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# Canonical smoothing of compact Alexandrov surfaces via Ricci flow.

Thomas Richard<sup>\*†</sup>

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In this paper, we show existence and uniqueness of Ricci flow whose initial condition is a compact Alexandrov surface with curvature bounded from below. This requires a weakening of the notion of initial condition which is able to deal with a priori non-Riemannian metric spaces. As a by-product, we obtain that the Ricci flow of a surface depends smoothly on Gromov-Hausdorff perturbations of the initial condition.

## Introduction

Ricci flow of smooth manifolds has had strong applications to the study of smooth Riemannian manifolds. It is therefore natural to ask if Ricci flow can be helpful in the study non-smooth geometric objects. A reasonable assumption to make on a metric space  $(X, d)$  that we want to deform by the Ricci flow is to require  $(X, d)$  to be approximated in some sense by a sequence  $(M_i, g_i)$  of smooth Riemannian manifolds. In [Sim09b] and [Sim09a], M. Simon studied a class of 3-dimensional metric spaces by this method. An important feature of such “Ricci flows of metric spaces” is that the notion of initial condition has to be weakened. In the work of M. Simon [Sim09b] and [Sim09a], and of the author [Ric11], a weak notion of initial condition has been used, which we call “metric initial condition” :

**Definition 0.1.** *A Ricci flow  $(M, g(t))_{t \in (0, T)}$  on a compact manifold  $M$  is said to have the metric space  $(X, d)$  as metric initial condition if the Riemannian distances  $d_{g(t)}$  uniformly converge as  $t$  goes to 0 (as functions  $M \times M \rightarrow \mathbb{R}$ ) to a distance  $\tilde{d}$  on  $M$  such that  $(M, \tilde{d})$  is isometric to  $(X, d)$ .*

*Remark 0.2.* The compactness assumption in the definition gives that  $(X, d)$  is homeomorphic to  $M$  with its manifold topology. This follows from the fact that  $\tilde{d}$  is continuous

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on  $M$ , which implies that the identity of  $M$  is continuous as an application from  $M$  with its usual topology to  $M$  with the topology defined by  $\tilde{d}$ , compactness of  $M$  then give that the identity is an homeomorphism.

The existence of such flows for some classes of metric spaces  $(X, d)$  has been proved in [Sim09b],[Sim09a] and [Ric11]. An interesting class of spaces for which existence holds is the class of compact Alexandrov surfaces whose curvature is bounded from below. In this paper we prove uniqueness for the Ricci flow with such surfaces as metric initial condition, more precisely :

**Theorem 0.3.** *Let  $(M_1, g_1(t))_{t \in (0, T]}$  and  $(M_2, g_2(t))_{t \in (0, T]}$  be two smooth Ricci flows which admit a compact Alexandrov surface  $(X, d)$  as metric initial condition. Assume furthermore that one can find  $K > 0$  such that :*

$$\forall (x, t) \in M_i \times (0, T] \quad K_{g_i(t)}(x) \geq -K.$$

*Then there exist a conformal diffeomorphism  $\varphi : M_1 \rightarrow M_2$  such that  $g_2(t) = \varphi^* g_1(t)$ .*

Note that the required bounds on the Ricci flow are those which are provided by the existence proof.

In the next few lines, we outline the proof of Theorem 0.3. Any of the two Ricci flows  $(M_i, g_i(t))$  stays in a fixed conformal class, and thus can be written  $g_i(t) = w_i(x, t)h_i(x)$  for some fixed background metric  $h_i$  which can be chosen to have constant curvature. We first show that the metric initial condition prescribes the conformal class of the flow, thus we can assume that  $h_1 = h_2 = h$ . The proof of this fact uses deep results from the theory of singular surfaces introduced by A. D. Alexandrov. This implies that our two Ricci flows can be seen as solutions of the following nonlinear PDE on  $(M, h)$  :

$$\frac{\partial w_i}{\partial t} = \Delta_h \log(w_i) - 2K_h.$$

One then shows that each of the  $w_i$  has an  $L^1$  initial condition as  $t$  goes to 0 and uses standard techniques to show uniqueness.

