}\in C(f)$ *as* $f_\bullet$ with a spot « $\bullet$ ». We check H0 for $f_\bullet$ provided to be invertible at $0_\bullet$ (see below). So, generally, all demonstrations for f are valid for all $f_\bullet$.

3- Notations of the fixed points and eigenvalues of the linear part at theses points

Let $0\in Fix(f): f(0)=0$. But, all $f^{(k)}$ can have $f^{(k)}(\alpha)=\alpha$ in C. These α_k are generally called cycles of k order. Let $Fix(f^{(k)})$ this set and $Fix(f_\bullet)$ the generic union for all k.

Under H0, the real eigenvalues λ ($\lambda_\bullet$) of the linear part of f ($f_\bullet$) ate 0 ($0_\bullet$) are determining for the convergence to some invariants: fixed points, cycles or distributions. We don't study here complex or multiplicity of λ. These cases are not very difficult. When exist integers $r\in N^d$ such as $\lambda^r=1$, there is resonance, and we present here only a trivial situation of this recurrent case. Let $\mathrm{Re}\,s^+=\left\{k\in N^d \,\middle|\, |\lambda|^k>1, |\lambda|^{k+1_\ell}>1, \ell=1,...d\right\}$ and $\mathrm{Re}\,s^-=\left\{k\in N^d \,\middle|\, |\lambda|^k<1, |\lambda|^{k+1_\ell}<1, \ell=1,...d\right\}$

4- General notations

If $a,x,y\in \mathrm{R}^{\mathrm{d}}$, xy is the scalar product; $x.y$ the vector with coordinates $x_\ell y_\ell$, $\ell=1,..,d$; in **bold** $\boldsymbol{x}=x_1...x_\ell.x_d$; $d\boldsymbol{x}=dx_1..dx_\ell.dx_d$ and $\partial h/\partial \boldsymbol{a}=\partial^d h/\partial a_1..\partial a_\ell.\partial a_d$.

II – Laplace-Steiltjes transform of the Perron Frobenius (PF) equation

1- Definitions : Perron Frobenius's operator (PF) and its Laplace-Steiltjes transformation (LPF)

Let P be an arbitrary probability measure and f an H0-application. For all Borelian set B, the transformation of P by the Perron Frobenius's operator (PF)is $P\circ f^{-1}(B)=P_f(B)$. The measure is f-invariant if $P(B)=P_f(B)$ for all Borelian set B. As f (resp. $f_\bullet$) apply a compact set C in C, P (resp. $P_\bullet$) exist [6]. All $f^{(k)}\in C(f)$ lets $P_f=P$ invariant.

2- Lemma 1

Under H0, for all f *and* p, $\phi_f(y)$ *is:* $\phi_f(y)=\int_{\mathrm{R}^{\mathrm{d}}} \mathrm{e}^{yf(x)}dP(x)$

By definition of $P\circ f^{-1}(x)$, this is true for all positive function of $L^1(P)$ (see [15]), hence for $\mathrm{e}^{yf(x)}$.

If P has a Lebesgue-Steiltjes density p, then $p_f=\Sigma_v p\circ f_v^{-1}\left|f_v^{-1\prime}\right|$ is the transformation of p by (PF). But this definition is not very useful, complicated and sometimes misleading. We prefer use the lemma 1. Nevertheless, if we suppose we start the iteration process from a point x_0, either $x_0=0$ and the Lebesgue-Steiltjes density is the Dirac's distribution $\delta(x)$, or $x_0\neq 0$ with the distribution $p(x)$ vanishing at 0.

Now, we take the Laplace-Steiltjes transform $\mathcal{L}$ of a probability measure $dP(x)$ with support contained in the compact C, is naturally written to capture the mass of the fixed point 0 $\phi(y)=\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\infty} \mathrm{e}^{yx}dP(x)$. Then, we have $\phi_f(y)=\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\infty} \mathrm{e}^{yx}dP_f(x)$ for all f with a fixed point at 0

and all probability measure P on C. Its existence is due to the compactness de C, which makes the set $\Omega = (y \mid \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\infty} \mathrm{e}^{yx} dP_f(x) < \infty \cap \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\infty} \mathrm{e}^{yx} dP(x) < \infty)$ not empty: $0 \in \Omega$ convex.

Under H0, the support of the distribution is contained in compact C with diameter D. Hence, $\phi(y)$ can be represent by analytic series in a poly-disk with infinite radius of convergence. $\phi(y) = \Sigma_n b_n y^n$ converges for all measure of probability P in $\mathrm{C}(|b_n| < D^n / n!)$. More, we can take $y = k(n+1)$ provided $Dk\mathrm{e}^{Dk} < 1/\mathrm{e}d$ to conserve the convergence of the series.

3-Definition: resolving equation L

PF equation gives a resolving equation L: $\theta_f(\phi) = \phi_f(y) - \phi(y) = 0$ *(resp. collection of equations* $\theta_{f.}(\phi) = \phi_{f.}(y) - \phi(y) = 0$*).*

More, if P is invariant, $\phi_f(y) = \int_{\mathrm{R}^d} \mathrm{e}^{yf\circ f(x)} dP(x) = \phi_{f\circ f}(y)$. *By* recurrence, $\phi_{f.}(y) = \phi(y)$.

III –Main Result

1-Notations of the sets of zeros of $e^n(y)$

Under H0, without resonance, let $\varphi(y) = \Sigma_n c_n y^n$ be a solution de of the resolving equation $\theta_f(\varphi) = \Sigma_n c_n e^n(y) = 0$ where $e^n(y) = (y^n - H_n(y)$ and $H_n(y) = \partial^n e^{yf(a)} / \partial a^n \,|\, a = 0$ at $a = 0$. Let φ_n be the expansion of order n de φ such as $\theta_f(\varphi_n) = 0$.

If $Z(e^n)$ *is the set of zeros of* $e^n(y)$*, let* $Z(\mathrm{E}_n) = Z(e^n) \cap_{\forall \ell} Z(e^{n+1_k})$ $\ell = 1,...d$*, be the set of common zeros of these polynomials.* E_n *is the ideal spanned by these* $d+1$ *polynomials.*

2-Theorem

When $n \to \infty$*:*

1*- Exists an integer* m *such as* $(\varphi_n(y) - 1)^m \in \mathrm{I}_n$*. With a probability 1, the distribution defined by* $\varphi_n(y)$ *is constant on the algebraic affine real manifolds defined by* E_n*.*

2*- If the manifolds are 0-dimensionnal and if* $\varphi(s)$ *is solution de the equation* $\theta_f(\varphi) = 0$ *for the density* $q(a)$*, we can write that the invariant density of Perron-Frobenius* $p(a)$ *is*

$$p(a) = \prod_{\ell=1}^{\ell=d} (-a_\ell) \partial^d q(a) / \partial a_1 .. \partial a_\ell .. \partial a_d$$

3*- These results are valid for* $\forall f_. \in C(f)$.

The demonstration needs some steps. $\delta(x)$ is the Dirac's distribution at 0;

IV- Effects of a translation on analytical expression of $\phi_f(y)$

1- Notations of the effects of a translation a **:**

Observe the effects of a translation a: $x = u + a \neq 0$ changing the origin from 0 to $-a$ in C. If $p(x)$ is a distribution density of a Lebesgue- Steiltjes measure for all $x \neq 0$, we have:

$f(x)$	becomes	$f_a(u) = f(u+a) - a$,
$p(x)$	becomes	$p(u) = p(x-a)$,
$p_f(x)$	becomes	$p_f(x-a)$,
$\phi(y)$	becomes	$\phi_a(y) = \phi(y,a) = \phi(y)\mathrm{e}^{ya}$,
$\phi_f(y) = E(\mathrm{e}^{yf(x)})$	becomes	$\phi_f(y,a) = E(\mathrm{e}^{yf(u+a)})$,
$\theta_f(\phi)$	becomes	$\theta_f(\phi_a) = \phi_f(y,a) - \phi(y,a)$.

The following proposition uses the notations: As $\phi(y)$ has an analytic expansion $\phi(y) = \Sigma_n b_n y^n$, then $\phi_f(y,a)$ has the expansion $\phi_f(y,a) = \Sigma_n b_n \partial^n e^{yf(a)} / \partial a^n$. All following series are uniformly convergent for all y (see Constantine [6] with $|b_n| < D^n / n!$). We shall see uniform valuations later.

2- Proposition 1

- Let $\phi(y) = \Sigma_n b_n y^n$ be the Laplace-Steiltjes transform of an arbitrary p. Under H0, the iteration f transforms all terms $y^n e^{ya}$ of $\phi_a(y) = \phi(y)e^{ya}$ in $H_n(y,a)e^{yf(a)} = \partial^n e^{yf(a)} / \partial a^n$, and, for all f with $f(0) = 0$ and p, we have:

$$\boxed{\phi_f(y,a) = \Sigma_n b_n \partial^n e^{yf(a)} / \partial a^n}$$

- Invariant density is solution of the resolving equation: $\theta_f(\phi) = \Sigma_n b_n (y^n - H_n(y)) = 0$ Moreover, for $\forall a \in C$ and $\forall y$, $\theta_f(\phi_a) = \Sigma_n b_n \partial^n (e^{ya} - e^{yf(a)}) / \partial a^n = 0$. For all multi-integers q, we already have $\partial^q \theta_f(\phi_a) / \partial a^q = \partial^q E(e^{ya} - e^{yf(u+a)}) / \partial a^q = \Sigma_n b_n \partial^{n+q} (e^{ya} - e^{yf(a)}) / \partial a^{n+q} = 0$ and $\partial^q \theta_f(\phi) = \Sigma_n b_n (y^{n+q} - H_{n+q}(y)) = 0$.

Density $\mathcal{L}^{-1}(P(x-a)) = \mathcal{L}^{-1}(\phi(z)e^{za})$ where $\phi(z)$ has the analytic expansion $\phi(z) = \Sigma_n b_n z^n$. We can use the Laplace-Steiltjes transform as a Fourier transform that is: $\mathcal{L}^{-1}(z^m e^{za}) = \partial^m(\delta(x-a) / \partial x^m = \delta^m(x-a)$, because, at 0, the density is invariant $\delta(f(0)) = \delta(0)$.

As $\phi_f(y,a) = \int e^{yf(x)} dP(x-a)$, we can write:

$\phi_f(y,a) = \Sigma_n b_n \partial^n \int (\delta_{(x-a)} d\boldsymbol{x}) / \partial a^n = \Sigma_n b_n \partial^n e^{yf(a)} / \partial a^n$

Hence, the resolving equation is: $\theta_f(\phi_a) = \phi(y,a) - \phi_f(y,a) = \Sigma_n b_n \partial^n (e^{ya} - e^{yf(a)}) / \partial a^n = 0$.

Then, at $a = 0$, we have $\theta_f(\phi) = \Sigma_n b_n (y^n - H_n(y)) = 0$. Observe the identity for $\theta_f(\phi_a) = 0$:

$\phi(y) = (\Sigma_n b_n H_n(y,a)) e^{y(f(a)-a)}$ for $\forall a \in C$ and $\forall y$, and the last result is obtained by derivation. Denote that, in combinatorial theory, these polynomials $H_n(y,a)$ are known as Bell's polynomials. (Notice: $H_n(y)$ are polynomials in y even if f is only analytical).

3- Remarks

- We can localise the results on a p - dimensional manifold taking a volume on the manifold instead of the Lebesgue-Steiltjes measure on R^d.

- A diffeomorphism g in C of infinite class defined by $x = g(u)$ induce $f_g(u) = g^{-1} \circ f \circ g(u)$, hence $\phi_f(y)$ becomes $\phi_{g^{-1} \circ f \circ g}(y)$.

- If $a = \alpha_k^p \in Fix(f^k)$ is a point of cycle of order k for $p = 1,...,k-1$ and $\alpha_k^{p+1} = f(\alpha_k^p)$, then $\phi(y) = (\Sigma_n b_n H_n(y, \alpha_k^p)) e^{y(\alpha_k^{p+1} - \alpha_k^p)}$ at α_k^p. So: $[\phi(y)]^k = \prod_{p=1}^{p=k} (\Sigma_n b_n H_n(y, \alpha_k^p))$.

4- Corollary 1

Polynomials $H_n(y,a)$ have some recurrent matrix equations (see Constantine [6]):

- $H_{n+1}(y,a) = \partial H_n(y,a) / \partial a + H_n(y,a) y \partial f(a) / \partial a$
- $\partial H_n(y,a) / \partial y = \Sigma_{k=0}^{k=n} C_n^k H_{n-k}(y,a) \partial^k f(a) / \partial a^k$.
- *At $a = 0$, the leading term of $H_n(y)$ is $(\lambda y)^n$ and polynomial $\lambda^{-n} H_n(y)$ can be seen like « unitary » in y.*

V- Relation between $\varphi(y)$ and $\phi(y)$ - Hypothesis on zeros real of $H_n(y)$

1- Relation between $\varphi(y)$ of (LPF) and $\phi(y)$ of (PF)

Let $\varphi(y)$ be the solution of (LPF) $\theta(\varphi)=0$ and $\phi(y)$ the transform of the solution of (PF). For conveniences, we suppose the random variable d-dimensional.

2- Proposition2

Under H0, $\phi(y,a)=\partial^2\varphi(y,a)/\partial y\partial a=\partial(y\varphi(y,a))/\partial y$

Let $\varphi(y)$ be a solution of $\theta_f(\varphi)=0$. In the expansion of $\varphi(y)$, b_n are fixed and we can construct $\varphi(y,a)$ and $\theta_f(\varphi_a)=0$. But, according to proposition 1, if $\varphi(y,a)$ is solution of (PF), $\theta_f(\varphi_a)=0$ will be an identity for all y and a. All derivatives must vanish. In fine: $\theta(\partial^2\varphi(y,a)/\partial y\partial a)=0$. $\phi(y,a)=\partial^2\varphi(y,a)/\partial y\partial a$. As $\varphi(y,a)=e^{ya}\int e^{ys}dP(s)$, we have the result.

It is not difficult to rewrite the result for a random vector p-dimensional $p\le d$ on a manifold.

We can verify the arguments are true for every $f_\bullet$. In the following, no resonance gets very important.

