A sequent calculus with procedure calls
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Abstract

In this paper, we extend the sequent calculus LKF [LM09] into a calculus $LK(T)$, allowing calls to a decision procedure. We prove cut-elimination of $LK(T)$. 

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1 The sequent calculus $\text{LK}(T)$

The sequent calculus $\text{LK}(T)$ manipulates the formulae of first-order logic, with the specificity that every predicate symbol is classified as either positive or negative, and boolean connectives come in two versions: positive and negative.

**Definition 1 (Formulae)** Literals are predicates (a predicate symbol applied to a list of first-order terms) or negations of predicates. Literals are equipped with the obvious involutive negation, and the negation of a literal $l$ is denoted $l^\bot$.

Let $P$ be the set of literals that are either predicates with positive predicate symbols, or negations of predicates with negative predicate symbols.

Positive formulae $P ::= p | A \land^+ B | A \lor^+ B | \exists x A$

Negative formulae $N ::= p^\bot | A \land^- B | A \lor^- B | \forall x A$

Formulae $A, B ::= P | N$

where $p$ ranges over $P$.

**Definition 2 (Negation)** Negation is extended from literals to all formulae:

\[
\begin{array}{c|c|c|c}
(p)^\bot & (A \land B)^\bot & (A \lor B)^\bot & (\exists x A)^\bot \\
\hline
p^\bot & A^\bot \land^+ B^\bot & A^\bot \lor^+ B^\bot & \forall x A^\bot \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
(p^\bot)^\bot & (A \land^+ B)^\bot & (A \lor^+ B)^\bot & (\exists x A)^\bot \\
\hline
p^\bot & A^\bot \land^- B^\bot & A^\bot \lor^- B^\bot & \forall x A^\bot \\
\end{array}
\]

**Definition 3 (LK($T$))** The sequent calculus $\text{LK}^p(T)$ manipulates two kinds of sequents:

Focused sequents $\Gamma \vdash [P]$

Unfocused sequents $\Gamma \vdash \Delta$

where $\Gamma$ is a multiset of negative formulae and positive literals, $\Delta$ is a multiset of formulae, and $P$ is said to be in the focus of the (focused) sequent. By $\text{lit}(\Gamma)$ we denote the sub-multiset of $\Gamma$ consisting of its literals.

The rules of $\text{LK}^p(T)$, given in Figure 1, are of three kinds: synchronous rules, asynchronous rules, and structural rules. These correspond to three alternating phases in the proof-search process that is described by the rules.

If $S$ is a set of literals, $\mathcal{T}(S)$ is the call to the decision procedure on the conjunction of all literals of $S$. It holds if the procedure returns $\text{UNSAT}$.

2 Admissible rules

**Definition 4 (Assumptions on the procedure)**

We assume that the procedure calls satisfy the following properties:

- **Weakening** If $\mathcal{T}(S)$ then $\mathcal{T}(S, S')$.
- **Contraction** If $\mathcal{T}(S, A, A)$ then $\mathcal{T}(S, A)$.
- **Instantiation** If $\mathcal{T}(S)$ then $\mathcal{T}(\{x\} \cup S)$.
- **Consistency** If $\mathcal{T}(S, p)$ and $\mathcal{T}(S, p^\bot)$ then $\mathcal{T}(S)$.

where $S$ is a set of literals.

**Lemma 1 (Admissibility of weakening and contraction)**

The following rules are admissible in $\text{LK}(T)$.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premiss</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma \vdash [B]$</td>
<td>$\frac{\Gamma, A \vdash [B]}{\Gamma, A \vdash \Delta}$</td>
<td>$\Gamma \vdash \Delta$</td>
</tr>
<tr>
<td>$\Gamma, A, A \vdash [B]$</td>
<td>$\frac{\Gamma, A \vdash [B]}{\Gamma, A, A \vdash \Delta}$</td>
<td>$\Gamma, A \vdash \Delta$</td>
</tr>
</tbody>
</table>