Our result can be stated in two other ways :

**Proposition 0.4.** *Let  $M$  be a smooth compact topological surface, and  $d$  be a distance on such that  $(M, d)$  is an Alexandrov surface with curvature bounded from below.*

*Let  $g_1(t)_{t \in (0, T)}$  and  $g_2(t)_{t \in (0, T)}$  be two Ricci flows on  $M$  which are smooth with respect to some differential structures on  $M$ . Assume furthermore that one can find  $K > 0$  such that :*

$$\forall (x, t) \in M_i \times (0, T] \quad K_{g_i(t)}(x) \geq -K$$

*and that for  $i = 1, 2$  the distances  $d_{g_i(t)}$  uniformly converge to  $d$  as  $t$  goes to 0.*

*Then  $g_1(t) = g_2(t)$  for  $t \in (0, T)$ .*

This proposition is not a consequence of Theorem 0.3, but just requires a minor adjustment in its proof, which will be indicated in Section 2.

**Proposition 0.5.** *Let  $(M_1, g_1(t))_{t \in (0, T]}$  and  $(M_2, g_2(t))_{t \in (0, T]}$  be two smooth Ricci flows such that for  $i = 1, 2$   $(M_i, g_i(t))$  Gromov-Hausdorff converges to a compact Alexandrov surface  $(X, d)$  with curvature bounded from below as  $t$  goes to 0. Assume furthermore that one can find  $K > 0$  such that :*

$$\forall (x, t) \in M_i \times (0, T] \quad K_{g_i(t)}(x) \geq -K.$$

*Then there exist a conformal diffeomorphism  $\varphi : M_1 \rightarrow M_2$  such that  $g_2(t) = \varphi^* g_1(t)$ .*

*Proof.* We just have to show that if  $(M^2, g(t))_{t \in (0, T]}$  is a smooth Ricci flow on a surface  $M^2$  such that for all  $t \in (0, T)$   $K_{g(t)} \geq -K$  and such that  $(M^2, g(t))$  Gromov-Hausdorff converges to  $(X, d)$  as  $t$  goes to 0, then  $(X, d)$  is the metric initial condition for the Ricci flow  $(M^2, g(t))$ .

Since the diameter and the volume are continuous with respect to Gromov-Hausdorff convergence with sectional curvature bounded from below, we have bounds on the diameter and the volume of  $(M, g(t))$  which are independent of  $t$ . Thanks to the lower bound on the curvature and Bishop-Gromov inequality, we thus have some  $v_0 > 0$  such that :

$$\forall t \in (0, T) \quad \forall x \in M \quad \text{vol}_{g(t)}(B_{g(t)}(x, 1)) \geq v_0.$$

Thanks to Lemma 4.2 in [Sim09a], we then have that, for some constant  $C > 0$  and all  $t \in (0, T)$  (for some possibly smaller  $T > 0$ ) :

$$\forall t \in (0, T) \quad |K_{g(t)}| \leq \frac{C}{T}.$$

One can then argue as in the proof of Theorem 9.2 of [Sim09a] to show that, as  $t$  goes to 0, the Riemannian distances uniformly converge to a distance  $\tilde{d}$  on  $M$  such that  $(M, \tilde{d})$  is isometric to  $(X, d)$ . Thus  $(X, d)$  is the metric initial condition of the Ricci flow  $(M, g(t))$ .  $\square$

As a corollary, we obtain the following statement, which says that for surfaces with curvature bounded from below Gromov-Hausdorff convergence of the initial conditions implies smooth convergence of the Ricci flows :

**Corollary 0.6.** *Let  $(M_i, g_i)_{i \in \mathbb{N}}$  be a sequence of compact surfaces with curvature bounded from below which converges to a compact Alexandrov surface  $(X, d)$  with curvature bounded from below, then there exist  $T > 0$  such that the Ricci flows  $(M_i, g_i(t))_{i \in \mathbb{N}}$  with initial condition  $(M_i, g_i)$  exist at least for  $t \in [0, T)$  and converges (as smooth Ricci flows on  $(0, T)$ ) to the unique Ricci flow with metric initial condition satisfying the bounds of Theorem 0.3.*