VI- Analysis of the solution $\varphi(y)$ of LPF

$\varphi(y)$ is now the Laplace-Steiltjes transform of a distribution with support in C and solution of LPF. We must study the expansion $\varphi_n(y)$ limited to order n and define the resolving deviation:

1- Definition 4: resolving deviation and the ideal I_n

We call resolving deviation: $e^n(y,a)=\partial^n(\mathrm{e}^{ya}-\mathrm{e}^{yf(a)})/\partial a^n$. *At the fixed point* $a=0$, $e^n(y)=e^n(y,0)=y^n-H_n(y)$ *is a polynomial of degree* n.

2- Proposition 3

If $(1\neq\lambda^n)$, *under H0 and* $\theta_f(\varphi_n)=0$, *we can construct* d *polynomials* φ_n *on* $Z(\mathrm{E}_n)$ *and* $\lim_{n\to\infty}Z(\mathrm{E}_n)=\left(y\middle|\theta_f(\varphi)=0\right)$.

For all analytic function $\psi(y)$, we can form the function $\theta_f(\psi)=\psi_f-\psi$ where $\psi_f(y)$ is obtained in substituting y^n by $H_n(y)$ in the analytic expansion of ψ. We perturb a solution φ of $\theta_f(\varphi)=0$ by an arbitrary polynomial $\varepsilon_n(y)=\sum_{k=0}^{k=n}\varepsilon_k y^k$. The coefficient of the leading term of $\varepsilon_n(y)$ is: $\varepsilon_n\neq 0$. The perturbed function $\psi^n(y)=\varphi(y)+\varepsilon_n(y)$ becomes under f: $\psi_f^n=\varphi_f+\sum_{k=0}^{k=n}\varepsilon_k H_k(y)$, yet convergent, and $\theta_f(\psi)=\sum_{k=0}^{k=n}\varepsilon_k e^k(y)$. As $\varepsilon_n\neq 0$ and $1\neq\lambda^n$, the coefficient of the leading term of $\theta_f(\psi^n)$ is the one of the leading term of $e^n(y)$: as $\varepsilon_n(1-\lambda^n)\neq 0$, we cannot vanish this coefficient. We must seek a polynomial with a greater degree, and so on.

By induction, to vanish $\theta_f(\varphi)$, we must take $e^n(y)$ null when $n\to\infty$.

Now, we take a large fixed n to construct φ_n by this way. Suppose $y\in Z(e^n)$ for this $n\in N^d$. Can we have $\varphi_n(y)=\Sigma_{p\le n}c_p y^p$ as solution of $\theta_f(\varphi_n)=\sum_{k=0}^{k=n}c_k e^k(y)=0$?

Response is yes: We find d solutions, but under some other conditions.

Here, polynomial $e^n(y)=\sum_{k=0}^{k=n}\alpha_{k,n}y^k$ is known, but vanished if $y\in Z(e^n)$. $\theta_f(\varphi_n)$ has a smaller degree than n: we have d candidates $e^{n-1_\ell}(y)$ for the leading term. Choosing one $e^{n-1_\ell}(y)$, we must identify the coefficients c_p to compute $\varphi_{\ell n}$. The term of degree $n-1_\ell$ is cancelled with c_{n-1_ℓ}: $(1-\lambda^{n-1_\ell})c_{n-1_\ell}+c_n\alpha_{n-1_\ell,n}=0$. Under the no-resonance hypothesis, notice that $1-\lambda^{n-1_\ell}\neq 0$ for $\ell=1,\ldots,d$. But c_n is arbitrary. Now, the polynomial $c_n(\alpha_{n-1_\ell}e^{n-1_\ell}(y)-(1-\lambda^{n-1_\ell})e^n(y))$ is known.

By successive linear identifications, we shorten the degree of unknown coefficients of $\varphi_{\ell n}$. If we denote coefficients $c_k = c_{k,\ell} c_n$, we have $P_{n-1_\ell}(y) = c_n(\Sigma_{k\le n} c_{k,\ell} e^k(y)) = 0$, then $\theta_f(\varphi_{\ell n}) = 0$. That is true for $\forall \ell$ and we have d solutions $\varphi_{\ell n}$. We take one denoted $\varphi_n(y)$.

We examine $\varphi_n(y)$ with a translation a of the origin: $\varphi_n(y)$ becomes $\varphi_n(y,a)$. As our previous computation for $\varphi(y)$, we have for all a : $\theta_f(\varphi_n(y,a)) = \varphi_n(y)\mathrm{e}^{ya} - \varphi_{fn}(y,a) = 0$. We expend θ_f : $\theta_f(\varphi_n(y,a)) = \Sigma_{k\le n} c_k e^k(y,a))$. Then, we can derive θ_f to obtain at $a = 0$: $\partial\theta_f(\varphi_n(y,a)) / \partial a_k\big|_{a=0} = \Sigma_{m\le n} c_m e^{m+1_k}(y) = 0$ for $k = 1,..,d$. A solution of this equation is obtain in taking the polynomial $e^{n+1_k}(y) = 0$. Hence, if $y \in Z(e^n) \cap Z(e^{n+1_k})$, $\varphi_n(y)$ is solution of the both equations $\theta_f(\varphi_n) = 0$ and $\theta_f(\varphi_{n+1_k}) = 0$. Thence, taking $Z(\mathrm{E}_n) = Z(e^n) \cap_k Z(e^{n+1_k})$, the d solutions $\varphi_{\ell n}$ .are valid.

3-Some inequalities

Let φ be the solution of $\theta_f(\varphi) = 0$ and $\varphi_n(y)$ the expansion of order n. We have: $\left|\varphi_n(y) - \varphi_{fn}(y)\right| < 2d\left|yD\right|^{n+1} \mathrm{e}^{|yD|} / (n+1)!$, $\left|\varphi(y) - \varphi_n(y,a)\right| < (\mathrm{e}^{|ya|} - 1)e^{|yD|} + \left|Dy\right|^{n+1} \mathrm{e}^{|yD|} / (n+1)!$

The general term of this series is less than $\left|yD\right|^n / n!$. The remaining Lagrange-Taylor's rest of order n of e^{ya} is less than $\left|\varphi(y) - \varphi_n(y)\right| < d\left|yD\right|^{n+1} \mathrm{e}^{|yD|} / (n+1)!$. As $\left|f(a)\right| < D$, we have also: $\left|\varphi_f(y) - \varphi_{fn}(y)\right| < d\left|yD\right|^{n+1} e^{|yD|} / (n+1)!$ for $\forall \left|a\right| < D$, and $\left|\varphi(y) - \varphi_n(y,a)\right| < (e^{|ya|} - 1)e^{|yD|} + \left|Dy\right|^{n+1} e^{|yD|} / (n+1)!$.

Now, for fixed n and $n - 1_\ell$, we study the solution φ_n, with $\varphi_n(0) = 1$ at $y = 0$.

4- Lemma 2

For large n, $\varphi_{\ell n}$ converge to φ when $n \to \infty$. Exist a number m such as $(\varphi_{\ell n} - 1)^m \in \mathrm{E}_n$.

Imbedding the problem in the algebraically closed field of the complex numbers. The Hilbert's theorem of zeros (nullstellensatz) says $(\varphi_{\ell n} - 1)^m \in \mathrm{E}_n$.

We can identify $\varphi_{\ell n}$ with $\varphi_n(y)$.

$\varphi_n(y)$ converges to φ for large n and $\forall y \le k(n+1)$. We take $y \in \left\{ Z(\mathrm{E}_n) \cap y \le k(n+1) \right\}$. As $\varphi_{\ell n}$ is a Laplace-Steiltjes transform of a measure in C, we have $\left|\varphi_n(y) - \varphi_{fn}(y)\right| < d\left|yD\right|^{n+1} e^{|yD|} / (n+1)!$, and then $\varphi_{\ell n}$ converge to φ when $n \to \infty$. In such case, the coefficient of the leading term of $\varphi_n(y)$ and the one of $\varphi_{fn}(y)$ are close to 0 when $n \to \infty$. But, $\theta(\varphi_{\ell n}) = \varphi_{\ell n} - \varphi_{f\ell n} = e^n(y)$, and the leading term of $e^n(y)$ is $1 - \lambda^n$ and tend toward 1 or infinite. Either $\varphi_{\ell n} - 1$has same zeros than $Z(\mathrm{E}_n)$ or $y = 0$.

Get the problem in the algebraically closed field of the complex numbers. The Hilbert's theorem of zeros (nullstellensatz) says $(\varphi_{\ell n} - 1)^m \in \mathrm{E}_n$. We can identify $\varphi_{\ell n}$ with $\varphi_n(y)$. We give a first characterisation of the distribution on I_n as Lucaks $[17]$or Wintner.

5- Proposition 4

With a probability 1, all real point of E_n belongs to the support of the distribution $\mathcal{L}^{-1}(\varphi_n)$ and tends asymptotically to the support of $\mathcal{L}^{-1}(\varphi)$. The distribution $P_n(a) = \mathcal{L}^{-1}(\varphi_n)$, defined for $y \in Z(\mathrm{E}_n) - \{0\}$, is constant on the real algebraic varieties affine (connected component) of E_n, null elsewhere.

6- Remarks

- In resonant case, exist $n \in \zeta_R = \left\{ r \in N^d \,\middle|\, 1 = \lambda^r \right\}$ where the leading term of $H_n(y)$ is y^n. To construct φ_n, we can identify the c_p for all $r \in \zeta_R$. But we shall see more interesting methods.

- If the measure $dP(a)$ is invariant, on have the identity $\int e^{ya} dP(a) \equiv \int e^{yf^{(p)}(a)} dP(a)$ for $\forall p \in N$. $\theta_p(y) = \phi_p(y) - \phi(y) = 0$. Then, the resolving deviation $f^{(p)}$ will asymptotically vanish.

7- Lemma 3

If $n \to \infty$ *with* $n \in \mathrm{Re}\, s^+$, *the zeros of* $e^n(y)$ *are very close to theses of* $H_n(y)$. *Conversely, if* $n \to \infty$ *with* $n \in \mathrm{Re}\, s^-$ *and* $n \to \infty$, *the solution is* $y = 0$. *We can say that the ideal* $\mathrm{E}_n - \{0\}$ *is asymptotically generated by the zeros of polynomials* $H_n(y)$ *and* $H_{n-1\ell}(y)$ $\ell = 1, 2, .., d$.

If $n \in \mathrm{Re}\, s^+$ and $n \to \infty$, we can write $e^n(y)$ as $e^n(y) = \lambda^n(\lambda^{-n} y^n - \lambda^{-n} H_n(y))$. But the leading term of $\lambda^{-n} H_n(y)$ is 1 and $\lambda^{-n} y^n \to 0$. By the continuity of the roots, we can say that the zeros of $e^n(y)$ are very close to theses of $H_n(y)$. Now, If $n \in \mathrm{Re}\, s^-$ and $n \to \infty$, $e^n(y)$ is has its leading term of y^n, and $e^n(y)$ is very close to y^n. Solution is $y = 0$

8-Normalisation of E_n

The number of zeros can increase with n. For instance, if $k \in R$, suppose that we have $yf(ka)$ such $yf(ka) \to -\infty$ when $k \to \pm\infty$ for a fixed couple (y, a), then, by Rolle's theorem $d^n e^{yf(ka)} / dk^n = \partial^n e^{yf(ka)} / \partial a^n \, a^n$ has n zeros. We normalise these zeros (think to the Hermitian polynomials) to obtain a continuum. Let $y = n.s$ ($y_\ell = s_\ell n_\ell$): $\mathrm{E}_n(y)$ becomes $\overline{\mathrm{E}}_n = \mathrm{E}_n(n.s)$.

9- Lemma 4

$\varphi_n(s) \le 1$ *and* $Z(\overline{\mathrm{E}}_n)$ *is bounded.*

- If $(\varphi_n(n.s) - 1) \in \overline{\mathrm{E}}_n$, $\varphi_n(ks)$ must converge on $Z(\overline{\mathrm{E}}_n)$. But logarithm φ_n is convex. For every coordinate $s_\ell n_\ell$, $\log \varphi_n(s_\ell n_\ell) \le \log \varphi_n(s_\ell(n_\ell - 1))/2 + \log \varphi_n(s_\ell(n_\ell + 1))/2$. Hence:
$\log \varphi_n(s_\ell n_\ell) - \log \varphi_n(s_\ell(n_\ell - 1)) \le \log \varphi_n(s_\ell(n_\ell + 1)) - \log \varphi_n(s_\ell n_\ell)$.
$\varphi_n(s_\ell n_\ell)) \ge \varphi_n(s_\ell)\varphi_n((n_\ell - 1)s_\ell) \ge (\varphi_n(s_\ell))^{n_\ell}$ et $\varphi_n(n.s) \ge \varphi_n(s)^n$ converge only if $\varphi_n(s) \le 1$.

VII- Random variables d-dimensional.

If $\varphi(y)$ is a solution of LPF $\theta(\varphi) = 0$ for a d-dimensional density $q(s)$ and $q(s) = 0$ if $s \in \partial C$ where ∂C is the edge of the distribution. Let $\phi(y)$ solution of the transform of PF with density $p(s)$.

1- Proposition 5

If H0, if $q(s) = 0$ on ∂C edge of the distribution, *we have* $p(s) = (-\boldsymbol{s})\partial q(s) / \partial \boldsymbol{s}$

From Proposition 2, we have $\phi(y, a) = \partial^2 \varphi(y, a) / \partial \boldsymbol{y} \partial \boldsymbol{a}$. Then, if $q(s) = 0$ on ∂C, we have the result.

2-Hypothesis H1 on real zeros of $H_n(y) = \partial^n e^{yf(a)} / \partial a^n e^{-yf(a)}\big|_{a=0}$

H1- Let $s_n \in S_n$ *be such as* $H_n(n.s_n) = 0$. *The repartition limit of* s_n *when* $n \to \infty$ *has a density* $q(s)$ *with respect to the* d*-dimensional Lebesgue-Steiltjes measure: in a volume* $d\boldsymbol{y} = n^d d\boldsymbol{s}$ *of* R^{+d} *we have asymptotically* $d\boldsymbol{y} = n^d q(s) d\boldsymbol{s}$ *zeros* s_n.

