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Absolute continuity of Markov chains ergodic measures by Dirichlet forms methods

Guillaume POLY

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Abstract

We study the absolute continuity of ergodic measures of Markov chains $X_{n+1} = F(X_n, Y_{n+1})$ for the discrete case, and $dX_t = b(X_t)dt + \sigma(X_t).dW_t$ for the continuous case. In the discrete case, we provide with a method enabling to deal with the case where the chains has several invariant measures whereas previous works (c.f. [10, 12]) made assumptions of contractivity, and hence unique ergodicity. Besides, the smoothness assumptions on F are weakened. In the continuous case, we make stronger smoothness assumptions than [3], but non-degeneracy assumptions are strongly weakened. The proofs are based on Dirichlet forms theory, and ergodic theory arguments.

1 Introduction

During the past two decades, the theory of Dirichlet forms has been extensively studied in the direction of improving regularity results of Malliavin calculus (cf. [14, 15]). The classical approach of Malliavin calculus consists of proving, for any test function ϕ, inequalities of the form $\left|\mathbb{E}(\phi^{(p)}(X))\right| \leq C\|\phi\|_{\mathcal{C}^0}$, from which one can deduce the smoothness of the law of X. Nevertheless, when applying this method, for instance to the case of a stochastic differential equation $dX_t = b(X_t)dt + \sigma(X_t).dW_t$, one always need strong regularity assumptions on the coefficients b and σ. With respect to the Malliavin calculus on the Wiener space, the Dirichlet forms method enables to take only Lipschitz hypotheses for the coefficients b and σ. Moreover, a general criterion exists, the energy image density (EID), proved for the Ornstein-Uhlenbeck form and many other local Dirichlet forms (cf [6, 9, 16]), which provides with an efficient tool for obtaining existence of densities. Let us recall that the (EID) criterion still remains a conjecture in the general case of a local Dirichlet form admitting a square field operator.

In this article, we will use the (EID) criterion to study the absolute continuity of an ergodic measure π of a Markov chain defined by $X_{N+1} = F(X_N, Y_{N+1})$, with $F : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ globally Lipschitz and Y_i an i.i.d. sequence. Previous works on this subject (cf [10, 12]) use the Malliavin approach which requires the mapping F being more regular. Besides, in [10, 12], F is assumed to be a contraction (e.g. $\sup_x \mathbb{E}\left(\|\partial_1 F(x, Y)\|\right) < 1$). Indeed, when performing the calculation of the square field operator, terms of the form $\nabla F_N(\cdot) \cdots \nabla F_1(x)$ (where $F_i(x) = F(x, Y_i)$ and ∇F_i is its Jacobian matrix) appear, which, without any assumption of

contractivity, grow exponentially fast. This very strong hypothesis prevents one from studying the case where the chain has several invariant measures. To face with this technical issue, we put fast enough decreasing weights in order to compensate the lack of contractivity. Several examples are given where the non degeneracy conditions are proven to be equivalent to geometric conditions. Finally, we provide with a method to endow the space $L^2(\pi)$ with a non trivial local Dirichlet form $\mathcal{E}_\pi$ whose square field operator Γ_π satisfies the (EID) criterion. In general, the Dirichlet form we build does not satisfy the usual sufficient (but non necessary) conditions entailing (EID) criterion (cf. [11], p105 or cf. [1] Theorems 3.2 and 5.3).

Our method is very general and, as we will see, applies also to the case of stochastic differential equations in order to provide sufficient conditions under which invariant measures are absolutely continuous with respect to the Lebesgue measure. This question has been extensively discussed during the two past decades using analytic methods and we refer to [3] for a rather complete and recent survey of the question. Let us mention the most general result already published in [3]. For an open subset Ω of $\mathbb{R}^d$ and all $f \in \mathcal{C}_c^\infty(\Omega)$, let us define $L_{A,b}f = \mathit{trace}(Af'') + (b, \nabla f)$, where A is a mapping from Ω taking values in the space of non-negative symmetric operators of $\mathbb{R}^d$ and b a vector field (A, $b \in L^1_{loc}(\Omega)$). Let μ a probability measure solution of $\int_\Omega L_{A,b} f d\mu = 0$ for all $f \in \mathcal{C}_c^\infty(\Omega)$. Let us assume that $\mu\{x \mid \det A(x) = 0\} = 0$, then μ is absolutely continuous with respect to the Lebesgue measure. Compared with this result, when the mappings A and b satisfy the usual Lipschitz conditions leading to the existence of a strong solution of the stochastic differential equation induced by $L_{A,b}$, and taking μ an ergodic probability measure we can strongly weaken the criterion: $\mu\{x \mid \det A(x) = 0\} = 0$ may be replaced by $\mu\{x \mid \det A(x) = 0\} < 1$. Roughly speaking, Lipschitz assumptions enable to "think" the stochastic equation as a dynamical system and use ergodic Theorems. The proof is essentially the same as in the discrete case. Thus, we discretize the stochastic flow in order to restrict ourselves to the case of a discrete Markov chain.

First, we recall some elementary facts of Dirichlet structure theory and precise our notations, especially the differential ones, next we state the main results. Finally we successively prove the results after-mentioned in the discrete case and in the continuous case. We are grateful to A.Coquio and E.Löcherbach for their helpful advices and notices, notably in concern with the proof of Theorem 1.1.

1.1 Preliminaries and notations

Originally introduced by Beurling and Deny (c.f. [2]), a Dirichlet form is a symmetric non-negative bilinear form $\mathcal{E}[\cdot,\cdot]$ acting on a dense subdomain $\mathcal{D}(\mathcal{E})$ of an Hilbert space $\mathcal{H}$, such that $\mathcal{D}(\mathcal{E})$ endowed with the norm $\sqrt{< X, X >_{\mathcal{H}} + \mathcal{E}[X, X]}$ is complete. We refer to [13, 11, 8, 4] for an exhaustive introduction to this theory. In the sequel we only focus on the particular case of local Dirichlet forms admitting square field operators.

Definition 1.1. *Following the terminology of [8], in this paper, a Dirichlet structure will design a term* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ *such that:*

(a) $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space.*

(b) $\mathbb{D}$ *is a dense sub-domain of* $L^2(\mathbb{P})$.

(c) $\Gamma[\cdot,\cdot] : \mathbb{D} \times \mathbb{D} \longrightarrow L^1(\mathbb{P})$ *is bilinear, symmetric, non-negative.*

(d) For all $m \geq 1$*, for all* $X = (X_1, \cdots, X_m) \in \mathbb{D}^m$*, and for all* $F \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R})$ *and* K*-Lipschitz:*

- $F(X) \in \mathbb{D}$,
- $\Gamma[F(X), F(X)] = \sum_{i=1}^{m} \sum_{j=1}^{m} \partial_i F(X) \partial_j F(X) \Gamma[X_i, X_j]$.

(e) Setting $\mathcal{E}[X,X] = \mathbb{E}\left(\Gamma[X,X]\right)$*, the domain* $\mathbb{D}$ *endowed with the norm:*

$$\|X\|_{\mathbb{D}} = \sqrt{\mathbb{E}(X^2) + \mathcal{E}[X,X]},$$

is complete. Thus, $\mathcal{E}$ *is a Dirichlet form with domain* $\mathbb{D}$ *on the Hilbert space* $L^2(\mathbb{P})$.

Remark 1.1. *One could work with a more general kind of Dirichlet structure, by choosing a measured space* $(\Omega, \mathcal{F}, m)$ *instead of a probability space. In order to avoid unessential difficulties we make here this restriction.*

The following definition is preponderant in this work.