*Proof of Corollary 0.6.* Let  $(M_i, g_i)_{i \in \mathbb{N}}$  be sequence satisfying the assumptions of Corollary 0.6. By continuity of the volume and the diameter with respect to Gromov-Hausdorff convergence of Alexandrov surfaces, we have constants  $V$  and  $D$  such that for any  $i \in \mathbb{N}$  :

- $K_{g_i} \geq -1$

- $\text{diam}(M_i, g_i) \leq D$
- $\frac{V}{2} \leq \text{vol}(M_i, g_i) \leq V$

The existence theory (Theorem 1.1) implies the Ricci flows  $(M_i, g_i(t))$  exist at least for  $t \in [0, T)$  and form a precompact sequence whose accumulation points can only be Ricci flows with metric initial condition  $(X, d)$  satisfying the bounds of Theorem 0.3. The uniqueness theorem then implies that there is only one accumulation point.  $\square$

Uniqueness and non-uniqueness issues have been previously considered for the Ricci flow of surfaces with “exotic” initial conditions in the works of Giesen and Topping ([GT11],[Top10]) and Ramos [Ram11].

The paper is organised as follows, in the first section, we sketch M. Simon’s existence proof in dimension 2. In the second section, we show that the metric initial condition uniquely specifies the conformal class. The last section completes the proof. In the appendix, we quickly summarize the results we need from the theory of Alexandrov surfaces.

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## 1. Existence

Here we briefly review the work of Miles Simon which shows the existence of a Ricci flow for compact Alexandrov surfaces with curvature bounded from below. Without loss of generality, we will assume that all Alexandrov surfaces with curvature bounded from below have curvature bounded from below by  $-1$ .

What allows us to flow these surfaces is that they can be approximated by smooth surfaces in a controlled way. This is what Theorem A.1 in the appendix says.

We will now construct a Ricci flow with metric initial condition  $(X, d)$  as limit of the Ricci flows of the  $(M_i, g_i)$ . In order to do this, we use the following estimates due to M. Simon :

**Theorem 1.1.** *For any  $V > 0$  and  $D > 0$ , there exists  $\kappa > 0$  and  $T > 0$  such that if  $(M, g)$  is compact Riemannian surface satisfying :*

- $K_g \geq -1$
- $\text{diam}(M, g) \leq D$
- $\frac{V}{2} \leq \text{vol}(M, g) \leq V$

*then the Ricci flow  $(M, g(t))$  with (classic) initial condition  $(M, g)$  exist at least for  $t \in [0, T)$  and satisfies :*

- $-1 \leq K_{g(t)} \leq \frac{\kappa}{t}$  for  $t \in [0, T]$
- $\text{diam}(M, g(t)) \leq 2D$  for  $t \in [0, T]$
- $\frac{V}{4} \leq \text{vol}(M, g(t)) \leq 2V$  for  $t \in [0, T]$
- $d_{g(s)} - \kappa(\sqrt{t} - \sqrt{s}) \leq d_{g(t)} \leq e^{\kappa(t-s)} d_{g(s)}$  for  $0 < s < t \leq T$ .

*Remark 1.2.* Note that in dimension 2 a lot of the arguments used by M. Simon to prove these estimates in dimension 3 become very simple. Only the existence of  $T$  and the  $\kappa/t$  bound require a delicate blowup analysis.

Using these estimates on each Ricci flow  $(M_i, g_i(t))$  with classic initial condition  $(M_i, g_i)$ , we have, using the compactness theorem of Hamilton for flows, a subsequence which converges to a Ricci flow  $(M, g(t))$  defined for  $t \in (0, T)$  which satisfies the estimates of Theorem 1.1. Using the estimate on the distances, we can argue as in [Sim09a] to show that  $(M, g(t))$  has  $(X, d)$  as metric initial condition.

## 2. Uniqueness of the conformal class

In this section, we prove that the metric initial condition determines the conformal class of the flow under the geometric estimates we have assumed.

