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# A quasi-optimal a priori error estimate for the two-dimensional Signorini problem approximated by linear finite elements

## *Une estimation d'erreur quasi-optimale pour l'approximation par éléments finis du problème de Signorini bidimensionnel*

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### A B S T R A C T

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The aim of this Note is to present a quasi-optimal a priori error estimate for the linear finite element approximation of the so-called two-dimensional Signorini problem, i.e. the equilibrium of a plane linearly elastic body in contact with a rigid foundation. Previous works on that subject give either non-optimal estimates or with a more restrictive supplementary condition on the solution.

### R É S U M É

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On présente dans cette Note une estimation optimale de l'erreur d'approximation par éléments finis affines du problème de Signorini, c'est à dire du problème de l'équilibre d'un corps élastique en contact avec une fondation rigide. Les travaux précédents sur ce sujet donnent soit des résultats non optimaux, soit avec des conditions supplémentaires plus contraignantes sur la solution.

## 1. Introduction

The a priori error estimate for the finite element approximation of Signorini's problem, i.e. the static contact problem of a linearly elastic structure (see [4]), has been addressed for the first time by J. Haslinger in [7]. Since then, it has been the subject of various works. If we limit ourselves to the approximation by linear finite elements, we can distinguish two types of approximations. The first one consists in directly approximating the variational inequality describing the Signorini problem, the only unknown being the elastic displacement [7,8,11]. The second approach, sometimes called hybrid formulation, involves a Lagrange multiplier to enforce the non-penetration in some weaker sense [9,2,12,10].

A priori error estimate in the  $H^1$ -norm for the displacement of order  $h^{3/4}$  have been established in [8], for the direct approximation, and in [2], for the hybrid one, under  $H^2$  regularity assumption on the solution. The order  $h$  has been shown in [9,2,12] with an additional regularity assumption on the contact stress ( $H^1(\Gamma_C)$ , where  $\Gamma_C$  is the contact boundary). Note that an order  $h$  was early proposed in [11] but with an erroneous estimate in negative norm for the Lagrange interpolation operator. More optimal estimates have been proven with the additional assumption that the number of transitions between contact and non-contact is finite on the contact boundary. Order  $h\sqrt{|\log h|}$  in obtained in [1] and  $h\sqrt[4]{|\log h|}$  in [2], and

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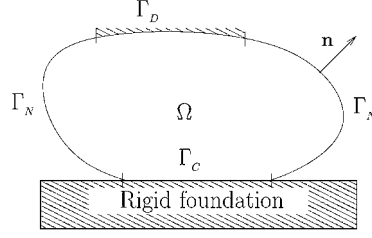


Fig. 1. Linearly elastic body  $\Omega$  in contact with a rigid foundation.

finally  $h$  in [10] where the result is also extended to three-dimensional problems. This condition may seem reasonable at first glance. However, there exist some infinitely oscillating solutions to the elasticity problem in the simple situation of a transition between Neumann and Dirichlet conditions [6]. Thus, infinitely oscillatory situation cannot be a priori excluded.

The result presented here concerns the direct approximation and is quasi-optimal in the sense that the only requirement for the solution to obtain the  $O(h)$  estimate is the  $H^{2+\varepsilon}$  regularity for an arbitrary small  $\varepsilon$ . The argument is limited to two-dimensional domains.

## 2. Signorini's problem

Let  $\Omega \subset \mathbb{R}^2$  be a bounded regular domain which represents the reference configuration of a linearly elastic body submitted to a Neumann condition on  $\Gamma_N$ , a Dirichlet condition on  $\Gamma_D$  and a unilateral contact condition on  $\Gamma_C$  between the body and a rigid foundation, where  $\Gamma_N$ ,  $\Gamma_D$  and  $\Gamma_C$  are non-overlapping open parts of  $\partial\Omega$ , the boundary of  $\Omega$  (see Fig. 1).

