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Invariant distributions and scaling limits for some diffusions in time-varying random environments

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Abstract. We consider a family of one-dimensional diffusions, in dynamical Wiener mediums, which are random perturbations of the Ornstein-Uhlenbeck diffusion process. We prove quenched and annealed convergences in distribution and under weighted total variation norms. We find two kind of stationary probability measures, which are either the standard normal distribution or a quasi-invariant measure, depending on the environment, and which is naturally connected to a random dynamical system. We apply these results to the study of a model of time-inhomogeneous Brox's diffusions, which generalizes the diffusion studied by Brox (1986) and those investigated by Gradinaru and Offret (2011). We point out two distinct diffusive behaviours and we give the speed of convergences in the quenched situations.

Key words: Time-dependent random environment; Time-inhomogeneous Brox's diffusion; Random dynamical system; Foster-Lyapunov drift condition; Fluctuating stationary distribution

Mathematics Subject Classification (2000): 60K37; 60J60; 60F99; 37H99; 60H25; 37B25

1 Introduction

Random walks (RWs) in random environments (REs) and their continuous-time counterparts, the diffusions in random environment, pave the way for the study of a multitude of interesting cases, which have been tackled since the 70's in a large section of the literature.

Concerning the genesis of the theory, we allude to [1, 2], as regards the discrete-time situation, and to [3, 4, 5], as regards the continuous-time one. For more recent refinements and generalizations, we refer to [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and for a general review of the topic, we refer to [16].

Here we investigate one-dimensional diffusions evolving in dynamical Wiener mediums, which has some common features with those studied in [4, 17]. We give, under weighted total variation norms, quenched and annealed diffusive scaling limits, may depend on the environment, and thus which are not always normal distributions. We also give the speeds of convergence under the quenched distributions. In addition, we bring out a phase transition phenomenon, which is the analogue in RE, to a particular situation considered in [17].

RWs in dynamical REs has been widely and intensively considered in the past few years under several assumptions. Initially, space-time i.i.d. REs has been introduced and studied in [18, 19, 20]. Further difficulties arise when the fluctuations of the REs are i.i.d. in space and Markovian in time, case addressed in [21, 22], and major one arise when we consider space-time mixing REs, case recently studied in [23, 24, 25]. However, continuous-time diffusions in time-varying random environment has been sparsely investigate. Nevertheless, we can mention [26, 27, 28, 29] concerning the homogenization of diffusions in time-dependent random flows.

1.1 The Wiener space

Introduce the space

$$\Theta := \left\{ \theta \in C(\mathbb{R}; \mathbb{R}) : \theta(0) = 0 \text{ and } \lim_{|x| \rightarrow \infty} x^{-2} \theta(x) = 0 \right\} \quad (1.1)$$

endowed by the standard σ -field \mathcal{B} generated by the Borel cylinder sets. It is classical that there exists a unique probability measure \mathscr{W} on (Θ, \mathcal{B}) such that $\{\theta(\pm x) : \theta \in \Theta, x \geq 0\}$ are two independent standard Brownian motions. The probability distribution \mathscr{W} is called the Wiener measure. We denote by $\{S_\lambda : \lambda > 0\}$ the scaling transformations on Θ defined by

$$S_\lambda \theta(\star) := \frac{\theta(\lambda \star)}{\sqrt{\lambda}}. \quad (1.2)$$

Note that Θ is naturally endowed with a structure of separable Banach space, such that \mathcal{B} coincide with the Borel σ -field \mathcal{B}_Θ .

1.2 Brox' results

In [4], Brox make sense to solution of the informal diffusion equation

$$dX_t = dB_t - \frac{1}{2} \theta'(X_t) dt, \quad (1.3)$$

where $\theta \in \Theta$ and B is a standard one-dimensional Brownian motion independent of $(\Theta, \mathcal{B}, \mathscr{W})$. Denoting by \mathbb{P}_θ the quenched distribution of such solution, the author shows that there exists a family of measurable functions $\{b_h : h > 0\}$ on (Θ, \mathcal{B}) such that, for \mathscr{W} -almost surely $\theta \in \Theta$, the following convergence holds in probability

$$\frac{X_t}{(\log t)^2} - b_1 \left(S_{(\log t)^2} \theta \right) = \frac{X_t - b_{\log t}(\theta)}{(\log t)^2} \xrightarrow[t \rightarrow \infty]{\mathbb{P}_\theta} 0. \quad (1.4)$$

The Wiener measure being invariant under the scaling transformations, if we denote by \hat{b}_1 the distribution of b_1 under \mathscr{W} , and by $\hat{\mathbb{P}}$ the annealed distribution (the expectation of \mathbb{P}_θ under \mathscr{W}), the following convergence holds in distribution

$$\frac{X_t}{(\log t)^2} \xrightarrow[t \rightarrow \infty]{(d)} \hat{b}_1. \quad (1.5)$$

The key to prove these results is to take full advantage of the representation of X in terms of a one-dimensional Brownian motion changed in scale and time, and of the invariance of the Brownian motions B and θ under the scaling transformations S_λ . The author prove that the diffusion is localized in the valleys of the potential θ , which are themselves characterized by b_1 .

1.3 Phase transition in a 2-stable deterministic environment

Furthermore, set $W(x) := |x|^{1/2}$ and consider, for any $\beta \in \mathbb{R}$, the particular time-inhomogeneous singular stochastic differential equation studied in [17] and which is given by

$$dY_t = dB_t - \frac{1}{2} \frac{W'(Y_t)}{t^\beta} dt. \quad (1.6)$$

The authors show in [17] the existence of a pathwise unique strong solution and prove diffusive and subdiffusive scaling limits in distribution, depending on the position of β with respect to $1/4$. More precisely, they prove that

$$\frac{Y_t}{\sqrt{t}} \xrightarrow[t \rightarrow \infty]{(d)} \begin{cases} \mathcal{N}(0, 1), & \text{when } \beta > 1/4, \\ k_c^{-1} e^{-\left[\frac{x^2}{2} + W(x)\right]} dx, & \text{when } \beta = 1/4, \end{cases} \quad (1.7)$$

and

$$\frac{Y_t}{t^{2\beta}} \xrightarrow[t \rightarrow \infty]{(d)} k_u^{-1} e^{-W(x)} dx, \quad \text{when } \beta < 1/4, \quad (1.8)$$

k_c and k_u being two normalization positive constants. In fact, to obtain the results in (1.7), they study the diffusion equation

$$dZ_t = dB_t - \frac{1}{2} [Z_t + e^{-rt} W'(Z_t)] dt. \quad (1.9)$$

This process is naturally related to equation (1.6) by setting $r := \beta - 1/4$, via a well chosen scaling transformation taking full advantage of the scaling property of the Brownian motion B and of the deterministic scaling property of the potential W . For more details we refer to [17]. We can expect to get similar results by replacing W in (1.6) by a typical Brownian path $\theta \in \Theta$, a 2-stable random process, and this is one of the main objects of this article.

The paper is organized as follows: in Section 2, we first introduce a diffusion equation in moving Wiener potential, which generalizes equation (1.9). Then we state our main results and we give the general strategy of the proofs. In Section 3, we apply these results to a model of time-inhomogeneous Brox's diffusions. This is a generalization of equation (1.6) and (1.3) and we get similar asymptotic behaviours as in (1.7). Thereafter, in Section 4, we prove existence, uniqueness and nonexplosion for the diffusion in time-dependent random environment that we consider (Theorem 2.1). In Section 5, we prove that this process is a strongly Feller diffusion, which satisfies the lower local Aronson estimate and a kind of cocycle property (Theorem 2.2). In Section 6, we prove some convergence results in order to prove Theorems 2.3 and 2.4 in the two last Sections.

2 Model and statement of results

2.1 Diffusions in a fluctuating Ornstein-Uhlenbeck potential

In the present paper, we study Brownian motions dynamics, in time-dependent Wiener mediums, given by the underlying dynamical random environment

$$\left\{ T_t \theta(x) := S_{e^{t/2}} \theta(x) = e^{-t/4} \theta(e^{t/2} x) : \theta \in \Theta, t, x \in \mathbb{R} \right\}. \quad (2.1)$$

The family $\{T_t : t \in \mathbb{R}\}$ is a one-parameter group of transformations leaving invariant \mathscr{W} and such that, under this probability measure, $\{T_t \theta(x) : t \in \mathbb{R}\}$ is a stationary Ornstein-Uhlenbeck process having $\mathcal{N}(0, x)$ as stationary distribution. Moreover, it can be proved that the dynamical system $(\Theta, \mathcal{B}, \mathscr{W}, (T_t)_{t \in \mathbb{R}})$ is ergodic (see Proposition 6.5).

We consider, for any $r \in \mathbb{R}$, the diffusion process Z , solution of the informal stochastic differential equation driven by a standard Brownian motion B , independent of $(\Theta, \mathcal{B}, \mathscr{W})$,

$$dZ_t = dB_t - \frac{1}{2} \partial_x V_\theta(t, Z_t) dt, \quad Z_s = z \in \mathbb{R}, t \geq s \geq 0, \theta \in \Theta, \quad \text{with } V_\theta(t, x) := \frac{x^2}{2} + e^{-rt} T_t \theta(x). \quad (2.2)$$

Note that when θ is equal to W , defined in (1.6), $T_t \theta$ in (2.2) is simply equal to θ and equation (2.2) is nothing but equation (1.9). The diffusion process Z can be seen as a Brownian motion immersed in the random time-varying potential $\{V_\theta(t, \cdot) : t \in \mathbb{R}\}$, as well as an Ornstein-Uhlenbeck diffusion process, whose potential is perturbed by the dynamical Wiener medium $\{e^{-rt} T_t \theta : t \in \mathbb{R}\}$. Moreover, one can

see Z as a distorted Brownian motion, whose drift is a Gaussian field $\{\Gamma(t, x) : t, x \in \mathbb{R}\}$ having mean function m_Γ and covariance function C_Γ (here a Dirac measure) given by

$$m_\Gamma(t, x) = -\frac{x}{2} \quad \text{and} \quad C_\Gamma(t, x; s, z) = \frac{1}{4} e^{-\left[r(t+s) + \frac{|t-s|}{4}\right]} \delta(e^{t/2}x - e^{s/2}z). \quad (2.3)$$

We need to give a correct sense to solution of equation (2.2). Formally, we can see Z as the diffusion process, whose conditional infinitesimal generator, given the environment $\theta \in \Theta$, is

$$L_\theta := L_{\theta, t} + \frac{\partial}{\partial t} := \left[\frac{1}{2} e^{V_\theta(t, x)} \frac{\partial}{\partial x} \left(e^{-V_\theta(t, x)} \frac{\partial}{\partial x} \right) \right] + \frac{\partial}{\partial t}. \quad (2.4)$$

The domain and the so-called generalized domain of L_θ are defined by

$$D(L_\theta) := \{F \in C^1 : e^{-V_\theta} \partial_x F \in C^1\} \quad \text{and} \quad \bar{D}(L_\theta) := \left\{ F \in W_{\text{loc}}^{1, \infty} : e^{-V_\theta} \partial_x F \in W_{\text{loc}}^{1, \infty} \right\} \quad (2.5)$$

where C^1 and $W_{\text{loc}}^{1, \infty}$ denote the space of real continuous functions $F(t, x)$ on $[s, \infty) \times \mathbb{R}$ such that the partial derivatives $\partial_t F$ and $\partial_x F$ (in the sense of distributions) exist and are respectively continuous functions and locally bounded functions. This kind of diffusion operators, with distributional drift, has been already study in [30, 31] in the case where the coefficients of the stochastic differential equation do not depend on time. Rigorously speaking, a weak solution to equation (2.2) is a solution to the martingale problem related to $(L_\theta, D(L_\theta))$.

Definition 2.1. *A continuous stochastic process $\{Z_t : t \geq s\}$ defined on a given filtered probability space is said to be a weak solution to equation (2.2) if $Z_s = z$ and if there exists an increasing sequence of stopping time $\{\tau_n : n \geq 0\}$ such that, for all $n \geq 0$ and $F \in D(L_\theta)$,*

$$F(t \wedge \tau_n, Z_{t \wedge \tau_n}) - \int_s^{t \wedge \tau_n} L_\theta F(u, Z_u) du, \quad t \geq s, \quad (2.6)$$

is a local martingale, with

$$\tau_e := \sup_{n \geq 0} \inf\{t \geq s : |Z_t| \geq n\} = \sup_{n \geq 0} \tau_n. \quad (2.7)$$

A weak solution is global when the explosion time satisfies $\tau_e = \infty$ a.s. and we said that the weak solution is unique if all weak solutions have the same distribution.

We are now able to state our first result.

Theorem 2.1. *For all $r \in \mathbb{R}$, $\theta \in \Theta$, $s \geq 0$ and $z \in \mathbb{R}$, there exists a unique global weak solution Z to equation (2.2). Moreover, there exists a standard Brownian motion B such that, for all $F \in \bar{D}(L_\theta)$,*

$$F(t, Z_t) = F(s, z) + \int_s^t L_\theta F(u, Z_u) du + \int_s^t \partial_x F(u, Z_u) dB_u, \quad t \geq s. \quad (2.8)$$

Since the one-dimensional equation (2.2) is not time-homogeneous, there are not simple conditions which characterized the nonexplosion as in [4, 30, 31]. Therefore, the main difficulty is to construct Lyapunov functions in order to prove the nonfiniteness of the explosion time (see Proposition 4.2). Existence and uniqueness are obtained by considering the equivalent one-dimensional stochastic differential equation with random continuous coefficients (4.3) (see Proposition 4.1), which is naturally connected to equation (2.2), via the pseudo-scale function S_θ defined in (4.1). This method is a generalization in the time-inhomogeneous setting of that employed in [4, 31, 30] and which uses the effective scale function.