Then, n^d points y_n verify $H_n(y_n) = 0$. Let $\varphi_n(y) = \frac{1}{n^d} \Sigma_n e^{yy_n}$

3- Proposition 6

Under H0 and H1, *If it is not masked the Perron Frobénius's density is asymptotically* $p(a) = \prod_{\ell=1}^{\ell=d} (-a_\ell) \partial^d q(a) / \partial a_1 .. \partial a_\ell .. \partial a_d$

Let $t = y / n$. Under H0, H1 $\varphi_n(t.n) \approx \frac{1}{n^d} \Sigma_n e^{ts_n} n^d q(s_n) ds_n \to \varphi(t) = \int e^{ts} q(s) ds$.

VIII- Applications

1- the logistic $f = \lambda(a - a^2)$ in dimension 1. if $4 \geq \lambda \geq 1$, f apply the compact set $\mathrm{C} = \left[f(\lambda/4), \lambda/4 \right]$ in C. Its resolving deviation is $: e_f^n(y)\big|_{a=0} = y^n - H_n(y)$ where $H_n(y) = \mathrm{H}_n(\sqrt{\lambda y/2})(\sqrt{2\lambda y})^n$ with the Hermitian polynomials of order n H_n. Hence, the distribution of the zeros of H_n, denoted as t, is asymptotically the semi-circular low of Wigner: $q(t)dt = k\sqrt{1-t^2}dt$. $q(1) = q(-1) = 0$. Thence $t = c\sqrt{y}/v$ with $c = \sqrt{\lambda/2}$ and $v = \sqrt{2n+1}$ has the semi-circular law. The distribution $p_1(s)$ of the variable $s = y/2n+1$ will verify: $t = r\sqrt{s}$ and: $p_1(s)ds = -(dq(t)/dt)dt = kr/\left(2s\sqrt{s(1-r^2 s)}\right)ds$. Then $p(s) = sp_1(s)$. The density of the invariant measure is a beta law $\beta(1/2,1/2)$: $p(s) = k'/\sqrt{s(1-r^2 s)}$, obtained by Kawamura with the Cunts-Krieger's algebra, which generalise the well-known result of Ulam and Von Neumann. But this distribution will be masked if exist attractive cycles according with the values of λ.

2- First Approach of the Julia's iteration

We prevent the readers that this example doesn't pretend characterise the Julia's sets. Many important works have been made. Here, we don't worry with cycles or resonances, or relations of domination.

Let $f(z) = z^2 + 1/4 - \alpha^2$ be the iteration de Julia in C [14]. It has 2 fixed points: $a_0 = 1/2 + \alpha$ and $a'_0 = 1/2 - \alpha$. We denote here with« **bold** » characters the vectors de R^2 for the complex numbers $\boldsymbol{a} = (a,b)$, $\boldsymbol{\alpha} = (\alpha,\beta)$ and $\boldsymbol{a'_0} = (a'_0, b'_0) = 1/2 - \boldsymbol{\alpha}$.

Consider the function $f(z)$ at the fixed point $\boldsymbol{a_0}$; we can write with $\boldsymbol{z} = \boldsymbol{a} + \boldsymbol{a}_0$: $\boldsymbol{f} = (a^2 - b^2 + 2a_0 a - 2b_0 b, 2ab + 2a_0 b + 2b_0 a)$. We can suppose that it applies a compact C in C by convergent reciprocal iterations. Let $\mu_0 = \sqrt{a_0^2 + b_0^2}$, $\mu = \sqrt{x^2 + y^2}$, $\gamma = \sqrt{(\mu + x)/2\mu}$ and $\delta = \sqrt{(\mu - x)/2\mu}$. We denote the orthogonal matrices:

$$\mu_0 \boldsymbol{O}_0 = \begin{pmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{pmatrix} \qquad \mu_y \boldsymbol{O}_y = \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \qquad \boldsymbol{o} = \begin{pmatrix} \gamma & -\delta \\ \delta & \gamma \end{pmatrix}.$$

Here, the sign ^{t} design the transposition. Quote that: ${}^t\boldsymbol{O}\boldsymbol{O}_0\boldsymbol{O} = \boldsymbol{O}_0$

We have the function: ${}^t\boldsymbol{y}\boldsymbol{f}(\boldsymbol{a}) = 2\,{}^t\boldsymbol{y}L(\boldsymbol{a}) + {}^t\boldsymbol{y}Q(\boldsymbol{a})$:

- the linear part is ${}^t\boldsymbol{y}L(\boldsymbol{a}) = x(a_0 a - b_0 b) + y(a_0 b + b_0 a) = \mu_0 {}^t\boldsymbol{y}\boldsymbol{O}_0\boldsymbol{a}$ where $\boldsymbol{O}_0$ is orthogonal.

-the Hessian is orthogonal ${}^t\boldsymbol{y}Q(\boldsymbol{a}) = xa^2 - xb^2 + 2yab = \mu_y\,{}^t\boldsymbol{a}\boldsymbol{O}_y\boldsymbol{a}$.

Suppose the eigenvalues of the linear part a_0 and $\overline{a_0}$ are conjugated with a module greater than 1.

La Hessian of ${}^t\boldsymbol{y}Q(\boldsymbol{a})$ has characteristic equation $x^2 + y^2 - \mu^2 = 0$ with 2 eigenvalues, one positive μ, and the other negative $-\mu$. Noting $\boldsymbol{y} = \boldsymbol{O}\boldsymbol{s}$ and $\boldsymbol{a} = \boldsymbol{O}\boldsymbol{u}$ with $\boldsymbol{u} = (u,v)$ and $\boldsymbol{s} = (r,s)$:

${}^t\boldsymbol{y}\boldsymbol{f}(\boldsymbol{a}) = 2\mu_0\,{}^t\boldsymbol{s}\,{}^t\boldsymbol{O}\boldsymbol{O}_0\boldsymbol{O}\boldsymbol{u} + \mu u^2 - \mu v^2 = 2r(a_0 u - b_0 v) + 2s(a_0 v + b_0 u) + \mu u^2 - \mu v^2$

Hence : ${}^t\boldsymbol{y}\boldsymbol{f}(\boldsymbol{a}) = (2ra_0 + 2tb_0)u + \mu_y u^2 + (2ta_0 - 2rb_0)v - \mu_y v^2$. Namely: ${}^t\boldsymbol{y}\boldsymbol{a} = {}^t\boldsymbol{s}\boldsymbol{u}$. In the orthogonal eigenspaces, the application split into two independent iterations: $(2ra_0 + 2sb_0)u + \mu u^2$ and $(2ta_0 - 2rb_0)v - \mu v^2$. As $\boldsymbol{O}$ is orthogonal: $\mu = \sqrt{r^2 + s^2}$

The resolving deviation gives two Hermitian polynomials: $\mathrm{H}_n((2ra_0 + 2sb_0)/i\sqrt{2\mu})(i\sqrt{2\mu})^n$ and $\mathrm{H}_n((2sa_0 - 2rb_0)/\sqrt{2\mu})(\sqrt{2\mu})^n$. The first is always positive. The zeros of the second follow

asymptotically a semi-circular Law. Thence, $(2s_1a_0 - 2r_1b_0)/\sqrt{2\mu}$ (with $r_1 = r/\sqrt{2n+1}$ and $s_1 = s/\sqrt{2n+1}$) follows a semi-circular Law. And $\boldsymbol{s}_1\sqrt{2n+1} = \boldsymbol{s}$ gives the distribution.
We change coordinates for $\boldsymbol{a'_0}$ without modify the distribution. They may be masked locally by cycles or distributions of $\boldsymbol{f}_.$.

Chapter 2: Approximation of the zeros

I- Notations, rappels and hypotheses

1-Leading idea

We search the asymptotic distribution of the real zeros of $H_n(y)$ when $y \to \infty$ with n. As Plancherel and Rotach [20] do for the Hermitian polynomials, we represent $H_n(y)$ in the field of the complex numbers with the Cauchy's d-dimensional integral : $H_{n-1}(y) = \partial^{n-1} e^{yf(a)} / \partial a^{n-1}\big|_{a=0} = c \int_\Gamma \frac{e^{yf(a)} da}{a^n}$

where Γ is a closed poly-disk around the fixed point 0 of f, $a \in C^d$. Note $n-1 = (n_1 - 1, n_2 - 1, ...n_d - 1)$. c is a finite non-null constant.

The steepest descent's method, due to Riemann-Debbie, gives us an approximation of this integral when $y \to \infty$. If y is a zero of $H_{n-1}(y)$, we will hope that the approximation will give this zero. The method needs many assumptions. If $yf(a)$ is polynomial in a, we can see Pham [19] or Delabaere [7] for a good formulation.

Definition 1

We call Plancherel-Rotach's function PRF: $\gamma(a) = yf(a) - n \log a$ *where* $\log a$ *is the main determination of the complex logarithm.* Then $H_{n-1}(y) = c \int_\Gamma e^{\gamma(a)} da$

2-Basic notations

y positive of R^d tend toward the infinite with n and we write $y = n.s$ where $y_\ell = n_\ell s_\ell$. All the $n_\ell \to \infty$. We shall see soon that all the must be equal. Now we can write $\gamma(a) = n.(sf(a) - \log a)$

3-Recalls on the steepest descent's method and assumptions

The approximation of $H_{n-1}(y)$ by the steepest descent's method is the finite sum of contributions $\Sigma_c Q_c(a)$ of all the critical points of $\gamma(a)$. Among these contributions, some of them are exponentially negligible. So, we keep only that one of which the real part is the most important. The approximation can be written (see [7]*):*

$H_{n-1}(ns) = Q_n(a) = c' e^{\gamma(a)}(1 + g(a,n,s)/n')$

Where the function g is bounded when n' depends on n and tends toward infinite. c' is never null.

A point a is critical point non-null of $\gamma(a)$ if we have $\partial\gamma(a)/\partial a = 0$ which can be written $a_\ell n.s\partial f(a)/\partial a_\ell - n_\ell = 0$. Then $a\partial\gamma(a)/\partial a$ is polynomial with the same degree than f.

4- Hypotheses: *general position*

H1- *The critical point* a *giving the maximum of* $e^{\gamma(a)}$ *is assumed be in* ***general position*** [7]**:**

Definition 2

The critical point in *general position if it is isolated with finite distance.*
Here, for all fixed s *and, it is isolated, unique, (except its complex conjugate) and from the fixed point 0. It is called Morse 's point, if it is not degenerated.*

H2- *A sufficient condition to obtain this maximum is that the Hessian (which is Hermitian) of* $\gamma(a)$ *is definite negative at* a*. Critical points not degenerated are isolated.*

Unfortunately, in a lot of problems, the Hessian is not definite negative (see V).

H3- $n.sf$ *tends toward* $-\infty$ *when the variable* $n \to \infty$ *to define the Laplace's integral.*

II- Asymptotic equivalence relation of the real zeros of $H_{n-1}(y)$ when $n \to \infty$

We first identify the equation of the zeros of the approximation, then we study under what conditions, the distribution of the approximation is asymptotical very near with the zeros of $H_n(y)$. The variety of the situations is so rich that we examine here only the most common.

As we are only interesting by the real zeros of $H_{n-1}(y)$, we can neglect in the representation $Q(a)$ all the factors $\neq 0$: that may be functions, constants, etc . Notice that $g/n' \to 0$ when $n \to \infty$ and $c' \neq 0$ in $Q(a) = c' \mathrm{e}^{\gamma(a)}(1 + g(a, n.s, 1/n')/n')$. Then $\mathrm{e}^{\gamma(a)}$ is the one and only one factor making $Q(a)$ null. We write this equivalence $H_{n-1}(y) \underset{0}{=} \mathrm{e}^{\gamma(a)}$. Then n' has no importance when $n \to \infty$.

1-Lemma 1

Almost everywhere (when Hessian is not null) : $H_{n-1}(y) \underset{0}{=} J_n(y) = \sin(\mathrm{Im}\,\gamma(a))$

- First, if $H_{n-1}(y) = 0$, all contributions $\mathrm{e}^{\gamma(a)}$ must be null at critical $a = a(y)$. The polynomial $a\partial\gamma/\partial a = 0$ has real coefficients. Hence, if a is complex, $\bar{a}$ is solution, but the contour gives a negative contribution, then the sum is $\mathrm{e}^{\gamma(a)} - \mathrm{e}^{\overline{\gamma(a)}} \underset{0}{=} J_n(y)$. Let p be the number of complex coordinates of a critical point of PRF.

- Conversely, if $J_n(a) = 0$, we can imagine that another contribution $Q_1(a)$ takes the place of Q. Hence, it can exist y and Q_1 such as $|H_{n-1}(y)| = |Q_1(a)| = \delta > 0$ when $Q(a) = 0$. But, it is impossible :

Let a' be a point in a small neighbourhood of a where Q is the dominating contribution and such as Q_1 is continuous. Suppose that exists Q_1 as defined above. At a', $Q_1(a') < Q(a') < \varepsilon/3$ because Q is dominating. Then $H_{n-1}(y') = Q(a')$. As (a, a') determines a unique couple (y, y') : $|H_{n-1}(y)| < |H_{n-1}(y) - H_{n-1}(y')| + |H_{n-1}(y')|$. In this inequality, we can substitute H_{n-1} by Q or Q_1 each time they are dominating: $|H_{n-1}(y)| < |Q_1(a) - Q(a')| + |Q(a')| < |Q_1(a) - Q(a')| + \varepsilon/3$, but $|Q_1(a) - Q(a')| < |Q_1(a) - Q_1(a')| + |Q_1(a') - Q(a')|$. The continuity of Q_1 induces the contradiction.

2- Come back to the ideal E_n

Recall that $Z(\mathrm{E}_n)$ be the set of common zeros of the polynomials $e^n(y) = 0$ and $e^{n+1_\ell}(y) = 0$, $\ell = 1, \ldots d$. E_n is ideal of these polynomials. And, if $n \to \infty$ with $n \in \mathrm{Re}\,s^+$, the zeros of $e^n(y)$ are very close to those have $H_n(y)$. But, we can study $H_n(y)$ even though $n \notin \mathrm{Re}\,s^+$. Let $Z(\mathrm{H}_n)$ the set of common zeros of the polynomials obtained after writing $H_n(y)$ instead of $e^n(y)$.

Let $Z(\mathrm{J}_n)$ be the set of the common zeros of $J_n(y)$ for n -1 and $n - 1 + 1_\ell$, $\ell = 1, 2, .., d$.

