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Abstract

A graph $G = (V, E)$ is arbitrarily partitionable if for any sequence τ of positive integers adding up to $|V|$, there is a sequence of vertex-disjoint subsets of V whose orders are given by τ, and which induce connected subgraphs. In this paper, we focus on the number and the sizes of the components of an arbitrarily partitionable graph after removing a cutset.

1 Introduction

1.1 Arbitrarily partitionable graphs

Consider a network of computers that you want to separate into disjoint subnetworks of given sizes. If you want to assume that the sizes of the subnetworks may be chosen completely arbitrarily, then the underlying graph, where computers are represented by vertices and links between two computers by edges, must be *arbitrarily partitionable*.

More formaly, let n, $\tau_1, \ldots, \tau_k$ be positive integers such that $\tau_1 + \ldots + \tau_k = n$. $\tau = (\tau_1, \ldots, \tau_k)$ is called a *decomposition* of n.

Let $G = (V, E)$ be a graph of order n and S a subset of V. $G[S]$ denotes the subgraph of G induced by S. Let $(\tau_1, \ldots, \tau_k)$ be a decomposition of n. G is *τ-partitionable* iff there exists a partition of V: $V_1, \ldots, V_k$ such that for each $i, 1 \leq i \leq k$

- $|V_i| = \tau_i$
- $G[V_i]$ is connected

In that case, τ is said to be *realizable* in G and $(V_1, \ldots, V_k)$ is a *realization* of τ in G.

A graph $G = (V, E)$ of order n is *arbitrarily partitionable* (AP for short) iff for each decomposition τ of n, G is τ-partitionable.

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1.2 On-line and recursive versions

Consider now that the sizes of the subnetworks are requested one by one instead of all together. Using the graph modeling, that means that you must provide a connected subgraph of given order and assume that the rest of the graph remains "on-line" partitionable or is empty. Graphs which have this property for any sequence of requests are called *on-line arbitrarily partitionable*.

The following characterization of such graphs has been given in [HTW07]:

A connected graph $G = (V, E)$ of order n is *on-line arbitrarily partitionable* (*OLAP* for short) iff for each integer $1 \leq \lambda \leq n-1$, there exists a subset V_λ of V such that

- $|V_\lambda| = \lambda$,
- $G[V_\lambda]$ is connected,
- $G[V \backslash V_\lambda]$ is OL-AVD.

Another family of arbitrarily partitionable graphs has also been considered in [BGW11]: *recursively arbitrarily partitionable* graphs. In that case, we want not only to provide connected subgraphs, but also that these subgraphs are themselves partitionable.

A graph $G = (V, E)$ of order n is *recursively arbitrarily partitionable* (*RAP* for short) iff

- $G = K_1$ or
- G is connected and for each decomposition $\tau = (\tau_1, \ldots, \tau_k)$ of n, $k \geq 2$, there exists a partition of V: $V_1, \ldots, V_k$ such that for all $i, 1 \leq i \leq k$,
  - $|V_i| = \tau_i$,
  - $G[V_i]$ is RAP.

In [BGW11], it has been shown that for all graph G, G is AP $\Rightarrow$ G is OLAP $\Rightarrow$ G is RAP and that it exists AP graphs that are not OLAP and OLAP graphs that are not RAP.

1.3 Previous results

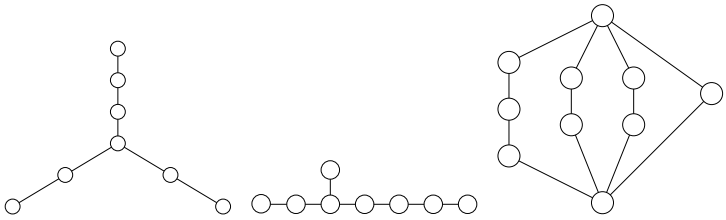
Most of the papers on AP, OLAP and RAP graphs consider quite simple classes of graphs like trees or a class of 2-connected graphs called balloons. Among the previous results, we will focus on those which help us to determine how many components may be obtained after removing a cutest and the sizes of these components.