Definition 1.2. *Let* $S = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ *be a Dirichlet structure. We say that* S *satisfies the energy image density criterion, if and only if, for all* $p \geq 1$*, for all* $X = (X_1, \cdots, X_p) \in \mathbb{D}^p$*:*

$$X_*(\mathbb{1}_{\{\det(\Gamma[X_i,X_j])>0\}} d\mathbb{P}) << d\lambda_p.$$

Conjecture. *(Bouleau-Hirsch)*
Every Dirichlet structure (in the sense of 1.1) satisfies the criterion E.I.D..

As already mentioned, we refer to [6, 9, 5] for examples and sufficient conditions entailing E.I.D..

The most illustrative example of this kind of structure is the Sobolev space $\mathcal{H}^1(\Omega, \lambda_d)$ where Ω is a bounded open subset of $\mathbb{R}^d$, and λ_d the d-dimensional Lebesgue measure In this case:

- $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{B}(\Omega), \frac{d\lambda_d}{\lambda_d(\Omega)})$,
- $\mathbb{D} = \mathcal{H}^1(\Omega)$,
- $\Gamma[\phi] = \nabla\phi \,.\, {}^t\nabla\phi$,
- $\mathcal{E}[\phi] = \dfrac{1}{\lambda_d(\Omega)} \displaystyle\int_\Omega \nabla\phi^t \nabla\phi \, d\lambda_d$.

Another fundamental structure is the Sobolev space $\mathcal{H}^1(\mathbb{R}^p, \mathcal{N}(0, I_p))$ where $\mathcal{N}(0, I_p)$ denotes the standard Gaussian distribution in $\mathbb{R}^p$. In this case:

- $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p), \mathcal{N}(0, I_p)$,

- $\mathcal{H}^1(\mathbb{R}^p, \mathcal{N}(0, I_p))$,
- $\Gamma[\phi] = \nabla\phi \,.\, {}^t\nabla\phi$,
- $\mathcal{E}[\phi] = \displaystyle\int_{\mathbb{R}^p} \nabla\phi^t\nabla\phi \, d\mathcal{N}(0, I_p)$.

Letting $p \to \infty$ in the previous structure leads to the fundamental structure of Ornstein-Uhlenbeck on the Wiener space. We refer to [4] for a short introduction and to [8] for more details. Let us mention that,in the structure $\mathcal{H}^1(\mathbb{R}^{\mathbb{N}}, \mathcal{N}(0, 1)^{\mathbb{N}})$, the square field operator Γ is the Malliavin square field operator on the Wiener space.

Let us enumerate the notations adopted in the present paper:

- for $X \in \mathbb{D}$, we set $\Gamma[X] = \Gamma[X, X]$ and $\mathcal{E}[X] = \mathcal{E}[X, X]$,
- for $X = (X_1, \cdots, X_m) \in \mathbb{D}^m$:

$$\Gamma[X] = \Gamma[X, {}^tX] = \begin{pmatrix} \Gamma[X_1, X_1] & \Gamma[X_1, X_2] & \cdots & \Gamma[X_1, X_n] \\ \Gamma[X_2, X_1] & \Gamma[X_2, X_2] & \cdots & \Gamma[X_2, X_n] \\ \vdots & \vdots & \cdots & \vdots \\ \Gamma[X_n, X_1] & \Gamma[X_n, X_2] & \cdots & \Gamma[X_n, X_n] \end{pmatrix},$$

- for $\phi \in \mathcal{C}^1(\mathbb{R}^p, \mathbb{R}^q)$, we set the Jacobian matrix: $\nabla\phi(x) = \begin{pmatrix} \partial_1\phi_1(x) & \cdots & \partial_p\phi_1(x) \\ \vdots & \cdots & \vdots \\ \partial_1\phi_q(x) & \cdots & \partial_p\phi_q(x) \end{pmatrix}$,
- for $\phi(\cdot, \cdot) \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^q)$, $\nabla_x\phi(x, y)$ is the partial Jacobian matrix of: $\begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R}^q \\ x \longrightarrow \phi(x, y) \end{cases}$, and $\nabla_y\phi(x, y)$ the Jacobian matrix of $y \longrightarrow \phi(x, y)$.
- when the mappings considered are only Lipschitz, $\nabla\phi$ will denotes an arbitrary representation of the Jacobian matrix. In the case of a particular representation (c.f. 2.4), we will precise it,
- in a topological space $(E, \mathcal{T})$, $x_n \xrightarrow[n\to\infty]{\mathcal{T}} x$ naturally means that x_n converges toward x in the topology $\mathcal{T}$.

Finally, we end this preliminary section by giving without proof, elementary calculus rules which are currently used in the paper.

Lemma 1.1. *Let $p, q \geq 1$. Let $X = (X_1, \cdots, X_p) \in \mathbb{D}^p$and let $F \in \mathcal{C}^1 \cap \mathcal{L}ip(\mathbb{R}^p, \mathbb{R}^q)$. Then,*

$$\Gamma[F(X)] = \nabla F(X)\Gamma[X]^t\nabla F(X).$$

(Where $\nabla F(X) \in \mathcal{M}_{q,p}(\mathbb{R})$ and $\Gamma[X] \in \mathcal{M}_q(\mathbb{R})$.)

Remark 1.2. *In the whole article, we assume that the Markov chains under consideration admit ergodic probability measures whereas it is not systematically true when the state space is not compact. Our approach consists simply of studying the regularity of an ergodic probability measure if it exists.*

1.2 Statement of the main results

1.2.1 Discrete case:

Theorem 1.1. *Let $F : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be globally Lipschitz and $\nabla F(\cdot)$ be one representation of its Jacobian matrix. Let $(Y_i)_{i\in\mathbb{N}^*}$ be the coordinates of a product Dirichlet structure $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p), \mu, \mathbb{D}, \Gamma)^{\mathbb{N}^*}$ which is assumed to satisfy (EID) criterion. Let finally π be an ergodic measure of the Markov chain $X_{N+1} = F(X_N, Y_{N+1})$. We make the two following non-degeneracy conditions:*

1. $\pi \otimes \mu \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid \det \nabla_x F(x, y) = 0\} = 0$,
2. *there exists $N_0 > 1$ such that $\pi \otimes \mu^{\mathbb{N}^*} \left\{\det \Gamma[X^x_{N_0}] > 0\right\} > 0$.*

Then π is absolutely continuous with respect to λ_n.

More concretely, the previous Theorem may be applied in the following way.

Corrolary 1.1. *Let $F : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be globally Lipschitz and $\nabla F(\cdot)$ be one representation of its Jacobian matrix. Let Y_i be a i.i.d. sequence with common law $d\mu(x) = \mathbb{1}_\Omega(x)\frac{d\lambda_p(x)}{\lambda_p(\Omega)}$ for some bounded open subset Ω of $\mathbb{R}^p$ and let π be an ergodic measure of the Markov chain $X_{N+1} = F(X_N, Y_{N+1})$. Let us assume the two non-degeneracy conditions:*

1. $\pi \otimes \mu \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid \det \nabla_x F(x, y) = 0\} = 0$,
2. *there exists $N_0 > 1$ such that $\pi \otimes \mu^{\mathbb{N}^*} \left\{\det \nabla_{Y_1,\cdots,Y_{N_0}}[X^x_{N_0}]^t \nabla_{Y_1,\cdots,Y_{N_0}}[X^x_{N_0}] > 0\right\} > 0$.*

   $\Big(\nabla_{Y_1,\cdots,Y_{N_0}}[X^x_{N_0}]$ is one representation of the Jacobian matrix of the (Lipschitz) mapping $(Y_1, \cdots, Y_{N_0}) \longrightarrow X^x_{N_0}\Big)$.