2.2 Strong Feller property, cocycle property and lower local Aronson estimate

In the following, we denote by $\mathbb{P}_{s,z}(\theta)$ the distribution of the weak solution of equation (2.2), called the quenched distribution, which existence is stated in Theorem 2.1. We introduce the canonical process $\{X_t : t \geq 0\}$ on the space of continuous functions from $[0, \infty)$ to \mathbb{R} , endowed with its natural Borel σ -field \mathcal{F} , and we denote by $P_\theta(s, z; t, dx)$ and $P_{s,t}(\theta)$, the probability transition kernel and the associated Markov kernel defined, for all measurable nonnegative function F on \mathbb{R} by

$$P_{s,t}(\theta)F(z) := \mathbb{E}_{s,z}(\theta)[F(X_t)] = \int_{\mathbb{R}} F(x)P_\theta(s, z; t, dx). \quad (2.9)$$

Theorem 2.2. *For all $r \in \mathbb{R}$ and $\theta \in \Theta$, $\{\mathbb{P}_{s,z}(\theta) : s \geq 0, z \in \mathbb{R}\}$ is a strongly Feller continuous family. Moreover, the time-inhomogeneous semigroups $\{P_{s,t}(\theta) : t \geq s \geq 0, \theta \in \Theta\}$ satisfy*

$$P_{s,s+t}(\theta) = P_{0,t}(e^{-rs} T_s \theta) \quad \text{and} \quad P_{0,s+t}(\theta) = P_{0,s}(\theta)P_{0,t}(e^{-rs} T_s \theta). \quad (2.10)$$

Besides, $P_\theta(s, z; t, dx)$ admits a density $p_\theta(s, z; t, x)$, which is measurable with respect to (θ, s, t, z, x) on $\Theta \times \{t > s \geq 0\} \times \mathbb{R}^2$, and which satisfies the lower local Aronson estimate: for all $\theta \in \Theta$, $T > 0$ and compact set $C \subset \mathbb{R}$, there exists $M > 0$ such that, for all $0 \leq s < t \leq T$ and $z, x \in C$,

$$p_\theta(s, z; t, x) \geq \frac{1}{\sqrt{M(t-s)}} \exp\left(-M \frac{|z-x|^2}{t-s}\right). \quad (2.11)$$

The idea is to study the equivalent stochastic differential equation (4.3) and to prove a similar theorem (see Theorem 5.1) by using standard technics.

Besides, the transition density being measurable with respect to θ , we can define the annealed distribution $\widehat{\mathbb{P}}_{s,z}$ and the associated Markov kernel $\widehat{P}_{s,t}$ as

$$\widehat{\mathbb{P}}_{s,z} := \mathbb{E}_{\mathcal{W}}[\mathbb{P}_{s,z}] := \int_{\Theta} \mathbb{P}_{s,z}(\theta) \mathcal{W}(d\theta) \quad \text{and} \quad \widehat{P}_{s,t} := \mathbb{E}_{\mathcal{W}}[P_{s,t}] := \int_{\Theta} P_{s,t}(\theta) \mathcal{W}(d\theta). \quad (2.12)$$

We point out that $\{\widehat{P}_{s,t} : t \geq s \geq 0\}$ is no longer a semigroup and that the canonical process is not a Markov process under the annealed distribution $\widehat{\mathbb{P}}_{s,z}$. Moreover, in the light of (2.10), we can assume without loss of generality that $s = 0$ in (2.2) and we set

$$\mathbb{P}_z(\theta) := \mathbb{P}_{0,z}(\theta), \quad P_\theta(z; t, dx) = P_\theta(0, z; t, dx), \quad p_\theta(z; t, x) = p_\theta(0, z; t, x), \\ P_t(\theta) := P_{0,t}(\theta) \quad \text{and} \quad \widehat{P}_t := \widehat{P}_{0,t}. \quad (2.13)$$

Furthermore, we can see that the case where $r = 0$ is of particular interest since relation (2.10) can be written in this situation

$$P_{s,s+t}(\theta) = P_t(T_s \theta) \quad \text{and} \quad P_{s+t}(\theta) = P_s(\theta)P_t(T_s \theta). \quad (2.14)$$

Roughly speaking, the diffusion equation (2.2) is time-homogeneous in distribution since from the scaling property \mathcal{W} is (T_t) -invariant. The relation (2.14) is called the cocycle property and it induces (see [32] for a definition) a random dynamical system (RDS) over $(\Theta, \mathcal{B}, \mathcal{W}, (T_t))$ on the set \mathcal{M} of signed measures on \mathbb{R} , by setting, for all $\nu \in \mathcal{M}$,

$$\nu P_t(\theta)(dx) := \int_{\mathbb{R}} P_\theta(z; t, dx) \nu(dz) = \left(\int_{\mathbb{R}} p_\theta(z; t, x) \nu(dz) \right) dx. \quad (2.15)$$

Note that the set of probability measure on \mathbb{R} , denoted by \mathcal{M}_1 , is invariant under this RDS.

2.3 Quasi-invariant and stationary probability measures

To state our next important results, we need to introduce some additional notations and definitions. We said that μ is a random probability measure on \mathbb{R} , over $(\Theta, \mathcal{B}, \mathcal{W})$, if $\mu_\theta \in \mathcal{M}_1$ for \mathcal{W} -almost surely θ , and if $\theta \mapsto \mu_\theta(A)$ is measurable for all Borel set A . For such random probability measure μ , we introduce the probability measure $\hat{\mu}$ defined by

$$\hat{\mu} := \mathbb{E}_{\mathcal{W}}[\mu] := \int_{\Theta} \mu_\theta \mathcal{W}(d\theta). \quad (2.16)$$

Let U_α and V_α be the functions on \mathbb{R} defined by

$$U_\alpha(x) := \exp\left(\alpha \frac{x^2}{2}\right) \quad \text{and} \quad V_\alpha(x) := \exp(|x|^\alpha). \quad (2.17)$$

The U_α -total variation norm and the V_α -total variation norm of $\mathbf{v} \in \mathcal{M}$ are defined as in [33] by

$$\|\mathbf{v}\|_{U_\alpha} := \sup_{|f| \leq U_\alpha} |\mathbf{v}(f)| \quad \text{and} \quad \|\mathbf{v}\|_{V_\alpha} := \sup_{|f| \leq V_\alpha} |\mathbf{v}(f)|, \quad (2.18)$$

the supremum being taken on the set of bounded measurable functions f on \mathbb{R} . Note that if $\mathbf{v} \in \mathcal{M}_1$ then $\|\mathbf{v}\|_{U_\alpha}$ and $\|\mathbf{v}\|_{V_\alpha}$ are nothing but $\mathbf{v}(U_\alpha)$ and $\mathbf{v}(V_\alpha)$. In addition, we set

$$\begin{aligned} \mathcal{M}_{U_\alpha} &:= \{\mathbf{v} \in \mathcal{M} : \|\mathbf{v}\|_{U_\alpha} < \infty\}, & \mathcal{M}_{V_\alpha} &:= \{\mathbf{v} \in \mathcal{M} : \|\mathbf{v}\|_{V_\alpha} < \infty\}, \\ \mathcal{M}_{1,U_\alpha} &= \mathcal{M}_1 \cap \mathcal{M}_{U_\alpha} & \text{and} & \quad \mathcal{M}_{1,V_\alpha} = \mathcal{M}_1 \cap \mathcal{M}_{V_\alpha}. \end{aligned} \quad (2.19)$$

Theorem 2.3. *Assume that $r = 0$. There exists a random probability measure μ on \mathbb{R} over $(\Theta, \mathcal{B}, \mathcal{W})$, unique up to a \mathcal{W} -null set, such that, for all $t \geq 0$,*

$$\mu_\theta P_t(\theta) = \mu_{T_t \theta} \quad \mathcal{W}\text{-a.s.} \quad (2.20)$$

Moreover, for all $\alpha \in (0, 1)$, the quasi-invariant measure satisfies

$$\mu_\theta \in \mathcal{M}_{1,U_\alpha} \quad \mathcal{W}\text{-a.s.} \quad \text{and} \quad \hat{\mu} \in \mathcal{M}_{1,V_\alpha}. \quad (2.21)$$

Furthermore, there exists $\lambda > 0$ such that, for all $\mathbf{v} \in \mathcal{M}_{1,U_\alpha}$ and $\hat{\mathbf{v}} \in \mathcal{M}_{1,V_\alpha}$,

$$\limsup_{t \rightarrow \infty} \frac{\log\left(\|\mathbf{v} P_t(\theta) - \mu_{T_t \theta}\|_{U_\alpha}\right)}{t} \leq -\lambda \quad \mathcal{W}\text{-a.s.} \quad (2.22)$$

and

$$\lim_{t \rightarrow \infty} \left\| \hat{\mathbf{v}} P_t - \hat{\mu} \right\|_{V_\alpha} = 0. \quad (2.23)$$

Linear RDSs has been studied in an extensive body of the literature. The dynamics (in particular the Lyapunov exponents) in the case where the discrete-time linear RDS acts on a finite dimensional space (the case of infinite products of random matrices) has been well understood for a long time, for instance in [34, 35], whereas the situation where the general linear RDS acts on a separable Banach space has been newly studied in [36].

Our goal in Theorem 2.3 is to obtain a quasi-invariant probability measure for the random Markov kernels $P_t(\theta)$ and to give convergence results in the separable Banach space \mathcal{M}_{U_α} (exponential convergence) and \mathcal{M}_{V_α} . We need a kind of random Perron-Frobenius theorem, which has been, for example, obtained in [37] for infinite products of nonnegative matrices, and more recently in [38] for infinite products of stationary Markov kernels over a compact set.

However, the Markov operators that we consider act on the infinite dimensional space \mathcal{M} and are defined over the noncompact set \mathbb{R} . To overcome this problem, we need to see that U_α and V_α are Foster-Lyapunov functions (see Propositions 6.2 and 6.3). More precisely, we show that Lyapunov exponents can be chosen independently of the environment θ , while keeping a control on the expectation of the U_α -norm and the V_α -norm. The classical method to construct Foster-Lyapunov functions for Markov kernels is to construct Lyapunov functions for the infinitesimal generators (see Lemma 6.1 and 6.2). Nonetheless, we stress that neither U_α nor V_α belong to the generalized domain $\overline{D}(L_\theta)$ and we need to approximate uniformly these functions by functions of this domain, while keeping a control on the expectation under the Wiener measure. This is possible by using the Hölder continuity of Brownian paths (see Proposition 6.1).

Then, we use the explicit bound on convergence of time-inhomogeneous Markov chains (see Proposition 6.4), obtained from [33], via coupling constructions, Foster-Lyapunov conditions and the cocycle property, together with the ergodicity of the underlying dynamical system $(\Theta, \mathcal{B}, \mathcal{W}, (T_t)_{t \in \mathbb{R}})$. We point out that the Aronson estimate (2.11) is necessary to the coupling constructions.

Furthermore, let us denote by $\{U_t : t \geq 0\}$ the canonical process on the space Ξ of continuous functions from $[0, \infty)$ to Θ , endowed with its natural σ -field \mathcal{G} , and introduce the Markov kernels $\Pi_{\theta, z}$ on $(\Xi \times \Omega, \mathcal{G} \otimes \mathcal{F})$, and the probability measure $\bar{\mu}$ on $(\Theta \times \mathbb{R}, \mathcal{B} \otimes \mathcal{B}(\mathbb{R}))$, respectively defined by the product and disintegration formula

$$\Pi_{\theta, z} := \delta_{\{T_t \theta : t \geq 0\}} \otimes \mathbb{P}_z(\theta) \quad \text{and} \quad \bar{\mu}(d\omega, dx) := \mathcal{W}(d\omega) \mu_\omega(dx). \quad (2.24)$$

Then we can see that $\{(U_t, X_t) : t \geq 0\}$ is a time-homogeneous Markov process under $\Pi_{\theta, z}$ such that $\bar{\mu}$ is an invariant initial distribution. This process is called the skew-product Markov process (see [39, 40] for the discrete-time situation). By applying standard results on general time-homogeneous Markov processes (see for instance [41]) we deduce that for all $F \in L^1(\Theta \times \mathbb{R}, \bar{\mu})$, $z \in \mathbb{R}$ and \mathcal{W} -a.s. $\theta \in \Theta$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t F(U_\tau, X_\tau) d\tau = \int_{\Theta \times \mathbb{R}} F(\omega, x) \bar{\mu}(d\omega, dx), \quad \Pi_{\theta, z}\text{-a.s.} \quad (2.25)$$

Moreover, we note that equation (2.21) provides some information on the tails of μ_θ and $\hat{\mu}$. Especially, one can see that for all $\alpha \in (0, 1)$, there exists a random variable $C : \Theta \rightarrow (0, \infty)$ and $\hat{C} > 0$ such that, for all $R \geq 0$,

$$\mu_\theta((R, \infty)) \leq C_\theta \exp\left(-\alpha \frac{R^2}{2}\right) \quad \mathcal{W}\text{-a.s.} \quad \text{and} \quad \hat{\mu}((R, \infty)) \leq \hat{C} \exp(-R^\alpha). \quad (2.26)$$

Theorem 2.4. *Assume that $r > 0$. For all $z \in \mathbb{R}$ and for \mathcal{W} -almost surely $\theta \in \Theta$, the following convergence holds under $\mathbb{P}_z(\theta)$,*

$$\lim_{t \rightarrow \infty} X_t \stackrel{(d)}{=} \mathcal{N}(0, 1). \quad (2.27)$$

Here the space-time mixing environment is, contrary to Theorem 2.20, asymptotically negligible and the diffusion behaves, in long time, as the underlying Ornstein-Uhlenbeck process. Since the cocycle property (2.14) is no longer satisfied, we lose the structure of linear RDS. To prove this result, we use once again Proposition 6.4 but we also need to apply [17, Lemma 4.5] to the equivalent stochastic differential equation with random continuous coefficients (4.3). Following the terminology used in [17], it is not difficult to see that this equation is asymptotically time-homogeneous and $S_*\Gamma$ -ergodic, with S the scale function of the Ornstein-Uhlenbeck diffusion process having $\Gamma \sim \mathcal{N}(0, 1)$ as stationary distribution and $S_*\Gamma$ the pushforward distribution of Γ by S . As they mention in [17], the main difficulty to apply this lemma is usually to show the boundedness in probability. To this end, we need to use again the Foster-Lyapunov functions U_α and V_α .