III – Main theorem

1- Theorem 2

If the critical point $a \in Z(\mathrm{J}_n)$ *is in general position, then :*

1 *-* $\gamma(a)$ *depends only on one common index of derivation* $n_\ell = n$. *We can write* $y_\ell = ns_\ell$ *and the PRF is* $n\gamma(a) = n(sf(a) - \log \boldsymbol{a})$ *with* $\boldsymbol{a} = \Pi_\ell a_\ell$. *The coordinates are defined by the system :*

$a_\ell s\partial f(a)/\partial a_\ell = 1$ $\ell = p+1, 2, .., d$. *As* f *is polynomial, solutions are algebraic affine manifolds.*

Moreover, the resolving deviation $e^n(y)$ *must be computed only for common derivation* n. *Thus, let* $\Delta = \boldsymbol{\lambda} = |f|$ *be the determinant of the linear part of* f. *We have*

If $|\Delta| < 1$ *the iteration converges to 0;*

If $|\Delta| > 1$ *the iteration, if it is not masked, can converge to a distribution defined by* $Z(\mathrm{J}_n)$

2 - *The imaginary parts of the complex coordinates of* a *satisfy a uniform code of the real zeros, which are algebraic affine manifolds for all complex coordinates* $\ell = 1, 2, .., p$. *These* $d - p$ *dimensional many folds are indexed by a code* $k \in [1, n-1]^p$ *and verify :*

$s_\ell \,\mathrm{Im}\, f_\ell(a) - \vartheta_\ell = k_\ell \pi / n$ *with the code* $k_\ell \in (1, \ldots, n-1)$. *Between two zeros (manifolds* $\mathbb{M}_f$*) at time* $n-1$ *there is one zero at time* n *: then we have a Rolle's foliation.*

When $n_\ell \to \infty$

if $\kappa_\ell = \lim_{n\to\infty} k_\ell / n$ *for* $\ell = 1,..,p$ *, then* κ *is asymptotically a random uniform vector on the unit cube* $[0,1]^p$ *.Then, The* $d-p$ *dimensional many folds are indexed by a random vector of* $[0,1]^p - \{0\}$ *and define the random equation* $\boxed{\pi\kappa_\ell = s_\ell \operatorname{Im} f_\ell(a) - \vartheta_\ell}$ $\ell = 1,2,..,p$

Remark : It seems possible to extend the results to the rational fractions and analytical functions.

IV- Proof: Analysis of the zeros of $J_n(y)$

Let $C(a) = \operatorname{Re}(\gamma(a))$ and $S(a) = \operatorname{Im}(\gamma(a))$ be the real and imaginary parts of $\gamma(a)$ at a. Let $p \neq 0$ be the number of complex coordinates.

We must find the zeros of $\sin(S(a))$. Hence $\operatorname{Im} n.(sf(a)) - \vartheta) = k\pi$.

Proof of point 1

Suppose we found a complex critical point $a(y)$ such as $J_n(y) = 0$. As $\sin kS(a) = 0$ for all integers $k \in N$, a is invariant if we multiply y by any integers k. Hence, $k\gamma(a)$ gives the same critical point a than $k'\gamma(a)$. We can write $k\gamma(a) = k'\Sigma_\ell n_\ell(s_\ell f_\ell(a) - \log a_\ell)$

But, if $y \in Z(\mathrm{J}_n)$, all change of $n_\ell - 1$ in n_ℓ gives the same point y for all ℓ:

then $a_\ell y\partial f(a)/\partial a_\ell - n_\ell = 0$ becomes $(a_\ell + \Delta a_\ell) y\partial f(a+\Delta a)/\partial a_\ell - n_\ell + 1 = 0$. Dividing y by a common $n = \min(n_\ell)$, we obtain by difference $\left[\partial(a_\ell(y/n)\partial f(a)/\partial a_\ell)/\partial a\right]\Delta a = 1/n$ and $\Delta a = c/n$. Then, the integrand is quite constant:

$k\gamma(a) = k'\Sigma_\ell(n_\ell + 1)(s_\ell f_\ell(a) - \log a_\ell) = k'\gamma(a+\Delta a) + k'\Sigma_\ell(s_\ell f_\ell(a+\Delta a) - \log a_\ell + \Delta a_\ell)$

$(k-k')\gamma(a) = k'\Sigma_\ell(s_\ell f_\ell(a) - \log a_\ell)$

Then, with $k - k' = 1$ all the n_ℓ must be equal: $n_\ell = n$ for $\ell = 1,2,..,d$.

Starting from a point a_0, the iterations $f^{(n)}$ can tends only toward 0 or $Z(\mathrm{J}_n)$. But they can be masked by any distribution of $Fix(f_\cdot)$.

2-Lemma 2

If $k_{c\ell} = k_\ell / n$ *, the zeros of* $J_n(a)$ *must verify the condition* $\Sigma_\ell n_\ell(s_\ell \operatorname{Im} f_\ell(a) - \vartheta_\ell - k_{c_\ell}\pi) = 0$

$s\operatorname{Im} f(a)$ contains only sinus, the arcs of which are linear combinations of multiple integers of ϑ, then $s\operatorname{Im} f(a)$ is periodic: $S(a(\vartheta + 2k'\pi)) = S(a(\vartheta)) + 2k'\pi$. Hence, all $NS(a(2k'\pi)) - k\pi = 0$ are solutions. Writing $k\pi = \Sigma_\ell n k_{c_\ell}\pi$ where $k_{c\ell} = k_\ell / n_\ell$ and $y_{c\ell} = s_\ell n$, the solutions can be written as $n\Sigma_\ell(s_\ell \operatorname{Im} f_\ell(a) - \vartheta - k_{c_\ell}\pi) = 0$ where p is the number of complex coordinates. Let k_ℓ runs over $1,2,..,n-1$, then the solution compounds $n-1$ distinct points for each complex coordinates.

In particular, $\vartheta = 0$ such as $S(a(0)) = 0$ gives the fixed point and a non-null solution $\Sigma_\ell(s_\ell \operatorname{Im} f_\ell(a) - k_\ell\pi) = 0$. But we don't know if there is other $0 \neq \vartheta \in (-\pi,+\pi)^p$. If such a solution exists, we cannot have $C(a) = 0$, because $a = 0$. As $a \in Z(\mathrm{J}_n)$, we examine now a complex coordinate a_ℓ, when $n_\ell - 1$ becomes n_ℓ, the others $k \neq \ell$ remaining constant.

3- Proof of point 2

If $y \in Z(\mathrm{J}_n)$, then y remains invariant when only $n_\ell - 1$ becomes n_ℓ. The equation of the complex a_ℓ, $a_\ell y\partial f(a)/\partial a_\ell = n_\ell$ becomes $a_\ell y\partial f(a)/\partial a_\ell - n_\ell + 1 = 0$, the other equations don't move. The critical point a becomes $a + \Delta a$. If we normalise the equations by the common n, we have again :

$\left[\partial(a_\ell y\partial f(a)/\partial a_\ell)/\partial a\right]\Delta a = \Delta$ and $|\Delta a| < c\Delta$

Now, the equation $\Sigma_j n_j(s_j \operatorname{Im} f_j(a) - \vartheta_j - k_{c_j}\pi) = 0$ becomes

$\Sigma_j n'_j(s_j \operatorname{Im} f_j(a') - \vartheta'_j - k'_{c_j}\pi) = 0$

where $n'_j = n_j$ if $j \neq \ell$ and $n'_\ell = n_\ell + 1$

$k'_{cj} = k_{cj}$ if $j \neq \ell$ and $k'_{c\ell} = n_\ell k_{c\ell} / (n_\ell + 1)$ for the range of $k_{c\ell}$ + a new point : $n_\ell / (n_\ell + 1)$

By difference we obtain ; $\partial S(a + \mu\Delta a) / \partial a \Delta a + s_\ell \operatorname{Im} f_\ell(a + \Delta a) - k'_{c\ell} \pi - \delta k'_{c_\ell} \pi = 0$

where $n_\ell \delta k_{c_\ell}$ is the point when the range change from $(1, n_\ell - 1)$ to $(1, n_\ell)$

Examine the quantity $\partial S(a + \mu\Delta a) / \partial a \Delta a$. As $\partial S(a) / \partial a = 0$ at the critical point a, then $\left|\partial S(a + \mu\Delta a) / \partial a\right| < \left|\partial^2 S(a + \mu\Delta a) / \partial a^2\right| \left|\mu\Delta a\right|$. We obtain $\left|\partial \operatorname{Im} S(a + \mu\Delta a) / \partial a \Delta a\right| < c' \left|\Delta a\right| < c'c / n$

Then, $s_\ell \operatorname{Im} f_\ell(a) - \theta_\ell - k_{c\ell}\pi \to 0$ with $n_\ell k_{c_\ell}$ running over $(1, n_\ell)$.

This is true for all complex coordinates and , when n is large, we have the result.

We obtain a uniform code by k_c of coordinates of real zeros of $e^{C(a)} \sin S(a)$. Note that a $n+1$ -th zero is between two n-th zeros. If 0 is the unique solution of $\operatorname{Im}(sf(a)) - \vartheta = 0$, the order $n+1$ manifolds can't intersect order n manifolds. Then we have a Rolle's foliation

Suppose now that $n \to \infty$

We have already $s_\ell = \lim_{n\to\infty} y_\ell / n_\ell$, $\kappa_\ell = \lim_{n\to\infty} k_\ell$ for $\ell = 1, .., p$;

Let κ a random vector with uniform repartition on the unit cube

equation $s_\ell \operatorname{Im} f_\ell(a) - \vartheta_\ell = k_\ell \pi / n_\ell$) tend to $s_\ell \operatorname{Im} f_\ell(a) - \vartheta_\ell = \kappa_\ell \pi$

4- Complex or real coordinates of a_c

The question is to know how vary the number p of complex coordinates of a_c with s. We know some results in the 1-dimensional case. Criterions are condition as $0 < s \leq c''z$. In the multidimensional case, we obtain criterions with via the Morse's theory [18], which can explain the Stokes's phenomena.

2- Examples

- One-dimensional case

a_c depends only on $s = y / n$: $a_c \partial f(a_c) / \partial a - 1 / s = 0$ and the equation of zeros is $(s \operatorname{Im} f(a_c) - \vartheta_\ell) / \pi = \kappa$. then $q(s)ds = \Pr ob\{1 \text{ zéro} \in (s, s + ds)\} = f(a_c)ds / \pi$

- Case of a cycle

Let a cycle : $f^{(p)}(0) = 0$ with $f^{(k)}(0) = \alpha_k \neq 0$ pour $k = 1, 2, .., p-1$. A point α_k of this cycle is fixed point for the application $f^{(p)}$ as $f^{(np)}(\alpha_k) = \alpha_k$. If exists a critical point a_{ck} of $\gamma(a) = \zeta s f^{(p)}(a) - \zeta \log a$ we can have a probability distribution. But this distribution will move at each iteration and be cyclical as in a Markov process.

- Hermitian case : $f = \lambda a - a^2 / 2$: $c\mathrm{H}_{n-1}(\lambda) = c\partial^{n-1} e^{(\lambda a - a^2/2)} / \partial a^{n-1}\big|_{a=0} = \int_\Gamma \dfrac{e^{(\lambda a - a^2/2)} da}{a^n}$

The PR function [11] gives for $t = \lambda / \sqrt{4n} = \cos\vartheta$: $\mathrm{H}_{n-1}(\lambda) \underset{0}{=} \sin\left[\vartheta - \sin(2\vartheta)/2\right]$ when $\lambda \to \infty$ under the condition $\lambda^2 - 4n \leq 0$. The density of zeros of t is the semi-circular low : $p(t)dt = (1 - \cos(2\vartheta))d\vartheta / \pi = (2/\pi)\sqrt{1 - t^2}\, dt$.

Demonstration that seems more explicative than the theorem of Sturm.

- r-Hermitian case: $f = \lambda a - a^r / r$.

We write the PR function $\gamma(z) = \lambda z - z^r / r - t \log z$ and the critical point is solution of the Lambert's [9] equation of $z\gamma'(z) = \lambda z - z^r - t = 0$.

If $r = 3$ and if we use the Fourier's transform, the characteristic function is the Airy's function.

- **Quadratic case in** R^d**:** $f(a) = \lambda a + Q(a)$: $Q(a)$ is quadratic. We can diagonalize yQ with transformation orthogonal $a = O_y u$. In this basis, the volume is invariant, then $n \log a = n \log u$. The function $\gamma(a)$ is $\gamma(u) = y\lambda O_y u_\ell + \Sigma(\mu_\ell u_\ell^2 - n \log u_\ell)$, the eigenvalues μ_ℓ are solutions of $|yQ(a) - \mu I| = 0$. The semi circular low is generated by each coordinate having a $\mu_\ell < 0$. Moreover, the distributions are independent in orthogonal eigenspaces. The equations $\mu_\ell = 0$ give zones where the number p of complex coordinates remains constant.

3- Relations of domination and distributions masked

- We have now a tool to explicit the relations of domination between the distributions imputed to each point $Fix(f)$. Consider without loss of generality two fixed points: α for f and α_m for $f^{(m)}$:

For fixed n, a point s of a manifold $\mathbb{M}_f$ solution at α verifies the $d + p$ following equations:

$\partial \gamma_f(a) / \partial a_\ell = 0$, $\ell = 1, 2, .., d$

$\mathrm{Im}\, \gamma_{f\ell}(a) = k_\ell \pi / n$, $\ell = 1, 2, .., p$

The first d equations are generally a bijection between a and s, The n^p (for each k_ℓ) others imply n^p manifolds $d - p$ dimensional $\mathbb{M}_f$.

We can do the same considerations to compute a point s_m on a manifold $\mathbb{M}_{f^{(m)}}$ solution at α_m, but with p_m instead of p.

Consider now the intersection of all these manifolds $\mathbb{M} = \mathbb{M}_f \cap \mathbb{M}_{f^{(m)}}$, supposing we are in the Rolle's foliation case. We can write now $\gamma_f(s)$ instead of $\gamma_f(a)$ on $\mathbb{M}$.

At a point s, one of 2 functions, $\exp(\gamma_f(s))$ or $\exp(\gamma_{f^{(k)}}(s))$, must exponentially dominate the other. Then, the dominated iteration is masked. Avery important case for classify the region with constant domination is the surfaces defined by $\gamma_f(s) = \gamma_{f^{(k)}}(s)$ We can classify in this category the Stokes phenomena or all random manifolds like the Julia's iteration.