**Proposition 2.1.** *Let  $(M_1, h_1)$  (resp.  $(M_2, h_2)$ ) be compact Riemannian surfaces of constant curvature,  $g_1(x, t) = w_1(x, t)h_1(x)$  (resp.  $g_2(x, t) = w_2(x, t)h_2(x)$ ) a smooth Ricci flow on  $M_1 \times (0, T]$  (resp.  $M_2 \times (0, T]$ ).*

*Assume that :*

1.  $-1 \leq K_{g_1}(x, t)$  and  $-1 \leq K_{g_2}(x, t)$
2.  $(M_1, g_1(t))$  and  $(M_2, g_2(t))$  have the same Alexandrov surface  $(X, d)$  as metric initial condition.

*Then there exist a conformal diffeomorphism  $\varphi : (M_1, h_1) \rightarrow (M_2, h_2)$ .*

Set  $u_i(x, t) = \frac{1}{2} \log w_i(x, t)$ . In the following lemmas,  $u$  denotes either  $u_1$  or  $u_2$ .

**Lemma 2.2.** *When  $t$  goes to 0,  $u(x, t)$  converges in  $L^1$  norm to an integrable function  $u_0(x)$ .*

*Proof.* Since  $\partial_t u = -2K_g \leq 2$ , we have that  $u(x, t) - 2t$  increases as  $t$  decreases to 0. This allows us to define the pointwise limit  $u_0(x)$  of  $u(t, x)$  as  $t$  goes to 0. If we fix  $t_0 > 0$ , this also gives us that, for  $t \in (0, t_0)$ ,  $u(x, t) \geq u(x, t_0) - 2(t_0 - t)$ . Thus  $u$  is uniformly bounded from below. Moreover, by Jensen's inequality :

$$\exp \left( 2 \int_M u(x, t) \frac{dv_h}{\text{vol}(M, h)} \right) \leq \int_M e^{2u(x, t)} \frac{dv_h}{\text{vol}(M, h)} = \frac{\text{vol}(M, g(t))}{\text{vol}(M, h)}$$

which gives that  $u(\cdot, t)$  is uniformly bounded in  $L^1$ , thus by monotone convergence,  $u_0$  is in  $L^1$  and the convergence is in  $L^1$  norm. □

**Lemma 2.3.**  $u_0$  belongs to the space  $Pot(M, h)$  defined in the appendix.

*Proof.* The previous lemma shows that  $u_0$  is an  $L^1$  function. We just need to check that the distributional Laplacian of  $u_0$  is a signed measure.

To see this, we write, for a smooth function  $\eta : M \rightarrow \mathbb{R}$  :

$$\begin{aligned} \int_M \eta(x) \Delta_h u(x, t) dv_h(x) &= \int_M \eta(x) (K_h - K_{g(t)} e^{2u(x, t)}) dv_h \\ &= \int_M \eta(x) K_h dv_h - \int_M \eta(x) d\omega_{g(t)} \end{aligned}$$

where  $d\omega_{g(t)} = K_{g(t)} e^{2u(x, t)} dv_h$  is the curvature measure of  $(M_i, g_i(t))$ . By Theorem A.2, since the distance  $d_{g(t)}$  uniformly converges to the distance  $d$ , the curvature measures weakly converges to the curvature measure of  $(M, d)$  which we call  $d\omega$ . We integrate by parts on the left side of the previous equality and let  $t$  go to 0, we get :

$$\int_{M_i} u_0(x) \Delta_h \eta(x) dv_h = \int_M \eta(x) K_h dv_h - \int_{M_i} \eta(x) d\omega.$$

This tells us that the distributional laplacian of  $u_0$  is the measure  $\mu = K_h dv_h - d\omega$ .  $\square$

As in the appendix, we define a new distance on  $M$  by  $d_0 = d_{h, u_0}$ . Since  $(M, d)$  has curvature bounded from below, the condition  $d\mu^+(\{x\}) < 2\pi$  is satisfied (see Remark A.5), and  $d_0$  is a distance on  $M$  whose induced topology is the usual manifold topology of  $M$ .

