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Lower-order logsums

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Abstract

The logsum formula, which provides the expected maximum utility for the multinomial logit model, is often used as a measure of welfare. We provide here a closed form formula of the welfare measure of an individual who has not access to his first-best choice, but has access to his r th-best choice, $r = 2, \dots, n$, where n is the number of alternatives. The derivation is based on a standard identity in order statistics.

JEL classification:

C25; D11

Keywords:

Discrete choice models; Gumbel distribution; Logit; Logsum; Order statistics

1 Introduction

The celebrated logsum formula, which is the expected maximum utility in the multinomial logit model (MNL) framework, can be used, under linear in income specifications of the conditional utilities, as a welfare measure (see the discussions of McFadden, 1981, and Small and Rosen, 1981). For a review of the theoretical and applied literature on the use of the logsum, we refer the reader to de Jong, Daly, Pieters, and van der Hoorn (2007).

If the individual is constrained to choose a lower-order alternative, the expected lower-order utility (lower-order logsum in the MNL framework) provides

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the corresponding welfare measure. These constraints occur when the best alternatives are not available. Such constraints are often binding in restaurants, for airline trips or when the market is thin. An application of discrete choice models with capacity constraints for the residential location choice is given in de Palma, Picard, and Waddell (2007).

For the MNL, the expected utility of the best alternative is the well-known logsum formula. In this paper, we derive an explicit formula for the r th-order logsum which represent the expected utility of a consumer who has access to his/her r th-best choice. We show that each lower-order logsum is an alternating weighted sum of logsum formulas. Our computation relies on a contribution to order statistics by Balakrishnan (1988).

Section 2 defines the order utilities and their distributions. Section 3 recalls standard identities for order statistics and adapt them to reverse order statistics. In Section 4, we apply these results to the MNL and derive an explicit formula for the r th-order expected utility (lower-order logsum). Section 5 presents the two and three alternatives cases and the formulas in the symmetric case.

2 Definition of order utilities and notations

Let $\mathcal{C}_n \equiv \{1, \dots, n\}$ be the total set of alternatives, $n \geq 2$, and $\{U_i\}_{i \in \mathcal{C}_n}$ a sequence of independent real random utilities with absolutely continuous CDF's $\{F_i(\cdot)\}_{i \in \mathcal{C}_n}$. For any subset $\mathcal{S}_m \subset \mathcal{C}_n$, where the subscript denotes the subset cardinality: $|\mathcal{S}_m| = m$, $1 \leq m \leq n$, the corresponding *order utilities* are the sequence $\{U_i\}_{i \in \mathcal{S}_m}$ ranked in a non-increasing order: $U_{\mathcal{S}_m}^{(1)} \geq \dots \geq U_{\mathcal{S}_m}^{(m)}$, $\mathcal{S}_m \subset \mathcal{C}_n$. More particularly, $U_{\mathcal{S}_m}^{(1)} = \max_{i \in \mathcal{S}_m} U_i$ is the best utility and $U_{\mathcal{S}_m}^{(m)} = \min_{i \in \mathcal{S}_m} U_i$ is the worst utility within \mathcal{S}_m .

Denote by $F_{\mathcal{S}_m}^{(r)}(x)$ the CDF of $U_{\mathcal{S}_m}^{(r)}$, $1 \leq r \leq m \leq n$. Straightforwardly, the maximum utility $U_{\mathcal{S}_m}^{(1)}$ has CDF given by

$$F_{\mathcal{S}_m}^{(1)}(x) = \prod_{i \in \mathcal{S}_m} F_i(x), \quad \mathcal{S}_m \subset \mathcal{C}_n. \quad (1)$$

As we will see below, the expression of the CDF's of the lower-order utilities can be obtained as a function of the first-best ones given by (1).