We shall apply this method of domination to EDP or EDO.

V- Linear degeneracy of the Hessian

$\partial^2 \gamma(a) / \partial a \partial \bar{a}$ definite implies that $y\partial^2 f(a) / \partial a \partial \bar{a}$ must be definite at the critical point. The discriminant of $yf(a)$: $|y\partial^2 f(a) / \partial a \partial \bar{a}|$ will degenerate in two distinguish situations: this arrive for particular values (time to time)of y or structurally, when the rank of $y\partial^2 f(a) / \partial a \partial \bar{a}$ is less than d.

1- The from time to time degeneracy

The determinant is a homogeneous polynomial in y $|y\partial^2 f(a) / \partial a^2|$ with a degree d. This polynomial can vanish when one or more eigenvalues vanish. This equation cut the space in regions where the eigenvalues of the Hessian keeps a constant sign. For instance, in dimension 2 without loss of generality, if f is quadratic, with two quadrics : $Q = Q(\alpha, \beta, \gamma) = \alpha a^2 + 2\beta ab + \gamma b^2$ and $Q' = Q(\alpha', \beta', \gamma')$. The Hessian of $\boldsymbol{y}f$ where $\boldsymbol{y} = (x, y)$: $xQ + yQ'$ has the determinant $(\alpha\gamma - \beta^2)x^2 + (\alpha'\gamma' - \beta'^2)y^2 + (\alpha\gamma + \alpha'\gamma' - 2\beta\beta')xy$. Its cancellation gives two straight lines of time-to-time degeneracy.

1- the structural degeneracy

More difficult is the structural degeneracy when, at the critical point, $\gamma(a)$ has a bloc of coordinates which depends on the others. We use here a **bold** notation of the vectors of R^d with two spaces R^p and R^{d-p} : $\boldsymbol{a} = (a, b)$ with $a \in R^p$ and $b \in R^{d-p}$. The structural degeneracy appears each time when b depend functionally on a, without a depends on b in the equations $\boldsymbol{a}_c \partial \gamma(\boldsymbol{a}_c) / \partial \boldsymbol{a} = 0$. The

Hessian is now functionally degenerated. Then the Hessian is function only of a of R^p. The method of the steepest descent must be modified. Unfortunately, we don't know a good book about that.
Unlike the literature on the chaos, which imputes these phenomena the non-linearity, linearity will induce many complicated situations, or exhibit some auto-similar structures and many effects masked by the probability's calculus. The most elementary case studied here is that where f is partly linear. This case gives a first explication of the " esthetical" chaos $[10]$.

2-Definition 3
The *iteration f is said partly linear if f is linear in b and non linear in a.*
Notations
We can write that: $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{f}(a,b) = \boldsymbol{h}(a)b + \boldsymbol{g}(a)$ where $\boldsymbol{g}$ is a polynomial application of R^p in R^{d} and $\boldsymbol{h}$ is a polynomial matrix $(d, d-p)$ in a each element of which apply R^p in R^{d}. Here polynomial means non linear. At the fixed point $0 : \boldsymbol{g}(0) = 0$. The linear part of $\boldsymbol{f}$ at $\boldsymbol{0}$ is supposed diagonalized real: $\boldsymbol{h}(0) = (0, \lambda')$ and $\partial \boldsymbol{g}(\boldsymbol{0}) / \partial a = (\lambda, 0)$. We can write $\boldsymbol{f}(\boldsymbol{a}) = (\lambda a + A(a)b + B(a), \lambda' b + C(a)b + D(a))$ where the $A(a), C(a)$ have all at least degree 1 and the $B(a), D(a)$ at least degree 2. Note $\boldsymbol{a} = (a,b)$ and $\boldsymbol{n} = (n,n')$, $\boldsymbol{y} = (x,y)$, $\boldsymbol{ay} = ax + by$ and $\boldsymbol{yf}(\boldsymbol{a}) = \boldsymbol{yh}(a)b + \boldsymbol{yg}(a)$. The function of Plancherel-Rotach in $\boldsymbol{a}$ is:
$N\gamma(\boldsymbol{a}) = \boldsymbol{yh}(a)b + \boldsymbol{yg}(a) - n\log a - n'\log b$. That relative a at the critical point $\boldsymbol{sh}(a_c)b_c = 1$ is:
$\gamma_1(a) = n\log(\boldsymbol{yh}(a)) + \boldsymbol{yg}(a) - \boldsymbol{n}\log a$.

Note $\boldsymbol{y} = n\boldsymbol{s}$ and $\boldsymbol{1}loga = \sum_{\ell=1}^{\ell=p} loga_\ell$ and $\boldsymbol{1}\log \boldsymbol{sh}(a) = \sum_{\ell=p+1}^{\ell=d} \log \boldsymbol{sh}_\ell(a)$.

The resolving deviation can be written: $e_f^{n-1,n'}(\boldsymbol{a},\boldsymbol{y})_{a=0,b=0} = x^{n-1}y^{n'} - H_{n-1,n'}(\boldsymbol{y})$ with $H_{n-1,n'}(\boldsymbol{y}) = \partial^{n-1}((\boldsymbol{yh}(a))^{n'} e^{\boldsymbol{yf}(a)}) / \partial a^{n-1}\big|_{a=0}$ where the term obtained by n' derivations in b is $\boldsymbol{yh}(a)^{n'} = y^{n'}\left[(\lambda' + xA(a)/y\right]^{n'}$. Then $e_f^{n-1,n'}(\boldsymbol{0},\boldsymbol{y}) = y^{n'}(x^{n-1} - H_{n-1,n'}(\boldsymbol{y}) / y^{n'})$.
The convergence is deeply modified: the density for the d- dimensional Lebesgue's measure is null.
3- Proposition 3
If $\boldsymbol{sh}(a)$ has no real zero, the zeros of $H_{n-1}(y)$ tend to a family of algebraic manifolds of dimension p depending on an random uniform vector of R^p for $\ell = 1, 2, ..., p$:

$$\pi k_\ell = \boldsymbol{s}_\ell \operatorname{Im} \boldsymbol{g}_\ell(a) + \boldsymbol{1}Arg(\boldsymbol{sh}_\ell(a)) - \vartheta_\ell$$

When, $n \to \infty$ the code κ_ℓ / n tends asymptotically to a distribution uniform on $(0,1)^p$ and gives the distribution of s by reciprocal image:
The main manifold contains par the fixed point, the others are obtained par iteration of the critical point.
An important case is that where $\boldsymbol{h}(a)$ is constant: $\pi k_\ell = \boldsymbol{s}_\ell \operatorname{Im} \boldsymbol{g}_\ell(a) - \vartheta_\ell$
We represent $H_{n-1,n'}(\boldsymbol{y})$ by the integral of contour:
$cH_{n-1,n'}(\boldsymbol{y}) = \int_\Gamma (\boldsymbol{yh}(a))^{n'} e^{\boldsymbol{yf}(a)} / (a^n b) dadb = \int_\Gamma e^{\gamma(a,b)} dadb$ where
$\gamma(a,b) = \boldsymbol{yf}(a) + n'\boldsymbol{yh}(a) - \boldsymbol{n}\log a - \log b$
Le polynomial $\boldsymbol{yh}(a)$ is holomorphic at a. We can construct $\log \boldsymbol{yh}(a)$ on a locally connected open space, but we have many problems with the singularities of $\boldsymbol{yh}(a)$, (number finite of zeros of $\boldsymbol{yh}(a)$ of null measure) and the monodromy. the equations of the critical point are :
$\partial\gamma(a,b) / \partial b = \partial \boldsymbol{yf}(\boldsymbol{a}) / \partial b - n'/b = \boldsymbol{yh}(\boldsymbol{a}) - n'/b = 0$
$\partial\gamma(a,b) / \partial a = \partial \boldsymbol{yf}(\boldsymbol{a}) / \partial a - \boldsymbol{n}/a = 0$
thence, if we write $\gamma_1(a) = n'\log(\boldsymbol{yh}(a)) + \boldsymbol{yg}(a) - \boldsymbol{n}\log a$, the equation is:
$\partial\gamma(a,b) / \partial a = \partial\gamma_1(a) = 0$

We follow with the coordinates of $\boldsymbol{n}$ are equal $n-1$ *and* $n'=n$ *:* if the Hessian of $\gamma_1(a)$ is definite negative and if $\boldsymbol{sh}(a)$ don't have real zero, the critical points a_c verify the polynomial equations $\partial\gamma_1(a_c)/\partial a=0$.

We take $\gamma_1(a)$ instead of the PRF $\gamma(a)=yf(a)-n\log a$. All is as if $n\{\log(\boldsymbol{sh}(a))+\boldsymbol{sg}(a)\}$ replace $nsf(a)$. For a fixed $\boldsymbol{s}$, if we can apply the steepest descent's method, the following demonstration will be the same than the precedent where $a\in R^p$.

note $a=(\rho e^{i\vartheta})$, $S(\vartheta,\rho)=\operatorname{Im}\gamma(a)=\boldsymbol{s}\operatorname{Im}\boldsymbol{g}(a)-\boldsymbol{1}\vartheta+\boldsymbol{1}Arg\boldsymbol{sh}(a)$

then, for fixed $\boldsymbol{s}$, we have a uniform code with $\sum_{\ell=1}^{\ell=p}\boldsymbol{s}_\ell\operatorname{Im}\boldsymbol{g}(a)-\vartheta_\ell+\boldsymbol{1}Arg\boldsymbol{sh}_\ell(a)-\pi k_\ell=0$

$\ell=1,2,...,p$. We have p equations making a as a function of the variables k, a manifold in R^{d} with the uniform code κ_ℓ : $\pi\kappa_\ell=\boldsymbol{s}_\ell\operatorname{Im}\boldsymbol{g}_\ell(a)+\boldsymbol{1}Arg\boldsymbol{sh}_\ell(a)-\vartheta_\ell$

When $n\to\infty$, we obtain a family of random manifolds depending a uniform random p-dimensional vector.

4-First analysis critical point $\boldsymbol{a}_c=(a_c,b_c)$

The Perron Frobenius's operator is defined by $\Pr ob(\boldsymbol{a})=\Sigma_\alpha\Pr ob(f_\alpha^{-1}(\boldsymbol{a}))$, then when we iterate n times, we apply $f^{(n)}$. But the critical points are the images by $f^{(n)}$ of the critical random point $\boldsymbol{a}_c=(a_c,b_c)$.

All the branches are obtained by iterations of the critical point. Thence, the distinct points of cycles $f^{(n)}(a)=a$ will be on these branches. But, heuristically, the iterations of $\boldsymbol{a}_c$ are more and more negligible : iterate f n times, then $\boldsymbol{a}_c$ is iterated n times, but $f^{(k)}(\boldsymbol{a}_c)$ only $PE(n/(k+1))$ times.

5- Example: the branches of random parabola of Henon [9]

The iteration of Henon without resonance $f(a,a')$ is characteristic of the phenomena:

$a_1=b+\gamma a-\alpha a^2$, $b_1=\beta a$. À $\boldsymbol{a}=(a,b)$, we take $\boldsymbol{y}=(x,y)>0$. We write $\boldsymbol{y}f(a,b)=x(b+\gamma a-\alpha a^2)+y\beta a$ and the resolving deviation for $n=n'$ $e_f^n(\boldsymbol{a},\boldsymbol{y})=\partial^n\left[e^{\boldsymbol{ay}}-e^{\boldsymbol{y}f(\boldsymbol{a})}\right]/\partial\boldsymbol{a}^n=x^n(y^n-\mathrm{H}_n((x\gamma+y\beta)/\sqrt{2\alpha x}))$ where H_n is a Hermitian polynomial . Under the condition $\left|x\gamma+y\beta)/\sqrt{2\alpha x}\right|>|y|$, then, $t=x\gamma+y\beta)/\sqrt{2\alpha x}$ follows the semi circular low. If the initial space we have branches of random parabola following $\beta(1/2,1/2)$ **.** We must iterate the critical point to obtain the whole random curve. Roughly speaking, in a neighbourhood of the point 0, we obtain random 'parabola' and we can calculate the mean and the variance. In the resonant cases, we must consider the intersection of parabola with the equation of resonance. Rest to examine the decreasing of the branches with the iterations…

Chapter III – Applications : PDE, ODE, etc.

I- Introduction, notations and definitions

Many physical problems can be written with the differential equation $\partial a / \partial t = F(a)$, where $F(a)$ is an application of R^{d} in R^{d}. In this equation, the unknown variables are a vector a of R^{d} and the known variables are a vector $t = (q, t_k)$ of $\mathrm{R}^{+\mathrm{k}}$. The vector $q = (q_1, .., q_i, ...q_{k-1}) \in \mathrm{R}^{+k-1}$ is the position and $t_k \in \mathrm{R}^{+}$ is the time. The number of equations equals the numbers of unknown variables a. We reserve the index $\ell = 1, 2, ...d$ to the unknown a and the index $i = 1, 2, ..k$ to the known t with $d \geq k$.

We translate the differential equation as differential iteration to use our methods. As in numerical calculus, we define :

1- Definition 1

A differential iteration is the application $f(a, \delta)$ *of* R^{d} *in* R^{d}, *defined by* $f(a, \delta) = a + \delta F(a)$ *for all fixed* $\delta_0 > \delta > 0$ *of* R^{+k}.

Starting from an initial position $a(0) = (a_\ell(t(0)))$ at $t(0) = (q(0), 0)$, we iterate $f(a, \delta)$ with a path δ in all directions, with the particularity that, when $f(a, \delta)$ is iterated n times, δ will depend on n. Let δ be a small fixed vector of $\mathrm{R}^{+\mathrm{k}}$. When we iterate $n \in N^d$ times $f(a, \delta)$, we link δ to $n : \delta_i = t_i / n$. The solution $a(t)$ of the differential equations can be obtained when $n \to \infty$. We don't study here the frontier's problems. The main reference on these questions is Arnold [1] and [2]. In order to use the previous results, we define:

2- Hypothesis H

We assume the previous hypothesis of chapter 1and 2. Quickly said: the polynomial application F *iterates a compact* set $C \subset \mathrm{R}^{\mathrm{d}}$ *in* C *and we can apply the steepest descent's method to* $f(a, \delta)$.