In [BF06], the following result gives an upper bound on the degree of AP trees:

Theorem 1 *[BF06] If a tree T is AP, then its maximum degree is 4. Moreover, every vertex of degree 4 in T is adjacent to a leaf.*

In [HTW07] and [BGW11], OLAP and RAP-trees have been completely characterized. For that, we need the following notations:

- A *k-pode* $T_k(t_1, \ldots, t_k)$ is a tree of order $1 + \sum_{i=1}^{k} t_i$ composed by k paths of respective orders $t_1, \ldots, t_k$, connected to a unique node, called the *root* of the k-pode (cf. Figure 1a).
- Let a and b be two positive integers. A caterpillar $Cat(a, b)$ is a tree of order $a + b$, composed by three paths of order a, b and 2, sharing exactly one node, called the *root* of the caterpillar. $Cat(a, b)$ is isomorphic to $T_3(a-1, b-1, 1)$ (cf. Figure 1b).



(a) 3-pode $T_3(3,2,2)$ (b) $Cat(3,5)$ (c) $B(3,2,2,1)$

Figure 1: Examples of Graphs

Theorem 2 *[HTW07] A tree T is OLAP if and only if either T is a path or T is a caterpillar $Cat(a,b)$ with a and b given in Table 1 or T is the 3-pode $T_3(2,4,6)$.*

a	b
2, 4	$\equiv 1 \pmod 2$
3	$\equiv 1,2 \pmod 3$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1,5 \pmod 6$

a	b
7	8, 9, 11, 13, 15
8	11, 19
9, 10	11
11	12

Table 1: Values a, b ($b \geq a$), such that $Cat(a,b)$ is OLAP

Theorem 3 *[BGW11] A tree T is RAP if and only if either T is a path or T is a caterpillar $Cat(a,b)$ with a and b given in Table 2 or T is the 3-pode $T_3(2,4,6)$.*

a	b
2, 4	$\equiv 1 \pmod 2$
3	$\equiv 1,2 \pmod 3$
5	6, 7, 9, 11, 14, 19
6	7
7	8, 9, 11, 13, 15

Table 2: Values a, b ($b \geq a$), such that $Cat(a,b)$ is RAP

We also have some results on the class of 2-connected graphs called k-balloons. Let $b_1, \ldots, b_k$ be positive integers. A *k-balloon* $B(b_1, \ldots, b_k)$ is a graph of order $2 + \sum_{i=1}^{k} b_i$ composed by two vertices (called *roots*) linked by k paths (called *branches*) of *widths* (the number of internal vertices) $b_1, \ldots, b_k$ (cf. Figure 1c).

Theorem 4 *[BGW11] If a k-balloon is RAP, then $k \leq 5$. This bound is tight.*

This result has been extended to OLAP-k-balloons:

Theorem 5 *[BBFP12] If a k-balloon is OLAP, then $k \leq 5$.*

Upper bounds for the size of the smallest branch of a RAP or OLAP k-balloon have also been given:

Theorem 6 *[BBFP12] Let $B(b_1, \ldots, b_k)$ be a k-balloon, with $k \geq 4$ and $b_1 \leq \cdots \leq b_k$. If $B(b_1, \ldots, b_k)$ is OLAP, then $b_1 \leq 11$. If $B(b_1, \ldots, b_k)$ is RAP, then $b_1 \leq 7$.*

2 Size and number of components after removing a cutset of size at most 2

Results given in subsection 1.3 allow us to easily deduce some bounds on the number of components and their sizes, after removing a custet of size 1 or 2, using the following remarks:

Remark 7 *If G contains a spanning subgraph which is AP (resp. OLAP, RAP), then G is AP (resp. OLAP, RAP). In particular, if G is traceable (containing an hamiltonian path), then G is RAP (and then OLAP and AP).*

Remark 8 *Let $G = (V, E)$ be a graph with a cutest S and let $V_1, \cdots V_k$ be the connected components of $G[V \backslash S]$:*

- *if $|S| = 1$ and G is AP (resp. OLAP, RAP), then the k-pode $T_k(|V_1|, \cdots, |V_k|)$ is AP (resp. OLAP, RAP) ;*
- *if $|S| = 2$ and G is AP (resp. OLAP, RAP), then the k-balloon $B(|V_1|, \cdots, |V_k|)$ is AP (resp. OLAP, RAP) ;*