**Proof:** By induction on the derivation of the premiss. □
### Synchronous rules

\[
\begin{array}{c}
\Gamma \vdash [A] \\
\Gamma \vdash [B] \\
\hline
\Gamma \vdash [A \land B]
\end{array}
\]

\[
\begin{array}{c}
\Gamma \vdash [A_1] \\
\Gamma \vdash [A_1 \lor A_2] \\
\hline
\Gamma \vdash [\exists x A]
\end{array}
\]

\[
\frac{\Gamma, p \vdash [p]}{\Gamma \vdash [p]} \text{ \(p\) positive literal}
\]

\[
\frac{\Gamma, p \vdash [p]}{\Gamma \vdash [p]} \text{ \(p\) positive literal}
\]

\[
\frac{\Gamma \vdash N}{\Gamma \vdash [N]} \text{ \(N\) negative}
\]

### Asynchronous rules

\[
\begin{array}{c}
\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta \\
\hline
\Gamma \vdash A \land B, \Delta
\end{array}
\]

\[
\begin{array}{c}
\Gamma \vdash A_1, A_2, \Delta \\
\Gamma \vdash A_1 \lor A_2, \Delta \\
\hline
\Gamma \vdash (\forall x A), \Delta \text{ \(x \notin \text{FV}(\Gamma, \Delta)\)}
\end{array}
\]

\[
\frac{\Gamma, A^+ \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ \(A\) positive or literal}
\]

### Structural rules

\[
\begin{array}{c}
\Gamma, P^+ \vdash [P] \\
\hline
\Gamma, P^+ \vdash P \text{ \(P\) positive}
\end{array}
\]

\[
\frac{\Gamma \vdash \mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash \_}
\]

---

**Lemma 2 (Admissibility of instantiation)** The following rules are admissible in \(LK(\mathcal{T})\).

\[
\begin{array}{c}
\Gamma \vdash [B] \\
\begin{array}{c}
\{x\} \Gamma \vdash \{x\} [B] \\
\{x\} \Gamma \vdash \{x\} \Delta
\end{array}
\end{array}
\]

**Proof:** By induction on the derivation of the premiss. \(\square\)

---

### 3 Invertibility of the asynchronous phase

**Lemma 3 (Invertibility of asynchronous rules)**

All asynchronous rules are invertible in \(LK(\mathcal{T})\).

**Proof:** By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of \(A \land \neg B\): by case analysis on the last rule actually used

\[
\begin{array}{c}
\Gamma \vdash A \land \neg B, \Delta \quad \Gamma \vdash A \land \neg B, C, \Delta' \quad \Gamma \vdash A \land \neg B, D, \Delta' \\
\hline
\Gamma \vdash A \land \neg B, C \land \neg D, \Delta' \quad \Gamma \vdash A \land \neg B, C, D, \Delta' \quad \Gamma \vdash A \land \neg B, \forall x C, \Delta', x \notin \text{FV}(\Gamma, \Delta', A \land \neg B)
\end{array}
\]

By induction hypothesis we get

\[
\begin{array}{c}
\Gamma \vdash A, C, \Delta' \\
\Gamma \vdash A, D, \Delta' \\
\hline
\Gamma \vdash A, C \land \neg D, \Delta' \quad \Gamma \vdash A, C, D, \Delta' \quad \Gamma \vdash B, C, D, \Delta'
\end{array}
\]

\[
\begin{array}{c}
\Gamma \vdash A \land \neg B, C \land \neg D, \Delta' \\
\Gamma \vdash A \land \neg B, C, D, \Delta' \\
\hline
\Gamma \vdash A \land \neg B, C \land \neg D, \Delta' \quad \Gamma \vdash A, C, D, \Delta'
\end{array}
\]

By induction hypothesis we get

\[
\begin{array}{c}
\Gamma \vdash A, C, D, \Delta' \\
\Gamma \vdash A, C, \Delta' \\
\hline
\Gamma \vdash A, C \land \neg D, \Delta' \quad \Gamma \vdash B, C, D, \Delta' \quad \Gamma \vdash B, C \land \neg D, \Delta'
\end{array}
\]