3 Application to time-inhomogeneous Brox's diffusions

3.1 Associated models

We turn now to our main application, the study of the so-called time-inhomogeneous Brox's diffusion. We consider, for any $\beta \in \mathbb{R}$, the informal stochastic differential equation driven by a standard Brownian motion B , independent of the environment $(\Theta, \mathcal{B}, \mathcal{W})$,

$$dY_t = dB_t - \frac{1}{2} \frac{\theta'(Y_t)}{t^\beta} dt, \quad Y_u = y \in \mathbb{R}, t \geq u > 0, \theta \in \Theta. \quad (3.1)$$

A solution to equation (3.1) is, in the same manner as in definition 2.1, the diffusion whose conditional generator, given $\theta \in \Theta$, is

$$\mathcal{L}_\theta := \left[\frac{1}{2} e^{\theta(x)/t^\beta} \frac{\partial}{\partial x} \left(e^{-\theta(x)/t^\beta} \frac{\partial}{\partial x} \right) \right] + \frac{\partial}{\partial t}, \quad (3.2)$$

with domain

$$D(\mathcal{L}_\theta) := \left\{ F(t, x) \in C^1 : e^{-\theta(x)/t^\beta} \partial_x F(t, x) \in C^1 \right\}. \quad (3.3)$$

As for equation (2.2), where we can assume without loss of generality that $s = 0$, we can assume that $u = 1$ in equation (3.1). Moreover, as in (1.9), we assume that $\beta = r - 1/4$ and we define, for all continuous functions ω on $[1, \infty)$ and all measurable function G on $[1, \infty) \times \mathbb{R}$, $\Phi_e(\omega)(t) := \omega(e^t)/e^{t/2}$ and $\mathcal{E}G(t, x) := G(e^t, e^{t/2}x)$. It is a simple calculation to see that $\mathcal{E} : D(\mathcal{L}_\theta) \rightarrow D(L_\theta)$ is a bijection and that $L_\theta = \mathcal{E} \circ \mathcal{L}_\theta \circ \mathcal{E}^{-1}$. In the same way as in [17, Proposition 2.1 and Section 2.2.1] we deduce that $\{Y_t : t \geq 1\}$ is a weak solution to equation (3.1) if and only if $\{Z_t := \Phi_e(Y_t) : t \geq 0\}$ is a weak solution to equation (2.2). Then a direct application of Theorem 2.1 gives that for all $\theta \in \Theta$, there exists a unique irreducible strongly Feller diffusion process solution to equation (3.1). Let $\mathbb{Q}_y(\theta)$ be its quenched distribution and denote by $\{R_t(\theta) : t \geq 1\}$, the time-inhomogeneous semigroup associated to $\{X_t/\sqrt{t} : t \geq 1\}$ under $\mathbb{Q}_y(\theta)$, and by $\widehat{\mathbb{Q}}_y$ and $\{\widehat{R}_t : t \geq 1\}$ there annealed counterparts.

3.2 Associated asymptotic behaviours

The following two corollaries are the analogous of Theorems 2.3 and 2.4. We recall that S_λ , $\lambda > 0$, defined in (1.2), denote the standard scaling transformations on the Wiener space.

Corollary 3.1. *Assume that $\beta = 1/4$. For all $\alpha \in (0, 1)$ there exists $\lambda > 0$ such that, for all $\nu \in \mathcal{M}_{1, U_\alpha}$ and $\widehat{\nu} \in \mathcal{M}_{1, V_\alpha}$,*

$$\limsup_{t \rightarrow \infty} \frac{\log \left(\left\| \nu R_t(\theta) - \mu_{S_{\sqrt{t}}\theta} \right\|_{U_\alpha} \right)}{\log t} \leq -\lambda \quad \mathcal{W}\text{-a.s.} \quad (3.4)$$

and

$$\lim_{t \rightarrow \infty} \left\| \widehat{\nu} \widehat{R}_t - \widehat{\mu} \right\|_{V_\alpha} = 0. \quad (3.5)$$

Corollary 3.2. *Assume that $\beta > 1/4$. For all $y \in \mathbb{R}$ and for \mathcal{W} -almost surely $\theta \in \Theta$, the following convergence holds under $\mathbb{Q}_y(\theta)$,*

$$\lim_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} \stackrel{(d)}{=} \mathcal{N}(0, 1). \quad (3.6)$$

The scaling limits (3.4), (3.5) and (3.6) are to be compared with the two convergences presented in (1.7) (the deterministic situation studied in [17]) and convergences (1.4) and (1.5) (the random time-homogeneous situation considered in [4]). These results have some common features with those presented in [17] and [4] and also with those presented in [10, 19, 11, 20, 26, 27, 28, 29] concerning the

quenched central limit theorem (3.6). There is still a phase transition phenomenon for $\beta = 1/4$ and we obtain distinct quenched and annealed scaling limits for the critical point. Moreover, we are more accurate concerning the speed of convergence, which is polynomial here, and exponential in Theorem 2.3.

Nevertheless, the case where $\beta < 1/4$ seems to be out of range of the present technics. In fact, we wait a stronger localization phenomenon and a subdiffusive behaviour of order $t^{2\beta} \log^2(t)$ when $\beta \geq 0$ and an almost sure convergence when $\beta < 0$ (which can be seen as a generalization and mixture of results presented in (1.4), (1.5) and (1.8)). Note that in the case where $\beta < 0$ equation (3.1) is (via a simple change of time) a damped stochastic differential equation in random environment. Furthermore, some methods elaborated in this paper can be used to study a similar interesting situation where we replace the Brownian environment θ in (3.1) by another self-similar process. These situations are object of some works in progress. The case of a multiplicative noise or similar equations in higher dimension seems to be more difficult.

4 Proof of Theorem 2.1

Theorem 2.1 will be a direct consequence of Propositions 4.1 and 4.2.

4.1 Equivalent SDE and martingale problem

We introduce an auxiliary stochastic differential equation, which is naturally connected to equation (2.2). Let S and H be the functions on $\Theta \times \mathbb{R}^2$ defined by

$$S_\theta(t, x) := \int_0^x e^{V_\theta(t, y)} dy = e^{-t/2} \int_0^{e^{t/2}x} \exp\left(\frac{e^{-t}z^2}{2} - e^{-(\frac{t}{4}+r)}\theta(z)\right) dz \quad \text{and} \quad S_\theta(t, H_\theta(t, x)) = x. \quad (4.1)$$

Recall that the time-dependent random potential V is defined in (2.2) and note that H is well defined since the so-called pseudo-scale function $x \mapsto S_\theta(t, x)$ is an increasing bijection of \mathbb{R} . Moreover, by using the second representation of S_θ , obtained by the change of time $z := e^{t/2}y$, we can see that $S_\theta(t, x)$ and $H_\theta(t, x)$ are continuously differentiable with respect to $(t, x) \in \mathbb{R}^2$ and we can set

$$\sigma_\theta(t, x) := (\partial_x S_\theta)(t, H_\theta(t, x)) \quad \text{and} \quad d_\theta(t, x) := (\partial_t S_\theta)(t, H_\theta(t, x)). \quad (4.2)$$

Then we can consider, for any $\theta \in \Theta$, the stochastic differential equation with continuous coefficients and driven by a standard Brownian motion B , independent of $(\Theta, \mathcal{B}, \mathcal{W})$,

$$dZ_t = \sigma_\theta(t, Z_t) dB_t + d_\theta(t, Z_t) dt, \quad Z_s = z \in \mathbb{R}, \quad t \geq s \geq 0. \quad (4.3)$$

Remark that, for all $(\theta, s, t, x) \in \Theta \times \mathbb{R}^3$,

$$S_\theta(s+t, x) = S_{(e^{-rs}T_s\theta)}(t, x), \quad H_\theta(s+t, x) = H_{(e^{-rs}T_s\theta)}(t, x) \\ \sigma_\theta(s+t, x) = \sigma_{(e^{-rs}T_s\theta)}(t, x) \quad \text{and} \quad d_\theta(s+t, x) = d_{(e^{-rs}T_s\theta)}(t, x). \quad (4.4)$$

Let $C^{1,2}$ be the space of continuous functions $F(t, x)$ on $[s, \infty) \times \mathbb{R}$ such that $\partial_t F$, $\partial_x F$ and $\partial_{xx}^2 F$ exist and are continuous functions and denote by L_θ the generator of (4.3) given by

$$L_\theta := L_{\theta, t} + \frac{\partial}{\partial t} := \left[\frac{\sigma_\theta^2(t, x)}{2} \frac{\partial^2}{\partial x^2} + d_\theta(t, x) \frac{\partial}{\partial x} \right] + \frac{\partial}{\partial t}. \quad (4.5)$$

Note that S_θ and H_θ induce two bijections from the space of measurable functions on $[s, \infty) \times \mathbb{R}$ into itself, inverse to each other, by setting

$$\mathcal{S}_\theta F(t, x) := F(t, S_\theta(t, x)) \quad \text{and} \quad \mathcal{H}_\theta F(t, x) := F(t, H_\theta(t, x)). \quad (4.6)$$

By restriction, we get that \mathcal{S}_θ and \mathcal{H}_θ induce bijections

$$\mathcal{S}_\theta : C^{1,2} \longrightarrow D(L_\theta), \quad \mathcal{H}_\theta : D(L_\theta) \longrightarrow C^{1,2} \quad \text{and} \\ \mathcal{S}_\theta : W_{\text{loc}}^{1,2,\infty} \longrightarrow \bar{D}(L_\theta), \quad \mathcal{H}_\theta : \bar{D}(L_\theta) \longrightarrow W_{\text{loc}}^{1,2,\infty}, \quad (4.7)$$

where $W_{\text{loc}}^{1,2,\infty}$ denote the Sobolev space of continuous functions $F(t, x)$ on $[s, \infty) \times \mathbb{R}$, such that the partial derivatives $\partial_t F$, $\partial_x F$, $\partial_t(\partial_x F)$ and $\partial_{xx}^2 F$ exist and are locally bounded functions. Moreover, the infinitesimal generators L_θ and \bar{L}_θ are equivalent. More precisely, they satisfy

$$\mathcal{S}_\theta^{-1} \circ L_\theta \circ \mathcal{S}_\theta = \bar{L}_\theta. \quad (4.8)$$

Proposition 4.1. *For all $r \in \mathbb{R}$, $\theta \in \Theta$, $s \geq 0$ and $z, z \in \mathbb{R}$ such that $z := S_\theta(s, z)$, the process $\{Z_t : t \geq s\}$ is a weak solution to equation (2.2) if and only if $\{Z_t := S_\theta(t, Z_t) : t \geq s\}$ is a weak solution (up to the explosion time τ_e) to the stochastic differential equation (4.3). Furthermore, there exists a unique weak solution (Z, B) and, for all $G \in W_{\text{loc}}^{1,2,\infty}$,*

$$G(t, Z_t) = G(s, z) + \int_s^t L_\theta G(u, Z_u) du + \int_s^t \partial_x G(u, Z_u) \sigma_\theta(u, Z_u) dB_u, \quad s \leq t < \tau_e. \quad (4.9)$$

Proof. Assume that Z be a weak solution to equation (4.3). By using the Ito formula, Z solves the martingale problem related to $(L_\theta, C^{1,2})$. Therefore, $Z_s = z$ and there exists an increasing sequence of stopping time $\{\tau_n : n \geq 0\}$ such that, for all $n \geq 0$ and $G \in C^{1,2}$,

$$G(t \wedge \tau_n, Z_{t \wedge \tau_n}) - \int_s^{t \wedge \tau_n} L_\theta G(u, Z_u) du, \quad t \geq s,$$

is a local martingale, with

$$\tau_e := \sup_{n \geq 0} \inf \{t \geq s : |Z_t| \geq n\} = \sup_{n \geq 0} \tau_n.$$

We deduce from relation (4.8) that $\{Z_t := H_\theta(t, Z_t) : t \geq s\}$ is a weak solution to (2.2) since $Z_s = z$, for all $n \geq 0$ and $F \in D(L_\theta)$, $G := \mathcal{H}_\theta F \in C^{1,2}$, and

$$F(t \wedge \tau_n, Z_{t \wedge \tau_n}) - \int_s^{t \wedge \tau_n} L_\theta F(u, Z_u) du = G(t \wedge \tau_n, Z_{t \wedge \tau_n}) - \int_s^{t \wedge \tau_n} L_\theta G(u, Z_u) du.$$

A similar reasoning allow us to show that if Z is a weak solution to equation (2.2) then $\{Z_t := S_\theta(t, Z_t) : t \geq s\}$ is a weak solution to equation (4.3). Moreover, equation (4.3) has continuous coefficients σ_θ and d_θ and is strictly elliptic (that is $\sigma_\theta > 0$) and we deduce, by using classical arguments of localization (see, for instance, [42, p. 250-251]), that there exists a unique weak solution (Z, B) . Furthermore, by using the classical Ito-Krylov formula (see, for instance, [43, Chapter 10] or [44, p. 134]), we obtain (4.9). \square

4.2 Chain rules

To construct Lyapunov functions for the infinitesimal generator L_θ we need to give the associated chain rules. For all $\theta \in \Theta$ and $\varphi \in W_{\text{loc}}^{1,\infty}$ (the space of real continuous functions such that the partial derivatives in the sense of distributions exist and are locally bounded functions) define

$$F_\theta^\varphi(t, x) := \int_0^x \exp[e^{-rt} T_t \theta(y)] \varphi(t, y) dy \in \bar{D}(L_\theta). \quad (4.10)$$

By standard computations, we get the following chain rules

$$L_{\theta,t}F_{\theta}^{\varphi}(t,x) = \frac{1}{2} \exp [e^{-rt} T_t \theta(x)] (\partial_x \varphi(t,x) - x\varphi(t,x)), \quad (4.11)$$

and

$$\begin{aligned} \partial_t F_{\theta}^{\varphi}(t,x) &= \frac{x}{2} \exp [e^{-rt} T_t \theta(x)] \varphi(t,x) - \frac{1}{2} F_{\theta}^{\varphi}(t,x) + \int_0^x \exp [e^{-rt} T_t \theta(y)] \left(\partial_t \varphi(t,y) - \frac{y}{2} \partial_x \varphi(t,y) \right) dy \\ &\quad - \left(\frac{1}{4} + r \right) \int_0^x \exp [e^{-rt} T_t \theta(y)] (e^{-rt} T_t \theta(y)) \varphi(t,y) dy. \end{aligned} \quad (4.12)$$