**Lemma 2.4.** For any  $x$  and  $y$  in  $M$ ,  $d(x, y) = d_0(x, y)$ .

*Proof.* For  $t > 0$ , consider the curvature measures :

$$d\omega_t = K_{g(t)} e^{2u(x, t)} dv_h.$$

By Theorem A.2, the curvature measures weakly converge to the curvature measure  $d\omega$  of  $(M, d)$ . Moreover, since the curvature of  $(M, d)$  is bounded from below by  $-1$ ,  $d\omega \geq -e^{2u_0} dv_h$ . Set :

$$d\mu_t = K_h dv_h - d\omega_t.$$

As  $t$  goes to 0,  $d\mu_t$  weakly converges to  $d\mu$ , since  $d\mu$  is bounded from below by an integrable function, we have that  $d\mu_t^+$  and  $d\mu_t^-$  weakly converge to  $d\mu^+$  and  $d\mu^-$ . We also have convergence of the volumes. We can then apply Theorem A.4 to get that  $d_{g(t)}$  uniformly converges to  $d_{h, u_0}$ . This gives the claimed result.  $\square$

We will write  $(M, e^{2u_0} h)$  for  $M$  equipped with the distance  $d_0$ .

We are now ready to prove Proposition 2.1 :

*Proof (of Proposition 2.1).* For each Ricci flow  $(M_i, e^{2u_i(x, t)} h_i(x))$ , we have constructed a  $u_{i,0}(x)$  such that  $(M_i, e^{2u_{i,0}} h_i)$  is isometric to  $(X, d)$ . Thus there exists an isometry  $\varphi$  from  $(M_1, e^{2u_{1,0}} h_1)$  to  $(M_2, e^{2u_{2,0}} h_2)$ . Theorem A.6 exactly gives that  $\varphi$  is conformal from  $(M, h_1)$  to  $(M, h_2)$ .  $\square$

### 3. End of the proof

Thanks to the results of the previous section, we can now assume that  $g_1(x, t) = w_1(x, t)h(x)$  and  $g_2(x, t) = w_2(x, t)h(x)$  are two Ricci flows on a surface  $(M, h)$  with metric initial condition  $(M, d)$  defined for  $t$  in  $(0, T]$

It is a standard fact that  $w_1$  and  $w_2$  satisfy the following equation of  $M \times (0, T]$  :

$$\frac{\partial w_i}{\partial t} = \Delta_h \log(w_i) - 2K_h. \quad (1)$$

The next lemma relate the metric initial condition with the behaviour of  $w_i$  as  $t$  goes to 0 :

**Lemma 3.1.**  $w_i(\cdot, t)dv_h$  weakly converges to the 2-dimensional area measure  $d\sigma$  associated with  $d$ .

This is Theorem A.2 in the appendix.

First we prove some estimates on  $w_i$  :

**Lemma 3.2.** One can find  $C > 0$  depending on  $K$ ,  $w_1$  and  $w_2$  only, such that :

$$Ce^t \leq w_i(x, t)$$

for all  $x$  in  $M \times (0, T]$ .

*Proof.* We set  $w = w_1$ , the proof is the same for  $w_2$ . We have :

$$\partial_t g = \partial_t wh = -2K_g g = -2K_g wh$$

which gives  $\partial_t w = -2K_g w$ . Using the geometric estimates on the curvature, we get :

$$\frac{\partial_t w}{w} \leq 2$$

Let  $0 < t_1 < t_2 < T$ , compute at some fixed  $x \in M$ , then :

$$[\log(w(x, t))]_{t_1}^{t_2} \leq 2(t_2 - t_1)$$

and :

$$\frac{w(x, t_2)}{w(x, t_1)} \leq e^{2(t_2 - t_1)}$$

thus :

$$\frac{w(x, t_1)}{w(x, t_2)} \geq e^{2(t_1 - t_2)}$$

Let  $t_1 = t$  and  $t_2 > 0$  be some fixed time in  $(0, T)$  and use that  $w(\cdot, t_2)$  is smooth on  $M$  compact, we get the required estimate.  $\square$

The weak convergence of  $w_i(\cdot, t)$  to  $d\sigma$  is not really pleasant to work with when dealing with uniqueness issues. In fact, the following lemma shows that the convergence is strong in  $L^1$ .