The displacement  $u(t, x)$  of the body obeys the following equations:

$$-\operatorname{div} \sigma(u) = f, \quad \text{in } \Omega, \quad \sigma(u) = \mathcal{A} \varepsilon(u), \quad \text{in } \Omega, \quad \sigma(u) \mathbf{n} = \mathbf{g}, \quad \text{on } \Gamma_N, \quad u = 0, \quad \text{on } \Gamma_D, \quad (1)$$

where  $\sigma(u)$  is the stress tensor,  $\varepsilon(u) = (\nabla u + \nabla u^T)/2$  is the linearized strain tensor,  $\mathcal{A}$  is the fourth order elasticity tensor which satisfies usual conditions of symmetry, coercivity and boundedness,  $\mathbf{n}$  is the outward unit normal to  $\Omega$  on  $\partial\Omega$  and  $f$ ,  $\mathbf{g}$  are given force densities. On  $\Gamma_C$ , it is usual to decompose the displacement and the stress in normal and tangential components as follows, assuming  $\Gamma_C$  to have the  $\mathcal{C}^1$  regularity:

$$u_N = u \cdot \mathbf{n}, \quad u_T = u - u_N \mathbf{n}, \quad \sigma_N(u) = (\sigma(u) \mathbf{n}) \cdot \mathbf{n}, \quad \sigma_T(u) = \sigma(u) \mathbf{n} - \sigma_N(u) \mathbf{n}.$$

Assuming also that there is no initial gap between the solid and the rigid foundation, the frictionless unilateral contact condition is expressed by the following complementary condition:

$$u_N \leq 0, \quad \sigma_N(u) \leq 0, \quad u_N \sigma_N(u) = 0, \quad \sigma_T = 0. \quad (2)$$

A classical weak form (see [4]) can be obtained introducing

$$V = \{v \in H^1(\Omega; \mathbb{R}^d), v = 0 \text{ on } \Gamma_D\}, \quad K = \{v \in V : v_N \leq 0 \text{ on } \Gamma_C\},$$

$$a(u, v) = \int_{\Omega} \mathcal{A} \varepsilon(u) : \varepsilon(v) \, dx, \quad l(v) = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} \mathbf{g} \cdot v \, d\Gamma.$$

Problem (1)–(2) is then formally equivalent to the variational inequality

$$\begin{aligned} &\text{Find } u \in K \text{ satisfying} \\ &a(u, v - u) \geq l(v - u), \quad \forall v \in K. \end{aligned} \quad (3)$$

## 3. Finite element discretization

A linear finite element approximation of Problem (3) is obtained given a regular family of triangulations  $\mathcal{T}^h$  of  $\Omega$  (in the sense of Ciarlet [3]). For the sake of simplicity, we assume  $\Omega$  to be polygonal (which means that  $\Gamma_C$  is a line segment). We consider the classical finite element space

$$V^h = \{v^h \in \mathcal{C}^0(\bar{\Omega}) : v|_T \in P^1(T) \forall T \in \mathcal{T}^h, v^h = 0 \text{ on } \Gamma_D\},$$

and the discrete convex set  $K^h = K \cap V^h$  of admissible displacements. Then, the finite element approximation of Problem (3) reads as

$$\begin{aligned} &\text{Find } u^h \in K^h \text{ satisfying} \\ &a(u^h, v^h - u^h) \geq l(v^h - u^h), \quad \forall v^h \in K^h. \end{aligned} \quad (4)$$

We consider the following classical adaptation of Falk's lemma (see [7,8,11]):

**Lemma 3.1.** *Provided  $f \in L^2(\Omega)$ ,  $g \in L^2(\Gamma_N)$ ,  $a(\cdot, \cdot)$  coercive, for  $u$  solution to (3) with  $\sigma_N \in L^2(\Gamma_C)$  and  $u^h$  solution to (4), there exists  $C > 0$  a constant such that*

$$\|u - u^h\|_{1,\Omega} \leq C \inf_{v^h \in K^h} \|u - v^h\|_{1,\Omega} + \int_{\Gamma_C} \sigma_N v_N^h d\Gamma + C \inf_{v \in K} \int_{\Gamma_C} \sigma_N (v_N - u_N^h) d\Gamma. \quad (5)$$