But here, we shall see how the steepest descent's method can be managed.

3- Difficulties for the probabilistic approach under H

We want determine the conditions of convergence toward an asymptotic distribution and the distribution induced. But, if we apply all the previous results to $f(a, \delta)$ under H, the distribution will depend on δ. What is the limit of the asymptotic invariant set of zeros S_δ when δ vary?

We shall proof in the following that, when the Hessian of F is definite, then the invariant set $S(\delta)$ is reduce to fixed points and cycles. The degeneracy of the Hessian can induce a random variable with a complicated behaviour. The case is only studied here when the iteration is partly linear $f(a, \delta)$.

For ODE, we recall that some deterministic theorems, as the Poincare-Bendixon's, give asymptotic behaviours, but can't be used beyond the dimension 2. One of these theorems says that, if F is Lipschitzian, starting from a point of this compact, we have a local solution, but does not give any information about the behaviour when $t \to \infty$.

4 - Problem of notation's coherence with definitions

We shall see a problem when the $t \to \infty$: as the convergence toward some set can depend on the direction τ of t, we must choose a direction $\tau = \lim_{t \to \infty} t / |t|$ with $|t| = \Sigma_i t_i$. Hence, $\delta = \tau |t| / n \to 0$

In the case of the ODE or when all the variables q are bounded, then $\tau = 1$.

5- Partly linear *F*

Definition 2

The iteration $f(a, \delta)$ *is said partly linear if* F *is partly linear.*

6- Notation

As in chapter 2-V, we denote: $\boldsymbol{F}(\boldsymbol{a}) = (\lambda a + A(a)b + B(a), \lambda' b + C(a)b + D(a))$ where $\boldsymbol{a} = (a,b)$ with $a \in \mathrm{R}^{\mathrm{p}}$ and $b \in \mathrm{R}^{\mathrm{d-p}}$. The linear part of $\boldsymbol{F}$ is (λa , $\lambda' b$) and all other polynomials have a degree greater than 2.

We denote $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{\alpha}$ and $\boldsymbol{u} = (u,v)$ in R^{p} and $\boldsymbol{y} = (x,y) \in \mathrm{R}^{\mathrm{d}}$. The PRF of $f(\boldsymbol{a},\delta)$ is: $\gamma_f(\boldsymbol{a}) = \boldsymbol{y} f(\boldsymbol{a},\delta) - n \log \boldsymbol{a}$.

For every zero $\boldsymbol{\alpha}$ of $\boldsymbol{F}$, we define a partly linear iteration: $\boldsymbol{G}_{\boldsymbol{\alpha}}(\boldsymbol{u}) = (u + \tau A(u+\alpha)v, v)$ at $\boldsymbol{\alpha}$ and the PRF γ_{G_α} of $\boldsymbol{G}_{\boldsymbol{\alpha}}(a)$ is $\gamma_{G_\alpha}(\boldsymbol{a}) = x(a + \tau A(u+\alpha)) + yb - n\log a - n \log b$.

At $\boldsymbol{0}$: $\boldsymbol{G}(\boldsymbol{a}) = (a + \tau A(a)b, b)$ and $\gamma_G(\boldsymbol{a}) = x(a + \tau A(a)) + yb - n\log a - n \log b$ $\boldsymbol{0}$

and have $\boldsymbol{G}_{\boldsymbol{\alpha}}$ *is* γ_{G_α}

II- Theorem 3

If the differential iteration $f(a,\delta) = a + \delta F(a)$, *applying* $C \subset \mathrm{R}^{\mathrm{d}}$ *in* C *for all* $\delta_0 > \delta > 0$ *with* $\delta = |t|\tau / n \to 0$ *in* $\mathrm{R}^{+\mathrm{k}}$ *is polynomial and apply a compact* C *in* C *with two fixed points* $\boldsymbol{0}, \alpha \in C$, *and verify the hypotheses H of chapter1 and chapter2:*

1--The invariant distribution dominates the fixed point according to $\lambda\tau$ *be positive or negative . In the EDO case, the condition* $\Sigma\lambda > 0$ *implies the* convergence toward *if there is no other dominating distribution. In the case of the PDE, the planes* $\lambda\tau = 0$ *get big gaps. If* $\tau = 1$, *we have the classical resonance.*

2-- If the Hessian of $yF(a)$ *is definite, there exists, besides the fixed points* of $\boldsymbol{F}$, *only cyclical orbits with uniform density and period* T *such as* $\int_0^T F(a(t+t_0)dt = 0, \ \forall t_0$.

3- If $\boldsymbol{F}$ *is partly linear, then asymptotically*

the critical point $\boldsymbol{a} = (a,b)$ *de* $\gamma_f(\boldsymbol{a})$ can *tend the critical point of either* $\gamma_G(\boldsymbol{a})$ *at* $\boldsymbol{0}$ *or* γ_{G_α} *at* $\boldsymbol{\alpha}$. *Thence, we can find the asymptotic invariant solution among the solutions of each iteration* $\boldsymbol{G}_{\boldsymbol{\alpha}}(\boldsymbol{a})$. *As* $\boldsymbol{G}_{\boldsymbol{\alpha}}$ *are partly linear, we have families of random manifolds by iteration of the critical point.*

4--According with $\mathrm{Re}(x\tau A(a)\beta)$ *is positive or negative at critical point, we shall take the distribution at* $\boldsymbol{0}$ *or at* $\boldsymbol{\alpha}$ *that gets get the greatest contribution of the steepest descent's method. The critical points realising* $\mathrm{Re}(x\tau A(a)\beta) = 0$ *can induce a route of communication between the two sets.*

The proof is guided by the geometry of the attractor of Lorenz.

1- Proof of point 1: conditions of convergence

The real eigenvalues ρ of the linear part of $f(a,\delta)$ denoted $\lambda \approx 1 + \rho\delta$ verify $|(\partial F(0)/\partial a) - \lambda I| = 0$. As in chapter 1, we write the resolving deviation of the iteration $f(a,\delta)$: $e_f^n(y,\delta) = y^n - H_n(y,\delta)$ with $H_n(y,\delta) = \partial^n e^{yf(a,\delta)} / \partial a^n \big|_{a=0}$. For all fixed $\delta > 0$, we can write : $e_f^n(y,\delta) = y^n - \rho^n H_n(y,\delta)/\rho^n \to 0$ when $n \to \infty$. $H_n(y,\delta)/\rho^n$ has its leading term equal 1.

when $n \to \infty$: $\rho^n = (1+\lambda\delta)^n \approx \mathrm{e}^{n\lambda\delta}$ with $t = \delta = \tau|t|/n$, provided that $|t|/n$ is bounded. That expression tend toward infinite , 0 or 1 according to $\lambda\tau = \Sigma_{\ell i}\lambda_{\ell i}\tau_{\ell i}$ be positive, negative or null. Then, the zeros of $H_n(s,\rho)$ give the limit or not.

In the case of the PDE, consider $\lambda\tau = 0$. The direction τ can vary and becomes $\lambda\tau = 0$. The process can enter in resonance and conversely. Then, we can observe big gaps in the behaviour.

Note that even though the fixed point is repulsive, it can exist attractive cycles which mask these facts.

2 – Proof of point 2: Cycles or fixed points when the Hessian does not degenerate.

- Non-cyclical case; we study now the zeros of $H_{n-1}(y,\delta)$ by the steepest descent's method. We search the critical points of the PRF with : $\gamma(a,\delta) = ya + \delta y f(a,\delta) - n\log a$ where $\delta = \tau|t|/n$. At

$0 \in Fix(f)$ the domain $S(\delta)$ is determined by the critical points of $\gamma(a,\delta)$, provided that $\gamma(a,\delta) \to -\infty$ when $(y,n) \to \infty$ and the Hessian be definite negative. The equations of the critical point a are $a_\ell \partial \gamma a_c, \delta) / \partial a_\ell = y_\ell a_\ell + \delta \partial y F(a) / \partial a_\ell - n = 0$, $\ell = 1,2,..,d$. if $\delta \to 0$ and $n \to \infty$, $y_\ell a_\ell - n \to 0$ and a_ℓ is real. There is no probabilistic invariant distribution.

- Cyclical case:

$f^{(n)}(a,\delta) = a$. We observe that, if $\delta \to 0$ independently of n, the cycles disappear .

But, we remark for any cycle K : $f^{(n)}(a,\delta) = f^{(n-1)}(a,\delta) + \delta F \circ f^{(n-1)}(a,\delta) = a$ for all $a \in K$, because we have the limit $f^{(n)}(a,0) = f^{(n-1)}(a,0) = a$ when $n \to \infty$. Nevertheless, for all $\delta = t/n > 0$, we can eventually found solutions with period T such as $\lim_{n\to\infty} f^{(n)}(a, T/n) = a$. In the ODE case, as each point is matched once per period T, the asymptotic density is uniform. Moreover, the recurrence's relation can be written: $\partial f^{(n)}(a,\delta)/\partial\delta = \left[1 + \delta F \circ f^{(k)}(a,\delta)\right] \partial f^{(n-1)}(a,\delta)/\partial\delta + F \circ f^{(n-1)}(a,\delta)$.

When $\delta \to 0$, we have by recurrence for all $a \in K$: $(1/n) \sum_{k==0}^{k=n-1} F \circ f^{(k)}(a,\delta) \to 0$ and asymptotically $\int_0^T F(a(t+t_0)dt = 0, \forall t_0$. In the PDE case, we obtain similar results.

3 - Proof of point3 : Partly linear differential iteration

We come back to the partly linear differential iteration (*IV-1-5*): $f(\boldsymbol{a},\delta) = \boldsymbol{a} + \delta F(\boldsymbol{a})$ with $\delta = |t|\tau / n$. The real zeros of F are fixed , isolated points of $f(a,\delta)$, and don't depend on δ.

We denote $\boldsymbol{a}_1 = (a, b_1)$ with $b_1 = b|t|/n$ and $\boldsymbol{y}_1 = (x, y_1)$ with $y_1|t|/n = y$ such as $y_1 b_1 = yb$

$f(\boldsymbol{a}_1,\delta) = (a + \tau A(a) b_1 + \delta(\lambda a + B(a)), b_1 n/|t| + (\tau\lambda' b_1 + \tau C(a) b_1) + \delta D(a))$. Then, with $\boldsymbol{y}_1 = (x, y_1)$:

$\boldsymbol{y} f(\boldsymbol{a},\delta) = x\left[a + \tau A(a) b_1 + \delta(\lambda a + B(a))\right] + y_1\left[b_1 + \delta(\lambda' b_1 + C(a) b_1) + (|t|/n) D(a))\right]$

As $\delta, |t|/n \to 0$, we obtain asymptotically $\boldsymbol{y} f(\boldsymbol{a},\delta) \to \boldsymbol{y}_1 \boldsymbol{G}(\boldsymbol{a}_1) = xa + \tau A(a) b_1 + y_1 b_1$ and , writing y, b instead of y_1, b_1 to lighten the notation, we have the PRF : $\gamma_f(\boldsymbol{a},\delta) \to \boldsymbol{y} G(\boldsymbol{a}) - n \log \boldsymbol{a}$

This phenomenon localizes the distributions around each zero of $\boldsymbol{F}$ and transforms $\boldsymbol{0}$ in a point fixed of $\boldsymbol{G}(\boldsymbol{a})$. Let $\boldsymbol{\alpha} = (\alpha, \beta)$ another zero of $\boldsymbol{F}$, distinct of $\boldsymbol{0}$: $\boldsymbol{F}(\boldsymbol{\alpha}) = (0,0)$, the translated iteration (proposition 1) of $\boldsymbol{\alpha}$ $f_{\boldsymbol{\alpha}}(\boldsymbol{a},\delta) = f(\boldsymbol{a} + \boldsymbol{\alpha}, \delta) - \boldsymbol{\alpha}$ will be yet partly linear.

4 - Proof of point4 : route of communication

Let $\boldsymbol{G}_{\boldsymbol{\alpha}}(\boldsymbol{a}) = (a + \tau A(a + \alpha) b, b)$ be the asymptotic iteration de $f_{\boldsymbol{\alpha}}(\boldsymbol{a},\delta)$ localized en $\boldsymbol{\alpha}$. We observe that, for $\delta = 0$, the asymptotic distribution $\boldsymbol{G}_{\boldsymbol{\alpha}}(\boldsymbol{a})$ en $\boldsymbol{\alpha}$ is not the translated function of $\boldsymbol{G}(\boldsymbol{a})$ en $\boldsymbol{0}$. As we translate $Ta = a + \boldsymbol{\alpha}$, $T f_{\boldsymbol{\alpha}} T^{-1}$ gives the same invariant measure, then, if $f(\boldsymbol{a},\delta) \to \boldsymbol{G}(\boldsymbol{a})$ and $f_{\boldsymbol{\alpha}}(\boldsymbol{a},\delta) \to \boldsymbol{G}_{\boldsymbol{\alpha}}(\boldsymbol{a})$, we should have $T\boldsymbol{G}_{\boldsymbol{\alpha}} T^{-1} = \boldsymbol{G}$. But, we observe that the iteration computed at $\boldsymbol{\alpha}$ and translated to $\boldsymbol{0}$: $T\boldsymbol{G}_{\boldsymbol{\alpha}} T^{-1}(\boldsymbol{a}) = (a + \tau A(a)(b - \beta), b) = \boldsymbol{G}(\boldsymbol{a}) - (\tau A(a)\beta, 0)$. That can be explained by the fact that $\boldsymbol{\alpha}$ is not fixed point of $\boldsymbol{G}$.

But, for all $\delta \neq 0$, as $\boldsymbol{\alpha}$ is fixed point of $f(\boldsymbol{a},\delta)$, $f(\boldsymbol{a},\delta)$ and $f_{\boldsymbol{\alpha}}(\boldsymbol{a},\delta)$ determine a distribution that we have examined in the domination problems: the sign of $\gamma(\boldsymbol{a},\delta) - \gamma_{\boldsymbol{\alpha}}(\boldsymbol{a},\delta) = x\tau A(a)\beta$ gives the most important contribution.

III- Examples

1- Deterministic solutions

- The logistic $da/dt = \alpha(a - a^2)$. $f(a,\delta) = a + \delta F(a) = a + \delta\alpha(a - a^2)$.