Then π is absolutely continuous with respect to λ_n.

Remark 1.3. *Before making the proof of Theorem 1.1, let us explain roughly the role played by the non-degeneracy conditions 1) and 2). The condition 1) ensures that from X_N to X_{N+1}, the system conserves all the "noise" already accumulated, just like a convolution would do. The condition 2) expresses the fact that, with some positive probability, some "noise" is added on the system. Finally, the ergodicity condition entails that the aforementioned process of "regularization" is repeated infinitely often so that after an "infinite" time the law of the Markov chain is absolutely continuous with respect to the Lebesgue measure λ_n.*

1.2.2 Continuous case:

In the continuous time setting, the previous results take the following form.

Theorem 1.2. *Let $b : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \longrightarrow \mathcal{M}_{n,p}(\mathbb{R})$ two globally Lipschitz mappings. Let us consider the stochastic differential equation:*

$$dX_t = b(X_t)dt + \sigma(X_t).dW_t \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^p$. Let π an ergodic measure of the diffusion. We assume that $\pi\left\{\exists t > 0 \mid \det \Gamma[X_t^x] > 0\right\} > 0$. Then π is absolutely continuous with respect to Lebesgue.

Corrolary 1.2. *Under the hypotheses of the above Theorem, if $\pi\left\{x \mid \det \sigma(x)^t\sigma(x) = 0\right\} < 1$, then π is absolutely continuous with respect to the Lebesgue measure λ_n.*

Remark 1.4. *In the continuous setting, because of the flow property, condition 1) of non-degeneracy is automatically fulfilled and does not appear in the hypotheses.*

2 Absolute continuity of ergodic measures, discrete case

2.1 Main Theorems

Before making the proof of Theorem 1.1, let us state some preliminary results.

Lemma 2.1. *Let $(S_i)_{1\leq i\leq N}$ be a finite family of symmetric nonnegative matrix of $\mathbb{R}^n$. Let $(\alpha_i)_{1\leq i\leq N}$ be a family of positive numbers. Then,*

$$\det\left(\sum_{i=1}^{N} S_i\right) > 0 \Leftrightarrow \det\left(\sum_{i=1}^{N} \alpha_i S_i\right) > 0.$$

Proof. Let $X \in \mathbb{R}^n$, since for all i, $\alpha_i > 0$ and ${}^tXS_iX \geq 0$, then:

$$\sum_{i=1}^{N} {}^tXS_iX = 0 \Leftrightarrow \sum_{i=1}^{N} \alpha_i {}^tXS_iX = 0.$$

□

Lemma 2.2. *In the Theorem (1.1) setting, we may define on an auxiliary probability space, random variables $(z_i, y_i, \Gamma[y_i])_{i\in\mathbb{N}^*}$ such that:*

1. *for all $i \geq 1$, $z_i = F(z_{i+1}, y_i)$,*
2. *for all $i \geq 1$, z_i is π distributed,*
3. *for all $P \geq 1$, $\sigma\left(z_i, y_i, \Gamma[y_i];\ i \geq P+1\right)$ is a sigma-algebra independent from $(y_i, \Gamma[y_i])_{1\leq i\leq P}$,*
4. *for all $i \geq 1$, $(y_i, \Gamma[y_i])$ has the same distribution as $(Y_i, \Gamma[Y_i])$.*

(Let us note η the joint law of the sequence of variables $(z_i, y_i, \Gamma[y_i])_{i\geq 1}$).

Proof. We set $S = (\mathbb{R}^n \times \mathbb{R}^p \times \mathcal{M}_p(\mathbb{R}))^{\mathbb{N}^*}$ equipped with $\mathcal{T}$ the product topology. Let X be a random variable π-distributed and independent from $(Y_i, \Gamma[Y_i])_{i\geq 1}$ and let $Z_P^N = F_P \circ F_{P+1} \circ F_N(X)$ the reversed iterations. Finally, we set ν_N the probability over the topological space $(S, \mathcal{T})$ being the joint law of the sequence of variables:

$$\left((Z_1^N, Y_1, \Gamma[Y_1]), (Z_2^N, Y_2, \Gamma[Y_2]), \cdots, (Z_N^N, Y_N, \Gamma[Y_N]), (0_{\mathbb{R}^n}, 0_{\mathbb{R}^p}, 0_{\mathcal{M}_p(\mathbb{R})}), (0_{\mathbb{R}^n}, 0_{\mathbb{R}^p}, 0_{\mathcal{M}_p(\mathbb{R})}), \cdots\right).$$

ν_N being a tight sequence of probabilities over $(S, \mathcal{T})$, for some subsequence N_k, μ_{N_k} converges in distribution toward a probability, namely η. We then set $(z_i, y_i, \Gamma[y_i])_{i\geq 1}$ the coordinates of the probability space $(S, \mathcal{T}, \eta)$. Checking the conditions 1), 2), 3), 4) aforementioned is straightforward. □

Remark 2.1. *We could construct theses auxiliary variables by reversing the iterations, that is to say making $F_1 \circ F_2 \circ \cdots \circ F_N(x)$ instead of $F_N \circ F_{N-1} \circ \cdots \circ F_1(x)$. The variable z_1 is then obtained with infinitely many iterations and is π distributed. All the strategy of the proof Theorem (1.1) consists of exploiting this fact with the (EID) criterion.*

Lemma 2.3. *Let us endow the space $(S, \mathcal{T})$ with the measure η being the joint law of the sequence of variables $(z_i, y_i, \Gamma[y_i])_{i\geq 1}$ aforementioned. Then the left shift τ in the topological space $(S, \mathcal{T})$ is η-ergodic.*
(Left shift means : $\tau(\omega_1, \omega_2, \ldots, \omega_N, \ldots) = (\omega_2, \omega_3, \ldots, \omega_{N+1}, \ldots)$).

Proof. For simplicity we set $(z_i, y_i, \Gamma[y_i])_{i\geq 1} = \omega = (\omega_1, \omega_2, \cdots)$, where $\omega_i = (z_i, y_i, \Gamma[y_i])$ and we define $(\forall P \geq 1)$ $\mathcal{F}_P = \sigma(y_1, \Gamma[y_1], y_2, \Gamma[y_2], \cdots, y_P, \Gamma[y_P])$. Let μ_1, μ_2 be two τ-invariant measures on the topological space S, such that $\eta = \frac{\mu_1+\mu_2}{2}$. By definition, η is ergodic if and only if it is extremal in the convex of τ-invariant probability measures, hence we want to prove that $\eta = \mu_1 = \mu_2$. Let us be given an integer $P \geq 1$ and a map ψ such that η-almost surely:

$$\psi(\omega_1, \cdots, \omega_P) = \psi(\omega_2, \cdots, \omega_{P+1}). \tag{2}$$