**Proof.** We give the proof for the self-consistency of the Note. Denoting  $\alpha > 0$  the coercivity constant of  $a(\cdot, \cdot)$  one has

$$\alpha \|u - u^h\|_{1,\Omega}^2 \leq a(u - u^h, u - u^h) = a(u, u) + a(u^h, u^h) - a(u, u^h) - a(u^h, u).$$

Using the two variational inequalities, one deduces for arbitrary  $v \in K$  and  $v^h \in K^h$

$$\alpha \|u - u^h\|_{1,\Omega}^2 \leq a(u, v - u^h) - l(v - u^h) + a(u, v^h - u) - l(v^h - u) + a(u^h - u, v^h - u).$$

Using now a Green formula (see [11]) one obtains  $a(u, v) - l(v) = \int_{\Gamma_C} \sigma_N v_N d\Gamma$ . From this and the complementarity condition, which gives  $\sigma_N u_N = 0$  a.e. on  $\Gamma_C$ , one obtains

$$\alpha \|u - u^h\|_{1,\Omega}^2 \leq \int_{\Gamma_C} \sigma_N (v_N - u_N^h) d\Gamma + \int_{\Gamma_C} \sigma_N v_N^h d\Gamma + M \|u - u^h\|_{1,\Omega} \|u - v^h\|_{1,\Omega},$$

where  $M > 0$  is the continuity constant of  $a(\cdot, \cdot)$ . One can conclude with Young's inequality.  $\square$

#### 4. Quasi-optimal error estimate

**Theorem 4.1.** *Provided  $f \in L^2(\Omega)$ ,  $g \in L^2(\Gamma_N)$ ,  $a(\cdot, \cdot)$  coercive, for  $u$  solution to (3) with  $u \in H^{2+\varepsilon}(\Omega)$  for some  $\varepsilon > 0$  and  $u^h$  solution to (4), there exists  $C > 0$  a constant independent of  $h$  such that*

$$\|u - u^h\|_{1,\Omega} \leq Ch \|u\|_{2+\varepsilon,\Omega}. \quad (6)$$

**Proof.** Since  $K^h \subset K$  the second infimum of (5) vanishes. Considering  $\Pi^h : H^2(\Omega) \rightarrow V^h$  the classical Lagrange interpolation operator and since  $\Pi^h u \in K^h$  one deduces from (5)

$$\|u - u^h\|_{1,\Omega}^2 \leq C \|u - \Pi^h u\|_{1,\Omega}^2 + \int_{\Gamma_C} \sigma_N (\Pi^h u)_N d\Gamma, \quad (7)$$

where, here and hereafter,  $C > 0$  denotes a constant whose value may vary from an occurrence to another but which is independent of  $h$  and of a possibly considered element. Since a classical interpolation error estimate [3] gives  $\|u - \Pi^h u\|_{1,\Omega} \leq Ch \|u\|_{2,\Omega}$  we can focus on the last term of (7). Let  $\mathcal{E}^h$  be the set of edges on  $\Gamma_C$  such that  $u_N(c) < 0$  for  $c$  one of the vertices of the edge. Then

$$\int_{\Gamma_C} \sigma_N (\Pi^h u)_N d\Gamma = \int_{e \in \mathcal{E}^h} \sigma_N (\Pi^h u)_N d\Gamma,$$

because  $(\Pi^h u)_N$  vanishes on each edge for which  $u_N(a) = 0$  on the two vertices. Now, for  $e \in \mathcal{E}^h$  and due to the complementarity condition, one has

$$\begin{aligned} \int_e \sigma_N (\Pi^h u)_N d\Gamma &= \int_e \sigma_N ((\Pi^h u)_N - u_N) d\Gamma \leq \|\sigma_N\|_{0,e} \|(\Pi^h u)_N - u_N\|_{0,e} \\ &\leq C \left( h \|\sigma_N\|_{0,e}^2 + \frac{1}{h} \|(\Pi^h u)_N - u_N\|_{0,e}^2 \right). \end{aligned}$$

