A sequent calculus with procedure calls

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Abstract

In this paper, we extend the sequent calculus LKF [LM09] into a calculus $\mathsf{LK}(\mathcal{T})$, allowing calls to a decision procedure. We prove cut-elimination of $\mathsf{LK}(\mathcal{T})$.

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1 The sequent calculus $\mathsf{LK}(\mathcal{T})$

The sequent calculus $\mathsf{LK}(\mathcal{T})$ manipulates the formulae of first-order logic, with the specificity that every predicate symbol is classified as either positive or negative, and boolean connectives come in two versions: positive and negative.

Definition 1 (Formulae) *Literals* are predicates (a predicate symbol applied to a list of first-order terms) or negations of predicates. Literals are equipped with the obvious involutive negation, and the negation of a literal l is denoted $l^\perp$.

Let $\mathcal{P}$ be the set of literals that are either predicates with positive predicate symbols, or negations of predicates with negative predicate symbols.

$$\begin{array}{lll} \text{Positive formulae} & P & ::= \ p \mid A\wedge^+B \mid A\vee^+B \mid \exists xA \\ \text{Negative formulae} & N & ::= \ p^\perp \mid A\wedge^-B \mid A\vee^-B \mid \forall xA \\ \text{Formulae} & A,B & ::= \ P \mid N \end{array}$$

where p ranges over $\mathcal{P}$.

Definition 2 (Negation) Negation is extended from literals to all formulae:

$$\begin{array}{|lcl|lcl|} \hline (p)^\perp & := & p^\perp & (p^\perp)^\perp & := & p \\ (A\wedge^+B)^\perp & := & A^\perp\vee^-B^\perp & (A\wedge^-B)^\perp & := & A^\perp\vee^+B^\perp \\ (A\vee^+B)^\perp & := & A^\perp\wedge^-B^\perp & (A\vee^-B)^\perp & := & A^\perp\wedge^+B^\perp \\ (\exists xA)^\perp & := & \forall xA^\perp & (\forall xA)^\perp & := & \exists xA^\perp \\ \hline \end{array}$$

Definition 3 ($\mathsf{LK}(\mathcal{T})$) The sequent calculus $\mathsf{LK}^p(\mathcal{T})$ manipulates two kinds of sequents:

$$\begin{array}{ll} \text{Focused sequents} & \Gamma \vdash_{\mathcal{T}} [P] \\ \text{Unfocused sequents} & \Gamma \vdash_{\mathcal{T}} \Delta \end{array}$$

where Γ is a multiset of negative formulae and positive literals, Δ is a multiset of formulae, and P is said to be in the *focus* of the (focused) sequent. By $\mathsf{lit}(\Gamma)$ we denote the sub-multiset of Γ consisting of its literals.

The rules of $\mathsf{LK}^p(\mathcal{T})$, given in Figure 1, are of three kinds: synchronous rules, asynchronous rules, and structural rules. These correspond to three alternating phases in the proof-search process that is described by the rules.

If S is a set of literals, $\mathcal{T}(S)$ is the call to the decision procedure on the conjunction of all literals of S. It holds if the procedure returns UNSAT.

2 Admissible rules

Definition 4 (Assumptions on the procedure)

We assume that the procedure calls satisfy the following properties:

Weakening If $\mathcal{T}(S)$ then $\mathcal{T}(S,S')$.

Contraction If $\mathcal{T}(S,A,A)$ then $\mathcal{T}(S,A)$.

Instantiation If $\mathcal{T}(S)$ then $\mathcal{T}(\left\{{}^t\!/\!_x\right\}S)$.

Consistency If $\mathcal{T}(S,p)$ and $\mathcal{T}(S,p^\perp)$ then $\mathcal{T}(S)$.

where S is a set of literals.