**Lemma 3.3.** *As  $t$  goes to 0,  $w(\cdot, t)$  converges in  $L^1$  norm to a function  $w_0$  which satisfies  $w_0 dv_h = \mathcal{H}^2$ .*

*Proof.* Let  $\tilde{w}(x, t) = e^{-2t}w(x, t)$ , then :

$$\partial_t \tilde{w}(x, t) = -2e^{-2t}w(x, t) + e^{-2t}\partial_t w(x, t)$$

As in the proof of the previous lemma :  $\partial_t w \leq 2w$ . So  $\partial_t \tilde{w} \leq 0$  and  $\tilde{w}(x, t)$  increases as  $t$  decreases to 0. Let  $w_0$  be the pointwise limit of  $\tilde{w}(\cdot, t)$  as  $t$  goes to 0. Since  $\int_M \tilde{w}(x, t) dv_h = e^{-2t} \text{vol}(M, g(t))$  is bounded, Lebesgue's monotone convergence theorem gives that  $w_0$  is in  $L^1$  and  $\tilde{w}(\cdot, t)$  (and  $w(\cdot, t)$ ) converges in  $L^1$  norm to  $w_0$ . Since  $L^1$  convergence implies weak convergence,  $w_0 dv_h = d\sigma$ .  $\square$

We now prove the uniqueness statement.

**Proposition 3.4.**  *$w_1(x, t) = w_2(x, t)$  for any  $x \in M$  and  $t \in (0, T]$ .*

*Proof.* We will prove that for any smooth nonnegative function  $\eta$  on  $M$  and any  $T' \in (0, T]$  :

$$\int_M (w_1(x, T') - w_2(x, T'))\eta(x) dv_h(x) = 0$$

Let  $\psi$  be a smooth function on  $M \times (0, T']$  and  $0 < s < T'$ , then :

$$\begin{aligned} & \int_M (w_2(x, T') - w_1(x, T'))\psi(x, t) dv_h - \int_M (w_2(x, s) - w_1(x, s))\psi(x, s) dv_h = \\ & \int_s^{T'} \int_M (w_2(x, \tau) - w_1(x, \tau))(A(x, \tau)\Delta_h \psi(x, \tau) + \partial_t \psi(x, \tau)) dv_h d\tau \end{aligned}$$

where  $A(x, \tau) = \frac{\log(w_2(x, \tau)) - \log(w_1(x, \tau))}{w_2(x, \tau) - w_1(x, \tau)}$ . Since  $w_1$  and  $w_2$  are smooth on  $M \times (0, T]$ ,  $A$  is smooth too. Moreover, by the mean value theorem and lemma 3.2, we have, for  $(x, t) \in M \times (0, T]$  :

$$\varphi(t) \leq A(x, t) \leq \frac{1}{C_1}$$

where  $\varphi$  is the positive continuous function defined by:

$$\varphi(t) = \inf_{x \in M} \min \left( \frac{1}{w_1(x, t)}, \frac{1}{w_2(x, t)} \right) > 0.$$

We now choose  $\psi$  to be the solution of the following backwark heat equation :

$$\begin{cases} \frac{\partial \psi}{\partial t}(x, t) = -A(x, t)\Delta_h \psi(x, t) \\ \psi(x, T') = \eta(x) \end{cases}$$

Thanks to the properties of  $A$ ,  $\psi$  is smooth on  $M \times (0, T']$  and the maximum principle shows that :  $0 \leq \psi(x, t) \leq \sup_{x \in M} \eta(x)$ . we get :

$$\int_M (w_2(x, T') - w_1(x, T'))\eta(x) dv_h = \int_M (w_2(x, s) - w_1(x, s))\psi(x, s) dv_h$$

We now let  $s$  go to 0, since  $w_1(\cdot, s) - w_2(\cdot, s)$  goes to 0 in  $L^1$  norm and  $\psi(x, s)$  is bounded, the right hand side of the previous equality goes to 0 and :

$$\int_M (w_2(x, T') - w_1(x, T')) \eta(x) dv_h = 0$$

Since this equality is true for any  $\eta$  and any  $T' > 0$ , we have that  $w_1$  and  $w_2$  are equal almost everywhere, since these functions are smooth, we get equality everywhere.  $\square$

## A. Facts from the theory of Alexandrov surfaces

This appendix gathers the results from the theory of Alexandrov surfaces with bounded integral curvature or curvature bounded from below that have been used in the paper. All these results can be found in the works of Alexandrov and Reshetnyak (see [Ale06], [AZ67] and [Res93]). A survey in a more modern language can be found in [Tro09].