Then η-almost surely, for all $Q \geq P$, since $\sigma(\omega_i \; ; \; i \geq P+1)$ is independent from $\mathcal{F}_P$ (c.f. Lemma (2.2)):

$$\begin{aligned}
\psi(\omega_1, \cdots, \omega_P) &= \psi(\omega_{1+Q}, \cdots, \omega_{Q+P}) \\
&= \int \psi(\omega_{1+Q}, \cdots, \omega_{Q+P}) d\mathbb{P}_{(y_1, \Gamma[y_1], \cdots, y_p, \Gamma[y_P])} \\
&= \int \psi(\omega_1, \cdots, \omega_P) d\mathbb{P}_{(y_1, \Gamma[y_1], \cdots, y_p, \Gamma[y_P])} \\
&= \mathbb{E}\left\{\psi(\omega_1, \cdots, \omega_P) \mid z_{P+1}\right\} \\
&= f(z_{P+1}),
\end{aligned}$$

with $f(x) = \mathbb{E}\left\{\psi(\omega_1, \cdots, \omega_P) \mid z_{P+1} = x\right\}$. But, according to (2), η-almost surely, $f(z_{P+1}) = f(z_{P+2})$. Since π is an ergodic measure of the Markov chain X_N, f is constant π-almost everywhere, which implies that ψ is constant η-almost surely. Setting $\eta_P = (\omega_1, \cdots, \omega_P)_* \eta$ the

projection of η on the P first coordinates, we have proved that the left shift τ_P on the space $S_P = (\mathbb{R}^n \times \mathbb{R}^p \times \mathcal{M}_p(\mathbb{R}))^P$ is η_P-ergodic. However, $\eta_P = \frac{(\omega_1,\cdots,\omega_P)_*\mu_1 + (\omega_1,\cdots,\omega_P)_*\mu_2}{2}$ where $(\omega_1,\cdots,\omega_P)_*\mu_i$ ($i \in \{1,2\}$) are τ_P-invariant. Necessarily:

$$\eta_P = (\omega_1,\cdots,\omega_P)_*\mu_1 = (\omega_1,\cdots,\omega_P)_*\mu_2.$$

Letting $P \to \infty$ we get $\eta = \mu_1 = \mu_2$. □

By functional calculus, the following result is true when F is $\mathcal{C}^1$. We need to extend it to the Lipschitz case, without using Lipschitz functional calculus which fails in this setting.

Lemma 2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ be a Dirichlet structure satisfying the (EID) criterion. Let $X = (X_1, \cdots, X_n)$ be in $\mathbb{D}^n$ and let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a Lipschitz mapping. Then, for any Borel representation $\nabla F(\cdot)$ of the Jacobian matrix of F we have:*

$$\det \Gamma[F(X)] = \det\left(\nabla F(X)\Gamma[X]^t \nabla F(X)\right).$$

Proof. Let ∇F be any Borel representation of the Jacobian matrix of F, we notice that:

$$\det\left(\nabla F(X)\Gamma[X]^t\nabla F(X)\right) = \det\left(\nabla F(X)\Gamma[X]^t\nabla F(X)\mathbb{1}_{\{\det\Gamma[X]>0\}}\right).$$

But, using (EID), $X_*\left(\mathbb{1}_{\{\det\Gamma[X]>0\}}d\mathbb{P}\right) \ll \lambda_n$ ensures that $\det\left(\nabla F(X)\Gamma[X]^t\nabla F(X)\mathbb{1}_{\{\det\Gamma[X]>0\}}\right)$ does not depend on the representation of ∇F. Hence, $\det\left(\nabla F(X)\Gamma[X]^t\nabla F(X)\right)$ neither depends on the representation of ∇F. Besides, using a suitable smoothing argument, one can prove (c.f.[8]) that there exists at least one representation $\tilde{\nabla F}$ for which the functional calculus $\Gamma[F(X)] = \tilde{\nabla F}(X)\Gamma[X]^t\tilde{\nabla F}(X)$ holds (but not necessarily for all representations). This implies:

$$\begin{aligned}\det\Gamma[F(X)] &= \det\left(\tilde{\nabla F}\Gamma[X]^t\tilde{\nabla F}\right)\\ &= \det\left(\nabla F(X)\Gamma[X]^t\nabla F(X)\right).\end{aligned}$$

□

Now, we make the proof Theorem (1.1).

Proof. Theorem (1.1)

Let K be the Lipschitz constant of the mapping F, let α be some positive number such that $0 < K^2\alpha < 1$, and let A be a Borel subset of $\mathbb{R}^n$ with $\lambda_n(A) = 0$. In order to use the Radon-Nykodym criterion, we wish to prove that $\pi(A) = 0$.

For some latter convenience, we set for all (N, P) with $1 \le N \le P$:

- $Z_N^P(x) = F_N \circ \cdots \circ F_P(x)$ (with $F_i(x) = F(x, Y_i)$),
- $\begin{cases} M_1(x) = \nabla_y F(x, Y_1), \\ M_P(x) = \nabla_x F(Z_2^P(x), Y_1)\cdots\nabla_x F(Z_P^P(x), Y_{P-1})\nabla_y F(x, Y_P) \ \ (P > 1), \end{cases}$

- $\left\{ \begin{array}{l} m_1 = \nabla_y F(z_2, y_1), \\ m_P = \nabla_x F(z_2, y_1) \cdots \nabla_x F(z_P, y_{P-1}) \nabla_y F(z_{P+1}, y_P) \ \ (P > 1). \end{array} \right.$

In application of the (EID) criterion, and thanks to Lemma (2.4):

$$\mathbb{E}\big\{\mathbb{1}_A(Z_1^P(x)) \det \Gamma[Z_1^P(x)]\big\} = \mathbb{E}\big\{\mathbb{1}_A(Z_1^P(x)) \det \Big(\sum_{k=1}^{P} M_k(x)\Gamma[Y_k]^t M_k(x)\Big)\big\} = 0. \quad (3)$$

Using Lemma (2.1) in (3), we get:

$$\mathbb{E}\big\{\mathbb{1}_A(Z_1^P(x)) \det \Big(\sum_{k=1}^{P} \alpha^k M_k(x)\Gamma[Y_k]^t M_k(x)\Big)\big\} = 0. \quad (4)$$

Now, integrating (4) on x with respect to π entails:

$$\mathbb{E}\big\{\mathbb{1}_A(z_1) \det \Big(\sum_{k=1}^{P} \alpha^k m_k\Gamma[y_k]^t m_k\Big)\big\} = 0. \quad (5)$$

Since $\mathbb{E}\left\{\sum_{k=1}^{P} \alpha^k \left\|m_k\Gamma[y_k]^t m_k\right\|\right\} \leq E\{\|\Gamma[Y_1]\|\} \sum_{k=1}^{P} (K^2\alpha)^k$ we can let $P \to \infty$ so that:

$$\mathbb{E}\big\{\mathbb{1}_A(z_1) \det \Big(\sum_{k=1}^{\infty} \alpha^k m_k\Gamma[y_k]^t m_k\Big)\big\} = 0. \quad (6)$$

Setting $J(\omega) = \Big(\sum_{k=1}^{\infty} \alpha^k m_k\Gamma[y_k]^t m_k\Big)$ (with $\omega = (z_i, y_i, \Gamma[y_i])_{i\geq 1}$), let us notice that:

$$\begin{aligned} \det J(\omega) &= \det \Big(\sum_{k=1}^{\infty} \alpha^k m_k\Gamma[y_k]^t m_k\Big) \\ &\geq \det \Big(\sum_{k=2}^{\infty} \alpha^k m_k\Gamma[y_k]^t m_k\Big) \\ &= \alpha^{2n} \det \nabla_x F(z_2, y_1) J(\tau\omega)^t \nabla_x F(z_2, y_1). \end{aligned}$$

But the non degeneracy assumption 1) ensures that $\eta\{\det F(z_2, y_1) \neq 0\} = 1$, so that we get $\{\det J(\omega) = 0\} \subset \{\det J(\tau\omega) = 0\}$. Since $\eta\{\det J(\omega) = 0\} = \eta\{\det J(\tau\omega) = 0\}$, then $\{\det J(\omega) = 0\} = \{\det J(\tau\omega) = 0\}$. Lemma (2.3) ensures that $\eta\{\det J = 0\} \in \{0, 1\}$. Using the non degeneracy assumption 2):

$$\eta\left\{\det\left(\sum_{k=1}^{N_0} \alpha^k m_k\Gamma[y_k]^t m_k\right) = 0\right\} < 1.$$

Then, $\eta\{\det J = 0\} < 1$ and so $\eta\{\det J = 0\} = 0$. Finally, (6) implies $\mathbb{E}\{\mathbb{1}_A(z_1)\} = \pi(A) = 0$. $\square$

Proof. Corollary (1.1)

Let us consider the Dirichlet structure :

$$S_\Omega = \big(\Omega, \mathcal{B}(\Omega), \mathbb{1}_\Omega(x)\frac{d\lambda_p(x)}{\lambda_p(\Omega)}, \mathcal{H}^1(\Omega), \Gamma[\phi] = \nabla\phi^t\nabla\phi\big).$$

Since $S_\Omega^{\mathbb{N}^*}$ satisfies the (EID) criterion (cf. [11], p.105), we may use Theorem (1.1) to get the wished result. □

Now let us give few applications.