We have 2 fixed points $a = 0$ and $a = 1$. But, only $a = 1$ has $1 - \delta\alpha < 1$. The limit is this point.

- Hamilton's equations in R^d

- Keeping the notations of the classical mechanics [1]:

$dp/dt = -\partial H(p,q)/\partial q$; $dq/dt = \partial H(p,q)/\partial p$.

These equations are an application of R^{2d} in R^{2d} when the Hamiltonian H is autonomous. With $F = (-\partial H(p,q)/\partial q, \partial H(p,q)/\partial p)$, we construct the iteration $f(\boldsymbol{a},\delta) = \boldsymbol{a} + \delta F(\boldsymbol{a})$, $\boldsymbol{a} = (p,q)$ and assume that it applies a compact C in C for all small $\delta > 0$.

We search the fixed points (of Lagrange or Trojans) $\partial H(p,q)/\partial q = 0$ and $\partial H(p,q)/\partial p = 0$. Then, we compute the eigenvalues of the linear part at these fixed points, denoting I the identity matrix of R^d and vanishing the determinant composed with 4 blocks $d \times d$:

$$J(\lambda) = \begin{vmatrix} -\partial^2 H/\partial q^2 - \lambda I & -\partial^2 H/\partial p \partial q \\ \partial^2 H/\partial p \partial q & \partial^2 H/\partial p^2 - \lambda I \end{vmatrix} = 0$$

Let $\gamma(p,q) = xp - x\delta\partial H(p,q)/\partial q + yq + y\delta\partial H(p,q)/\partial p - n\log p - n\log q$ be the PRF. The invariant density will be seek among imaginary critical points of:

$n/q = y - x\delta\partial^2 H/\partial q^2 + y\delta\partial^2 H/\partial p\partial q$

$n/p = x - x\delta\partial^2 H/\partial q\partial p + y\delta\partial^2 H/\partial p^2$

giving the distribution if the Hessian is degenerated except the case of the cycles with a uniform distribution. Thence $\int_0^T \partial H/\partial q dt = 0$ and $\int_0^T \partial H/\partial p dt = 0$.

- k bodies with mass m_i in a field of forces derived of a potential $U(q)$

Hamilton's equations are : $dp/dt = -\partial U/\partial q$, $dq/dt = p/m$ with coordinates p_i/m_i ;

If we can check the validation of the calculus, the PRF of the differential iteration is: $\gamma(p,q) = xp + yq + \delta(-x\partial U/\partial q + yp/m) - n\log p - n\log q$ which is composed by two functions independent: $\gamma(p,q) = \gamma_0(p) + \gamma_1(q)$ with $\gamma_0(p) = (xp + \delta yp/m - n\log p)$ and $\gamma_1(q) = (yq - \delta x\partial U/\partial q - n\log q)$.

Thence the equations of the critical points (p,q) are:

$\partial\gamma/\partial p = x - \delta x\partial^2 U/\partial p\partial q + \delta y/m - n/p = x + \delta y/m - n/p = 0$

$\partial\gamma/\partial q = y - \delta x\partial^2 U/\partial q^2 - n/q = 0$.

If $\gamma(p,q)$ has a non degenerate Hessian, the system does not have a random solution. After checking calculus ($U(q)$ is not polynomial), if the potential is Newtonian, we have cycles or points.

2- Probabilistic solutions

2-1- Lorenz equations (see [16] for instance)

This equation is very important because the partial linearity of its differential iteration induce an asymptotic probabilistic solution. It's the best example to illustrate the previous results.

- Presentation of the differential iteration and fixed points

We write these equations with our notations $d\boldsymbol{a}/dt = F(\boldsymbol{a})$, where $\boldsymbol{a} = (a,b,c)$: ($da/dt = \sigma(b-a)$; $db/dt = \rho a - b - ac$; $dc/dt = -\beta c + ab$).

With δ being the time, the differential iteration is quadratic and partly linear in a :

($a + \delta\sigma(b-a)$; $b + \delta(\rho a - b - ac)$; $c + \delta(-\beta c + ab)$) The iteration applies a compact C in C For $\delta_0 > \delta > 0$ (the phenomenon happens between a hot plane and a cold plane). The fixed points are zeros of $F(\boldsymbol{a}) = 0$. When $\rho > 1$ and $\alpha = \sqrt{\beta(\rho-1)}$, we have three: the point $\boldsymbol{0} = (0,0,0)$, and two symmetrically with the c axis: $\boldsymbol{\alpha}_+ = (\alpha, \alpha, \alpha^2/\beta)$ and $\boldsymbol{\alpha}_- = (-\alpha, -\alpha, \alpha^2/\beta)$.

The eigenvalues equation of the linear part is : $(\beta+\lambda)\left[(\sigma+\lambda)(1+\lambda) - \sigma\rho\right] = 0$ at $\boldsymbol{0}$, but $\lambda(\beta+\lambda)(1+\sigma+\lambda) - \alpha^2(\lambda+2\sigma) = 0$ at $\boldsymbol{\alpha}_+$ or at $\boldsymbol{\alpha}_-$. The coefficients (β,ρ,σ) are assumed such the three fixed points are repulsive. Hence, we have random distributions around each point fixed. We don't speak here about attractive cycles or resonances. It remains many things to clarify.

-PRF at the fixed point $\boldsymbol{0}$

When $\delta \to 0$, the differential iteration tends toward: $G(\boldsymbol{a})$ =(a ; $b + \rho a - ac$; $c + ab$).

If $\boldsymbol{y} = (x, y, z)$, the PRF is:

$\gamma(\boldsymbol{a}) = \boldsymbol{y}G(\boldsymbol{a}) - nLoga - nLogb - nLogc = \boldsymbol{y}G(\boldsymbol{a}_1) - nLogabc$

Denoting $\boldsymbol{y} = n\boldsymbol{s}$, with $\boldsymbol{s} = (r, s, t)$ and $\boxed{\varpi = r + s\rho}$,

we can write: $\boldsymbol{y}G(\boldsymbol{a}) / n = a\varpi + bs + ct - sac + tba = L(\boldsymbol{a}) + Q(\boldsymbol{a})$

with a linear part $L(\boldsymbol{a}) = a\varpi + bs + ct$ and a quadratic part $Q(\boldsymbol{a}) = -sac + tba$

As the application is quadratic, we can compute directly with Hermitian polynomials. The Hessian of $G(\boldsymbol{a}) / n$ has the constant quadric Q and the eigenvalues matrix of which T is orthogonal for all $\boldsymbol{a}$:

$$Q = \begin{pmatrix} 0 & t & -s \\ t & 0 & 0 \\ -s & 0 & 0 \end{pmatrix}, \text{ with : } T = \frac{1}{\mu\sqrt{2}} \begin{pmatrix} 0 & \mu & \mu \\ s\sqrt{2} & -t & t \\ t\sqrt{2} & s & -s \end{pmatrix}$$

μ is the positive eigenvalue of the characteristic equation of Q: $\mu(\mu^2 - s^2 - t^2) = 0$.

- Changing basis

The orthogonal application $\boldsymbol{a} = T\boldsymbol{u}$ avec $\boldsymbol{u} = (u, v, w)$ transforms the 3 factors of the PRF: $\gamma(\boldsymbol{a}) / n = \boldsymbol{s}G(\boldsymbol{a}) - Logabc = L(\boldsymbol{a}) + Q(\boldsymbol{a}) - Logabc$:

- $Q(\boldsymbol{a})$ becomes $-\mu v^2 + \mu w^2$,

- $L(\boldsymbol{a})$ becomes $LT\boldsymbol{u} = \mu u + \varpi v / \sqrt{2} + \varpi w / \sqrt{2}$.

- As the orthogonal transformation T lets invariant the volume of the rectangular parallelepiped abc , we have $Logabc = \log uvw$.

In the new basis $\boldsymbol{u}$, $\gamma(\boldsymbol{u}) / n = \gamma_1(u) + \gamma_2(v) + \gamma_3(w)$ is the sum of three independent functions: $\gamma_1(u) = \varpi u / \sqrt{2} - \mu u^2 - \log u$, $\gamma_2(v) = \mu v - \log v$ and $\gamma_3(w) = \varpi w / \sqrt{2} + \mu w^2 - \log w$

becomes in base $\boldsymbol{u}$: $\gamma(\boldsymbol{u}) / n$ the n derivates of which are with H_n the Hermitian polynomial:

$\partial^n \exp(\varpi u / \sqrt{2} - \mu u^2) / \partial u^n \big|_{u=0} = \mathrm{H}_n(\varpi / 2\sqrt{\mu})(-\sqrt{2\mu})^n$; $\partial^n \exp(\mu v) / \partial v^n \big|_{v=0} = \mu^n$;

$\partial^n \exp(\varpi w / \sqrt{2} + \mu w^2) / \partial w^n \big|_{w=0} = \mathrm{H}_n(\varpi / 2i\sqrt{\mu})(-i\sqrt{2\mu})^n$

Thence, the real zeros of $\partial^n \mathrm{e}^{\gamma(\boldsymbol{u})} / \partial \boldsymbol{u}^n \big|_{\boldsymbol{u}=\boldsymbol{0}}$ are zeros of $\mathrm{H}_n(\varpi / 2\sqrt{\mu})$.

- Computing the solution

As we note : $\partial\gamma(\boldsymbol{u}) / \partial \boldsymbol{u} = \partial\gamma(\boldsymbol{a}) / \partial \boldsymbol{a}.\partial \boldsymbol{a} / \partial \boldsymbol{u} = \partial\gamma(\boldsymbol{a}) / \partial \boldsymbol{a} T$, critical points of $\gamma(\boldsymbol{a})$ are theses of $\gamma(\boldsymbol{u})$ and conversely. Hence, it is equivalent to compute the resolving deviation $e_f^n(a, y)\big|_{a=0} = \partial^n(\mathrm{e}^{ya} - \mathrm{e}^{yf(a,\delta)}) / \partial a^n \big|_{a=0}$ in basis $\boldsymbol{a}$ or in basis $\boldsymbol{u}$. The question is now: $\partial^n e^{nsf(a,\delta)} / \partial a^n \big|_{a=0} >> \partial^n (e^{nsa}) / \partial a^n \big|_{a=0}$ in the base $\boldsymbol{u}$? The point fixed $\boldsymbol{0}$ doesn't be attractive.

As $\boldsymbol{ya} / n = \boldsymbol{s}T\boldsymbol{u} = \mu u + (r / \sqrt{2})(v + w)$, the derivates third n of $\exp \boldsymbol{s}T\boldsymbol{u}$ are $\mu^n (r / \sqrt{2})^{2n}$. Thence, the resolving deviation: $\mu^n \left[(r^2 / 2)^n - \mathrm{H}_n(\varpi / 2\sqrt{\mu}) \mathrm{H}_n(\varpi / 2i\sqrt{\mu})(i2\mu)^n \right]$

The leading term of the second member of the deviation must be greater than $(r^2 / 2)^n$ to obtain the probabilistic solution. The condition is $|\varpi| = |r + s\rho| > r$. Then $\varpi / (2\sqrt[4]{(s^2 + t^2)})$ follows asymptotically the Wigner's law. We have a family of random ovals of law beta $\beta(1/2, 1/2)$,

- Computation at the other fixed points and dominating distributions

We search the distributions around the 2 other fixed points. To compute $\boldsymbol{s}\boldsymbol{G}_+(\boldsymbol{a})$ at $\boldsymbol{\alpha}_+$ (resp. $\boldsymbol{s}\boldsymbol{G}_-(\boldsymbol{a})$ at $\boldsymbol{\alpha}_-$) from $\boldsymbol{s}\boldsymbol{G}(\boldsymbol{a})$ at $\boldsymbol{0}$, it is enough to change in the differential iteration $\boldsymbol{a} = (a, b, c)$ in $\boldsymbol{a} + \boldsymbol{\alpha}_+ = (a, b + \alpha, c + \alpha^2 / \beta)$ (resp. $\boldsymbol{a} + \boldsymbol{\alpha}_- = (a, b - \alpha, c + \alpha^2 / \beta)$):

$(a+\delta\sigma(b-a)\ ;\ b+\delta(\rho a-b-ac)\ ;\ c+\delta(-\beta c+ab))$. Then $\mathbf{sG_+(a)} = \mathbf{sG(a)} - (s(\rho - c)+tb)\alpha$ and $\mathbf{sG_-(a)} = \mathbf{sG(a)} + (s(\rho - c)+tb)\alpha$.

The two distributions centred in α_+ and α_- dominate the one centred in $\mathbf{0}$. $(s(\rho - c)+tb) = 0$ is route of communication between the two others. At α_+ we take $\varpi_+ = r + s\rho + s\alpha^2/\beta + t\alpha$ instead of ϖ, and $\varpi_- = r + s\rho + s\alpha^2/\beta - t\alpha$ at α_-, to obtain the asymptotic geometry.

- Geometrical interpretation

The variable random $(r+s\rho)/(2\sqrt[4]{(s^2+t^2)}) = \chi$ gives the distribution of the random ovals at $\mathbf{0}, \alpha_+, \alpha_-$. The plane $c = r + s\rho$ intersects the cylinder $(c/2\chi)^4 = s^2 + t^2$ with random radius $(c/2\chi)^2$. Such are the first approach of a mathematical study. To complete, we must compute also the critical points iterates. Thence:

-Proposition 2

An asymptotic solution of the equation of Lorenz is composed by 2 family of random ovals random $(r+s\rho)/(2\sqrt[4]{(s^2+t^2)}) = \chi$ *with beta law* $\beta(1/2,1/2)$ *, centred on the 2 fixed points* α_+ *and* α_- *, linked by the route of communication* $(s(\rho - c)+tb) = 0$ *and the distributions of iterates of critical points.*

2-2 - Navier Stokes equation of and its random ovals

This example is the same as the Lorenz's case except the fact that δ is not uni dimensional.

Fefferman [8], writer of the Clay's foundation, feels uncomfortable to formalize mathematically The Navier Stokes equation : prudently, he keeps the concepts of the XIX century where the equations differentials are expended by Fourier's series in order to compact the solution. Moreover, the presentation is trapped by physical considerations, introducing many parameters. In the intention to be as close as possible with the physical reality? For fear of forgetting something, we neglect a difficulty? We must admit that the mathematicians have matched many difficulties with the dimension 3 .