Using Remark 8 and Theorems 1, 2, 3, 4, 5, 6, we obtain the following results:

Corollary 9 *Let G be a graph with a cutset S, and $V_1, \cdots V_k$ the connected components of $G[V \backslash S]$, with $|V_1| \leq \cdots \leq |V_k|$:*

- *if $|S| = 1$ then*
  - *if G is AP, then $k \leq 4$ and if $k = 4$, then $|V_1| = 1$;*
  - *if G is OLAP (resp. RAP), then $k \leq 3$, and if $k = 3$, then either $(|V_1|, |V_2|, |V_3|) = (1, a, b)$, with values a and b taken in Table 1 (resp. 2) , or $(|V_1|, |V_2|, |V_3|) = (2, 4, 6)$;*
- *if $|S| = 2$ then*
  - *if G is OLAP, then $k \leq 5$;*
  - *if G is OLAP (resp. RAP) and $k \in \{4, 5\}$ then $|V_1| \leq 11$ (resp. ≤ 7).*

3 Number of components after removing a cutset of size $k \geq 2$ in AP graphs

In the previous section, we have proved that if we remove a cutset of size 1 from a graph G, then the number of the remaining connected components is at most 4 if G is AP, and 3 if G is OLAP. Similarly, if we remove a cutset of size 2 from an OLAP graph, then the number of components is at most 5.

In this section, we will prove that for any $k \geq 2$, a similar statement does not hold for AP graphs.

Theorem 10 *For any integers $k \geq 2$ and $c \geq 2$, there exists an AP graph $G = (V, E)$ with a cutest S of size k such that the number of connected components of $G[V \backslash S]$ is c.*

Proof.

The proof of this theorem is based on examples of AP graphs having a cutest S of size k and such that the number of the remaining connected components after deleting vertices of S is c.

We first consider the case where $c \leq k$. Let $G = (S_1 \cup P \cup S_2, E)$ be the graph with $2k$ vertices constructed as follows:

- $G[S_1]$ is a stable set with $c-1$ vertices ;
- $G[P]$ is a path with $k-c+1$ vertices ;
- $G[S_2]$ is a stable set with k vertices ;
- every vertex of S_1 or P is adjacent to all the vertices of S_2.

Clearly, S_2 is a cutest of size k. The components of $G[V \backslash S_2]$ are the $c-1$ vertices of S_1 plus the path induced by P. Because G admits the complete bipartite graph $K_{k,k}$ has partial graph and $K_{k,k}$ is hamiltonian, then G is AP.

Now, we assume that $c > k$.

We denote by $K_k(a_1, \ldots, a_c)$ the graph formed by $c+1$ cliques, one of size k, the others of size $a_1, \ldots a_c$. Moreover, we add all the edges between the vertices of the clique of size k and the vertices of the other cliques (see Figure 2). Clearly, the clique K_k is a cutset of $K_k(a_1, \ldots, a_c)$.

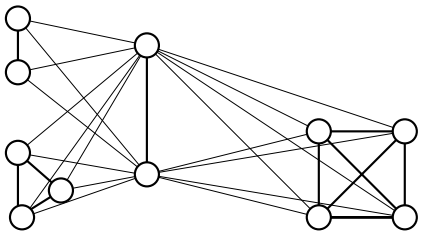


Figure 2: Graph $K_2(2,3,4)$

We consider the case of the graph $K_k(a_1, \ldots, a_c)$ where values $a_1, \ldots, a_c$ are chosen as follows:

1. $1 \leq a_1 \leq \ldots \leq a_c$
2. $\forall i, 1 \leq i \leq c$, we denote $n_i = i + \sum_{1 \leq j \leq i} a_j$.
3. $\forall i, 1 \leq i \leq c-1$, choose a_{i+1} such that
   (a) $\forall j, 2 \leq j \leq n_i, a_{i+1} \equiv 0 \pmod{j}$
   (b) $a_{i+1} \geq n_i a_i$

   For example, $a_i \cdot n_i!$ is a possible value for a_{i+1}.