Application 1. *Let $A \in \mathcal{GL}_n(\mathbb{R})$ be an invertible matrix, let Ω be a bounded and connected open subset of $\mathbb{R}^p$, let $d\mu(x) = \mathbb{1}_\Omega(x)\frac{d\lambda_p(x)}{\lambda_p(\Omega)}$ be the uniform measure on Ω and let $G : \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be globally Lipschitz. Let us given $(Y_i)_{i\geq 1}$ a i.i.d sequence with common distribution μ and let π be an ergodic probability measure of the Markov chain $X_{N+1} = AX_N + G(Y_{N+1})$. If we assume in addition that with π positive probability, $X_N(x)$ converges in distribution toward π then we have the alternative:*
either π is absolutely continuous, either $supp(\pi)$ is contained in a strict vector subspace.

Let us assume, that non-degeneracy condition 2) fails, we have:

$$\det J(\omega) \quad = \quad \det\left(\sum_{k=1}^{\infty}\alpha^k A^{k-1}\Gamma[G(y_k)]^t A^{k-1}\right) = 0.$$

Let us define $H(\omega) = \ker\big(\sum_{k=1}^{\infty}\alpha^k A^{k-1}\Gamma[G(y_k)]^t A^{k-1}\big)$ and let us take $X \in H(\omega)$. We have:

$$\begin{aligned} 0 &= {}^tX\left(\sum_{k=1}^{\infty}\alpha^k A^{k-1}\Gamma[G(y_k)]^t A^{k-1}\right)X \\ &\geq {}^tX\left(\sum_{k=2}^{\infty}\alpha^k A^{k-1}\Gamma[G(y_k)]^t A^{k-1}\right)X \\ &= \alpha\ {}^t(AX)J(\tau\omega)(AX). \end{aligned}$$

From the above equation, and since $\det A \neq 0$, we deduce that $A\left(H(\omega)\right) \subset H(\tau\omega)$ which by ergodicity (2.3) entails that $\dim(H(\omega)) = \dim(H(\tau\omega))$ and so that $AH(\omega) = H(\tau\omega)$. Consequently, $A^{-N}H(\omega) = H(\tau^N\omega)$. Therefore for all $N \geq 1$, $H(\omega)$ is independent from $(\Gamma[G(y_1)], \cdots, \Gamma[G(y_N)])$, this ensures that $H(\omega)$ is a deterministic vector subspace of $\mathbb{R}^n$. Let us fix $(\xi_1, \cdots, \xi_d)$ a deterministic basis of $H(\omega)$, where $d = \dim H(\omega)$. Then, by construction,

for all $1 \leq i \leq d$ and all $N \geq 1$, almost surely:

$$\begin{aligned} 0 &= {}^t\xi_i \sum_{k=1}^{\infty} \alpha^k A^{k-1} \Gamma[G(y_k)]^t A^{k-1} \xi_i \\ &= {}^t\xi_i \sum_{k=1}^{N} \alpha^k A^{k-1} \Gamma[G(y_k)]^t A^{k-1} \xi_i \\ &= {}^t\xi_i \sum_{k=1}^{N} A^{k-1} \Gamma[G(y_k)]^t A^{k-1} \xi_i \\ &= \int_{\mathbb{R}^n} \Gamma[\sum_{j=1}^{n} \xi_i^{(j)} X_N^{(j)}(x)] d\pi_x. \end{aligned} \tag{7}$$

Nevertheless, since Ω is connected, we can prove that the Dirichlet structure $(\Omega, \mathcal{B}(\Omega), d\mu, \mathcal{H}^1(\Omega), \nabla^t\nabla)^{\mathbb{N}_*}$ satisfies the recurrence property :

$$\forall X \in \mathbb{D},\ \mathcal{E}[X] = 0 \Rightarrow X \text{ is constant}.$$

Thus, (7) implies that, π_x almost surely, there exists a finite family $(C_{i,N}(x))_{1 \leq i \leq d}$ such that:

$$\sum_{j=1}^{n} \xi_i^{(j)} X_N^{(j)}(x) = C_{i,N}(x). \tag{8}$$

Let us assume that $X_N(x) \xrightarrow[N \to \infty]{Law} \pi$. Then, letting $N \to \infty$, (8) implies that there exists a finite family of numbers $(C_i)_{1 \leq i \leq d}$ such that π_x almost everywhere:

$$\forall i \in \{1, \cdots, d\},\ \sum_{j=1}^{n} \xi_i^{(j)} x_j = C_i.$$

More geometrically, the support of π is contained in the intersection of d hyperplanes of $\mathbb{R}^n$.

Remark 2.2. *Without this additional assumption we failed in getting an analogous conclusion. For instance, such an assumption is fulfilled when the matrix A contracts, but also in a more general context since we only need the convergence in distribution of X_N^x for some $x \in supp(\pi)$.*

Application 2. *The above method applies to the case of a stochastic differential equation $dX_t = AX_t dt + \sigma.dW_t$ where $A \in \mathcal{M}_n(\mathbb{R})$ is dissipative ($< Ax, x > \leq -c\|x\|^2$), $\sigma \in \mathcal{M}_{n,p}(\mathbb{R})$ and $(W_t)_{1 \leq i \leq p}$ a p-dimensional standard Brownian motion. In particular:*

$$X_t^x = e^{At} \left(\int_0^t e^{-As} \sigma.dW_s + x \right).$$

Discretizing, we may consider the Markov chain:

$$X_{(N+1)T} = e^{AT} X_{NT} + e^{AT} \int_0^T e^{-As} \sigma dW_s^{(N)},$$

where $(W_s^{(i)})_{s\in[0,T]}$ *are independent standard Brownian motions. Then we are exactly in the setting of application (1) and the following alternative holds: either the invariant probability measure is absolutely continuous, either it is supported by a strict vector subspace of* $\mathbb{R}^n$*. Of course, in this Gaussian context, we might get the same result, studying the covariance matrix of* X_N^x*. However the method is more general and could apply to s.d.e. of the form* $dX_t = AX_t dt + \sigma.d\xi_t$*,* ξ_t *being a suitable Levy process.*

Application 3. *Let us be given* $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ *two contractions. Let* $(U_i)_{i\geq 1}$ *be a i.i.d. sequence of uniformly distributed variables in* $[0,1]$*. Let* π *be the invariant probability of the Markov chain* $X_{N+1} = U_{N+1}f(X_N) + (1 - U_{N+1})g(X_N)$*. We fix* f', g' *two Borel representations of the derivatives of* f *and* g*. We assume that:*