4.3 Nonexplosion

Proposition 4.2. *For all $r \in \mathbb{R}$, $\theta \in \Theta$, $s \geq 0$ and $z \in \mathbb{R}$, the explosion time of any weak solution Z to equation (2.2) is infinite a.s.*

Proof. Let $0 < \alpha < 1/2$ and U_{α} be the function defined in (2.17) and set

$$U_{\theta}(t,x) := 1 + \int_0^x \exp [e^{-rt} T_t \theta(y)] U'_{\alpha}(y) dy \in \overline{D}(L_{\theta}).$$

We shall prove that U_{θ} is a Lyapunov function, in the sense that, for all $T > 0$, there exists $\lambda > 0$ such that, for all $0 \leq t \leq T$ and $x \in \mathbb{R}$,

$$L_{\theta}U_{\theta}(t,x) \leq \lambda U_{\theta}(t,x) \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \inf_{0 \leq t \leq T} U_{\theta}(t,x) = \infty. \quad (4.13)$$

Note that, since θ is continuous and $\lim_{|x| \rightarrow \infty} \theta(x)/x^2 = 0$, the second relation in (4.13) is clear. Moreover, by using (4.11) and (4.12), we can see that

$$L_{\theta,t}U_{\theta}(t,x) = -\alpha(1-\alpha) \left(1 - \frac{1}{(1-\alpha)x^2} \right) x^2 \exp [e^{-rt} T_t \theta(x)] U_{\alpha}(x) \quad (4.14)$$

and

$$\begin{aligned} \partial_t U_{\theta}(t,x) &= \frac{\alpha}{2} x^2 \exp [e^{-rt} T_t \theta(x)] U_{\alpha}(x) - (U_{\theta}(t,x) - 1) - \int_0^x \frac{\alpha y^2}{2} \exp [e^{-rt} T_t \theta(y)] U'_{\alpha}(y) dy \\ &\quad - \int_0^x \left(\frac{1}{4} + r \right) \exp [e^{-rt} T_t \theta(y)] (e^{-rt} T_t \theta(y)) U'_{\alpha}(y) dy. \end{aligned} \quad (4.15)$$

We can see that for x sufficiently large

$$-\alpha(1-\alpha) \left(1 - \frac{1}{(1-\alpha)x^2} \right) + \frac{\alpha}{2} < 0. \quad (4.16)$$

Then we get from (4.14) that there exist $L_1 > 0$ and a compact set C such that, for all $0 \leq t \leq T$ and $x \in \mathbb{R}$,

$$L_{\theta,t}U_{\theta}(t,x) + \frac{\alpha}{2} x^2 \exp [e^{-rt} T_t \theta(x)] U_{\alpha}(x) \leq L_1 \mathbb{1}_C(x) \leq L_1 U_{\theta}(t,x). \quad (4.17)$$

Besides, can see that there exists $L_2 > 0$ such that, for all $0 \leq t \leq T$ and $y \in \mathbb{R}$,

$$\frac{\alpha y^2}{2} + \left(\frac{1}{4} + r \right) \geq \frac{\alpha y^2}{4} - L_2 \quad (4.18)$$

We deduce from (4.18), (4.17) and (4.15) that (4.13) is satisfied with $\lambda := L_1 + L_2$. By using a classical argument (see for instance [42, Theorem 10.2.1]), we can complete the proof. \square

5 Proof of Theorem 2.2

We show that it suffices to prove the analogous theorem for diffusion equation (4.3).

Theorem 5.1. *For all $r \in \mathbb{R}$ and $\theta \in \Theta$, $\{\mathbf{P}_{s,z}(\theta) : s \geq 0, z \in \mathbb{R}\}$ is a strongly Feller continuous family. Moreover, the time-inhomogeneous semigroups $\{P_{s,t}(\theta) : t \geq s \geq 0, \theta \in \Theta\}$ satisfy*

$$P_{s,s+t}(\theta) = P_{0,t}(e^{-rs}T_s\theta) \quad \text{and} \quad P_{0,s+t}(\theta) := P_{0,s}(\theta)P_{0,t}(e^{-rs}T_s\theta). \quad (5.1)$$

Besides, $P_\theta(s, z; t, dx)$ admits a density $p_\theta(s, z; t, x)$, which is measurable with respect to (θ, s, t, z, x) on $\Theta \times \{t > s \geq 0\} \times \mathbb{R}^2$, and which satisfies the lower local Aronson estimate: for all $\theta \in \Theta$, $T > 0$ and compact set $C \subset \mathbb{R}$, there exists $M > 0$ such that, for all $0 \leq s < t \leq T$ and $z, x \in C$,

$$p_\theta(s, z; t, x) \geq \frac{1}{\sqrt{M(t-s)}} \exp\left(-M \frac{|z-x|^2}{t-s}\right). \quad (5.2)$$

Indeed, let $\mathbf{P}_{s,z}(\theta)$ be the distribution of the solution to equation (4.3) and denote by $P_\theta(s, z; t, dx)$ and by $P_{s,t}(\theta)$ the associated transition kernels and Markov kernels. Firstly, assume that $\{\mathbf{P}_{s,z}(\theta) : s \geq 0, z \in \mathbb{R}\}$ is strongly Feller continuous. One get by using Proposition 4.1 that, for all bounded measurable function F on $[0, \infty) \times \mathbb{R}$, $t \geq s \geq 0$ and $z \in \mathbb{R}$,

$$\mathbb{E}_{s,z}(\theta)[F(t, X_t)] = \mathbf{E}_{s, S_\theta(s,z)}(\theta)[F(t, H_\theta(t, X_t))].$$

Since S_θ is continuous on \mathbb{R}^2 , we deduce that $\{\mathbf{P}_{s,z}(\theta) : s \geq 0, z \in \mathbb{R}\}$ is also strongly Feller continuous. Secondly, assume that $\{P_{s,t}(\theta) : t \geq s \geq 0\}$ satisfies relations (5.1). We get from (4.4) and Proposition 4.1 that, for all nonnegative function F on \mathbb{R} , $s, t \geq 0$ and $z \in \mathbb{R}$,

$$\begin{aligned} P_{s,s+t}(\theta)F(z) &= P_{s,s+t}(\theta)[F(H_\theta(s+t, \star))](S_\theta(s, z)) \\ &= P_{0,t}(e^{-rs}T_s\theta)[F(H_{(e^{-rs}T_s\theta)}(t, \star))](S_{(e^{-rs}T_s\theta)}(0, z)) = P_{0,t}(e^{-rs}T_s\theta)F(z). \end{aligned}$$

By using the Markov property, we obtain relations (2.14). Finally, if $P_\theta(s, z; t, dx)$ admits a measurable density $p_\theta(s, z; t, x)$ which satisfies the lower local Aronson estimate (5.2). Once again, Proposition 4.1 applies and gives that $P_\theta(s, z; t, dx)$ admits a density p such that

$$p_\theta(s, z; t, x) = p_\theta(s, S_\theta(s, z); t, S_\theta(t, x))e^{V_\theta(t, x)}.$$

Since S_θ is a locally Lipschitz function, we deduce that $p_\theta(s, z; t, dx)$ is also measurable and satisfies (2.11). This completes the proof, excepted for Theorem 5.1.

Proof. Since equation (4.3) is strictly elliptic ($\sigma_\theta > 0$) and has continuous coefficients, it is classical (see for instance [42, Corollary 10.1.4]) that its unique weak solution is a strongly Feller continuous diffusion, which admits transition densities $p_\theta(s, z; t, x)$ measurable with respect to $(s, t, z, x) \in \{t > s \geq 0\} \times \mathbb{R}^2$ for each $\theta \in \Theta$. Moreover, we can see that relations (5.1) are direct consequences of the Markov property and (4.4). We need to prove the measurability of p on $\Theta \times \{t > s \geq 0\} \times \mathbb{R}^2$ and the lower local Aronson estimate (5.2). To this end, set for all $\delta \geq 0$,

$$P_{\delta, \theta}(s, z; t, dx) := \mathbf{P}_{s,z}(\theta)(X_t \in dx, \tau_\delta(s) > t) = \mathbf{P}_{s,z}^{(\delta)}(\theta)(X_t \in dx, \tau_\delta > t)$$

with

$$\tau_\delta(s) := \inf\{t \geq s : |X_t| \geq \delta\} \wedge T.$$

Here $\mathbf{P}_{s,z}^{(\delta)}(\theta)$ denotes the distribution of the truncated diffusion process, whose coefficients are given on $[s, \infty) \times \mathbb{R}$ by

$$d_{\theta}^{(\delta)}(t,x) := d_{\theta}(t \wedge T, (x \wedge \delta) \vee -\delta) \quad \text{and} \quad \sigma_{\theta}^{(\delta)}(t,x) := \sigma_{\theta}(t \wedge T, (x \wedge \delta) \vee -\delta).$$

It is not difficult to see that the fundamental solution $p_{\theta}^{(\delta)}$ of the associated partial differential equation (PDE) satisfies the global Aronson estimates. Indeed, even if the associated partial differential operator is not of divergence form, we can see that it is equivalent to a uniformly elliptic divergence type operator, with bounded coefficients, employing the change of scale defined on $[s, \infty) \times \mathbb{R}$ by

$$k_{\theta}^{(\delta)}(t,x) := \int_0^x \frac{1}{(\sigma_{\theta}^{(\delta)}(t,y))^2} \exp\left(2 \int_0^y \frac{d_{\theta}^{(\delta)}(t,z)}{(\sigma_{\theta}^{(\delta)}(t,z))^2} dz\right) dy.$$

Therefore the results in [45] apply and we deduce that the fundamental solution $q_{\theta}^{(\delta)}$ of the associated PDE satisfies the global lower local Aronson estimate. Since $k_{\theta}^{(\delta)}$ is locally Lipschitz and

$$p_{\theta}^{(\delta)}(s,z,t,x) = q_{\theta}^{(\delta)}(s, k_{\theta}^{(\delta)}(s,z), t, k_{\theta}^{(\delta)}(t,x)) \partial_x k_{\theta}^{(\delta)}(t,x),$$

it is the same for $p_{\theta}^{(\delta)}$. Then, following exactly the same lines as the proof of [46, Theorem II.1.3] in the time-homogeneous situation, we can prove that, for all $0 < \eta < 1$, there exists $M > 0$ such that, for all $0 \leq s < t \leq T$, $|z| \leq \eta\delta$, $|x| \leq \eta\delta$ and $|t-s| \leq (\eta\delta)^2$,

$$p_{\theta}^{(\delta)}(s,z,t,x) \geq \frac{1}{\sqrt{M(t-s)}} \exp\left(-M \frac{|x-z|^2}{t-s}\right).$$

Since $p \geq p_{\delta}$ we deduce that p satisfies (5.2) by taking δ sufficiently large. It remains to prove the measurability of p . Since $(\theta, t, x) \mapsto \sigma_{\theta}(t, x)$ and $(\theta, t, x) \mapsto d_{\theta}(t, x)$ are continuous on $\Theta \times \mathbb{R}^2$, we can apply [42, Theorem 11.1.4] and we deduce that, for all bounded continuous function G on the canonical space Ω , $(\theta, s, z) \mapsto \mathbf{E}_{s,z}(\theta)[G]$ is continuous on $\Theta \times [0, \infty) \times \mathbb{R}$. Then we can see that, for all bounded measurable function F on \mathbb{R} , $(\theta, s, z, t) \mapsto \mathbf{E}_{s,z}(\theta)[F(X_t)]$ is measurable on $\Theta \times \{t > s \geq 0\} \times \mathbb{R}$. Indeed, assume that F is L -lipschitz. We can write, for all compact set K of the canonical space Ω ,

$$\begin{aligned} |\mathbf{E}_{s,z}(\theta)[F(X_t)] - \mathbf{E}_{s_0,z_0}(\theta_0)[F(X_{t_0})]| &\leq L \mathbf{P}_{s,z}(\theta)(\Omega \setminus K) + L \mathbf{E}_{s,z}(\theta)[\mathbb{1}_K |X_t - X_{t_0}|] \\ &\quad + |\mathbf{E}_{s,z}(\theta)[F(X_{t_0})] - \mathbf{E}_{s_0,z_0}(\theta_0)[F(X_{t_0})]|. \end{aligned}$$

By letting $(s, z, \theta, t) \longrightarrow (s_0, z_0, \theta_0, t_0)$ and by using the tightness of the family of probability measure $\{\mathbf{P}_{s,z}(\theta) : s \geq 0, z \in \mathbb{R}, \theta \in \Theta\}$ we get the continuity and we deduce our claim since any measurable bounded function is the pointwise limit of a sequence of lipschitzian functions. Therefore, we can define the measure ν on the product measurable space $\Theta \times \mathbb{R}^4$ by setting, for all $B \in \mathcal{B}$ and $I_k \in \mathcal{B}(\mathbb{R})$, $k \in \{1, \dots, 4\}$,

$$\nu\left(B \times \prod_{k=1}^4 I_k\right) := \int_{B \times (\prod_{k=1}^3 I_k)} P_{\theta}(s, z, t, I_4) \mathbb{1}_{t > s \geq 0} \mathscr{W}(d\theta) ds dz dt$$

By standard results on disintegration of measures, the Radon-Nykodym derivative of ν with respect to $\mathscr{W}(d\theta) ds dz dt dx$, which is nothing but $p_{\theta}(s, z, t, x)$, is measurable. \square