A standard estimate (see [8]) gives  $\sum_{e \in \mathcal{E}^h} \frac{1}{h} \|(\Pi^h u)_N - u_N\|_{0,e}^2 \leq \frac{1}{h} \|(\Pi^h u)_N - u_N\|_{0,\Gamma_C}^2 \leq Ch^2 \|u\|_{2,\Omega}^2$ . We use a scaling argument to estimate the remaining term  $\|\sigma_N\|_{0,e}^2$ . Let  $\hat{e} = (0, 1)$  be a reference segment and let us denote by  $\hat{\sigma}_N$  the normal stress obtained from  $e$  on  $\hat{e}$  by a linear transformation. Since  $u_N < 0$  on one of the vertices of  $e$  and due to the complementarity condition, there exists a small interval  $(0, \varepsilon)$  on which  $\hat{\sigma}_N$  is a.e. equal to 0, eventually adapting the orientation of the linear transformation to ensure the correspondence of 0 with the vertex on which  $u_N < 0$ . In particular, this means  $\hat{\sigma}_N \in H_0^{1/2+\varepsilon}(\hat{e})$ , where  $H_0^{1/2+\varepsilon}(\hat{e}) = \{v \in H^{1/2+\varepsilon}(\hat{e}), v(0) = 0\}$ . We will prove later on that the semi-norm

$$|v|_{1/2+\varepsilon, \hat{e}} = \left( \int_{\hat{e}} \int_{\hat{e}} \frac{(v(\hat{x}) - v(\hat{y}))^2}{|\hat{x} - \hat{y}|^{2+2\varepsilon}} d\hat{x} d\hat{y} \right)^{1/2}$$

is equivalent on  $H_0^{1/2+\varepsilon}(\hat{e})$  to the classical norm  $\|v\|_{1/2+\varepsilon, \hat{e}} = (\|v\|_{0, \hat{e}}^2 + |v|_{1/2+\varepsilon, \hat{e}}^2)^{1/2}$ . This leads to

$$\|\sigma_N\|_{0, e}^2 = |e| \|\hat{\sigma}_N\|_{0, \hat{e}}^2 \leq Ch |\hat{\sigma}_N|_{1/2+\varepsilon, \hat{e}}^2 = Ch^{1+2\varepsilon} |\sigma_N|_{1/2, e}^2.$$

Since  $\sum_{e \in \mathcal{E}^h} |\sigma_N|_{1/2+\varepsilon, e}^2 \leq |\sigma_N|_{1/2+\varepsilon, \Gamma_C}^2 \leq C \|u\|_{2+\varepsilon, \Omega}^2$ , one finally obtains (6). The proof of equivalence on  $H_0^{1/2+\varepsilon}(\hat{e})$  between the two norms  $|v|_{1/2+\varepsilon, \hat{e}}$  and  $\|v\|_{1/2+\varepsilon, \hat{e}}$  is a straightforward application of the Petree–Tartar lemma we recall here (see [5] for instance).

**Lemma 4.2.** *Let  $X, Y, Z$  be three Banach spaces,  $A \in \mathcal{L}(X, Y)$  be injective and  $T \in \mathcal{L}(X, Z)$  be compact, if there exists  $c > 0$  a constant such that  $c\|x\|_X \leq \|Ax\|_Y + \|Tx\|_Z \quad \forall x \in X$  then the range of  $A$  is closed or, equivalently, there exists  $\alpha > 0$  a constant such that*

$$\alpha\|x\|_X \leq \|Ax\|_Y \quad \forall x \in X.$$

The equivalence is obtained by taking  $X = H_0^{1/2+\varepsilon}(\hat{e})$ ,  $Y = L^2(\hat{e} \times \hat{e})$ ,  $Z = L^2(\hat{e})$ ,  $Av = \frac{v(\hat{x}) - v(\hat{y})}{|\hat{x} - \hat{y}|^{1+\varepsilon}}$  and  $T$  the compact injection between  $X$  and  $Z$ .  $A$  is injective since  $Av = 0$  implies  $v$  is a constant a.e. on  $\hat{e}$  and since the only constant in  $H_0^{1/2+\varepsilon}(\hat{e})$  is zero.  $\square$

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