3 (Reverse) order statistics identities

Let $\{X_i\}_{i \in \mathcal{C}_n}$ be a sequence of absolutely continuous real independent non-identically distributed random variables. For any non-empty subset $\mathcal{S}_m \subset \mathcal{C}_n$, let $X_{(1), \mathcal{S}_m} \leq \dots \leq X_{(m), \mathcal{S}_m}$ be the corresponding order statistics ranked in a non-decreasing order. Let denote by $h_{(r), \mathcal{S}_m}(t)$ the PDF of $X_{(r), \mathcal{S}_m}$, $1 \leq r \leq m \leq n$.

Balakrishnan (1988, Relation 2) has shown this important identity

$$h_{(r),\mathcal{C}_n}(t) = \sum_{m=r}^n (-1)^{m-r} \binom{m-1}{r-1} \sum_{\mathcal{S}_m \subset \mathcal{C}_n} h_{(m),\mathcal{S}_m}(t), \quad 1 \leq r \leq n-1, \quad (2)$$

for any $t \in (-\infty, \infty)$.

Since a reverse ranking is adopted for the order utilities, define instead the reverse order statistics $X_{\mathcal{S}_m}^{(1)} \geq \dots \geq X_{\mathcal{S}_m}^{(m)}$, and let $\phi_{\mathcal{S}_m}^{(r)}(x)$, $1 \leq r \leq m \leq n$, the corresponding PDF's. The r th-reverse order statistics corresponds to the $(n-r+1)$ th-order statistics: $X_{\mathcal{C}_n}^{(r)} = X_{(n-r+1),\mathcal{C}_n}$. Moreover, the last reverse order statistics corresponds to the first order statistics: $X_{(m),\mathcal{S}_m} = X_{\mathcal{S}_m}^{(1)}$. Therefore, Identity (2) yields, for any $t \in (-\infty, \infty)$, the following identity for the reverse order statistics PDF's

$$\phi_{\mathcal{C}_n}^{(r)}(t) = \sum_{m=n-r+1}^n (-1)^{m-1-n+r} \binom{m-1}{n-r} \sum_{\mathcal{S}_m \subset \mathcal{C}_n} \phi_{\mathcal{S}_m}^{(1)}(t), \quad 2 \leq r \leq n. \quad (3)$$

Integration of both sides of the above equation yields a similar identity for the CDF's

$$\Phi_{\mathcal{C}_n}^{(r)}(x) = \sum_{m=n-r+1}^n (-1)^{m-1-n+r} \binom{m-1}{n-r} \sum_{\mathcal{S}_m \subset \mathcal{C}_n} \Phi_{\mathcal{S}_m}^{(1)}(x), \quad 2 \leq r \leq n, \quad (4)$$

where $\Phi_{\mathcal{S}_m}^{(r)}(x) \equiv \int_{-\infty}^x \phi_{\mathcal{S}_m}^{(r)}(t) dt$, $1 \leq r \leq m \leq n$, for any $x \in (-\infty, \infty)$.

Assume that $\{X_i\}_{i \in \mathcal{C}_n}$ have finite expectation, and let $\mu_{\mathcal{S}_m}^{(r)} \equiv \int_{-\infty}^x t \phi_{\mathcal{S}_m}^{(r)}(t) dt$, $1 \leq r \leq m \leq n$, the expectations of the reverse order statistics. The expectation of the r th-reverse order statistics in the total set \mathcal{C}_n verifies ¹ the identity (see Eq. 3)

$$\mu_{\mathcal{C}_n}^{(r)} = \sum_{m=n-r+1}^n (-1)^{m-1-n+r} \binom{m-1}{n-r} \sum_{\mathcal{S}_m \subset \mathcal{C}_n} \mu_{\mathcal{S}_m}^{(1)}, \quad 2 \leq r \leq n. \quad (5)$$

We now apply the above identities (4) and (5) to the MNL framework.