Lemma 1 (Admissibility of weakening and contraction)

The following rules are admissible in $\mathsf{LK}(\mathcal{T})$.

$$\frac{\Gamma \vdash [B]}{\Gamma, A \vdash [B]} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

$$\frac{\Gamma, A, A \vdash [B]}{\Gamma, A \vdash [B]} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Proof: By induction on the derivation of the premiss. $\square$

Synchronous rules

$$\frac{\Gamma \vdash [A] \qquad \Gamma \vdash [B]}{\Gamma \vdash [A \wedge^{+} B]} \qquad \frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^{+} A_2]} \qquad \frac{\Gamma \vdash [\{{}^{t}\!/\!{}_{x}\} A]}{\Gamma \vdash [\exists x A]}$$

$$\frac{}{\Gamma, p \vdash [p]}\ p \text{ positive literal} \qquad \frac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp})}{\Gamma \vdash [p]}\ p \text{ positive literal}$$

$$\frac{\Gamma \vdash N}{\Gamma \vdash [N]}\ N \text{ negative}$$

Aynchronous rules

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge^{-} B, \Delta} \qquad \frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \vee^{-} A_2, \Delta} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash (\forall x A), \Delta}\ x \notin \mathsf{FV}(\Gamma, \Delta)$$

$$\frac{\Gamma, A^{\perp} \vdash \Delta}{\Gamma \vdash A, \Delta}\ A \text{ positive or literal}$$

Structural rules

$$\frac{\Gamma, P^{\perp} \vdash [P]}{\Gamma, P^{\perp} \vdash}\ P \text{ positive} \qquad \frac{\mathcal{T}(\mathsf{lit}(\Gamma))}{\Gamma \vdash}$$

Figure 1: System $\mathsf{LK}(\mathcal{T})$

Lemma 2 (Admissibility of instantiation) *The following rules are admissible in $\mathsf{LK}(\mathcal{T})$.*

$$\frac{\Gamma \vdash [B]}{\{{}^{t}\!/\!{}_{x}\}\Gamma \vdash [\{{}^{t}\!/\!{}_{x}\} B]} \qquad \frac{\Gamma \vdash \Delta}{\{{}^{t}\!/\!{}_{x}\}\Gamma \vdash \{{}^{t}\!/\!{}_{x}\}\Delta}$$

Proof: By induction on the derivation of the premiss. □

3 Invertibility of the asynchronous phase

Lemma 3 (Invertibility of asynchronous rules)
All asynchronous rules are invertible in $\mathsf{LK}(\mathcal{T})$.

Proof: By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of $A \wedge^{-} B$: by case analysis on the last rule actually used
  - $$\frac{\Gamma \vdash A \wedge^{-} B, C, \Delta' \quad \Gamma \vdash A \wedge^{-} B, D, \Delta'}{\Gamma \vdash A \wedge^{-} B, C \wedge^{-} D, \Delta'}$$
    By induction hypothesis we get
    $$\frac{\Gamma \vdash A, C, \Delta' \qquad \Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, C \wedge^{-} D, \Delta'} \text{ and } \frac{\Gamma \vdash B, C, \Delta' \quad \Gamma \vdash B, D, \Delta'}{\Gamma \vdash B, C \wedge^{-} D, \Delta'}$$
  - $$\frac{\Gamma \vdash A \wedge^{-} B, C, D, \Delta'}{\Gamma \vdash A \wedge^{-} B, C \vee^{-} D, \Delta'}$$
    By induction hypothesis we get $\dfrac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^{-} D, \Delta'}$ and $\dfrac{\Gamma \vdash B, C, D, \Delta'}{\Gamma \vdash B, C \vee^{-} D, \Delta'}$
  - $$\frac{\Gamma \vdash A \wedge^{-} B, C, \Delta'}{\Gamma \vdash A \wedge^{-} B, (\forall x C), \Delta'}\ x \notin \mathsf{FV}(\Gamma, \Delta', A \wedge^{-} B)$$

By induction hypothesis we get

$$\frac{\Gamma \vdash A, C, \Delta'}{\Gamma \vdash A, (\forall x C), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta', A) \text{ and } \frac{\Gamma \vdash B, C, \Delta'}{\Gamma \vdash B, (\forall x C), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta', B)$$

- $\dfrac{\Gamma, C^{\perp} \vdash A \wedge^{-} B, \Delta'}{\Gamma \vdash A \wedge^{-} B, C, \Delta'}\, C$ positive or literal

  By induction hypothesis we get

  $$\frac{\Gamma, C^{\perp} \vdash A, \Delta'}{\Gamma \vdash A, C, \Delta'}\, C \text{ positive or literal and } \frac{\Gamma, C^{\perp} \vdash B, \Delta'}{\Gamma \vdash B, C, \Delta'}\, C \text{ positive or literal}$$