1. $\forall x \in \mathbb{R}$*,* $f'(x) \neq 0$ *or* $g'(x) \neq 0$*,*
2. f *and* g *do not fix the same point,*

then π *is absolutely continuous with respect to the Lebesgue measure.*

Let us fix $x \in \mathbb{R}$*, and let us assume that for two distinct numbers* u, v *in* $[0,1]$ *we have:*

$$uf'(x) + (1-u)g'(x) = vf'(x) + (1-v)g'(x) = 0.$$

Then, $f'(x) = g'(x) = 0$*. Hence condition 1) ensures that* $\pi_x \otimes \lambda_y$ *almost everywhere* $yf'(x) + (1-y)g'(x) \neq 0$*. Besides, let us assume that condition 2) of Corollary (1.1) fails. We then have that* π_x *almost everywhere,* $f(x) = g(x)$*. Since supp(*π*) is stable by* f *and* g*, we deduce that* f *and* g *fix the same point which contradicts our second assumption. It follows that the non-degeneracy condition 2) of Corollary (1.1)is fulfilled and that* π *is absolutely continuous with respect to the Lebesgue measure.*

2.2 Building Dirichlet structures satisfying (EID)

Let us be placed in the setting of Theorem (1.1), we wish to exhibit a way to construct in the probability space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \pi)$ a Dirichlet form $\mathcal{E}_\pi$ with square field operator Γ_π such that the structure satisfies the (EID) criterion. More generally, we prove that $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \pi, \mathbb{D}_\pi, \Gamma_\pi)^{\mathbb{N}^*}$ also satisfies (EID). This process of construction enables to create easily Dirichlet structures satisfying (EID) criterion whereas the usual sufficient conditions entailing (EID) are not necessarily fulfilled. Another interest of this process of construction, consists of iterating the square field operator Γ_π in order to give quantitative estimates of $\frac{d\pi}{d\lambda}(x)$, nevertheless we decided to restrict our attention to the absolute continuity and the aforementioned estimates will be performed in a forthcoming paper. We keep the notations adopted in the proof of Theorem (1.1).

Definition 2.1. *Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we set:*

$$\begin{aligned}
\Gamma_\pi[\phi](x) &= \mathbb{E}\left\{\nabla\phi(z_1)J(\omega)^t\nabla\phi(z_1) \mid z_1 = x\right\} \\
&= \nabla\phi(x)\mathbb{E}\{J(\omega) \mid z_1 = x\}^t\nabla\phi(x), \\
\mathcal{E}_\pi[\phi] &= \int_{\mathbb{R}^n} \Gamma_\pi[\phi](x)d\pi_x \\
&= \mathbb{E}\left\{\nabla\phi(z_1)\left(\sum_{k=1}^{\infty}\alpha^k m_k\Gamma[y_k]^t m_k\right)^t\nabla\phi(z_1)\right\}. \qquad (9)
\end{aligned}$$

Theorem 2.1. *The symmetric, bilinear form $\mathcal{E}_\pi[.]$ defined in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is closable.*

Proof. Let ϕ_N be sequence of maps in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfying:

(a) $\int_{\mathbb{R}^n}\phi_N^2(x)d\pi_x \xrightarrow[N\to\infty]{} 0$,

(b) $\mathcal{E}_\pi[\phi_N - \phi_M] \xrightarrow[N\to\infty,\, M\to\infty]{} 0$.

In order to prove the closability of the form, we must prove that $\mathcal{E}_\pi[\phi_N] \xrightarrow[N\to\infty]{} 0$.

For $N \geq 1$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ we set:

$$\mathcal{E}_\pi^N[\phi] = \mathbb{E}\left\{\nabla\phi(z_1)\left(\sum_{k=1}^{N}\alpha^k m_k\Gamma[y_k]^t m_k\right)^t\nabla\phi(z_1)\right\} \leq \mathcal{E}_\pi[\phi].$$

Then for all $P, Q \in \mathbb{N}^*$, setting $C = \sup_k\sqrt{\mathcal{E}_\pi[\phi_k]}$, we have:

$$\begin{aligned}
\mathcal{E}_\pi[\phi_P] &\leq \left|\mathcal{E}_\pi[\phi_P] - \mathcal{E}_\pi^N[\phi_P]\right| + \mathcal{E}_\pi^N[\phi_P] \\
&\leq \left|\mathcal{E}_\pi[\phi_P] - \mathcal{E}_\pi^N[\phi_P]\right| + \left|\mathcal{E}_\pi^N[\phi_P] - \mathcal{E}_\pi^N[\phi_Q]\right| + \mathcal{E}_\pi^N[\phi_Q] \\
&\leq \left|\mathcal{E}_\pi[\phi_P] - \mathcal{E}_\pi^N[\phi_P]\right| + \sqrt{\mathcal{E}_\pi^N[\phi_P - \phi_Q]}\sqrt{\mathcal{E}_\pi^N[\phi_P + \phi_Q]} + \mathcal{E}_\pi^N[\phi_Q] \\
&\leq \left|\mathcal{E}_\pi[\phi_P] - \mathcal{E}_\pi^N[\phi_P]\right| + 2C\sqrt{\mathcal{E}_\pi[\phi_P - \phi_Q]} + \mathcal{E}_\pi^N[\phi_Q]. \\
\mathcal{E}_\pi[\phi_P] &\leq \limsup_{N\to\infty}\left|\mathcal{E}_\pi[\phi_P] - \mathcal{E}_\pi^N[\phi_P]\right| + \limsup_{Q\to\infty}\mathcal{E}_\pi^N[\phi_Q] + 2C\limsup_{Q\to\infty}\sqrt{\mathcal{E}_\pi[\phi_P - \phi_Q]}.
\end{aligned} \qquad (10)$$

- According to (9), for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have $\mathcal{E}_\pi^N[\phi] \xrightarrow[N\to\infty]{} \mathcal{E}_\pi[\phi]$. This implies that $\limsup_{N\to\infty}\left|\mathcal{E}_\pi[\phi_P] - \mathcal{E}_\pi^N[\phi_P]\right| = 0$.
- According to assumption b), we have:

$$\mathcal{E}_\pi^N[\phi_P - \phi_Q] \leq \mathcal{E}_\pi[\phi_P - \phi_Q] \xrightarrow[P,Q\to\infty]{} 0. \qquad (11)$$

Taking into account that $\int_{\mathbb{R}^n} \Gamma[\phi(Z_1^N(x))]d\pi_x \leq \frac{1}{\alpha^N}\mathcal{E}_\pi^N[\phi]$ and using assumption b) and (11):

$$\int_{\mathbb{R}^n} \mathbb{E}\left\{\left(\phi_P(Z_1^N(x))\right)^2\right\} d\pi_x \xrightarrow[P\to\infty]{} 0, \tag{12}$$
$$\int_{\mathbb{R}^n} \mathbb{E}\left\{\Gamma[\phi_P(Z_1^N(x)) - \phi_Q(Z_1^N(x))]\right\} d\pi_x \xrightarrow[P,Q\to\infty]{} 0.$$

Using (12) we may construct a subsequence N_k such that:

$$\int_{\mathbb{R}^n} \mathbb{E}\left\{\sum_{k=1}^{\infty}\left(\phi_{N_k}(Z_1^N(x))\right)^2\right\} d\pi_x < \infty, \tag{13}$$
$$\int_{\mathbb{R}^n} \sum_{k=1}^{\infty}\left|\mathbb{E}\left\{\Gamma[\phi_{N_{k+1}}(Z_1^N(x))]\right\} - \mathbb{E}\left\{\Gamma[\phi_{N_k}(Z_1^N(x))]\right\}\right| d\pi_x < \infty,$$
$$\int_{\mathbb{R}^n} \sum_{k=1}^{\infty} \mathcal{E}[\phi_{N_{k+1}}(Z_1^N(x)) - \phi_{N_k}(Z_1^N(x))]d\pi_x < \infty.$$