\[
\begin{array}{c}
\Gamma \vdash A \land \neg B, C, \Delta' \\
\Gamma \vdash A \land \neg B, (\forall x C), \Delta', x \notin \text{FV}(\Gamma, \Delta', A \land \neg B)
\end{array}
\]
By induction hypothesis we get
\[ \Gamma \vdash A \land C, \Delta' \land \neg B \\implies \forall x. (\forall x. C), \Delta' \\land \neg A \land B, \Delta' \\land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \\land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, \forall x. C, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', B) \]
\[
\Gamma, C \vdash A \land \neg B, \Delta' \land \neg C \\land \neg D, \Delta'
\]
\[ C \text{ positive or literal } \]
By induction hypothesis we get
\[ \Gamma \vdash A \land \neg B, C, \Delta' \land \neg C \land \neg D, \Delta' \land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
\[
\Gamma, C \vdash A \land \neg B, \Delta' \land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
\[ C \text{ positive or literal } \]
By induction hypothesis we get
\[ \Gamma, C \vdash A, B \land C, \Delta' \land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma \vdash A, B, C, \Delta' \land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma \vdash A, B, C \land \neg D, \Delta' \land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
\[ C \text{ positive or literal } \]
By induction hypothesis we get
\[ \Gamma, C \vdash A, B, C \land \neg D, \Delta' \land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma, C \vdash A, B, C \land \neg D, \Delta' \land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma, C \vdash A, B, C \land \neg D, \Delta' \land \neg \neg \forall x. (A \land C) \land \neg B, \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma \vdash A, C, \Delta' \land \neg \neg \forall x. (\forall x. A) \land \neg \neg \forall x. (\forall x. A), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma \vdash A, C, \Delta' \land \neg \neg \forall x. (\forall x. A) \land \neg \neg \forall x. (\forall x. A), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma \vdash A, B, C, \Delta' \land \neg \neg \forall x. (\forall x. A) \land \neg \neg \forall x. (\forall x. A), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma \vdash A, B, C, \Delta' \land \neg \neg \forall x. (\forall x. A) \land \neg \neg \forall x. (\forall x. A), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
By induction hypothesis we get
\[ \Gamma \vdash A, B, C, \Delta' \land \neg \neg \forall x. (\forall x. A) \land \neg \neg \forall x. (\forall x. A), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \text{ and } \Gamma \vdash B, (\forall x. C), \Delta' \land \neg \neg x \notin \text{FV}(\Gamma, \Delta') \]
4 Cut-elimination

Theorem 4 (cut₁ and cut₂) The following rules are admissible in LK(Γ).

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+) \quad \Gamma, p \vdash \Delta}{\Gamma \vdash \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+) \quad \Gamma, p \vdash [B]}{\Gamma \vdash [B]} \text{cut₂}
\end{array}
\]

Proof: By simultaneous induction on the derivation of the right premiss.

We reduce cut₂ by case analysis on the last rule used to prove the right premiss.

\[
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁}
\]

reduces to

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash C, \Delta} \text{cut₁}
\end{array}
\]

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash C, \Delta} \text{cut₁}
\end{array}
\]

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash C, \Delta} \text{cut₁}
\end{array}
\]

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash C, \Delta} \text{cut₁}
\end{array}
\]

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash C, \Delta} \text{cut₁}
\end{array}
\]

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash C, \Delta} \text{cut₁}
\end{array}
\]

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash C, \Delta} \text{cut₁}
\end{array}
\]

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash B, \Delta} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash C, \Delta} \text{cut₁}
\end{array}
\]

We have \( T(\text{lit}(\Gamma),p^+,B^+) \) as we assume the procedure to satisfy weakening.

If \( P^+ \in (\Gamma,p) \),

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash [P]} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma \vdash [P]} \text{cut₂}
\end{array}
\]

as \( P^+ \in (\Gamma) \).

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash [P]} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma \vdash [P]} \text{cut₂}
\end{array}
\]

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma, p \vdash [P]} \text{cut₁} \\
\frac{T(\text{lit}(\Gamma),p^+)}{\Gamma \vdash [P]} \text{cut₂}
\end{array}
\]
Proof: By simultaneous induction on the following lexicographical measure:

\[
\begin{align*}
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \quad \text{cut}_1 \quad \text{reduces to} \quad \Gamma \vdash \\
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \text{[B]} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[B]} \\
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \text{[∃xB]} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[∃xB]} \\
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \text{[N]} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[N]} \\
\end{align*}
\]

using the assumption of consistency.