- Inversion of $A \vee^{-} B$
  - $\dfrac{\Gamma \vdash A \vee^{-} B, C, \Delta' \quad \Gamma \vdash A \vee^{-} B, D, \Delta'}{\Gamma \vdash A \vee^{-} B, C \wedge^{-} D, \Delta'}$

    By induction hypothesis we get $\dfrac{\Gamma \vdash A, B, C, \Delta' \quad \Gamma \vdash A, B, D, \Delta'}{\Gamma \vdash A, B, C \wedge^{-} D, \Delta'}$
  - $\dfrac{\Gamma \vdash A \vee^{-} B, C, D, \Delta'}{\Gamma \vdash A \vee^{-} B, C \vee^{-} D, \Delta'}$

    By induction hypothesis we get $\dfrac{\Gamma \vdash A, B, C, D, \Delta'}{\Gamma \vdash A, B, C \vee^{-} D, \Delta'}$
  - $\dfrac{\Gamma \vdash A \vee^{-} B, C, \Delta'}{\Gamma \vdash A \vee^{-} B, (\forall x C), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta')$

    By induction hypothesis we get $\dfrac{\Gamma \vdash A, B, C, \Delta'}{\Gamma \vdash A, B, (\forall x C), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta')$
  - $\dfrac{\Gamma, C^{\perp} \vdash A \vee^{-} B, \Delta'}{\Gamma \vdash A \vee^{-} B, C, \Delta'}\, C$ positive or literal

    By induction hypothesis we get $\dfrac{\Gamma, C^{\perp} \vdash A, B, \Delta'}{\Gamma \vdash A, B, C, \Delta'}\, C$ positive or literal
- Inversion of $\forall x A$
  - $\dfrac{\Gamma \vdash (\forall x A), C, \Delta' \quad \Gamma \vdash (\forall x A), D, \Delta'}{\Gamma \vdash (\forall x A), C \wedge^{-} D, \Delta'}$

    By induction hypothesis we get $\dfrac{\Gamma \vdash A, C, \Delta'}{\Gamma \vdash A, C \wedge^{-} D, \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta')$ and $\Gamma \vdash A, D, \Delta'$
  - $\dfrac{\Gamma \vdash (\forall x A), C, D, \Delta'}{\Gamma \vdash (\forall x A), C \vee^{-} D, \Delta'}$

    By induction hypothesis we get $\dfrac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^{-} D, \Delta'}$
  - $\dfrac{\Gamma \vdash (\forall x A), D, \Delta'}{\Gamma \vdash (\forall x A), (\forall x D), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta')$

    By induction hypothesis we get $\dfrac{\Gamma \vdash A, C, \Delta'}{\Gamma \vdash A, (\forall x C), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta')$
  - $\dfrac{\Gamma, C^{\perp} \vdash (\forall x A), \Delta'}{\Gamma \vdash (\forall x A), C, \Delta'}\, C$ positive or literal

    By induction hypothesis we get $\dfrac{\Gamma, C^{\perp} \vdash A, \Delta'}{\Gamma \vdash A, C, \Delta'}\, C$ positive or literal
- Inversion of literals and positive formulae (A)
  - $\dfrac{\Gamma \vdash A, C, \Delta' \quad \Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, C \wedge^{-} D, \Delta'}$

    By induction hypothesis we get $\dfrac{\Gamma, A^{\perp} \vdash C, \Delta' \quad \Gamma, A^{\perp} \vdash D, \Delta'}{\Gamma, A^{\perp} \vdash C \wedge^{-} D, \Delta'}$

– $\dfrac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^- D, \Delta'}$

By induction hypothesis $\dfrac{\Gamma, A^\perp \vdash C, D, \Delta'}{\Gamma, A^\perp \vdash C \vee^- D, \Delta'}$

– $\dfrac{\Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, (\forall x D), \Delta'}\ x \notin \mathsf{FV}(\Gamma, \Delta')$

By induction hypothesis we get $\dfrac{\Gamma, A^\perp \vdash C, \Delta'}{\Gamma, A^\perp \vdash (\forall x C), \Delta'}\ x \notin \mathsf{FV}(\Gamma, \Delta')$