6 Preliminaries of Theorems 2.3 and 2.4

6.1 Uniform affine approximations of the environment

In the following, set for all $\gamma \in (0, 1/2)$ and $\theta \in \Theta$,

$$H_\gamma(\theta) := \sup_{n \geq 0} \frac{\|\theta^+\|_{\gamma,n} + \|\theta^-\|_{\gamma,n}}{L(n)}, \quad \Theta_\gamma := \{0 < H_\gamma < \infty\}, \quad (6.1)$$

with, for all $n \geq 0$ and $x \in \mathbb{R}$,

$$\|\theta^\pm\|_{\gamma,n} := \sup_{n \leq x < y \leq n+1} \frac{|\theta(\pm y) - \theta(\pm x)|}{|y-x|^\gamma} \quad \text{and} \quad L(x) := \sqrt{1 + \log(1 + |x|)}. \quad (6.2)$$

In addition, denote for all $\varepsilon > 0$ by $A_{\gamma,\varepsilon}(\theta)$ (see figure 1 below) the piecewise linear approximation of θ , associated to the subdivision $S_{\gamma,\varepsilon} := \{x_{n,k} : n \in \mathbb{Z}, 0 \leq k \leq m_n\}$, defined by $m_n := h_n^{-1} := \lceil L^{1/\gamma}(n)\varepsilon^{-1} \rceil + 1 \in \mathbb{N}$, $x_{n,k} := n + kh_n$ and $x_{-n,k} := -x_{n,k}$. Then introduce the random affine approximation $W_{\gamma,\varepsilon}$ defined, for all $\theta \in \Theta_\gamma$ by

$$W_{\gamma,\varepsilon}(\theta) := A_{\gamma,\eta_{\gamma,\varepsilon}(\theta)}(\theta), \quad \text{with} \quad \eta_{\gamma,\varepsilon}(\theta) := \left(\frac{\varepsilon}{H_\gamma(\theta)}\right)^{1/\gamma}, \quad (6.3)$$

and set

$$\Delta_{\gamma,\varepsilon}(\theta)(x) := \theta(x) - W_{\gamma,\varepsilon}(\theta)(x) \quad \text{and} \quad D_{\gamma,\varepsilon}(\theta) := \sup_{x \in \mathbb{R}} \frac{|W'_{\gamma,\varepsilon}(\theta)(x)|}{L^{1/\gamma}(x)}. \quad (6.4)$$

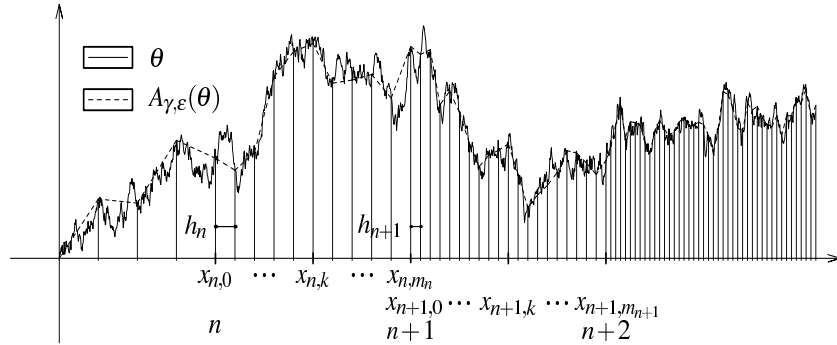


Figure 1: Affine approximation of a typical Brownian path θ

Proposition 6.1. For all $\gamma \in (0, 1/2)$, Θ_γ is (T_t) -invariant and there exists $\alpha > 0$ such that

$$\mathbb{E}_{\mathscr{W}} [\exp(\alpha H_\gamma^2)] := \int_{\Theta} \exp(\alpha H_\gamma^2(\theta)) \mathscr{W}(d\theta) < \infty. \quad (6.5)$$

Moreover, for all $\varepsilon > 0$ and $\theta \in \Theta_\gamma$,

$$\sup_{x \in \mathbb{R}} |\Delta_{\gamma,\varepsilon}(\theta)(x)| \leq \varepsilon \quad \text{and} \quad D_{\gamma,\varepsilon}(\theta) \leq \varepsilon \left(1 + (\varepsilon^{-1} H_\gamma(\theta))^{1/\gamma}\right). \quad (6.6)$$

Proof. Clearly $H_\gamma : \Theta \rightarrow [0, \infty]$ is a seminorm and to get inequality (6.5) it suffices to apply the Fernique theorem presented in [47]. We need to check that $H_\gamma < \infty$ \mathscr{W} -a.s. By using the Hölder continuity of the Brownian motion on compact sets, the seminorm defined on Θ by $N(\theta) := \|\theta^+\|_{\gamma,1} + \|\theta^-\|_{\gamma,1}$ is finite

a.s. Moreover, by using the Fernique theorem and the Markov inequality, we deduce that there exists $c, \beta > 0$ such that, for r sufficiently large,

$$F(r) := \mathscr{W}(\{N \geq r\}) \leq \mathbb{E}_{\mathscr{W}}[\exp(\beta N^2)]e^{-\beta r^2} \leq ce^{-\beta r^2}.$$

Besides, the random variables $(\theta \mapsto \|\theta^+\|_{\gamma,n} + \|\theta^-\|_{\gamma,n})$, $n \geq 0$, being i.i.d. by using the Markov property, we get that

$$\lim_{h \rightarrow \infty} \mathscr{W}(\{H_\gamma \leq h\}) = \lim_{h \rightarrow \infty} \prod_{n=0}^{\infty} (1 - F(hL(n))) \geq \lim_{h \rightarrow \infty} \prod_{n=0}^{\infty} \left(1 - \frac{c}{n\beta h^2}\right) = 1.$$

Fernique's theorem applies and we deduce (6.5). The fact that Θ_γ is (T_t) -invariant is obtained by noting that, for all $\theta \in \Theta$ and $t \in \mathbb{R}$,

$$H_\gamma(T_t \theta) \leq 2e^{[(\gamma - \frac{1}{2})\frac{t}{2}]} (e^{t/2} + 1) \sup_{n \geq 0} \left[\frac{L((n+1)e^{t/2} + 1)}{L(n)} \right] H_\gamma(\theta).$$

Furthermore, let $\varepsilon > 0$, $n \geq 0$ and $x, y \in \mathbb{R}$ be such that $n \leq x, y \leq n+1$ and $|y-x| \leq h_n$, where h_n denotes the step of the subdivision $S_{\gamma,\varepsilon}$ defined in Figure 1. We can see that

$$|\theta^\pm(y) - \theta^\pm(x)| \leq L(n)H_\gamma(\theta) h_n^\gamma \leq H_\gamma(\theta) \varepsilon^\gamma$$

and, when $|y-x| = h_n$, we get

$$\frac{|\theta^\pm(y) - \theta^\pm(x)|}{|y-x|} \leq L(n)H_\gamma(\theta) |y-x|^{\gamma-1} \leq H_\gamma(\theta) \varepsilon^\gamma h_n^{-1} \leq H_\gamma(\theta) \varepsilon^\gamma \left(\varepsilon^{-1} L^{1/\gamma}(n) + 1 \right).$$

Therefore, we obtain that

$$\sup_{x \in \mathbb{R}} |\theta(x) - A_{\gamma,\varepsilon}(\theta)(x)| \leq H_\gamma(\theta) \varepsilon^\gamma \quad \text{and} \quad \sup_{x \in \mathbb{R}} \frac{|A'_{\gamma,\varepsilon}(\theta)(x)|}{L^{1/\gamma}(x)} \leq H_\gamma(\theta) \varepsilon^{\gamma-1} (1 + \varepsilon).$$

Replacing in the two last inequalities ε by $\eta_{\gamma,\varepsilon}(\theta)$, defined in (6.3), we deduce the proposition. \square

6.2 Random Foster-Lyapunov drift conditions

6.2.1 For the infinitesimal generators

Let φ be a twice continuously differentiable function from $[1, \infty)$ into itself such that, $\varphi(v) = 1$ on $[1, 2]$, $\varphi(v) = v$ on $[3, \infty)$ and $\varphi(v) \leq v$ on $[0, \infty)$. In the sequel, we set

$$F_\theta^{\gamma,\varepsilon}(t, x) := 1 + \int_0^x \exp[e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(y)] U'_\alpha(y) dy \in \overline{D}(L_\theta) \quad (6.7)$$

and

$$G_\theta^{\gamma,\varepsilon}(t, x) := 1 + \int_0^x \exp[e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(y)] G'_\alpha(y) dy \in \overline{D}(L_\theta), \quad \text{with } G_\alpha(x) := \varphi(V_\alpha(x)). \quad (6.8)$$

Here we use $G_\alpha = \varphi(V_\alpha)$ in (6.8) instead of V_α because V'_α do not belong to $W_{loc}^{1,\infty}$ (there is a singularity in 0) contrary to U'_α in (6.7).

Lemma 6.1. *For all $r \in \mathbb{R}$, $\alpha \in (0, 1)$, $\gamma \in (0, 1/2)$, $T > 0$ and $\lambda > 0$, there exists $\overline{\varepsilon} > 0$ such that, for all $0 < \varepsilon < \overline{\varepsilon}$, there exist a random variable $B : \Theta \rightarrow [1, \infty)$ and $p, k, c > 0$ such that, for all $\theta \in \Theta_\gamma$, $0 \leq t \leq T$ and $x \in \mathbb{R}$,*

$$L_\theta F_\theta^{\gamma,\varepsilon}(t, x) \leq -\lambda F_\theta^{\gamma,\varepsilon}(t, x) + B_\theta, \quad \text{with } B_\theta \leq k \exp(c H_\gamma^p(\theta)). \quad (6.9)$$

Proof. The proof will be a consequence of the following two steps.

Step 1. For all $0 < \delta < 1$ and $R \geq 1$, there exists $\varepsilon_1 > 0$ such that, for all $0 < \varepsilon < \varepsilon_1$ and $0 < \ell < 1$, there exist a map $R_1 : \Theta \rightarrow [R, \infty)$ and $c_1 > 0$ such that, for all $\theta \in \Theta_\gamma$, $0 \leq t \leq T$ and $|x| \geq R_1(\theta)$,

$$L_{\theta,t}F_\theta^{\gamma,\varepsilon}(t,x) \leq -\delta\alpha(1-\alpha)x^2F_\theta^{\gamma,\varepsilon}(t,x), \quad \text{with } R_1(\theta) \leq c_1 \left(H_\gamma^{\frac{1}{\gamma(1-\delta)}}(\theta) \vee 1 \right). \quad (6.10)$$

By using chain rule (4.11) we obtain that

$$L_{\theta,t}F_\theta^{\gamma,\varepsilon}(t,x) = (-\alpha(1-\alpha)x^2 - \alpha x e^{-rt} (T_t W_{\gamma,\varepsilon}(\theta))'(x) + \alpha) \exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(x)] x^2 U_\alpha(x), \quad (6.11)$$

which can be written

$$L_{\theta,t}F_\theta^{\gamma,\varepsilon}(t,x) = -\alpha(1-\alpha) \left[1 - \frac{1}{(1-\alpha)x^2} + \frac{e^{-rt} (T_t W_{\gamma,\varepsilon}(\theta))'(x)}{(1-\alpha)x} \right] \exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(x)] x^2 U_\alpha(x). \quad (6.12)$$

Moreover, one can see by using the left hand side of (6.6) that

$$|(T_t W_{\gamma,\varepsilon}(\theta))'(x)| \leq \varphi_\gamma(t) D_{\gamma,\varepsilon}(\theta) L^{1/\gamma}(x), \quad \text{with } \varphi_\gamma(t) := (1+t/2)^{1/2\gamma} e^{t/4}. \quad (6.13)$$

In order to simplify our calculations, introduce

$$q := 1 \vee e^{-rT} \quad \text{and} \quad \Psi(\varepsilon) := \exp[q\varepsilon]. \quad (6.14)$$

Note that

$$\Psi^{-1}(\varepsilon) U_\alpha(x) \leq F_\theta^{\gamma,\varepsilon}(t,x) \leq \Psi(\varepsilon) U_\alpha(x). \quad (6.15)$$

We can choose $\varepsilon_1 > 0$ and $D \geq R$ such that

$$\left(1 - \frac{1}{(1-\alpha)D^2} - \frac{q}{(1-\alpha)D} \right) \Psi^{-2}(\varepsilon_1) \geq \delta. \quad (6.16)$$

Then we deduce the left hand side of (6.10) by using (6.16), (6.15), (6.13), (6.12) and by setting, for any $0 < \varepsilon < \varepsilon_1$,

$$R_1(\theta) := [\varphi_\gamma(T) D_{\gamma,\varepsilon}(\theta) c_{\gamma,\ell} \vee 1]^{\frac{1}{1-\ell}} D^{\frac{1}{1-\ell}}, \quad \text{with } c_{\gamma,\ell} := \sup_{|x| \geq 1} \frac{L^{1/\gamma}(x)}{|x|^\ell} < \infty. \quad (6.17)$$

Furthermore, the right hand side is obtained by using the right hand side of (6.6) and by choosing c_1 sufficiently large. This ends the proof of Step 1.