We reduce \text{cut}_2 again by case analysis on the last rule used to prove the right premiss.

\[
\begin{align*}
\Gamma, p \vdash \text{[B]} \quad \Gamma, p \vdash \text{[C]} \\
\end{align*}
\]

reduces to

\[
\begin{align*}
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \text{[B]} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[B]} \\
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \text{[C]} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[C]} \\
\end{align*}
\]

If \( p' \in \Gamma, p \),

\[
\begin{align*}
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \text{[p']} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[p']} \\
\end{align*}
\]

reduces to

\[
\begin{align*}
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma \vdash \text{[p']} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[p']} \\
\end{align*}
\]

if \( p' \in \Gamma \),

\[
\begin{align*}
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \text{[p']} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[p']} \\
\end{align*}
\]

if \( p' = p \).

Finally,

\[
\begin{align*}
\text{T} & \left(\text{lit}(\Gamma), p^+\right) \quad \Gamma, p \vdash \text{[p']} \quad \text{cut}_2 \quad \text{reduces to} \quad \Gamma \vdash \text{[p']} \\
\end{align*}
\]

since weakening gives \( \text{T}(\text{lit}(\Gamma), p^+, p'^+) \) and consistency then gives \( \text{T}(\text{lit}(\Gamma), p'^+) \). \( \Box \)

**Theorem 5 (cut\textsubscript{3}, cut\textsubscript{4} and cut\textsubscript{5})** The following rules are admissible in \( \text{LK(T)} \).

\[
\begin{align*}
\Gamma & \vdash \text{[A]} \quad \Gamma \vdash A^+, \Delta \quad \text{cut}_3 \\
\Gamma & \vdash \Delta \\
\Gamma & \vdash N \quad \Gamma, N \vdash \Delta \quad \text{cut}_4 \\
\Gamma & \vdash N \quad \Gamma, N \vdash \text{[B]} \\
\Gamma & \vdash \text{[B]} \quad \text{cut}_5 \\
\end{align*}
\]

**Proof:** By simultaneous induction on the following lexicographical measure:

- the size of the cut-formula (\( A \) or \( N \))
- the fact that the cut-formula (\( A \) or \( N \)) is positive or negative
  (if of equal size, a positive formula is considered smaller than a negative formula)
- the height of the derivation of the right premiss
Weakenings and contractions (as they are admissible in the system) are implicitly used throughout this proof.

In order to eliminate \( \text{cut}_3 \), we analyse which rule is used to prove the left premiss. We then use invertibility of the negative phase so that the last rule used in the right premiss is its dual one.

\[
\begin{array}{ccc}
\Gamma \vdash A & \Gamma \vdash [B] & \Gamma \vdash [A \wedge B] \\
\hline
\Gamma \vdash \Delta & \Gamma \vdash A^+, B^+, \Delta & \Gamma \vdash [A] \quad \Gamma \vdash A^+, B^+, \Delta \\
\end{array}
\]

\( \text{cut}_3 \) reduces to

\[
\begin{array}{ccc}
\Gamma \vdash [A_1] & \Gamma \vdash A_1, \Delta & \Gamma \vdash A_1 \wedge A_2, \Delta \\
\hline
\Gamma \vdash \Delta & \Gamma \vdash A_1^+, \Delta & \Gamma \vdash A_1^+, \Delta \\
\end{array}
\]

\( \text{cut}_3 \) reduces to

\[
\begin{array}{ccc}
\Gamma \vdash \exists x.A & \Gamma \vdash \forall x.A^+, \Delta & \Gamma \vdash \exists x.A, \Delta \\
\hline
\Gamma \vdash \Delta & \Gamma \vdash A_1^+, \Delta & \Gamma \vdash (\exists x.A^+, \Delta) \\
\end{array}
\]

\( \text{cut}_3 \) using the admissibility of instantiation.

\[
\begin{array}{ccc}
\Gamma \vdash N & \Gamma, N \vdash A & \Gamma \vdash [N] \\
\hline
\Gamma \vdash \Delta & \Gamma, N \vdash \Delta & \Gamma \vdash [N] \\
\end{array}
\]

We will describe below how \( \text{cut}_4 \) is reduced.

\[
\begin{array}{ccc}
\Gamma, p, p \vdash [p] & \Gamma, p, p \vdash (p^+), \Delta & \Gamma, p, p \vdash (p^+), \Delta \\
\hline
\Gamma, p \vdash \Delta & \Gamma, p, p \vdash \Delta & \Gamma, p \vdash \Delta \\
\end{array}
\]

\( \text{cut}_3 \) reduces to

\[
\begin{array}{ccc}
\Gamma \vdash \text{lit}(\Gamma) & \Gamma, p \vdash \Delta & \Gamma \vdash \text{lit}(\Gamma) \\
\hline
\Gamma \vdash \Delta & \Gamma, p \vdash \Delta & \Gamma \vdash \Delta \\
\end{array}
\]