– $\dfrac{\Gamma, B^\perp \vdash A, \Delta'}{\Gamma \vdash A, B, \Delta'}$ B positive or literal

By induction hypothesis we get $\dfrac{\Gamma, A^\perp, B^\perp \vdash \Delta'}{\Gamma, A^\perp \vdash B, \Delta'}$ B positive or literal

□

4 Cut-elimination

Theorem 4 (cut_1 and cut_2) *The following rules are admissible in $\mathit{LK}(\mathcal{T})$.*

$$\dfrac{\mathcal{T}(\mathit{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash \Delta}{\Gamma \vdash \Delta}\,\mathit{cut}_1 \qquad \dfrac{\mathcal{T}(\mathit{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [B]}{\Gamma \vdash [B]}\,\mathit{cut}_2$$

Proof: By simultaneous induction on the derivation of the right premiss.

We reduce cut_8 by case analysis on the last rule used to prove the right premiss.

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \dfrac{\Gamma, p \vdash B, \Delta \quad \Gamma, p \vdash C, \Delta}{\Gamma, p \vdash B \wedge^- C, \Delta}}{\Gamma \vdash B \wedge^- C, \Delta}\,\mathsf{cut}_1$$

reduces to

$$\dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B, \Delta}{\Gamma \vdash B, \Delta}\,\mathsf{cut}_1 \quad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash C, \Delta}{\Gamma \vdash C, \Delta}\,\mathsf{cut}_1}{\Gamma \vdash B \wedge^- C, \Delta}$$

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \dfrac{\Gamma, p \vdash B_1, B_2, \Delta}{\Gamma, p \vdash B_1 \vee^- B_2, \Delta}}{\Gamma \vdash B_1 \vee^- B_2, \Delta}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B_1, B_2, \Delta}{\Gamma \vdash B_1, B_2, \Delta}\,\mathsf{cut}_1}{\Gamma \vdash B_1 \vee^- B_2, \Delta}$$

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \dfrac{\Gamma, p \vdash B, \Delta}{\Gamma, p \vdash \forall x B, \Delta}}{\Gamma \vdash \forall x B, \Delta}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B, \Delta}{\Gamma \vdash B, \Delta}\,\mathsf{cut}_1}{\Gamma \vdash \forall x B, \Delta}$$

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \dfrac{\Gamma, p, B^\perp \vdash \Delta}{\Gamma, p \vdash B, \Delta}}{\Gamma \vdash B, \Delta}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma, B^\perp), p^\perp) \quad \Gamma, p, B^\perp \vdash \Delta}{\Gamma, B^\perp \vdash \Delta}\,\mathsf{cut}_1}{\Gamma \vdash B, \Delta}$$

We have $\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp, B^\perp)$ as we assume the procedure to satisfy weakening.

If $P^\perp \in (\Gamma, p)$,

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \dfrac{\Gamma, p \vdash [P]}{\Gamma, p \vdash}}{\Gamma \vdash}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp) \quad \Gamma, p^\perp \vdash [P]}{\Gamma \vdash [P]}\,\mathsf{cut}_2}{\Gamma \vdash}$$

as $P^\perp \in (\Gamma)$.

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \qquad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p)}{\Gamma, p \vdash}}{\Gamma \vdash}\,\mathsf{cut}_1 \qquad \text{reduces to} \qquad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma))}{\Gamma \vdash}$$

using the assumption of consistency.

We reduce cut_2 again by case analysis on the last rule used to prove the right premiss.

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \qquad \dfrac{\Gamma, p \vdash [B] \quad \Gamma, p \vdash [C]}{\Gamma, p \vdash [B \wedge^+ C]}}{\Gamma \vdash [B \wedge^+ C]}\,\mathsf{cut}_2$$

reduces to

$$\dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash [B]}{\Gamma \vdash [B]}\,\mathsf{cut}_2 \qquad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash [C]}{\Gamma \vdash [C]}\,\mathsf{cut}_2}{\Gamma \vdash [B \wedge^+ C]}$$