Step 2. For all $0 < \delta < 1$ and $R \geq 1$, there exists $\varepsilon_2 > 0$ such that, for all $0 < \varepsilon < \varepsilon_2$, there exists a constant $R_2 \geq R$ such that, for all $\theta \in \Theta_\gamma$, $0 \leq t \leq T$ and $|x| \geq R$,

$$\partial_t F_\theta^{\gamma,\varepsilon}(t,x) \leq (1-\delta) \frac{\alpha}{2} x^2 F_\theta^{\gamma,\varepsilon}(t,x). \quad (6.18)$$

By using chain rule (4.12) we get that

$$\begin{aligned} \partial_t F_\theta^{\gamma,\varepsilon}(t,x) &= \frac{\alpha}{2} x^2 \exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(x)] U_\alpha(x) - (F_\theta^{\gamma,\varepsilon}(t,x) - 1) \\ &\quad - \left(\frac{1}{4} + r \right) \int_0^x (e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(y)) \exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(x)] U'_\alpha(y) dy \\ &\quad - \int_0^x \left(\frac{\alpha y^2}{2} \right) \times \left(\exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(x)] U'_\alpha(y) \right) dy. \end{aligned}$$

By integration by parts we obtain that

$$\begin{aligned} \partial_t F_\theta^{\gamma, \varepsilon}(t, x) &= \frac{\alpha}{2} x^2 \exp [e^{-rt} T_t \Delta_{\gamma, \varepsilon}(\theta)(x)] U_\alpha(x) - (F_\theta^{\gamma, \varepsilon}(t, x) - 1) \\ &\quad - \left(\frac{1}{4} + r \right) \int_0^x (e^{-rt} T_t \Delta_{\gamma, \varepsilon}(\theta)(y)) \exp [e^{-rt} T_t \Delta_{\gamma, \varepsilon}(\theta)(x)] U'_\alpha(y) dy \\ &\quad - \frac{\alpha}{2} x^2 F_\theta^{\gamma, \varepsilon}(t, x) + \int_0^x F_\theta^{\gamma, \varepsilon}(t, y) \alpha y dy. \end{aligned} \quad (6.19)$$

Recall that q and Ψ are defined in (6.14) and note by using (6.15) and the right hand side of (6.6) that

$$\left| \int_0^x (e^{-rt} T_t \Delta_{\gamma, \varepsilon}(\theta)(y)) \exp [e^{-rt} T_t \Delta_{\gamma, \varepsilon}(\theta)(x)] U'_\alpha(y) dy \right| \leq q \varepsilon \Psi^2(\varepsilon) F_\theta^{\gamma, \varepsilon}(t, x) \quad (6.20)$$

and

$$\left| \int_0^x F_\theta^{\gamma, \varepsilon}(t, y) \alpha y dy \right| \leq \Psi^2(\varepsilon) F_\theta^{\gamma, \varepsilon}(t, x). \quad (6.21)$$

Therefore, we get from (6.21), (6.20), (6.19) and (6.15) that

$$\partial_t F_\theta^{\gamma, \varepsilon}(t, x) \leq \left([\Psi^2(\varepsilon) - 1] \frac{\alpha}{2} x^2 + [1 + \kappa q \varepsilon] \Psi^2(\varepsilon) \right) F_\theta^{\gamma, \varepsilon}(t, x), \quad \text{with } \kappa := |r| + \frac{1}{4}. \quad (6.22)$$

Inequality (6.18) is then a simple consequence of (6.22) by taking $\varepsilon_2 > 0$ and $R_2 \geq R$ such that, for all $x \geq R_2$,

$$[\Psi^2(\varepsilon_2) - 1] \frac{\alpha}{2} x^2 + [1 + q \kappa \varepsilon_2] \Psi^2(\varepsilon_2) \leq (1 - \delta) \frac{\alpha}{2} x^2.$$

This completes the proof of Step 2.

We deduce Lemma 6.1 from (6.18) and (6.10). Indeed, we can choose $0 < \delta < 1$ and $R \geq 1$ such that

$$\left(\delta \alpha (1 - \alpha) - (1 - \delta) \frac{\alpha}{2} \right) > 0 \quad \text{and} \quad \left(\delta \alpha (1 - \alpha) - (1 - \delta) \frac{\alpha}{2} \right) R^2 \geq \lambda. \quad (6.23)$$

We get the left hand side of (6.9) by using (6.23) and by setting $\bar{\varepsilon} := \varepsilon_1 \wedge \varepsilon_2$ and

$$B_\theta := \sup_{|x| \leq R_1(\theta) \vee R_2, 0 \leq t \leq T} L_\theta F_\theta^{\gamma, \varepsilon}(t, x), \quad (6.24)$$

Moreover, by using inequalities (6.22), (6.15), (6.13) and (6.11), we can see that there exists $C > 0$ such that

$$L_\theta F_\theta^{\gamma, \varepsilon}(t, x) \leq C \left(1 + D_{\gamma, \varepsilon}(\theta) |x| L^{1/\gamma}(x) + x^2 \right) U_\alpha(x). \quad (6.25)$$

Then it is not difficult to get the right hand side of (6.9) by taking $p := 2/(\gamma(1 - \ell))$, k, c sufficiently large and by using (6.25), (6.24) and the right hand sides of (6.10) and (6.6). The proof of Lemma 6.1 is done. \square

Lemma 6.2. *For all $r \in \mathbb{R}$, $\alpha \in (0, 1)$, $\gamma \in (\alpha/2, 1/2)$, $T > 0$, $\varepsilon > 0$ and $\lambda > 0$, there exist a random variable $B : \Theta \rightarrow [1, \infty)$, $k, c > 0$ and $0 < p < 2$ such that, for all $\theta \in \Theta_\gamma$, $0 \leq t \leq T$ and $x \in \mathbb{R}$,*

$$L_\theta G_\theta^{\gamma, \varepsilon}(t, x) \leq -\lambda G_\theta^{\gamma, \varepsilon}(t, x) + B_\theta, \quad \text{with } B_\theta \leq k \exp(c H_\gamma^p(\theta)). \quad (6.26)$$

Proof. This proof uses similar ideas as the proof of Lemma 6.1 and we only give the main lines. Once again, the proof will be a consequence of the following two steps.

Step 1. For all $0 < \delta < 1$, $R \geq 1$ and $0 < \ell < 1$, there exist $R_1 : \Theta \rightarrow [R, \infty)$ and $c_1 > 0$ such that, for all $\theta \in \Theta_\gamma$, $0 \leq t \leq T$ and $|x| \geq R_1(\theta)$,

$$L_{\theta,t} G_\theta^{\gamma,\varepsilon}(t,x) \leq -\alpha(1-\delta)|x|^\alpha G_\theta^{\gamma,\varepsilon}(t,x), \quad \text{with } R_1(\theta) \leq c_1 \left(H_\gamma^{\frac{1}{\gamma(1-\ell)}}(\theta) \vee 1 \right). \quad (6.27)$$

By using chain rule (4.11) we can see that, for all $x \in \{V_\alpha > 3\}$,

$$L_{\theta,t} G_\theta^{\gamma,\varepsilon}(t,x) = -\alpha \left(1 + \frac{e^{-rt} (T_t W_{\gamma,\varepsilon}(\theta))'(x)}{x} - \frac{\alpha}{|x|^{2-\alpha}} + \frac{1-\alpha}{x^2} \right) \times \exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(x)] |x|^\alpha G_\alpha(x).$$

Moreover, we can choose $D \geq 1$ such that $\{V_\alpha \leq 3\} \subset [-D, D]$ and

$$\left(1 - \frac{q}{D} - \frac{\alpha}{D^{2-\alpha}} \right) \Psi^{-2}(\varepsilon) \geq (1-\delta).$$

By setting R_1 as in (6.17) we can obtain Step 1.

Step 2. For all $\delta > 0$ and $R \geq 1$, there exists a constants $R_2 \geq R$ such that, for all $\theta \in \Theta_\gamma$, $0 \leq t \leq T$ and $|x| \geq R_2$,

$$\partial_t G_\theta^{\gamma,\varepsilon}(t,x) \leq \left(\frac{\alpha}{2} + \delta \right) |x|^\alpha G_\theta^{\gamma,\varepsilon}(t,x). \quad (6.28)$$

By using chain rule (4.12) we can see that, for all $x \in \{V_\alpha > 3\}$,

$$\begin{aligned} \partial_t G_\theta^{\gamma,\varepsilon}(t,x) &= \frac{\alpha}{2} |x|^\alpha \exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(x)] V_\alpha(x) - (G_\theta^{\gamma,\varepsilon}(t,x) - 1) \\ &\quad - \left(\frac{1}{4} + r \right) \int_0^x [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(y)] \exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(y)] G'_\alpha(y) dy \\ &\quad - \frac{1}{2} \int_0^x \exp [e^{-rt} T_t \Delta_{\gamma,\varepsilon}(\theta)(y)] y G''_\alpha(y) dy. \end{aligned}$$

Then we can deduce (6.28) by using similar methods as in the proof of (6.18).

We deduce Lemma 6.2 from (6.28) and (6.27) in the same manner as we get Lemma 6.1 from (6.18) and (6.10). The main variation is that we need to choose ℓ in (6.27) such that $p := \alpha/(\gamma(1-\ell)) < 2$. \square

6.2.2 For the Markov kernels

Proposition 6.2. For all $r \in \mathbb{R}$, $\alpha \in (0, 1)$, $\gamma \in (0, 1/2)$ and $\eta, \tau, T > 0$, there exists a random variable $B : \Theta \rightarrow [1, \infty)$ and $k, c, p > 0$ such that, for all $\kappa > 0$, $\theta \in \Theta_\gamma$, $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}$,

$$P_{s,t}(\theta) U_\alpha(x) \leq (\eta + \kappa + \mathbb{1}_{s \leq t \leq s+\tau}) U_\alpha(x) + B_\theta \mathbb{1}_{x \in \{U_\alpha \leq \kappa^{-1} B_\theta\}}, \quad (6.29)$$

with

$$B_\theta \leq k \exp(c H_\gamma^p(\theta)).$$

Proof. Let $0 < \bar{\varepsilon} < 1$ be as in Lemma 6.1, $\lambda > e^q$ and $0 < \varepsilon < \bar{\varepsilon}$ be such that $e^{-\lambda \tau + 2q} \leq \eta$ and $e^{2q\varepsilon} \leq \eta + 1$, where q is defined in (6.14). One can see by using Ito's formula (2.8) that there exists a Brownian motion W such that, under $\mathbb{P}_{s,x}$,

$$e^{\lambda t} F_\theta^{\gamma,\varepsilon}(t, X_t) = e^{\lambda s} F_\theta^{\gamma,\varepsilon}(s, x) + \int_s^t e^{\lambda u} (L_\theta F_\theta^{\gamma,\varepsilon} + \lambda F_\theta^{\gamma,\varepsilon})(u, X_u) du + \int_s^t e^{\lambda u} \partial_x F_\theta^{\gamma,\varepsilon}(u, X_u) dW_u. \quad (6.30)$$

Besides, we get from Lemma 6.1 there exist a random variable $B : \Theta \rightarrow [1, \infty)$, $k, c, p > 0$ such that, for all $\theta \in \Theta_\gamma$, $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}$,

$$L_\theta F_\theta^{\gamma, \varepsilon}(t, x) \leq -\lambda F_\theta^{\gamma, \varepsilon}(t, x) + B_\theta, \quad \text{with } B_\theta \leq k \exp(c H_\gamma^p(\theta)).$$

Then one can see by taking the expectation in (6.30) and by using (6.15) that, for all $\theta \in \Theta_\gamma$, $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}$,

$$P_{s,t}(\theta) U_\alpha(x) \leq e^{-\lambda(t-s)+2q\varepsilon} U_\alpha(x) + \lambda^{-1} e^{q\varepsilon} B_\theta \leq (\eta + \mathbb{1}_{s \leq t \leq s+\tau}) U_\alpha(x) + B_\theta.$$

Therefore, it follows that for all $\kappa > 0$,

$$P_{s,t}(\theta) U_\alpha(x) \leq (\eta + \kappa + \mathbb{1}_{s \leq t \leq s+\tau}) U_\alpha(x) + B_\theta \mathbb{1}_{x \in \{U_\alpha \leq \kappa^{-1} B_\theta\}}$$

and this ends the proof, excepted for Lemma 6.1. \square

Proposition 6.3. *For all $r \in \mathbb{R}$, $\alpha \in (0, 1)$, $\gamma \in (\alpha/2, 1/2)$ and $\eta, \tau, T > 0$, there exist a random variable $B : \Theta \rightarrow [1, \infty)$, $k, c > 0$ and $0 < p < 2$ such that, for all $\kappa > 0$, $\theta \in \Theta_\gamma$, $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}$,*

$$P_{s,t}(\theta) V_\alpha(x) \leq (\eta + \kappa + \mathbb{1}_{s \leq t \leq s+\tau}) V_\alpha(x) + B_\theta \mathbb{1}_{x \in \{V_\alpha \leq \kappa^{-1} B_\theta\}}, \quad (6.31)$$

with

$$B_\theta \leq k \exp(c H_\gamma^p(\theta)).$$

Proof. The proof follows the same lines as the proof of Proposition 6.2 and we only give the main ideas. Once again, by using Ito's formula and Lemma 6.2, we can prove that there exist a random variable $B : \Theta \rightarrow [0, \infty)$, $k, c > 0$ and $0 < p < 2$ such that, for all $\theta \in \Theta_\gamma$, $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}$,

$$P_{s,t}(\theta) G_\alpha(x) \leq (\eta + \mathbb{1}_{s \leq t \leq s+\tau}) G_\alpha(x) + B_\theta, \quad \text{with } B_\theta \leq k \exp(c H_\gamma^p(\theta)).$$

Moreover, since $G_\alpha \leq V_\alpha$ and

$$\mathbb{E}_{s,x}(\theta) [V_\alpha(X_t) \mathbb{1}_{\{V_\alpha(X_t) \geq 3\}}] = \mathbb{E}_{s,x}(\theta) [G_\alpha(X_t) \mathbb{1}_{\{V_\alpha(X_t) \geq 3\}}] \leq (\eta + \mathbb{1}_{s \leq t \leq s+\tau}) G_\alpha(x) + B_\theta,$$

we obtain that

$$P_{s,t}(\theta) V_\alpha(x) \leq (\eta + \mathbb{1}_{s \leq t \leq s+\tau}) V_\alpha(x) + (B_\theta + 3).$$

This is enough to complete the proof, excepted for Lemma 6.2. \square

6.3 Coupling method

6.3.1 Coupling construction

We said that C is a random $(1, \varepsilon)$ -coupling set associated to the random Markov kernel P and the random probability measure ν over $(\Theta, \mathcal{B}, \mathcal{W})$ on \mathbb{R} , if $\varepsilon : \Theta \rightarrow (0, 1/2]$ is a measurable map, C_θ is a compact set of \mathbb{R} for \mathcal{W} -almost surely $\theta \in \Theta$ and

$$\inf_{z \in C_\theta} P_\theta(z; \star) \geq \varepsilon_\theta \nu_\theta(\star) \quad \mathcal{W}\text{-a.s.} \quad (6.32)$$