In the Navier Stokes equation, the speed u and the pressure p of a fluid are the unknown variables and the known variables are the position $x \in \mathrm{R}^n$ and the time $\tau \in \mathrm{R}^+$. The divergence must be null.

We have not such considerations here. We take our hypothesis to determine the invariant measure in a compact set. Nevertheless, we must translate the equations [6] into iteration, thence obtain $\partial a/\partial t = F(a)$ where the unknown variables are denoted by a, and the known variables by t. The multiplicity of the variables makes complicated the translation.

-Translating [6] into $\partial a/\partial t = F(a)$

At the beginning, we have as many equations than unknown variables with the divergence null, then, we introduce as many equations than unknown intermediary variables :

- the matrix $\partial u/\partial x = b$ introduces the unknown intermediary $b \in \mathrm{R}^n \times \mathrm{R}^n$ with an equal number of equations. Thence : $\sum_{j=1}^{j=n} u_j \partial u_i/\partial x_j = ab_i$ with the matrix U_1, composed of 0 and 1, we have:

- $div\ u = \sum_{j=1}^{j=n} b_{jj} = U_1 b = 0$

- the field of the external forces $f_i(x,\tau) = f_i(c)$ is here polynomial (or well approximated by polynomials) verifying our hypothesis.

We group together all these unknown intermediary linear variables so various as $\partial u/\partial x = b$, $\Delta u_i = \sum_{j=1}^{j=n} \partial^2 u_i/\partial x_j^2 = \sum_{j=1}^{j=n} \partial b_{ij}/\partial x_j$, $\partial p/\partial x_j = d_j$ or $\partial c/\partial t = 1$ and we denote that in a block: $\partial d/\partial t = B$ where d and B are linear en (a,b,c)

- The Navier Stokes equation [6] is written in terms of $\partial a/\partial t = F(a)$:

$\partial a / \partial \tau = -ab + Ad + f(c)$ where A is a matrix with constant coefficients.
$\partial d / \partial t = B$

- Solution

We have obtained a good equation: $\partial \boldsymbol{a} / \partial t = F(\boldsymbol{a})$ with $\boldsymbol{a} = (a,b,c,d)$ and $t = (x,\tau)$, with a lot of unknown variables linked by linear relations. In the main equations, the only non-linear polynomials are ab and $f(c)$. Thence, the iteration $f(\boldsymbol{a},\delta) = \boldsymbol{a} + \delta F(\boldsymbol{a})$ with linearity. Assume that iteration apply a compact C in C for all $\delta > 0$. With $\boldsymbol{y} = (x,y)$, we can write $P = \boldsymbol{y}F(a,b,c,d) : P = x\left[-ab + Ad + f(c)\right] + zc + td + ux$ and $\gamma(\boldsymbol{a}) = P - n\log \boldsymbol{a}$

The Hessian of P is , if ${}^t x$ is the transposed vector of x : $Q = \begin{pmatrix} 0 & {}^t x & 0 \\ x & 0 & 0 \\ 0 & 0 & x\partial^2 f / \partial t^2 \end{pmatrix}$

$x\partial^2 f / \partial t^2$ being computed at 0. The characteristic equation is : $\mu^k(\mu^2 - |x|^2)\left|x\partial^2 f / \partial c^2 - \mu I\right| = 0$ where k is such as the total degree of the polynomial equalizes the dimension of the application. The Hessian is degenerated and has a positive eigenvalue $\mu = |x|$) and a negative $\mu = -|x|$; the movement is situated in the eigenspaces as in the Lorenz case. In dimension 3, we have an orthogonal matrix as T. We have groups of solutions with ovals depending on the beta law $\beta(1/2,1/2)$, but we can have independently random sets resulting of the forces f. The analysis of this equation is very connected to the Lorenz case: the only new elements are the increasing number of fixed points and the analysis of f. However, the more important new things are rifts, which can perturb the behaviour.

IV- Reflections on the recurrence of Poincaré or the horizon of Lyapounov ?

Definition

*Let A be a measurable set. A point a is said **recurrent** for A, iif for all integers p, exists an integer $k \geq p$ such as: $f^k(a) \in A$.*

1- **Poincare's theorem** says:

If $f(a)$ is a bijection and preserve the measure μ (for all $A \subset \sup p(\mu)$: $\mu(A) = \mu(f^{-1}(A)) > 0$),

Then, for all $A \subset \sup p(\mu)$, the points $a \in A$ are almost recurrent.

For all small $\delta_0 > \delta > 0$, $f(a,\delta)$ has a reciprocal image. The PF solution is recurrent if exists an invariant measure. As $f(a,\delta)$ applies a compact set C of R^d in C, this measure exists and all the points of the support of the invariant PF distribution are almost recurrent.

Assume that $F(a)$ verify our hypothesis and be partly linear such as the differential iteration be asymptotically equivalent to $\boldsymbol{G}(\boldsymbol{a}) = (a + \tau A(a)b, b)$. The critical points a and their repartition are reciprocal images of the uniform repartition of the unit hypercube $\mathrm{K} = (0,1)^p$ compacted with the point 0. We divide K in cubes κ_ε with edges $\varepsilon = 1/m$.

Taking a small path δ such as $f(a,\delta)$ is invertible in the compact. We have an invariant measure. We define the variables t_δ such as the points $a(t_\delta)$ of $f(a,\delta)$ belong at least one time in each κ_ε. t_δ is the variable of first matching for each cube κ_ε. For all $t << t_\delta$, the orbit is deterministic .

If t_β is sufficiently large such as we have many points in each cells κ_ε, then we can apply the probability calculus and, for all $t >> t_\beta$, the behaviour will be randomized. We can define $\theta = (t_\delta, t_\beta)$ as the range between the recurrence of Poincaré and the horizon of Lyapounov. The ergodism allow us to make calculus as if we draw randomly points in K.

This conception, nearby the quantum mechanics, gives a transition between determinism and probability. (see Ghys [8]).

2- Remarks for the mathematics

- Some asymptotic solutions of a differential equation can be randomized. The probabilistic theory brings information where the deterministic approach is can't explain something.
- Curiously, the asymptotic distribution is the same, as for ODE as for PDE, but the conditions of convergence are very different. In the PDE case, a little variation of the direction τ can induce great perturbations.
- Many ideas on the chaos must be corrected: the definition of the chaos, as a situation between determinism and probability, seems devoid of interest. The multiple errors of approximation making unpredictable the behaviour are not a good mathematical argument. The notion of sensibility to the initial conditions is without interest as for a Markov process.
- On the other hand, the degeneracy of the Hessian gets a great importance. This degeneracy remains a difficult problem. As example, if we have the equation $da/dt = F(a,t)$ instead of $da/dt = F(a)$, we can add a new coordinate $dt/dt' = 1$ and a new variable t', but that induce linearity with a degeneracy of the Hessian, which modifies the results as, we see for the non-autonomous phenomena.

3- Remarks for the physics

Limits and bounds are essential for the compactness of the iteration in physics. In many problems, the position q in $t = (q,t')$ is bounded. The work of Delabaere and Howls [5] induce these conditions without difficulty. In other problems, the spatial periodicity (par example, equation of Navier-Stokes [6]) of the solution $a(q,t') = a(q+e,t')$ is a good circumstance for bound the recurrence.

Bounds concern the majority of the applications of the physics at distance finite.

4- Other equations : with decay $a(\lambda t) = f \circ a(t)$ or $f \circ a(t) = \lambda a(t)$

These equations determine frequently the semi-invariant curves of iterations of $f(a)$ where λ are the eigenvalues of the linear part of f. They often present very curious structures (Henon's iteration).

Chapter 4: Resonance

I- Introduction

We speak about an important case, we have avoided until now.
It is well known that the resonance makes rigid any mechanical system. It is the same for the solutions of the Perron Frobenius's operator. It is a first approach of this delicate question.

II- Hypotheses, definition and notations

Hypothesis: the same as the previous one.
We diagonalize the linear part of f at the fixed point 0: $\partial f(0)/\partial a = \lambda$ real.

Definition

We say that there is resonance if exists an integer vector such as $\lambda^k = 1$. *We call direction of resonance* ζ_k, *the direction* $\zeta_k \log|\lambda| = 0$ *in the plane of resonance.*

We call group of resonance G^k *of order* k, *the multiplicative subgroup of* R^{d}, *the elements of which* μ *verify* $\mu^k = 1$

As soon as the dimension 3, we can have many vectors k which verify such a relation, and we obtain a lattice in the plane of resonance $x \log|\lambda| = 0$. We present here the more simple case where there is one k minimal $\lambda^k = 1$. Thence, we can extend the group G^k on R^{d} with denoting 1 for the null k_ℓ.

The problem

If f apply a compact C in C exists a unique invariant measure, whatever there is resonance or not. We use the resolving deviation $e^n(y) = e^n(y,0) = y^n - H_n(y)$, then, apply the steepest descent to study the zeros of $H_n(y)$. Then We have to compute the set $Z(\mathrm{J}_n)$.

When resonance, we observe the resolving deviation $e_f^{qk}(y) = y^{qk} - H_{qk}(y)$ where $n = qk$ with $q \in N$. It has the leading term null. We can obtain by identification d functions $\phi_q(y) = \sum_{n=0}^{n=kq} b_{n,q} y^n$ vanishing the resolving deviation such as the condition $\phi_q(y) = \phi_{qf}(y)$.

III -Theorem 4

Let k *be the vector of integers such as* $\lambda^k = 1$.
The resolving deviation $e_f^{qk}(y)$ *remains invariant under the action of the group* G^k.

The point critical a *of the PRF verifies the condition of resonance* $1 = a^{k-1}$.

- Let the resolving deviation $e_f^{qk}(y) = y^{qk} - H_{qk}(y)$ when $n = qk$ in the direction of resonance k. Let $\mu \in G^k$, and $a = \mu u$, then, as $\mu^{qk} = 1$, we have:

$$\partial^{kq}(e^{y\mu u} - e^{yf(\mu u)})/\partial u^{kq} = \partial^{qk}(\mathrm{e}^{ya} - \mathrm{e}^{yf(a)})/\partial a^{qk}(\partial a^{kq}/\partial u^{kq}) = \partial^{qk}(\mathrm{e}^{ya} - \mathrm{e}^{yf(a)})/\partial a^{qk}$$

- The distribution will verify simultaneously $\partial^{qk} e^{yf(a)}/\partial a^{qk} = 0$ for every integer q and $\partial^{n-1} e^{yf(a)}/\partial a^{n-1} = 0$ for all multi indexes such as $n_\ell = n$. The PRF for $\partial^{qk} e^{yf(a)}/\partial a^{qk} = 0$ can be written: $\gamma_{qk}(a) = yf(a) - (qk+1)\log a = yf(a) - q(\log a^k + 1/q \log a)$. When $q \to \infty$, $\gamma_{qk}(a) \to sf(a) - \log a^k$ with $s = y/q$. If we compare this result with the PRF for the function $\gamma_n(a) \to \gamma(a) = sf(a) - \log a$, we must take $\log a^k = \log a$ for the same distribution, and conversely.

Bibliography

[1] V. Arnold, *méthodes mathématiques de la mécanique classique*. Mir, Moscou 1976.

[2] V. Arnold, *Chapîtres supplémentaires* de la *théorie des EDO*. Mir, Moscou 1980.

[3] G. Cirier, *Sur la densité invariante d'un opérateur of Perron Frobenius-I* –hal-00522882,v1.

[4] G. Cirier, *Sur la densité invariante d'un opérateur of Perron Frobenius -II* – hal-00527832, v1

[5] G. Cirier, *Sur la densité invariante d'un opérateur of Perron Frobenius -III* – hal-00527832, v1

[6] G.M. Constantine & T.H.Savits, *a multivariate Faa di Bruno formula*, Trans. of AMS.vol**348**,

[7] E. Delabaere and C. J. Howls, *Global asymptotics for multiple integrals with boundaries*, Duke Math. J. 112, n° 2 (2002), p199-264. Voir aussi : www-irma.u-strasbg.fr/~blanloei/delabaere.ps

[8] C. L. Fefferman, *http://www.claymath.org/millennium/Navier-Stokes_Equations/navierstokes.pdf*

[9] H. Fell, *The geometry of zeros of trinomial equations*, Rend. Circ. Mat. Palermo (2) 29 no. 2 (1980),303–336.

[10] I. Gumowski, C. Mira, *Dynamique chaotique…*, CEPADUES ed. , Toulouse (1980).

[11] M. C. Gutzwiller, *Chaos in classical & quantum mechanics*, IAM 1, Springer-Verlag (1990).

[12] E. Ghys, *Variations autour du théorème de récurrence de Poincaré*. J. de M. ENS Lyon, Belin 2006.

[13] M. Henon, *A two dimensional mapping with a strange attractor* . Com. Math. Phys. 50 (1976)

[14] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, Topological Methods in Modern Mathematics, ed. Goldberg (1993). 53:1120, (1947).

[15] A. Lasota and M.C. Mackey, *Chaos, Fractals and Noise* , Second Edition, Springer-Verlag (1991).

[16] E.N. Lorenz. Deterministic nonperiodic flow. Journal of Atmospheric Sciences, Massachusetts Institute of Technology, Vol. 20, p130-141, 1963

[17] E.Lukacs, *On certain periodic characteristic functions*, Comp. Math.T**13**, p75-80,(1956)

[18] J. Milnor, *Morse Theory*, Princeton University Press, 1963

[19] F. Pham, *La descente des cols par les onglets de Lefschetz, avec vues sur Gauss-Manin*. Asterisque, 130 (1985) p11–47

[20] M. Plancherel et W. Rotach, *Sur les valeurs asymptotiques des polynômes d'Hermite* $\mathrm{H}_n(x) = (-1)^n e^{x^2/2} d^n e^{-x^2/2} / dx^n$. Commentarii Mathematici Helvetici 1, (1929)p 227 - 254.

[21] H. Poincaré ; *Sur le problème des trois corps et les équations de la dynamique*, Acta Math. **13** (1890), p1-270.

[22] D. Ruelle, *formalisme thermodynamique*, Lect. notes in mathematics, Vol. **567**, Springer(1977)

[23] S.M. Ulam and J. von Neumann, *On combination of stochastic and deterministic processes*, Bull. AMS,