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \qquad \dfrac{\Gamma, p \vdash [B_i]}{\Gamma, p \vdash [B_1 \vee^+ B_2]}}{\Gamma \vdash [B_1 \vee^+ B_2]}\,\mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash [B_i]}{\Gamma \vdash [B_i]}\,\mathsf{cut}_2}{\Gamma \vdash [B_1 \vee^+ B_2]}$$

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \qquad \dfrac{\Gamma, p \vdash [\{{}^{t}\!/\!{}_{x}\} B]}{\Gamma, p \vdash [\exists x B]}}{\Gamma \vdash [\exists x B]}\,\mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash [\{{}^{t}\!/\!{}_{x}\} B]}{\Gamma \vdash [\{{}^{t}\!/\!{}_{x}\} B]}\,\mathsf{cut}_2}{\Gamma \vdash [\exists x B]}$$

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \qquad \dfrac{\Gamma, p \vdash N}{\Gamma, p \vdash [N]}}{\Gamma \vdash [N]}\,\mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash N}{\Gamma \vdash N}\,\mathsf{cut}_1}{\Gamma \vdash [N]}$$

If $p' \in \Gamma, p$,

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \qquad \dfrac{}{\Gamma, p \vdash [p']}}{\Gamma \vdash [p']}\,\mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{}{\Gamma \vdash [p']} \qquad \text{if} \qquad p' \in \Gamma$$

$$\text{reduces to} \qquad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp})}{\Gamma \vdash [p']} \qquad \text{if} \qquad p' = p$$

Finally,

$$\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}) \qquad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p, {p'}^{\perp})}{\Gamma, p \vdash [p']}}{\Gamma \vdash [p']}\,\mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), {p'}^{\perp})}{\Gamma \vdash [p']}$$

since weakening gives $\mathcal{T}(\mathsf{lit}(\Gamma), p^{\perp}, {p'}^{\perp})$ and consistency then gives $\mathcal{T}(\mathsf{lit}(\Gamma), {p'}^{\perp})$. □

Theorem 5 (cut_3, cut_4 and cut_5) *The following rules are admissible in* $\mathsf{LK}(\mathcal{T})$.

$$\dfrac{\Gamma \vdash [A] \quad \Gamma \vdash A^{\perp}, \Delta}{\Gamma \vdash \Delta}\,\mathit{cut}_3$$

$$\dfrac{\Gamma \vdash N \quad \Gamma, N \vdash \Delta}{\Gamma \vdash \Delta}\,\mathit{cut}_4 \qquad \dfrac{\Gamma \vdash N \quad \Gamma, N \vdash [B]}{\Gamma \vdash [B]}\,\mathit{cut}_5$$

Proof: By simultaneous induction on the following lexicographical measure:

- the size of the cut-formula (A or N)
- the fact that the cut-formula (A or N) is positive or negative
  (if of equal size, a positive formula is considered smaller than a negative formula)
- the height of the derivation of the right premiss

Weakenings and contractions (as they are admissible in the system) are implicitly used throughout this proof.

In order to eliminate cut_3, we analyse which rule is used to prove the left premiss. We then use invertibility of the negative phase so that the last rule used in the right premiss is its dual one.

$$\dfrac{\dfrac{\Gamma\vdash[A]\quad\Gamma\vdash[B]}{\Gamma\vdash[A\wedge^+B]}\qquad\dfrac{\Gamma\vdash A^\perp,B^\perp,\Delta}{\Gamma\vdash A\vee^-B,\Delta}}{\Gamma\vdash\Delta}\,\mathsf{cut}_3\quad\text{reduces to}\quad\dfrac{\Gamma\vdash[B]\qquad\dfrac{\Gamma\vdash[A]\quad\Gamma\vdash A^\perp,B^\perp,\Delta}{\Gamma\vdash B^\perp,\Delta}\,\mathsf{cut}_3}{\Gamma\vdash\Delta}\,\mathsf{cut}_3$$

$$\dfrac{\dfrac{\Gamma\vdash[A_i]}{\Gamma\vdash[A_1\vee^+A_2]}\qquad\dfrac{\Gamma\vdash A_1^\perp,\Delta\quad\Gamma\vdash A_2^\perp,\Delta}{\Gamma\vdash A_1\wedge^-A_2,\Delta}}{\Gamma\vdash\Delta}\,\mathsf{cut}_3\quad\text{reduces to}\quad\dfrac{\Gamma\vdash[A_i]\qquad\Gamma\vdash A_i^\perp,\Delta}{\Gamma\vdash\Delta}\,\mathsf{cut}_3$$