Given a random $(1, \varepsilon)$ -coupling set C associated to the random probability measure ν , we construct a random Markov kernel P^* on $\mathbb{R} \times \mathbb{R}$ as follows. Let \bar{R} and \bar{P} be two random Markov kernels on $\mathbb{R} \times \mathbb{R}$ satisfying, for all $x, y \in C_\theta$ and $A, B \in \mathcal{B}(\mathbb{R})$,

$$\bar{R}_\theta(x, y; A \times \mathbb{R}) := \frac{P_\theta(x; A) - \varepsilon_\theta \nu_\theta(A)}{1 - \varepsilon_\theta}, \quad \bar{P}_\theta(x, y; \mathbb{R} \times A) := \frac{P_\theta(y; A) - \varepsilon_\theta \nu_\theta(A)}{1 - \varepsilon_\theta}$$

and

$$\bar{P}_\theta(x, y; A \times B) := (1 - \varepsilon_\theta) \bar{R}_\theta(x, y; A \times B) + \varepsilon_\theta \nu_\theta(A \cap B). \quad (6.33)$$

Note that we can assume that \bar{P} is a random coupling Markov kernel over P , in the sense that, for all $\theta \in \Theta$, $x, y \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$,

$$\bar{P}_\theta(x, y; A \times \mathbb{R}) = P_\theta(x; A) \quad \text{and} \quad \bar{P}_\theta(x, y; \mathbb{R} \times A) = P_\theta(y; A). \quad (6.34)$$

Then we define,

$$P_\theta^*(x, y; \star) := \begin{cases} \bar{R}_\theta(x, y; \star), & \text{if } (x, y) \in C_\theta \times C_\theta, \\ \bar{P}_\theta(x, y; \star), & \text{if } (x, y) \notin C_\theta \times C_\theta. \end{cases} \quad (6.35)$$

6.3.2 The Douc-Moulines-Rosenthal bound

In order to simplify our claims, we set

$$P_\theta := P_1(\theta), \quad P_\theta(z; dx) := P_\theta(0, z; 1, dx), \quad p_\theta(z, x) := p_\theta(0, z; 1, x) \quad T\theta := T_1\theta \quad (6.36)$$

and

$$\bar{U}_\alpha(x, y) := \frac{U_\alpha(x) + U_\alpha(y)}{2}.$$

Moreover, we denote for any random function $F : \Theta \rightarrow (0, \infty)$, $n \in \mathbb{N}$ and $j \in \{0, \dots, n\}$,

$$F_{j,n}^+(\theta) := \max_{0 \leq n_1 < \dots < n_j \leq n-1} \prod_{k=1}^j F(e^{-r n_k} T^{n_k} \theta) \quad \text{and} \\ F_{j,n}^-(\theta) := \max_{1 \leq n_1 < \dots < n_j \leq n} \prod_{k=1}^j F(e^{-r(n-n_k)} T^{-n_k} \theta) = F_{j,n}^+(T^{-n} \theta). \quad (6.37)$$

Proposition 6.4. *For all $r \in \mathbb{R}$, $\alpha \in (0, 1)$, $\gamma \in (0, 1/2)$ and $\rho \in (0, \infty)$, there exist a random variable $B : \Theta \rightarrow [1, \infty)$, with $\log(B) \in L^1(\Theta, \mathcal{B}, \mathcal{W})$, and a random $(1, \varepsilon)$ -coupling set C over $(\Theta, \mathcal{B}, \mathcal{W})$ on \mathbb{R} such that, for all $\theta \in \Theta_\gamma$ and $x, y \in \mathbb{R}$,*

$$P_\theta^* \bar{U}_\alpha(x, y) \leq \rho \bar{U}_\alpha(x, y) + B_\theta \mathbf{1}_{(x,y) \in C_\theta \times C_\theta} \quad \text{and} \quad \sup_{(x,y) \in C_\theta \times C_\theta} \bar{R}_\theta \bar{U}_\alpha(x, y) \leq \frac{\rho B_\theta}{1 - \varepsilon_\theta}. \quad (6.38)$$

Moreover, for all $n \in \mathbb{N}$, $j \in \{1, \dots, n+1\}$ and $\nu_1, \nu_2 \in \mathcal{M}_1$,

$$\|\nu_1 P_n(\theta) - \nu_2 P_n(\theta)\|_{U_\alpha} \leq 2\rho^n \left[(1 - \varepsilon)_{j,n}^+(\theta) \mathbf{1}_{j \leq n} + B_{j-1,n}^+(\theta) \right] \|\nu_1\|_{U_\alpha} \|\nu_2\|_{U_\alpha} \\ + 2(1 - \varepsilon)_{j,n}^+(\theta) \mathbf{1}_{j \leq n} \sum_{k=0}^{n-1} \rho^k B \left(e^{-r(n-k-1)} T^{n-k-1} \theta \right). \quad (6.39)$$

Proof. Let η and κ be two positive constants such that $\rho = \eta + 2\kappa$ and use Proposition 6.2 to obtain $B : \Theta \rightarrow [1, \infty)$ and $k, c, p > 0$ such that, for all $\theta \in \Theta_\gamma$,

$$P_\theta U_\alpha \leq (\eta + \kappa) U_\alpha + B_\theta \mathbf{1}_{\{U_\alpha \leq \kappa^{-1} B_\theta\}}, \quad \text{with } B_\theta \leq k \exp(c H_\gamma^p(\theta)).$$

The same arguments as in the proof of [33, Proposition 11, p. 1660] applies and we can see that, for any random Markov kernel \bar{P} satisfying (6.34),

$$\bar{P}_\theta \bar{U}_\alpha \leq \rho \bar{U}_\alpha + B_\theta \mathbf{1}_{C_\theta \times C_\theta}, \quad \text{with } C_\theta := \{U_\alpha \leq \kappa^{-1} B_\theta\}.$$

We get that (6.38) is satisfied by setting $B_\theta := ((\rho \kappa^{-1} B_\theta + B_\theta) \rho^{-1}) \vee B_\theta$ and by using (6.33). Besides, we can see by using (6.5) that $\log(B) \in L^1(\Theta, \mathcal{B}, \mathcal{W})$ and thus similarly for $\log(B)$. Moreover, for all $\theta \in \Theta_\gamma$, C_θ is a compact set and we deduce from the lower local Aronson estimate (2.11) that C is a random $(1, \varepsilon)$ -coupling set associated to the random probability measure ν defined, for all $\theta \in \Theta_\gamma$ and $A \in \mathcal{B}(\mathbb{R})$ by

$$\varepsilon_\theta := \left(\int_{\mathbb{R}} \inf_{z \in C_\theta} p_\theta(z, x) dx \right) \wedge \frac{1}{2} > 0 \quad \text{and} \quad \nu_\theta(A) := \frac{\int_A \inf_{z \in C_\theta} p_\theta(z, x) dx}{\int_{\mathbb{R}} \inf_{z \in C_\theta} p_\theta(z, x) dx}.$$

Furthermore, we can write by using (2.14) that

$$P_n(\theta) = P(\theta) \cdots P(e^{-r(n-1)} T^{n-1} \theta)$$

and therefore a direct application of [33, Theorem 8, p. 1656] gives (6.39). \square

6.4 Ergodicity and exponential stability of the RDS

6.4.1 Ergodicity

Proposition 6.5. *The dynamical system $(\Theta, \mathcal{B}, \mathcal{W}, (T_t)_{t \in \mathbb{R}})$ is ergodic.*

Proof. Introduce three measurable maps $U^\pm : \Theta \rightarrow \Theta$ and $S_t : \Theta \rightarrow \Theta$ defined by

$$U^\pm(\theta) := \left(s \mapsto e^{-s/4} \theta(\pm e^{s/2}) \right) \quad \text{and} \quad S_t(\theta) := (s \mapsto \theta(s+t)).$$

It is classical that the distribution of U^\pm under the Wiener measure \mathcal{W} , denoted by Γ , is the distribution of the stationary Ornstein-Uhlenbeck process having the standard normal distribution as stationary distribution. This process is ergodic and as a consequence the dynamical system $(\Theta, \mathcal{B}, \Gamma, (S_t)_{t \in \mathbb{R}})$ is ergodic (see for instance [48, Theorem 20.10]). Besides, it is a simple calculation to see that the following diagram is commutative,

$$\begin{array}{ccc} (\Theta, \mathcal{B}, \mathcal{W}) & \xrightarrow{U^\pm} & (\Theta, \mathcal{B}, \Gamma) \\ T_t \downarrow & & \uparrow S_t^{-1} \\ (\Theta, \mathcal{B}, \mathcal{W}) & \xrightarrow{U^\pm} & (\Theta, \mathcal{B}, \Gamma) \end{array}$$

Let $A \in \mathcal{B}$ be such that $T_t^{-1}(A) = A$, with $t \neq 0$. By using the commutativity of the previous diagram and the ergodicity of the dynamical system $(\Theta, \mathcal{B}, \Gamma, (S_t)_{t \in \mathbb{R}})$, it follows that

$$S_t^{-1}(U^\pm(A)) = U^\pm(T_t^{-1}(A)) = U^\pm(A) \quad \text{and} \quad \Gamma(U^\pm(A)) = 0 \quad \text{or} \quad = 1.$$

Moreover, we can see that

$$U^\pm(A) = (U^\pm)^{-1}(U^\pm(A)) \quad \text{and} \quad (U^+)^{-1}(U^+(A)) \cap (U^-)^{-1}(U^-(A)) = A.$$

We conclude that $\mathcal{W}(A) = 0$ or $= 1$ and the proof is finished. \square

6.4.2 Exponential stability

Lemma 6.3. *Assume that $r = 0$ and let F as in (6.37) such that $(\log(F) \vee 0) \in L^1(\Theta, \mathcal{B}, \mathcal{W})$.*

1. *If $\mathcal{W}(F < 1) = 1$ then, for all $L \geq 1$, there exists $\lambda > 0$ such that*

$$\limsup_{n \rightarrow \infty} e^{\lambda n} F_{\lfloor \frac{n}{L} \rfloor, n}^\pm(\theta) = 0 \quad \mathcal{W}\text{-a.s.}$$

2. If $\mathscr{W}(F \geq 1) > 0$ then, for all $\eta > 0$, there exists $L > 0$ such that

$$\limsup_{n \rightarrow \infty} e^{-\eta n} F_{\lfloor \frac{n}{L} \rfloor, n}^{\pm}(\theta) = 0 \quad \mathscr{W}\text{-a.s.}$$

Proof. We prove the lemma only for F^+ since the proof for F^- is obtained replacing T by T^{-1} . We set, for all $c \geq 0$ and $k \geq 1$,

$$\log \left[F_k^{(c)}(\theta) \right] := \log \left[F(T^{k-1}\theta) \right] \mathbb{1}_{F(T^{k-1}\theta) \geq c} \quad \text{and} \quad F^{(c)} := F_1^{(c)}.$$

Assume that $\mathscr{W}(F < 1) = 1$. We can see that there exist $0 < c < 1$ and $\ell > 0$ such that

$$\mathbb{E}_{\mathscr{W}} [\mathbb{1}_{F \geq c}] < L^{-1} \quad \text{and} \quad \mathbb{E}_{\mathscr{W}} \left[\log(F^{(c)}) \right] < -\ell.$$

By applying the ergodic theorem to the ergodic dynamical system $(\Theta, \mathcal{B}, \mathscr{W}, T)$ we obtain that, for \mathscr{W} -almost surely $\theta \in \Theta$ and all integer n sufficiently large,

$$\sum_{k=1}^n \mathbb{1}_{F(T^{k-1}\theta) \geq c} \leq \left\lceil \frac{n}{L} \right\rceil \quad \text{and} \quad F_{\lfloor \frac{n}{L} \rfloor, n}^+(\theta) \leq \prod_{k=1}^n F_k^{(c)}(\theta) \leq e^{-\ell n}.$$

Then we deduce the first point by taking $0 < \lambda < \ell$. Furthermore, assume that $\mathscr{W}(F \geq 1) > 0$. Note that if F is bounded \mathscr{W} -a.s. the second point of the lemma is obvious. Moreover, when F is unbounded with positive probability, it is not difficult to see that there exist $0 < \kappa < \eta$, $c \geq 1$ and $L \geq 1$ such that

$$\mathbb{E}_{\mathscr{W}} \left[\log(F^{(c)}) \right] < \kappa \quad \text{and} \quad \mathbb{E}_{\mathscr{W}} [\mathbb{1}_{F \geq c}] > L^{-1}.$$

Once again, the ergodic theorem allow us to obtain the second point since, for \mathscr{W} -almost surely $\theta \in \Theta$ and all integer n sufficiently large,

$$\sum_{k=1}^n \mathbb{1}_{F(T^{k-1}\theta) \geq c} \geq \left\lfloor \frac{n}{L} \right\rfloor \quad \text{and} \quad F_{\lfloor \frac{n}{L} \rfloor, n}^+(\theta) \leq \prod_{k=1}^n F_k^{(c)}(\theta) \leq e^{\kappa n}.$$

□

Proposition 6.6. Assume that $r = 0$. For all $\alpha \in (0, 1)$ there exists $\lambda > 0$ such that, for all families $\{v_t^{\pm} : t \geq 0\}$ of random probability measure on \mathbb{R} over $(\Theta, \mathcal{B}, \mathscr{W})$ satisfying

$$\lim_{t \rightarrow \infty} \frac{\log(\|v_t^{\pm}\|_{U_{\alpha}})}{t} = 0 \quad \mathscr{W}\text{-a.s.}, \quad (6.40)$$

the following forward and backward discrete-time convergences hold:

$$\limsup_{t \rightarrow \infty} \frac{\log(\|v_t^+(\theta)P_{[t]}(\theta) - v_t^-(\theta)P_{[t]}(\theta)\|_{U_{\alpha}})}{t} \leq -\lambda \quad \mathscr{W}\text{-a.s.} \quad (6.41)$$

and

$$\limsup_{t \rightarrow \infty} \frac{\log(\|v_t^+(\theta)P_{[t]}(T^{-[t]}\theta) - v_t^-(\theta)P_{[t]}(T^{-[t]}\theta)\|_{U_{\alpha}})}{t} \leq -\lambda \quad \mathscr{W}\text{-a.s.} \quad (6.42)$$

Proof. We prove only (6.42) since the proof of (6.41) follows the same lines by replacing T^{-1} by T . Let $0 < \rho < 1$ be and, following Proposition 6.4, write that, for all $\theta \in \Theta_{\gamma}$, $t \geq 0$ and $j \in \{0, \dots, [t] + 1\}$,

$$\begin{aligned} \left\| v_t^+ P_{[t]}(T^{-[t]}\theta) - v_t^- P_{[t]}(T^{-[t]}\theta) \right\|_{U_{\alpha}} &\leq 2\rho^{[t]} \left[(1 - \varepsilon)_{j, [t]}^-(\theta) \mathbb{1}_{j \leq [t]} + B_{j-1, [t]}^-(\theta) \right] \|v_t^+\|_{U_{\alpha}} \|v_t^-\|_{U_{\alpha}} \\ &\quad + 2(1 - \varepsilon)_{j, [t]}^-(\theta) \mathbb{1}_{j \leq [t]} \sum_{k=0}^{[t]-1} \rho^k B(T^{n-k-1}\theta). \end{aligned} \quad (6.43)$$

Since $\log B \in L^1(\Theta, \mathcal{B}, \mathcal{W})$, the ergodic theorem allows us to see that

$$\lim_{k \rightarrow \infty} \frac{\log [B(T^{-k+1}\theta)]}{k} = 0,$$

and we get, for all $\eta > 0$,

$$\limsup_{n \rightarrow \infty} e^{-\eta n} \left(\sum_{k=0}^{n-1} \rho^k B(T^{-k+1}\theta) \right) = 0 \quad \mathcal{W}\text{-a.s.}$$

Besides, one can see by using Lemma 6.3 that there exist $L \geq 1$ and $\ell > 0$ such that

$$\lim_{n \rightarrow \infty} e^{-\eta n} B_{[\frac{n}{L}], n}^-(\theta) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} e^{\ell n} (1 - \varepsilon)_{[\frac{n}{L}], n}^-(\theta) = 0.$$

Therefore we deduce from (6.40) the exponential convergence (6.42). \square

7 Proof of Theorem 2.3

Theorem 2.3 will be a consequence of Propositions 7.1 and 7.2.