\( \text{cut}_1 \)

In order to reduce \( \text{cut}_4 \), we analyse which rule is used to prove the right premiss.

\[
\begin{array}{ccc}
\Gamma \vdash \text{lit}(\Gamma) & \Gamma, p \vdash \Delta & \Gamma \vdash \text{lit}(\Gamma) \\
\hline
\Gamma \vdash \Delta & \Gamma, p \vdash \Delta & \Gamma \vdash \Delta \\
\end{array}
\]

\( \text{cut}_1 \)

if \( N \) is not an literal (hence, it is not passed on to the procedure).

\[
\begin{array}{ccc}
\Gamma, p \vdash \text{lit}(\Gamma), p^+ & \Gamma, p \vdash p^+ & \Gamma \vdash \text{lit}(\Gamma), p^+ \\
\hline
\Gamma \vdash \Delta & \Gamma, p \vdash \Delta & \Gamma \vdash \Delta \\
\end{array}
\]

\( \text{cut}_4 \) reduces to

\[
\begin{array}{ccc}
\Gamma \vdash \text{lit}(\Gamma), p^+ & \Gamma, p \vdash \Delta & \Gamma \vdash \text{lit}(\Gamma), p^+ \\
\hline
\Gamma \vdash \Delta & \Gamma, p \vdash \Delta & \Gamma \vdash \Delta \\
\end{array}
\]

\( \text{cut}_1 \)

if \( p^+ \) is an literal passed on to the procedure.

\[
\begin{array}{ccc}
\Gamma, N \vdash [N^+] & \Gamma \vdash N & \Gamma, N \vdash [N^+] \\
\hline
\Gamma \vdash \Delta & \Gamma \vdash [N^+] & \Gamma \vdash N \\
\end{array}
\]

\( \text{cut}_5 \) reduces to

\[
\begin{array}{ccc}
\Gamma \vdash \Delta & \Gamma \vdash [N^+] & \Gamma \vdash N \\
\hline
\Gamma, P^+ \vdash N & \Gamma, P^+ \vdash [P] & \Gamma \vdash [P] \\
\end{array}
\]

\( \text{cut}_5 \) reduces to

\[
\begin{array}{ccc}
\Gamma, P^+ \vdash N & \Gamma, P^+ \vdash [P] & \Gamma \vdash [P] \\
\hline
\Gamma, P^+ \vdash \Delta & \Gamma, P^+ \vdash \Delta & \Gamma \vdash \Delta \\
\end{array}
\]
\[
\frac{
\Gamma, N \vdash B, \Delta \quad \Gamma, N \vdash C, \Delta
}{
\Gamma \vdash N \quad \Gamma, N \vdash B \wedge C, \Delta
\} \text{cut}_4
\]

reduces to
\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta \quad \Gamma, N \vdash C, \Delta
}{
\Gamma \vdash B, \Delta \quad \Gamma \vdash C, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma, N \vdash B, C, \Delta
}{
\Gamma \vdash N \quad \Gamma, N \vdash B \wedge C, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B \vee C, \Delta
}{
\Gamma \vdash B \vee C, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash \forall x B, \Delta
}{
\Gamma \vdash \forall x B, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B^+ \vdash \Delta
}{
\Gamma \vdash B^+ \vdash \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta
}{
\Gamma \vdash B, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B, C, \Delta
}{
\Gamma \vdash B, C, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta
}{
\Gamma \vdash B, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash \forall x B, \Delta
}{
\Gamma \vdash \forall x B, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B^+ \vdash \Delta
}{
\Gamma, B^+ \vdash N \quad \Gamma, N, B^+ \vdash \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta
}{
\Gamma \vdash B, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B, C, \Delta
}{
\Gamma \vdash B, C, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B \vee C, \Delta
}{
\Gamma \vdash B \vee C, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash \forall x B, \Delta
}{
\Gamma \vdash \forall x B, \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B^+ \vdash \Delta
}{
\Gamma, B^+ \vdash N \quad \Gamma, N, B^+ \vdash \Delta
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta
}{
\Gamma \vdash B, \Delta
\} \text{cut}_4
\]

using weakening, and if \( B \) is positive or a negative literal.