Let $\tau = (\tau_1, \ldots, \tau_l)$ be a decomposition of $n = k + \sum_{1 \leq i \leq c} a_i$, with $\tau_1 \leq \ldots \leq \tau_l$. We consider two cases.

First case: $\tau_l \geq a_{c-1}$

In that case, $\tau_l \geq n_{c-2} = c - 2 + \sum_{1 \leq i \leq c-2} a_i \geq k - 1 + \sum_{1 \leq i \leq c-2} a_i$. Thus the part of size τ_l may cover all the cliques K_{a_i} with $i \leq c-2$, plus $k-1$ vertices of K_k and eventually some vertices of the cliques $K_{a_{c-1}}$ and K_{a_c}. The remaining graph is formed by at most two cliques joined by the last vertex of K_k and is traceable.

Second case: $\tau_l < a_{c-1}$

For each $i, 1 \leq i \leq \tau_l$, we denote by q_i the number of terms of τ having value i.

We have $n = \sum_{1 \leq i \leq \tau_l} i \cdot q_i$

Then there exists an integer $\alpha, 1 \leq \alpha \leq \tau_l$, such that $\alpha \cdot q_\alpha \geq \frac{n}{\tau_l}$. Thus, $\alpha \cdot q_\alpha > \frac{n}{a_{c-1}} \geq \frac{n_c}{a_{c-1}} \geq \frac{n_{c-1} a_{c-1}}{a_{c-1}} = n_{c-1}$.

We denote by s the index such that for all $i \leq s, a_i \not\equiv 0 \pmod{\alpha}$ and for all $i > s, a_i \equiv 0 \pmod{\alpha}$. Note that s may be equal to 0.

Because $a_s \not\equiv 0 \pmod{\alpha}$, we have $\alpha > n_{s-1}$ by property 3a of the definition of $K_k(a_1, \ldots, a_c)$. That means that one part of size α may cover all the cliques $K_{a_1}, \ldots, K_{a_{s-1}}$ plus a vertex of K_k and maybe some vertices of K_{a_s}.

Because $\forall i > s, a_i \equiv 0 \pmod{\alpha}$, and $\alpha \cdot q_\alpha > n_{c-1}$, we may cover also the cliques $K_{a_{s+1}}, \ldots, K_{a_{c-1}}$ with parts of size α.

Thus the remaining graph is formed by K_{a_c}, the remaining vertices of K_{a_s} and $k-1$ vertices of K_k. This graph is traceable. □

Balloons The previous result (Theorem 10) can be adapted to the special case of balloons. The interest of this result is that it gives (for the case $k = 2$) an example of graphs with a linear number of edges, contrarily to the example built for the proof of Theorem 10 where the number of edges may be quadratic.

Theorem 11 *$\forall k \geq 1$, there exists a k-balloon AP.*

Proof. We consider a k-balloon $B(b_1, \ldots, b_k)$ where branches have the same size than the cliques of $K_k(a_1, ..., a_c)$ given in the proof of Theorem 10:
$b_1 \leq \ldots \leq b_k$ and we denote $n_i = 1 + \sum_{1 \leq j \leq i} b_j$.
$B(b_1, \ldots, b_k)$ is constructed as follow:

- $b_1 \geq 1$
- $\forall i \leq k-1$, choose b_{i+1} such that
  1. $\forall j, 2 \leq j \leq n_i, b_{i+1} \equiv 0 \pmod{j}$
  2. $b_{i+1} \geq n_i b_i$

Using the same arguments than those used for $K_k(a_1, ..., a_c)$, it is easy to see that such a balloon is AP. □

4 Size of components

In this section, we will show that, if the number of connected components after removing a cutset of size at least 2, may be as large as wanted, the size of these components grows exponentially with their number.