7.1 Exponential weak ergodicity and quasi-invariant measure

Proposition 7.1. *For all $\alpha \in (0, 1)$ there exists $\lambda > 0$ such that, for all $\nu_1, \nu_2 \in \mathcal{M}_{1, U_\alpha}$,*

$$\limsup_{t \rightarrow \infty} \frac{\log (\|\nu_1 P_t(\theta) - \nu_2 P_t(\theta)\|_{U_\alpha})}{t} \leq -\lambda \quad \mathcal{W}\text{-a.s.} \quad (7.1)$$

Furthermore, there exists a unique (up to a \mathcal{W} -null set) random probability measure μ over $(\Theta, \mathcal{B}, \mathcal{W})$ on \mathbb{R} such that, for all $\alpha \in (0, 1)$ there exists $\lambda > 0$ such that, for all $\nu \in \mathcal{M}_{1, U_\alpha}$,

$$\limsup_{t \rightarrow \infty} \frac{\log (\|\nu P_t(T_{-t}\theta) - \mu_\theta\|_{U_\alpha})}{t} \leq -\lambda \quad \mathcal{W}\text{-a.s.} \quad (7.2)$$

Moreover, for all $t \geq 0$,

$$\mu_\theta \in \mathcal{M}_{1, U_\alpha} \quad \text{and} \quad \mu_\theta P_t(\theta) = \mu_{T_t\theta} \quad \mathcal{W}\text{-a.s.} \quad (7.3)$$

Proof. The cocycle property (2.14) allows us to write,

$$\nu P_t(\theta) = \nu P_{[t]}(\theta) P_{\{t\}}(T^{[t]}\theta)$$

and we deduce that

$$\|\nu_1 P_t(\theta) - \nu_2 P_t(\theta)\|_{U_\alpha} \leq \left(\sup_{0 \leq u \leq 1} \|P_u(T^{[t]}\theta)\|_{U_\alpha} \right) \|\nu_1 P_{[t]}(\theta) - \nu_2 P_{[t]}(\theta)\|_{U_\alpha}. \quad (7.4)$$

Moreover, by using Proposition 6.2, (6.5) and the ergodic theorem, we get that

$$\lim_{n \rightarrow \infty} \frac{\log (\sup_{0 \leq u \leq 1} \|P_u(T^n\theta)\|_{U_\alpha})}{n} = 0 \quad \mathcal{W}\text{-a.s.} \quad (7.5)$$

Besides, a direct application of Lemma 6.6 gives that there exists $\lambda > 0$, independent of ν_1 and ν_2 , such that

$$\limsup_{n \rightarrow \infty} \frac{\log (\|\nu_1 P_n(\theta) - \nu_2 P_n(\theta)\|_{U_\alpha})}{n} \leq -\lambda \quad \mathcal{W}\text{-a.s.} \quad (7.6)$$

We deduce inequality (7.1) from (7.6), (7.5) and (7.4). Furthermore, one can see by using Lemma 6.6 and similar arguments that

$$\sum_{n=0}^{\infty} \|\nu P_{n+1}(T^{-n-1}\theta) - \nu P_n(T^{-n}\theta)\|_{U_\alpha} < \infty \quad \mathscr{W}\text{-a.s.}$$

We obtain that, for \mathscr{W} -almost $\theta \in \Theta$, $\{\nu P_n(T^{-n}\theta) : n \geq 0\}$ is a Cauchy sequence in the separable Banach space \mathcal{M}_{U_α} . Then by using again Lemma 6.6 we get that there exist $\lambda > 0$ and a random probability measure $\mu_\theta \in \mathcal{M}_{U_\alpha}$ such that, for all $\nu \in \mathcal{M}_{1, U_\alpha}$,

$$\limsup_{n \rightarrow \infty} \frac{\log(\|\nu P_n(T^{-n}\theta) - \mu_\theta\|_{U_\alpha})}{n} \leq -\lambda \quad \mathscr{W}\text{-a.s.} \quad (7.7)$$

We deduce (7.2) from (7.7) in the same way as we obtain (7.1) from (7.6). Finally, we can see by using (7.2) that

$$\mu_\theta P_t(\theta) \stackrel{\mathcal{M}_{U_\alpha}}{=} \lim_{s \rightarrow \infty} \nu P_s(T_{-s}\theta) P_t(\theta) \stackrel{\mathcal{M}_{U_\alpha}}{=} \lim_{s \rightarrow \infty} \nu P_{t+s}(T_{-(s+t)} T_t \theta) \stackrel{\mathcal{M}_{U_\alpha}}{=} \mu_{T_t \theta} \quad \mathscr{W}\text{-a.s.}$$

□

7.2 Annealed convergences

Proposition 7.2. *For all $\alpha \in (0, 1)$ and $\hat{\nu} \in \mathcal{M}_{1, V_\alpha}$,*

$$\hat{\mu} \in \mathcal{M}_{1, V_\alpha} \quad \text{and} \quad \lim_{t \rightarrow \infty} \left\| \hat{\nu} \hat{P}_t - \hat{\mu} \right\|_{V_\alpha} = 0. \quad (7.8)$$

Proof. Let $0 < \rho < 1$ be and apply Proposition 6.3 to write, for all $0 \leq u \leq 1$,

$$P_u(\theta) V_\alpha \leq (\rho + 1) V_\alpha + B_\theta \quad \text{and} \quad P_\theta V_\alpha \leq \rho V_\alpha + B_\theta \quad \mathscr{W}\text{-a.s.} \quad (7.9)$$

We get from the latter inequality and (7.3) that $\mu_{T_\theta}(V_\alpha) \leq \rho \mu_\theta(V_\alpha) + B_\theta$ \mathscr{W} -a.s. and by taking the expectation of the last inequality, we obtain the left hand side of (7.8). Besides, since the Wiener measure is (T_t) -invariant, we can write,

$$\left\| \hat{\nu} \hat{P}_t - \hat{\mu} \right\|_{V_\alpha} \leq \mathbb{E}_{\mathscr{W}} \left[\left\| \hat{\nu} P_t(T^{-[t]}\theta) - \mu_{T_{[t]}\theta} \right\|_{V_\alpha} \right]. \quad (7.10)$$

Moreover, by using (7.3) and the cocycle property we can see that

$$\nu P_t(T^{-[t]}\theta) = \nu P_{[t]}((T^{-[t]})P_{\{t\}}(\theta)) \quad \text{and} \quad \mu_\theta P_{\{t\}}(\theta) = \mu_{T_{[t]}\theta}$$

and we get from (7.2) and (7.10) that

$$\begin{aligned} \lim_{t \rightarrow \infty} \left\| \hat{\nu} \hat{P}_t - \hat{\mu} \right\|_{V_\alpha} &\leq \lim_{t \rightarrow \infty} \left\| \hat{\nu} P_t(T^{-[t]}\theta) - \mu_{T_{[t]}\theta} \right\|_{V_\alpha} \\ &\leq \lim_{t \rightarrow \infty} \left(\sup_{0 \leq u \leq 1} \|P_u(\theta)\|_{V_\alpha} \right) \left\| \hat{\nu} P_{[t]}(T^{-[t]}\theta) - \mu_\theta \right\|_{V_\alpha} = 0 \quad \mathscr{W}\text{-a.s.} \end{aligned} \quad (7.11)$$

Furthermore, by using (7.9) and the cocycle property, it is not difficult to see that

$$\begin{aligned} \left\| \nu P_t(T^{-[t]}\theta) \right\|_{V_\alpha} &\leq (\rho + 1) \left(\rho \|\nu\|_{V_\alpha} + \sum_{k=0}^{\infty} \rho^k B(T^k \theta) \right) + B_\theta \\ &\quad \text{and} \quad \left\| \mu_{T_{[t]}\theta} \right\|_{V_\alpha} \leq (\rho + 1) \|\mu_\theta\| + B_\theta. \end{aligned}$$

Noting that the two previous bounds belong to $L^1(\Theta, \mathcal{B}, \mathscr{W})$ (see Proposition 6.3) and are independent of $t \geq 0$, the dominate convergence theorem applies and we deduce from (7.11) the right hand side of (7.8). □

8 Proof of Theorem 2.4

Proof. First, recall that under $\mathbb{P}_z(\theta)$ (see Proposition 4.1) $\{S_\theta(t, X_t) : t \geq 0\}$ is a solution of the SDE (4.3). Moreover, since $r > 0$, we can see by using (4.4) that

$$\lim_{t \rightarrow \infty} S_\theta(t, x) = S(x) := \int_0^x e^{\frac{y}{2}} dy, \quad \lim_{t \rightarrow \infty} H_\theta(t, x) = S^{-1}(x) := \int_0^x e^{\frac{y}{2}} dy, \\ \lim_{t \rightarrow \infty} \sigma_\theta(t, x) = S' \circ S^{-1}(x) \quad \text{and} \quad \lim_{t \rightarrow \infty} d_\theta(t, x) = 0, \quad (8.1)$$

uniformly on compact sets of \mathbb{R} . Following [17, Lemma 4.5] and denoting by Γ the standard normal distribution, it is not difficult to see that $\{S_\theta(t, X_t) : t \geq 0\}$ is asymptotically time-homogeneous and $S_*\Gamma$ -ergodic. According to the cited Lemma, if in addition $\{S_\theta(t, X_t) : t \geq 0\}$ is bounded in probability, we can deduce the convergence in distribution towards $S_*\Gamma$:

$$\left(\forall \varepsilon > 0, \exists R > 0, \sup_{t \geq 0} \mathbb{P}_z(\theta)(|S_\theta(t, X_t)| \geq R) \leq \varepsilon \right) \implies \lim_{t \rightarrow \infty} \left(S_\theta(t, X_t) \stackrel{(d)}{=} S_*\Gamma \right). \quad (8.2)$$

In fact, we shall prove that $\{X_t : t \geq 0\}$ is bounded in probability for \mathscr{W} -a.s. $\theta \in \Theta$, which shall imply the boundedness in probability of $\{S_\theta(t, X_t) : t \geq 0\}$ for \mathscr{W} -a.s. $\theta \in \Theta$. To this end, by using Proposition 6.2, we can find $0 < \rho < 1, L > 0, B : \Theta \rightarrow [1, \infty)$ and $k, c, p > 0$ such that, for all $0 \leq u \leq 1$,

$$P_u(\theta)U_\alpha \leq LU_\alpha + B_\theta, \quad P_\theta U_\alpha \leq \rho U_\alpha + B_\theta \quad \text{and} \quad B_\theta \leq k \exp[cH_\gamma^p(\theta)] \quad \mathscr{W}\text{-a.s.} \quad (8.3)$$

Relations (2.10) and the ergodic theorem allow us to write that, for all $t \geq 0$,

$$\sup_{t \geq 0} \|P_\theta(t, x, dy)\|_{U_\alpha} \leq \sup_{t \geq 0} L \left(\rho^{[t]} U_\alpha(x) + k \sum_{m=0}^{[t]-1} \rho^{[t]-m} \exp[ce^{-rpm} H_\gamma^p(T^m \theta)] \right) + B_\theta \\ \leq L \left(\rho U_\alpha(x) + \frac{k}{1-\rho} \exp \left[\sup_{m \geq 0} (ce^{-rpm} H_\gamma^p(T^m \theta)) \right] \right) + B_\theta < \infty \quad \mathscr{W}\text{-a.s.}$$

Then the Markov inequality implies that

$$\sup_{t \geq 0} \mathbb{P}_x(\theta)(|X_t| \geq R) \leq \frac{\sup_{t \geq 0} \|P_\theta(t, x, dy)\|_{U_\alpha}}{U_\alpha(R)} \quad \mathscr{W}\text{-a.s.}$$

and we obtain the fact that $\{X_t : t \geq 0\}$ is bounded in probability. Moreover, since

$$\lim_{|x| \rightarrow \infty} \inf_{t \geq 0} S_\theta(t, x) = \infty$$

we deduce the boundedness in probability of $\{S_\theta(t, X_t) : t \geq 0\}$. Therefore [17, Lemma 4.5] applies and this completes the proof. \square

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