$$\dfrac{\dfrac{\Gamma\vdash[\{{}^{t}\!/_{x}\}A]}{\Gamma\vdash[\exists xA]}\qquad\dfrac{\Gamma\vdash A^\perp,\Delta}{\Gamma\vdash(\forall xA^\perp),\Delta}}{\Gamma\vdash\Delta}\,\mathsf{cut}_3\quad\text{reduces to}\quad\dfrac{\Gamma\vdash[\{{}^{t}\!/_{x}\}A]\qquad\dfrac{\Gamma\vdash A^\perp,\Delta}{\Gamma\vdash(\{{}^{t}\!/_{x}\}A^\perp),\Delta}\,x\notin\mathsf{FV}(\Gamma,\Delta)}{\Gamma\vdash\Delta}\,\mathsf{cut}_3$$

using the admissibility of instantiation.

$$\dfrac{\dfrac{\Gamma\vdash N}{\Gamma\vdash[N]}\qquad\dfrac{\Gamma,N\vdash\Delta}{\Gamma\vdash(N^\perp),\Delta}}{\Gamma\vdash\Delta}\,\mathsf{cut}_3\quad\text{reduces to}\quad\dfrac{\Gamma\vdash N\qquad\Gamma,N\vdash\Delta}{\Gamma\vdash\Delta}\,\mathsf{cut}_4$$

We will describe below how cut_4 is reduced.

$$\dfrac{\dfrac{}{\Gamma,p\vdash[p]}\qquad\dfrac{\Gamma,p,p\vdash\Delta}{\Gamma,p\vdash(p^\perp),\Delta}}{\Gamma,p\vdash\Delta}\,\mathsf{cut}_3\quad\text{reduces to}\quad\dfrac{\Gamma,p,p\vdash\Delta}{\Gamma,p\vdash\Delta}$$

using the admissibility of contraction.

$$\dfrac{\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma),p^\perp)}{\Gamma\vdash[p]}\qquad\dfrac{\Gamma,p\vdash\Delta}{\Gamma\vdash(p^\perp),\Delta}}{\Gamma\vdash\Delta}\,\mathsf{cut}_3\quad\text{reduces to}\quad\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma),p^\perp)\qquad\Gamma,p\vdash\Delta}{\Gamma\vdash\Delta}\,\mathsf{cut}_1$$

In order to reduce cut_4, we analyse which rule is used to prove the right premiss.

$$\dfrac{\Gamma\vdash N\qquad\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma))}{\Gamma,N\vdash}}{\Gamma\vdash}\,\mathsf{cut}_4\quad\text{reduces to}\quad\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma))}{\Gamma\vdash}$$

if N is not an literal (hence, it is not passed on to the procedure).

$$\dfrac{\dfrac{\Gamma,p\vdash}{\Gamma\vdash p^\perp}\qquad\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma),p^\perp)}{\Gamma,p^\perp\vdash}}{\Gamma\vdash}\,\mathsf{cut}_4\quad\text{reduces to}\quad\dfrac{\mathcal{T}(\mathsf{lit}(\Gamma),p^\perp)\qquad\Gamma,p\vdash}{\Gamma\vdash}\,\mathsf{cut}_1$$

if $p^\perp$ is an literal passed on to the procedure.

$$\dfrac{\Gamma\vdash N\qquad\dfrac{\Gamma,N\vdash[N^\perp]}{\Gamma,N\vdash}}{\Gamma\vdash}\,\mathsf{cut}_4\quad\text{reduces to}\quad\dfrac{\dfrac{\Gamma\vdash N\quad\Gamma,N\vdash[N^\perp]}{\Gamma\vdash[N^\perp]}\,\mathsf{cut}_5\qquad\Gamma\vdash N}{\Gamma\vdash}\,\mathsf{cut}_3$$

$$\dfrac{\Gamma,P^\perp\vdash N\qquad\dfrac{\Gamma,P^\perp,N\vdash[P]}{\Gamma,P^\perp,N\vdash}}{\Gamma,P^\perp\vdash}\,\mathsf{cut}_4\quad\text{reduces to}\quad\dfrac{\dfrac{\Gamma,P^\perp\vdash N\quad\Gamma,P^\perp,N\vdash[P]}{\Gamma,P^\perp\vdash[P]}\,\mathsf{cut}_5}{\Gamma,P^\perp\vdash}$$