We use two notions of surfaces with special curvature properties in this work. Our main objects of interest are compact surfaces with curvature bounded from below by  $-k$ , which are surfaces with an intrinsic metric  $(X, d)$  whose geodesic triangles are “fatter” than those in the complete simply-connected surface of constant curvature  $-k$  (see [BBI01], chapter 4 and 10).

A wider class of surfaces is the class of surfaces with bounded integral curvature in the sense of Alexandrov. The definition we give in the next few lines stays informal, precise definition can be found in [AZ67] and [Res93]. The excess of a geodesic triangle  $T$  in an intrinsic is defined by  $e(T) = (\alpha + \beta + \gamma) - \pi$  where  $\alpha$ ,  $\beta$  and  $\gamma$  are the upper angles of  $T$ . A compact surface with an intrinsic metric  $(X, d)$  is said to have bounded integral curvature if there is a constant  $C$  such that for any finite family  $(T_i)$  of disjoint “nice enough” triangles,  $\sum_i |e(T_i)| \leq C$ .

Compact Alexandrov surfaces with curvature bounded from below are compact Alexandrov surfaces with bounded integral curvature, a proof of this fact can be found in [Mac98]. Alexandrov surfaces with bounded integral curvature have well defined notions of area and curvature, which are measures on the surface (signed measure for the curvature). In the case of compact smooth surfaces  $(M, g)$ , these measures coincide with the usual notions of volume form  $dv_g$  and curvature measure  $K_g dv_g$ , see [AZ67], chapters 5 and 8.

First we need a theorem on the approximation of compact Alexandrov surfaces with curvature bounded from below by smooth surfaces :

**Theorem A.1.** *For any compact Alexandrov surface with curvature bounded from below by  $k$   $(X, d)$ , there exist a sequence of smooth compact Riemannian surfaces  $(M_i, g_i)_{i \in \mathbb{N}}$  satisfying :*

- $K_{g_i} \geq k$
- $\text{diam}(M_i, g_i) \leq D$
- $\frac{V}{2} \leq \text{vol}(M_i, g_i) \leq V$

which Gromov-Hausdorff converges to  $(X, d)$ .

This theorem doesn't seem to have been explicitly stated before. When  $k = 0$ , it follows from the theorem of Alexandrov on the approximation of convex surfaces by convex polyhedra, which is proved in chapter 7, section 6 of [Ale06], and the fact that convex polyhedra can be approximated by smooth convex surfaces. When the curvature bound is not 0, one has to approximate the surface by polyhedra whose faces are geodesic triangle in a space form of curvature  $k$ .

The next theorem shows that the curvature measure and the area measure depend continuously on the distance, this is Theorem 6, p. 240 and Theorem 9 p. 269 in [AZ67].

**Theorem A.2.** *Let  $(d_i)_{i \in \mathbb{N}}$  and  $d$  be distances on a compact surface  $M$  such that :*

- *$(M, d)$  and each of the  $(M, d_i)$  are Alexandrov surfaces of bounded integral curvature.*
- *as functions on  $M \times M$ , the distances  $d_i$  uniformly converges to  $d$ .*

*Then the curvature measures  $d\omega_i$  of  $(M, d_i)$  weakly converges to the curvature measure  $d\omega$  of  $(M, d)$ , that is, for any continuous  $\varphi$  function on  $M$  :*

$$\int_M \varphi d\omega_i \xrightarrow{i \rightarrow \infty} \int_M \varphi d\omega.$$

*Moreover, the area measure  $d\sigma_i$  of  $d_i$  weakly converge to the area measure  $d\sigma$  of  $d$ .*

Our aim now is to present a partial converse of the previous theorem. In the sequel,  $h$  is a fixed smooth Riemannian metric on  $M$ . We consider the space  $Pot(M, h)$  of  $L^1$  functions  $u$  on  $M$  whose distributional laplacian with respect to  $h$  is a signed measure  $d\mu$  on  $M$ , we say that  $u$  is the potential of  $d\mu$ . Such a  $u$  is the difference of two subharmonic functions and has a representative which is well defined outside a set of Hausdorff dimension 0 in  $M$ .