4 Logit specification

4.1 First-best logsum

We first recall some classical results for a celebrated discrete choice model: the multinomial logit model. The MNL is obtained when the preferences are described by an additive random utility model (ARUM; for a survey, see Anderson,

¹Obviously, a similar identity holds for any moment, but only the first moment is of interest here for the computation of the lower-order logsums.

de Palma, and Thisse, 1992). The utility of alternative i can be written as

$$U_i = v_i + \epsilon_i, \quad i \in \mathcal{C}_n, \quad (6)$$

where v_i 's are the systematic components of the utilities and ϵ_i 's are their disturbance stochastic terms assumed i.i.d. Gumbel variables with CDF given by

$$G(x) = \exp(-e^{-x-\gamma}), \quad x \in \mathbb{R}. \quad (7)$$

The Euler's constant $\gamma \simeq 0.5772$ is introduced for normalization purpose so that the ϵ_i 's have zero expectation: $\mathbb{E}(\epsilon_i) = 0$, $i \in \mathcal{C}_n$.

Under the above assumptions, U_i has CDF $F_i(x) = G(x - v_i)$, and expectation $\mathbb{E}(U_i) = v_i$, $i \in \mathcal{C}_n$. Applying Eq. (1), it is easy to see that the maximum utility $U_{\mathcal{S}_m}^{(1)}$ in any subset \mathcal{S}_m turns out to be a Gumbel distribution (see, e.g., Anderson et al., 1992, pp. 60-61) with CDF

$$F_{\mathcal{S}_m}^{(1)}(x) = G(x - \lambda_{\mathcal{S}_m}^{(1)}), \quad \mathcal{S}_m \subset \mathcal{C}_n, \quad (8)$$

where $\lambda_{\mathcal{S}_m}^{(1)}$, the *expected maximum utility*, is given by the logsum formula

$$\lambda_{\mathcal{S}_m}^{(1)} \equiv \mathbb{E}(U_{\mathcal{S}_m}^{(1)}) = \ln \sum_{i \in \mathcal{S}_m} e^{v_i}, \quad \mathcal{S}_m \subset \mathcal{C}_n. \quad (9)$$

The invariance property of the distribution of the maximum utility is due to the fact that the Gumbel distribution is one of the three types of max-stable distributions (see, e.g., Galambos, 1987). Notice also this surprising invariance property: the expected maximum utility given the observation of the best choice is still given by the logsum formula (9). This property is only shared by the Gumbel distribution as shown by de Palma and Kilani (2007).

The purpose of this paper is to derive, in the MNL framework, a formula for the remaining expected lower-order utilities $\lambda_{\mathcal{C}_n}^{(r)} \equiv \mathbb{E}(U_{\mathcal{C}_n}^{(r)})$, $2 \leq r \leq n$, that we refer to the *lower-order logsums*. To show this result, we rely on Eq. (5).

4.2 r th-order logsum

For the MNL, substituting $F_{\mathcal{S}_m}^{(1)}(x)$ by its value given by Eq. (8) into Eq. (4), the CDF of the r th-order utility is obtained

$$F_{\mathcal{C}_n}^{(r)}(x) = \sum_{m=n-r+1}^n (-1)^{m-1-n+r} \binom{m-1}{n-r} \sum_{\mathcal{S}_m \subset \mathcal{C}_n} G(x - \lambda_{\mathcal{S}_m}^{(1)}), \quad 2 \leq r \leq n, \quad (10)$$

where $\lambda_{\mathcal{S}_m}^{(1)}$ is given by the logsum formula (Eq. 9), for any $x \in (-\infty, \infty)$. It is no longer a Gumbel distribution but has a CDF which is an alternating weighted sums of Gumbel CDF's. The expectation of $U_{\mathcal{C}_n}^{(r)}$ is given in the following theorem:

Theorem 1 For the MNL with set of alternatives \mathcal{C}_n , the r th-order expected utility, $\lambda_{\mathcal{C}_n}^{(r)} \equiv \mathbb{E}\left(U_{\mathcal{C}_n}^{(r)}\right)$, is given by

$$\lambda_{\mathcal{C}_n}^{(r)} = \sum_{m=n-r+1}^n (-1)^{m-1-n+r} \binom{m-1}{n-r} \sigma_m^n, \quad 2 \leq r \leq n, \quad (11)$$

where

$$\sigma_m^n \equiv \sum_{\mathcal{S}_m \subset \mathcal{C}_n} \ln \sum_{i \in \mathcal{S}_m} e^{v_i}, \quad 1 \leq m \leq n. \quad (12)$$

Proof. Using Eq. (5), we get Eq. (11), where $\sigma_m^n \equiv \sum_{\mathcal{S}_m \subset \mathcal{C}_n} \lambda_{\mathcal{S}_m}^{(1)}$ can be written, using Eq. (9), as in (12). ■

The lower-order logsums (see Eq. 11) are alternating weighted sums of logsums given by σ_m^n (see Eq. 12), computed for all the m -tuples \mathcal{S}_m of the choice set \mathcal{C}_n , $n-r+1 \leq m \leq n$. More particularly, using the fact that $\sigma_n^n = \ln \sum_{i=1}^n e^{v_i}$, the second-best logsum can be written as

$$\lambda_{\mathcal{C}_n}^{(2)} = \sum_{i=1}^n \ln \sum_{k \neq i} e^{v_k} - (n-1) \ln \sum_{i=1}^n e^{v_i}. \quad (13)$$

On the other hand, using the fact that $\sigma_1^n = \sum_{i=1}^n v_i$, the worst logsum is

$$\lambda_{\mathcal{C}_n}^{(n)} = \sum_{i=1}^n v_i - \sum_{m=2}^n (-1)^m \sigma_m^n. \quad (14)$$

5 Applications of the theorem

5.1 The two and three alternatives cases

Consider the two alternatives case. The logsum is $\ln(e^{v_1} + e^{v_2})$ while the second-best (worst) logsum is $v_1 + v_2 - \ln(e^{v_1} + e^{v_2})$. This can be explained as follows: $\min_{k=1,2} U_k + \max_{k=1,2} U_k = U_1 + U_2$ implies $\mathbb{E}(\min_{k=1,2} U_k) + \mathbb{E}(\max_{k=1,2} U_k) = v_1 + v_2$. Then: $\mathbb{E}(\min_{k=1,2} U_k) = v_1 + v_2 - \mathbb{E}(\max_{k=1,2} U_k)$, the required result. Little algebra shows: $\mathbb{E}(\max_{k=1,2} U_k) > \mathbb{E}(\min_{k=1,2} U_k)$.

Next, consider the three alternatives case. Applying Eq. (13), the second-best logsum is

$$\lambda_{\mathcal{C}_3}^{(2)} = \ln(e^{v_1} + e^{v_2}) + \ln(e^{v_1} + e^{v_3}) + \ln(e^{v_2} + e^{v_3}) - 2 \ln(e^{v_1} + e^{v_2} + e^{v_3}).$$

Applying Eq. (14), the third-best (worst) logsum is

$$\begin{aligned} \lambda_{\mathcal{C}_3}^{(3)} &= v_1 + v_2 + v_3 - \ln(e^{v_1} + e^{v_2}) - \ln(e^{v_1} + e^{v_3}) - \ln(e^{v_2} + e^{v_3}) \\ &\quad + \ln(e^{v_1} + e^{v_2} + e^{v_3}). \end{aligned}$$

It can be verified that: $\ln(e^{v_1} + e^{v_2} + e^{v_3}) > \lambda_{\mathcal{C}_3}^{(2)} > \lambda_{\mathcal{C}_3}^{(3)}$.

5.2 The symmetric case

In the symmetric case: $v_i = v$, so that the random utilities are i.i.d. Gumbels. Wlog we rescale the utilities and set: $v = 0$.