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\Gamma, N \vdash B, \Delta \quad \Gamma, N \vdash C, \Delta}{\Gamma, N \vdash B \wedge^- C, \Delta}}{\Gamma \vdash B \wedge^- C, \Delta}\,\mathsf{cut}_4$$

reduces to

$$\dfrac{\dfrac{\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta}{\Gamma \vdash B, \Delta}\,\mathsf{cut}_4 \qquad \dfrac{\Gamma \vdash N \quad \Gamma, N \vdash C, \Delta}{\Gamma \vdash C, \Delta}\,\mathsf{cut}_4}{\Gamma \vdash B \wedge^- C, \Delta}$$

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\Gamma, N \vdash B, C, \Delta}{\Gamma, N \vdash B \vee^- C, \Delta}}{\Gamma \vdash B \vee^- C, \Delta}\,\mathsf{cut}_4 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\Gamma \vdash N \quad \Gamma, N \vdash B, C, \Delta}{\Gamma \vdash B, C, \Delta}\,\mathsf{cut}_4}{\Gamma \vdash B \vee^- C, \Delta}$$

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\Gamma, N \vdash B, \Delta}{\Gamma, N \vdash \forall x B, \Delta}}{\Gamma \vdash \forall x B, \Delta}\,\mathsf{cut}_4 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta}{\Gamma \vdash B, \Delta}\,\mathsf{cut}_4}{\Gamma \vdash \forall x B, \Delta}$$

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\Gamma, N, B^\perp \vdash \Delta}{\Gamma, N \vdash B, \Delta}}{\Gamma \vdash B, \Delta}\,\mathsf{cut}_4 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\Gamma, B^\perp \vdash N \quad \Gamma, N, B^\perp \vdash \Delta}{\Gamma, B^\perp \vdash \Delta}\,\mathsf{cut}_4}{\Gamma \vdash B, \Delta}$$

using weakening, and if B is positive or a negative literal.

We have reduced all cases of cut_4; we now reduce the cases for cut_5 (again, by case analysis on the last rule used to prove the right premiss).

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\Gamma, N \vdash [B] \quad \Gamma, N \vdash [C]}{\Gamma, N \vdash [B \wedge^+ C]}}{\Gamma \vdash [B \wedge^+ C]}\,\mathsf{cut}_5 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\Gamma \vdash N \quad \Gamma, N \vdash [B]}{\Gamma \vdash [B]}\,\mathsf{cut}_5 \qquad \dfrac{\Gamma \vdash N \quad \Gamma, N \vdash [C]}{\Gamma \vdash [C]}\,\mathsf{cut}_5}{\Gamma \vdash [B \wedge^+ C]}$$

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\Gamma, N \vdash [B_i]}{\Gamma, N \vdash [B_1 \vee^+ B_2]}}{\Gamma \vdash [B_1 \vee^+ B_2]}\,\mathsf{cut}_5 \qquad \text{reduces to} \qquad \dfrac{\Gamma \vdash N \quad \Gamma, N \vdash [B_i]}{\Gamma \vdash [B_i]}\,\mathsf{cut}_5$$

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\Gamma, N \vdash [\{{}^{t}\!/_{x}\} B]}{\Gamma, N \vdash [\exists x B]}}{\Gamma \vdash [\exists x B]}\,\mathsf{cut}_5 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\Gamma \vdash N \quad \Gamma, N \vdash [\{{}^{t}\!/_{x}\} B]}{\Gamma \vdash [\{{}^{t}\!/_{x}\} B]}\,\mathsf{cut}_5}{\Gamma \vdash [\exists x B]}$$