We have reduced all cases of \text{cut}_4; we now reduce the cases for \text{cut}_5 (again, by case analysis on the last rule used to prove the right premise).

\[
\frac{
\Gamma, N \vdash [B] \quad \Gamma, N \vdash [C]
}{
\Gamma \vdash N \quad \Gamma, N \vdash [B \wedge C]
\} \text{cut}_5
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash [B \wedge C]
}{
\Gamma \vdash [B] \quad \Gamma \vdash [C]
\} \text{cut}_5
\]

\[
\frac{
\Gamma, N \vdash [B_1]
}{
\Gamma \vdash N \quad \Gamma, N \vdash [B_1]
\} \text{cut}_5
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash [B_1 \vee B_2]
}{
\Gamma \vdash [B_1 \vee B_2]
\} \text{cut}_5
\]

\[
\frac{
\Gamma, N \vdash [\exists x B]
}{
\Gamma \vdash [\exists x B]
\} \text{cut}_5
\]

\[
\frac{
\Gamma, N \vdash N'
}{
\Gamma \vdash N'
\} \text{cut}_4
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash [p]
}{
\Gamma \vdash [p]
\} \text{cut}_5
\]

since \( p \) has to be in \( \Gamma \).

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash [p]
}{
\Gamma \vdash [p]
\} \text{cut}_5
\]

\[
\frac{
\Gamma \vdash N \quad \Gamma, N \vdash [p]
}{
\Gamma \vdash [p]
\} \text{cut}_5
\]

\[
\frac{T(\text{lit}(\Gamma), p^+)}{
\Gamma \vdash [p]
}\]

\[
\frac{T(\text{lit}(\Gamma), p^+)}{
\Gamma \vdash [p]
}\]

\[
\frac{T(\text{lit}(\Gamma), p^+)}{
\Gamma \vdash [p]
}\]
Theorem 6 (cut$_6$, cut$_7$, cut$_8$, and cut$_9$) The following rules are admissible in LK($T$).

\[
\begin{align*}
\Gamma \vdash N, \Delta & \quad \Gamma, N \vdash \Delta \quad \text{cut}_6 \\
\Gamma, l \vdash \Delta & \quad \Gamma, l^\perp \vdash \Delta \quad \text{cut}_8 \\
\Gamma, l_1, \ldots, l_n \vdash \Delta & \quad \Gamma, (l_1^\perp \lor \cdots \lor l_n^\perp) \vdash \Delta \quad \text{cut}_9
\end{align*}
\]

Proof: cut$_6$ is proved admissible by induction on the multiset $\Delta$: the base case is the admissibility of cut$_4$, and the other cases just require the inversion of the connectives in $\Delta$.

For cut$_7$, we can assume without loss of generality (swapping $A$ and $A^\perp$) that $A$ is negative. Applying inversion on $\Gamma \vdash A^\perp, \Delta$ gives a proof of $\Gamma, A \vdash \Delta$, and cut$_7$ is then obtained by cut$_6$:

\[
\Gamma \vdash A, \Delta \quad \Gamma \vdash \Delta \quad \text{cut}_6
\]

For cut$_8$, we can assume without loss of generality (swapping $A$ and $A^\perp$) that $A$ is negative. Applying inversion on $\Gamma \vdash A^\perp, \Delta$ gives a proof of $\Gamma, A \vdash \Delta$, and cut$_7$ is then obtained by cut$_6$:

\[
\Gamma \vdash A, \Delta \quad \Gamma \vdash \Delta \quad \text{cut}_6
\]

5 Conclusion

It is worth noting that an instance of such a theory is the theory where $T(S)$ holds if and only if there is a literal $p \in S$ such that $p^\perp \in S$.

We proved the admissibility of cut$_4$ and cut$_9$ as they are used to simulate the DPLL($T$) procedure [NOT06] as the proof-search mechanism of LK($T$).

Further work will consist in using the cut-admissibility results to:

- show that changing the polarities of the connectives and predicates that are present in a sequent, does not change the provability of that sequent in LK($T$);
- prove the completeness of LK($T$) with respect to the standard notion of provability in first-order logic, working in a particular theory $T$ for which we have a (sound and complete) decision procedure;
- show how the DPLL($T$) procedure can be simulated in LK($T$) (with backtracking as well as with backjumping and lemma learning).

References