The volume of  $u$  is defined by  $V(u) = \int_M e^{2u} dv_h$ . Given a zero mass signed measure  $d\mu$  and  $V > 0$ ,  $d\mu$  has a unique potential  $u_{\mu, V}$  of volume  $V$ . We will denote by  $d\mu = d\mu^+ - d\mu^-$  the Jordan decomposition of  $\mu$ . Reshetnyak has studied the non-smooth Riemannian metric  $e^{2u}h$ . We have ([Res93] Theorem 7.1.1, [Tro09] Proposition 5.3) :

**Theorem A.3.** *Let  $u \in Pot(M, h)$  be a potential of  $d\mu$ . Assume that  $d\mu^+(\{x\}) < 2\pi$  for any  $x \in M$ . Define :*

$$d_{h,u}(x, y) = \inf_{\gamma \in \Gamma(x,y)} \int_0^1 e^{u(\gamma(\tau))} |\dot{\gamma}(\tau)|_h d\tau$$

*where  $\Gamma(x, y)$  is the space of  $C^1$  paths  $\gamma$  from  $[0, 1]$  to  $M$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ . Then  $d_{h,u}$  is a distance on  $M$  such that  $(M, d_{h,u})$  has bounded integral curvature. The curvature measure of this surface is given by :*

$$d\omega = K_h dv_h + d\mu.$$

We are now ready to state the converse of Theorem A.2. This is Theorem 7.3.1 in [Res93], see also [Tro09], Theorem 6.2.

**Theorem A.4.** *Let  $(M, h)$  be a smooth Riemannian surface and  $(d\mu_i^+)_{i \in \mathbb{N}}$   $(d\mu_i^-)_{i \in \mathbb{N}}$  be two sequences of (nonnegative) measures which weakly converge to  $d\mu^+$  and  $d\mu^-$  and such that  $d\mu_i^+(M)$  and  $d\mu_i^-(M)$  are equal and bounded independently of  $i$ .*

*Let  $V_i$  be a sequence of positive numbers converging to  $V$ . Let  $u_i$  be the potential of  $d\mu_i = d\mu_i^+ - d\mu_i^-$  of volume  $V_i$  and  $u$  be the potential of  $d\mu = d\mu^+ - d\mu^-$  of volume  $V$ .*

*Assume that  $d\mu(\{x\}) < 2\pi$  for all  $x \in M$ . Then the distances  $d_{h, u_i}$  uniformly converge as  $i$  goes to infinity to the distance  $d_{h, u}$ .*

*Remark A.5.* In the case of surfaces with curvature bounded from below, the condition  $d\mu(\{x\}) < 2\pi$  is automatically fulfilled. In fact, it follows from the discussion on “complete angles at a point” in [AZ67] (Chapter 2 Section 5 and Chapter 4 Section 4) that if the curvature  $d\omega(\{x\})$  of a point  $x$  in  $(M, d)$  is  $2\pi$ , then any two shortest paths  $\gamma_1$  and  $\gamma_2$  emanating of  $x$  will make a 0 angle at  $x$ . Since when the curvature is bounded from below this angle has to be greater than the comparison triangle, this is impossible.

Then next theorem, due to Huber says that the distance  $d_{h, u}$  determines the conformal class of  $h$ , see [Res93] Theorem 7.1.3 or [Tro09] Theorem 6.4.

**Theorem A.6.** *Let  $(M, h)$  and  $(M', h')$  be two compact Riemannian surfaces,  $u \in \text{Pot}(M, h)$  and  $u' \in \text{Pot}(M', h')$ . Assume  $f$  is an isometry from  $(M, d_{h, u})$  to  $(M', d_{h', u'})$ , the  $f$  is a conformal diffeomorphism from  $(M, h)$  to  $(M', h')$ .*

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