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\Gamma, N \vdash N'}{\Gamma, N \vdash [N']}}{\Gamma \vdash [N']}\,\mathsf{cut}_5 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\Gamma \vdash N \quad \Gamma, N \vdash N'}{\Gamma \vdash N'}\,\mathsf{cut}_4}{\Gamma \vdash [N']}$$

$$\dfrac{\Gamma \vdash N \qquad \dfrac{}{\Gamma, N \vdash [p]}}{\Gamma \vdash [p]}\,\mathsf{cut}_5 \qquad \text{reduces to} \qquad \dfrac{}{\Gamma \vdash [p]}$$

since p has to be in Γ.

$$\dfrac{\Gamma \vdash N \qquad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp)}{\Gamma, N \vdash [p]}}{\Gamma \vdash [p]}\,\mathsf{cut}_5 \qquad \text{reduces to} \qquad \dfrac{\mathcal{T}(\mathsf{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]}$$

$\square$

Theorem 6 (cut_6, cut_7, cut_8, and cut_9) *The following rules are admissible in $\mathsf{LK}(\mathcal{T})$.*

$$\frac{\Gamma \vdash N, \Delta \quad \Gamma, N \vdash \Delta}{\Gamma \vdash \Delta}\,\mathit{cut}_6 \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash A^\perp, \Delta}{\Gamma \vdash \Delta}\,\mathit{cut}_7$$

$$\frac{\Gamma, l \vdash \Delta \quad \Gamma, l^\perp \vdash \Delta}{\Gamma \vdash \Delta}\,\mathit{cut}_8 \qquad \frac{\Gamma, l_1, \ldots, l_n \vdash \Delta \quad \Gamma, (l_1^\perp \vee^- \ldots \vee^- l_n^\perp) \vdash \Delta}{\Gamma \vdash \Delta}\,\mathit{cut}_9$$

Proof: cut_6 is proved admissible by induction on the multiset Δ: the base case is the admissibility of cut_4, and the other cases just require the inversion of the connectives in Δ.

For cut_7, we can assume without loss of generality (swapping A and $A^\perp$) that A is negative. Applying inversion on $\Gamma \vdash A^\perp, \Delta$ gives a proof of $\Gamma, A \vdash \Delta$, and cut_7 is then obtained by cut_6:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,\mathsf{cut}_6$$

cut_8 is obtained as follows:

$$\frac{\dfrac{\Gamma, l^\perp \vdash \Delta}{\Gamma \vdash l, \Delta} \qquad \dfrac{\Gamma, l \vdash \Delta}{\Gamma \vdash l^\perp, \Delta}}{\Gamma \vdash \Delta}\,\mathsf{cut}_7$$

cut_9 is obtained as follows:

$$\frac{\dfrac{\Gamma, (l_1^\perp \vee^- \ldots \vee^- l_n^\perp) \vdash \Delta}{\Gamma \vdash l_1 \wedge^+ \ldots \wedge^+ l_n, \Delta} \qquad \dfrac{\dfrac{\Gamma, l_1, \ldots, l_n \vdash \Delta}{\Gamma \vdash l_1^\perp, \ldots, l_n^\perp, \Delta}}{\Gamma \vdash (l_1^\perp \vee^- \ldots \vee^- l_n^\perp), \Delta}}{\Gamma \vdash \Delta}\,\mathsf{cut}_7$$

□

5 Conclusion

It is worth noting that an instance of such a theory is the theory where $\mathcal{T}(S)$ holds if and only if there is a literal $p \in S$ such that $p^\perp \in S$.

We proved the admissibility of cut_8 and cut_9 as they are used to simulate the $\mathsf{DPLL}(\mathcal{T})$ procedure [NOT06] as the proof-search mechanism of $\mathsf{LK}(\mathcal{T})$.

Further work will consist in using the cut-admissibility results to:

- show that changing the polarities of the connectives and predicates that are present in a sequent, does not change the provability of that sequent in $\mathsf{LK}(\mathcal{T})$;
- prove the completeness of $\mathsf{LK}(\mathcal{T})$ with respect to the standard notion of provability in first-order logic, working in a particular theory $\mathcal{T}$ for which we have a (sound and complete) decision procedure;
- show how the $\mathsf{DPLL}(\mathcal{T})$ procedure can be simulated in $\mathsf{LK}(\mathcal{T})$ (with backtracking as well as with backjumping and lemma learning).

References

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