Consequently, π_x-almost everywhere, $\mathbb{E}\left\{\left(\phi_{N_k}(Z_1^N(x))\right)^2\right\} \xrightarrow[k\to\infty]{} 0$ and $\mathcal{E}[\phi_{N_k}(Z_1^N(x)) - \phi_{N_{k'}}(Z_1^N(x))] \xrightarrow[k,k'\to\infty]{} 0$. Nevertheless, $\phi_{N_k}(Z_1^N(x))$ belongs to the domain $\mathbb{D}$ of the Dirichlet structure $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p), \mu, \mathbb{D}, \Gamma)^{\mathbb{N}^*}$ which is complete for the Dirichlet norm. Hence, π_x-almost everywhere:

$$\mathcal{E}[\phi_{N_k}(Z_1^N(x))] \xrightarrow[k\to\infty]{} 0. \tag{14}$$

According to (13):

$$\begin{aligned}\mathcal{E}[\phi_{N_k}(Z_1^N(x))] &\leq \sum_{q=k}^{\infty}\left|\mathbb{E}\left\{\Gamma[\phi_{N_{q+1}}(Z_1^N(x))]\right\} - \mathbb{E}\left\{\Gamma[\phi_{N_q}(Z_1^N(x))]\right\}\right| \\ &\leq \sum_{q=1}^{\infty}\left|\mathbb{E}\left\{\Gamma[\phi_{N_{q+1}}(Z_1^N(x))]\right\} - \mathbb{E}\left\{\Gamma[\phi_{N_q}(Z_1^N(x))]\right\}\right| \in L^1(\pi).\end{aligned}$$

Hence by dominated convergence, (14) entails:

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} \mathcal{E}[\phi_{N_k}(Z_1^N(x))]d\pi_x = 0.$$

Up to any extraction, assumptions a), b) remain. Thus, we have proved that:

$$\lim_{Q\to\infty}\int_{\mathbb{R}^n} \mathcal{E}[\phi_Q(Z_1^N(x))]d\pi_x = 0.$$

Taking into account that $\mathcal{E}_\pi^N[\phi_Q] \leq \int_{\mathbb{R}^n} \mathcal{E}[\phi_Q(Z_1^N(x))]d\pi_x$, we have established that

$$\limsup_{Q\to\infty} \mathcal{E}_\pi^N[\phi_Q] = 0.$$

- Thanks to (10), we have:

$$\mathcal{E}_\pi[\phi_P] \leq 2C \limsup_{Q\to\infty} \sqrt{\mathcal{E}_\pi[\phi_P - \phi_Q]} \xrightarrow[P\to\infty]{} 0.$$

□

Theorem 2.2. *Let $\mathbb{D}_\pi$ be the closure of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ for the Dirichlet form $\mathcal{E}_\pi$. Under the non-degeneracy assumptions of Theorem (1.1), the resulting Dirichlet structure:*

$$S_\pi = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \pi, \mathbb{D}_\pi, \Gamma_\pi)$$

satisfies the (EID) criterion. Moreover, $S_\pi^{\mathbb{N}^*}$ also satisfies the (EID) criterion.

Proof. Let us be given $(\phi_1, \cdots, \phi_k) \in \mathbb{D}_\pi^k$ and let A be a Borel subset of $\mathbb{R}^k$, negligible for the Lebesgue measure λ_k. Since $\mathbb{D}_\pi$ is the closure of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ for $\mathcal{E}_\pi$, there exist k sequences $(\phi_1^{(N)}, \cdots, \phi_k^{(N)})$ in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ such that for all $i \in \{1, \cdots, k\}$ and $R \geq 1$:

$$\int_{\mathbb{R}^n} \left(\phi_i(x) - \phi_i^{(N)}(x)\right)^2 d\pi_x \xrightarrow[N\to\infty]{} 0, \tag{15}$$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathbb{E}\left\{\nabla\left(\phi_i^{(P)}(z_1) - \phi_i^{(Q)}(z_1)\right) J(\omega)^t \nabla\left(\phi_i^{(P)}(z_1) - \phi_i^{(Q)}(z_1)\right)\right\} \geq \\
&\int_{\mathbb{R}^n} \mathbb{E}\left\{\nabla\left(\phi_i^{(P)}(z_1) - \phi_i^{(Q)}(z_1)\right)\left(\sum_{j=1}^R \alpha^j m_j \Gamma[y_j]^t m_j\right)^t \nabla\left(\phi_i^{(P)}(z_1) - \phi_i^{(Q)}(z_1)\right)\right\} \geq \\
&\frac{1}{\alpha^R}\int_{\mathbb{R}^n} \mathbb{E}\left\{\Gamma[\phi_i^{(P)}(Z_1^R(x)) - \phi_i^{(Q)}(Z_1^R(x))]\right\} d\pi_x \xrightarrow[P,Q\to\infty]{} 0.
\end{aligned} \tag{16}$$

Using both (15) and (16), for all $i \in \{1, \cdots, k\}$, π_x-almost every where: $\phi_i(Z_1^R(x)) \in \mathbb{D}$. We set $\phi(x) = (\phi_1(x), \cdots, \phi_k(x))$. Using the (EID) criterion which is valid in the structure S, π_x-almost every where:

$$\mathbb{E}\left\{\mathbb{1}_A(\phi_1(Z_1^R(x)), \cdots, \phi_k(Z_1^R(x))) \det\left(\nabla\phi(Z_1^R(x))\Gamma[Z_1^R(x)]^t \nabla\phi(Z_1^R(x))\right)\right\} = 0.$$

Integrating x with respect to π and using Lemma(2.1) we have:

$$\mathbb{E}\left\{\mathbb{1}_A(\phi_1(z_1), \cdots, \phi_k(z_1)) \det\left(\nabla\phi(z_1)\left(\sum_{j=1}^R \alpha^j m_j \Gamma[y_j]^t m_j\right)^t \nabla\phi(z_1)\right)\right\} = 0.$$

Letting $R \to \infty$ we get:

$$\begin{aligned}
\mathbb{E}\left\{\mathbb{1}_A(\phi_1(z_1), \cdots, \phi_k(z_1)) \det\left(\nabla\phi(z_1) J(\omega)^t \nabla\phi(z_1)\right)\right\} &= \\
\mathbb{E}\left\{\mathbb{1}_A(\phi_1(z_1), \cdots, \phi_k(z_1)) \mathbb{1}_{\{\det(\nabla\phi(z_1)J(\omega)^t\nabla\phi(z_1))>0\}}\right\} &= 0.
\end{aligned} \tag{17}$$

Using [8] (p.188 Lemma 1.1.6.2):

$$\{\det \mathbb{E}\{J(\omega) \mid z_1\} = 0\} \subset \{\det J(\omega) = 0\} = \emptyset.$$

That is why:

$$\begin{aligned}
\{\det \Gamma_\pi[\phi](z_1) > 0\} &= \left\{\det\left(\nabla\phi(z_1)\mathbb{E}\{J(\omega) \mid z_1\}\,{}^t\nabla\phi(z_1)\right) > 0\right\} \\
&= \left\{\det\left(\nabla\phi(z_1)^t\nabla\phi(z_1)\right) > 0\right\} \\
&= \left\{\det\left(\nabla\phi(z_1)J(\omega)^t\nabla\phi(z_1)\right) > 0\right\}.
\end{aligned}$$

Finally, (17) leads to

$$\int_{\mathbb{R}^n} \mathbb{1}_A(\phi(x))\mathbb{1}_{\{\det\Gamma_\pi[\phi]>0\}}(x)d\pi_x = 0.$$

In order to prove that $S_\pi^{\mathbb{N}^*}$ satisfies the (EID) criterion, it is enough proving that for all $P \geq 1$, S_π^P satisfies (EID). In this case, we can use a coupling argument by saying that π^P is an ergodic invariant measure of the coupled Markov chain :

$$W_{N+1}(x_1, \cdots, x_P) = (X_{N+1}^{(1)}(x_1), \cdots, X_{N+1}^{(P)}(x_P)) = (F(X_N^{(1)}(x_1), Y_{N+1}^{(1)}), \cdots, F(X_N^{(P)}(x_P), Y_{N+1}^{(P)})).$$

(where $(Y_i^{(j)})$ is still i.i.d.)