Recall that the symmetric logsum is: $\lambda_{\mathcal{C}_n}^{(1)} = \ln n$. Note that the sum of logsums (12) involved in the computation of the lower-order logsums (11) are given by

$$\sigma_m^n = \binom{n}{m} \ln m, \quad 1 \leq m \leq n.$$

Therefore, in the symmetric case, the lower-order logsums are given by

$$\lambda_{\mathcal{C}_n}^{(r)} = \sum_{m=n-r+1}^n (-1)^{m-1-n+r} \binom{m-1}{n-r} \binom{n}{m} \ln m, \quad 2 \leq r \leq n. \quad (15)$$

Notice that since, in the symmetric case, utilities are i.i.d., the values of $\lambda_{\mathcal{C}_n}^{(r)}$ follow a triangle rule in order statistics² (see Arnold, Balakrishnan, and Nagaraja, 1992, Theorem 5.3.1)

$$r\lambda_{\mathcal{C}_n}^{(r+1)} + (n-r)\lambda_{\mathcal{C}_n}^{(r)} = n\lambda_{\mathcal{C}_{n-1}}^{(r)}, \quad 1 \leq r \leq n. \quad (16)$$

which yields the following recurrence relation

$$\lambda_{\mathcal{C}_n}^{(r+1)} = \lambda_{\mathcal{C}_n}^{(r)} - \frac{n}{r} \left(\lambda_{\mathcal{C}_n}^{(r)} - \lambda_{\mathcal{C}_{n-1}}^{(r)} \right). \quad (17)$$

Eq. (17) means that the $(r+1)$ th-order expected maximum utility with n alternatives can be deduced from the r th-order expected maximum utility with n and $n-1$ alternatives, and can be computed iteratively. The expression will be a combination of standard logsums (logarithms): $\ln 2, \dots, \ln n$. The numerical results are presented in the figure below where the upper (red) curve represents the standard logsums, which is used to deduce the lower figures which represent the lower-order logsums.

²The triangle rule is stated for the moments of order statistics. Noticing that $-U^{(r)}$ are order statistics associated to $\{-U_i\}_{i \in \mathcal{C}_n}$, the rule also holds for the moments of the order utilities.

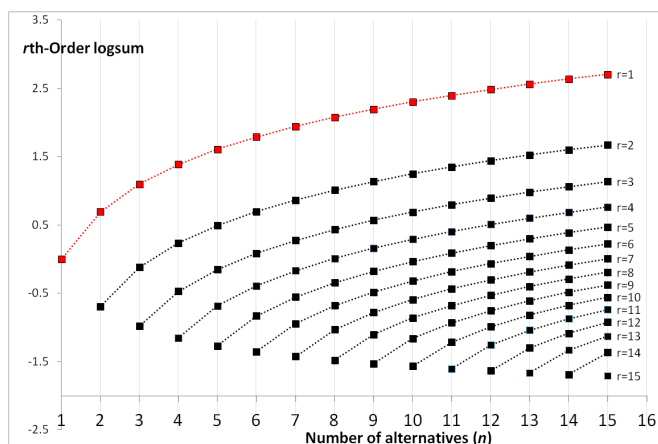


Fig. 1: Symmetric order logsums for $n = 2, \dots, 15$.

Several comments are in order. The expected maximum utility is increasing and concave while the expected minimum utility is decreasing and convex. For a given number of alternatives, the r th-order utility decreases with r and the utility loss when losing one rank is first decreasing and then increasing. This means that the loss from the penultimate to the last rank is significant. The expected utility of the r th-rank (computed from the top) increases with n while the expected utility of the r' th-rank (computed from the bottom) decreases with n . This means in particular that the penultimate for n alternatives is better than the penultimate for $n+1$ alternatives. Finally, about the of Julius Caesar's quote "I had rather be the first in this village than second in Rome," we see that the answer depends on the relative sizes of Rome and the village. For example, the first among six alternatives is better than the second among fifteen.

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