The Jacobian matrix of the two mappings:

1. $(x_1, \cdots, x_P) \longrightarrow (F(x_1, Y_1^1), \cdots, F(x_1, Y_1^P))$,
2. $(Y_1, \cdots, Y_P) \longrightarrow (F(x_1, Y_1^1), \cdots, F(x_1, Y_1^P))$,

are bloc-diagonal. So the non-degeneracy conditions of Theorem(1.1) remain for the coupled Markov chain $W_N(x_1, \cdots, x_P)$. Besides, if we apply the same process of construction of S_π to the coupled Markov chain, thanks to the bloc-diagonal forms of the Jacobian matrix, the Dirichlet structure obtained coincide with S_π^P. Since the conclusions of Theorem (2.2) hold, S_π^P checks the (EID) property. Finally, letting $P \to \infty$, $S_\pi^{\mathbb{N}^*}$ also satisfies (EID). □

3 Absolute continuity of ergodic measures, continuous case

Proof. (Theorem 1.2)
We wish to apply the same strategy as those previously used in the proof of Theorem 1.1. Let us be given for that, $T > 0$ and a family of independent Brownian motions $(\mathcal{W}_t^i)_{t\in[0,T]}$. We will note Γ_i the Malliavin calculus operator with respect to the Wiener space $\sigma(\mathcal{W}_t^i;\, t \in [0, T])$. ($\Gamma_i$ plays exactly the role of "∇_{Y_i}" in the proof of Theorem 1.1). Setting $\phi_i(x)$ the solution of the s.d.e. $dX_t^i = b(X_t)^i + \sigma(X_t^i)d\mathcal{W}_t^i$ starting from x at the time T, we consider the Markov chain:

$$W_{N+1}^x = \phi_{N+1}(W_N^x),\ W_0 = x.$$

Since the s.d.e. 1 under consideration is homogeneous, and thanks to the flow property, we can assert that π is invariant for the chain (W_n). Besides, thanks to [7], we know that $x \longrightarrow \phi_i(x)$

is a Lipschitz mapping, which Jacobian matrix almost surely and almost everywhere belongs to $\mathcal{G}L_n(\mathbb{R})$ (the set of invertible matrix of $\mathbb{R}^n$). Hence we apply the same procedure as those used in the proof of Theorem 1.1, the starting equation (consequence of (EID) in the Wiener space) being :

$$\mathbb{E}\left\{\chi_A(z_1)\det\left(\Gamma_1[z_1]+\nabla\phi_1(z_2)\Gamma_2[z_2]^t\nabla\phi_1(z_2)\Gamma_2+\cdots+\right.\right.$$
$$\left.\left.\nabla\phi_1(z_2)\cdots\nabla\phi_p(z_{p+1})\Gamma_{p+1}[z_{p+1}]^t\nabla\phi_p(z_{p+1})\cdots{}^t\nabla\phi_1(z_2)\right)\right\}=0,$$

where A is a negligible set and where $(z_i)_{i\geq 1}$ are defined in the same way as in the Lemma 2.2, in particular $z_i=\phi_{i+1}(z_{i+1})$. □

Proof. (Corollary 1.2)

In order to apply Theorem 1.2, it is enough checking that $\pi\left\{x \mid \exists t>0 \text{ with } \det\Gamma[X_t^x]>0\right\}>0$. But according to [6] or (Theorem 2.3.1 p.127 [15]), starting from x where $\det\sigma(x)^t\sigma(x)>0$, we know that for all $t>0$, $\det\Gamma[X_t^x]>0$. (X_t^x being the solution of the s.d.e. (1)). That is why:

$$\left\{x \mid \det(\sigma(x)^t\sigma(x))>0\right\}\subset\left\{x \mid \exists t>0 \text{ with } \det\Gamma[X_t^x]>0\right\}.$$

Hence:

$$\pi\left\{x \mid \exists t>0 \text{ with } \det\Gamma[X_t^x]>0\right\}>0.$$

□

References

[1] S. Albeverio and M. Röckner. Classical Dirichlet forms on topological vector spaces–closability and a Cameron-Martin formula. *Journal of Functional Analysis*, 88(2):395–436, 1990.

[2] A. Beurling and J. Deny. Espaces de Dirichlet I. le cas élémentaire. *Acta Mathematica*, page 203, 1958.

[3] V.I. Bogachev, N.V. Krylov, and M. Röckner. Elliptic and parabolic equations for measures. *Russian Mathematical Surveys*, 64:973, 2009.

[4] N. Bouleau. *Error calculus for finance and physics: The language of Dirichlet forms*, volume 37. de Gruyter, 2003.

[5] N. Bouleau and L. Denis. Energy image density property and the lent particle method for Poisson measures. *Journal of Functional Analysis*, 257(4):1144–1174, 2009.

[6] N. Bouleau and F. Hirsch. Formes de Dirichlet générales et densité des variables aléatoires réelles sur l'espace de Wiener. *Journal of Functional analysis*, 69(2):229–259, 1986.

[7] N. Bouleau and F. Hirsch. On the derivability, with respect to the initial data, of the solution of a stochastic differential equation with Lipschitz coefficients. *Séminaire de Théorie du Potentiel Paris, No. 9*, pages 39–57, 1989.

[8] N. Bouleau and F. Hirsch. Dirichlet forms and analysis on Wiener space, vol. 14 of de Gruyter Studies in Mathematics, 1991.

[9] A. Coquio. Formes de Dirichlet sur l'espace canonique de Poisson et application aux équations différentielles stochastiques. *Ann. Inst. Henri Poincaré*, 19(1):1–36, 1993.

[10] A. Coquio and JB Gravereaux. Calcul de Malliavin et régularité de la densité d'une probabilité invariante d'une chaîne de Markov. volume 28, pages 431–478. Elsevier, 1992.

[11] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19. Walter de Gruyter, 2010.

[12] JB Gravereaux. Calcul de Malliavin et probabilité invariante d'une chaîne de Markov. *Ann. Inst. Henri Poincaré*, 24(2):159–188, 1988.

[13] Z.M. Ma and M. Röckner. *Introduction to the theory of (non-symmetric) Dirichlet forms*. Springer-Verlag Berlin, 1992.

[14] P. Malliavin. *Stochastic analysis*, volume 313. Springer Verlag, 1997.

[15] D. Nualart. *The Malliavin calculus and related topics*. Springer, 1995.

[16] S. Song. Admissible vectors and their associated Dirichlet forms. *Potential Analysis*, 1(4):319–336, 1992.