Theorem 12 *Let $G = (V, E)$ be a graph with n vertices, S a cutset of G of size k, $c_1, \ldots, c_l$ the orders of the connected components of $G_{V \setminus S}$, such that $l > k$ and $1 \leq c_1 \leq c_2 \leq \ldots \leq c_l$. Then the values of the sequence $(c_i)_{i \geq 1}$ grow exponentially with i.*

To prove this theorem, we use Lemma 13 to 15:

Lemma 13 *Let $G = (V, E)$ be a graph with n vertices, S a cutset of G of size k, $c_1, \ldots, c_l$ the orders of the connected components of $G_{V \setminus S}$, such that $l > k$ and $1 \leq c_1 \leq c_2 \leq \ldots \leq c_l$.*

Let $a, q_1, \ldots, q_l, r_1, \ldots, r_l$ be integers such that

- $2 \leq a \leq n-1$
- $\forall i, 1 \leq i \leq l, c_i = q_i a + r_i$ *with* $r_i < a$

If G is AP, then

$$\sum_{1 \leq i \leq l} r_i \leq (k+1) \cdot (a-1)$$

Proof.

Observe that, by the assumptions we made, G is a partial graph of $K_k(c_1, \ldots, c_l)$. If a decomposition $\tau = (\tau_1, \ldots, \tau_p)$ of n is realizable in G, and $V_1, \ldots, V_p$ is a realization of τ in G, then τ is realizable in $K_k(c_1, \ldots, c_l)$ and $V_1, \ldots, V_p$ is a realization of τ in $K_k(c_1, \ldots, c_l)$. Moreover, if G is AP, then $K_k(c_1, \ldots, c_l)$ is AP.

Hence, to prove Lemma 13, we may consider only graphs of the form $K_k(c_1, \ldots, c_l)$.

First, we remark that for any decomposition τ of $n = k + \sum_{1 \leq i \leq l} c_i$, there exists a realization of τ in $K_k(c_1, \ldots, c_l)$ where the vertices of the clique K_k are covered by the greatest parts of τ (note that one part may contain more than one vertex of the clique K_k). In fact, suppose that we have a part V_1 which does not cover any vertex of K_k and a vertex v in K_k covered by a part V_2 such that $|V_2| < |V_1|$. Then we may always exchange $|V_1| - |V_2|$ vertices from V_1 to V_2 inside the clique containing V_1.

We consider now the decomposition $\tau = (a, \ldots, a, r)$ of n with $r < a$ and its realization in $K_k(c_1, \ldots, c_l)$. Because $a > r$, we may suppose that all the parts covering the vertices of K_k are of size a. The cliques $K_{c_1}, \ldots, K_{c_l}$are also mainly covered by parts of size a. The remaining vertices must be covered by the at most k parts of size a which cover also the k vertices of K_k, and the part of size r. Because $r \leq a - 1$, we obtain $\sum_{1 \leq i \leq k} r_i + k \leq k \cdot a + a - 1$. □

Corollary 14 *Let $G = (V, E)$ be a graph with n vertices, S a cutset of G of size k, $c_1, \ldots, c_l$ the orders of the connected components of $G_{V \setminus S}$, such that $l > k$ and $1 \leq c_1 \leq c_2 \leq \ldots \leq c_l$. If G is AP, then $\forall i, 1 \leq i \leq k$,*

$$c_i \geq \frac{1}{k} \sum_{1 \leq j < i} c_j$$

Proof. We consider $a = c_i + 1$. In that case, for all $j \leq i$, we have $r_j = c_j$. Thus, Lemma 13 implies $\sum_{i < j \leq k} r_j + c_i + \sum_{1 \leq j < i} r_j \leq (k+1) \cdot c_i$. Because $\sum_{i < j \leq k} r_j > 0$, we obtain the result. □

The following lemma completes the proof of Theorem 12

Lemma 15

$$\forall i \geq 1, c_i \geq (1 + \frac{1}{k})^{i-2} \times \frac{c_1}{k}$$

Proof. Consider the sequence $(v_i)_{i \geq 1}$ defined by $v_1 = c_1$ and $\forall i \geq 2, v_i = \frac{1}{k} \sum_{1 \leq j < i} v_j$. Corollary 14 implies that $\forall i, i \geq 1, c_i \geq v_i$.
We have $v_2 = \frac{v_1}{k} = \frac{c_1}{k}$ and $v_3 = \frac{1}{k}(v_1 + v_2) = v_2 + \frac{1}{k} v_2 = (1 + \frac{1}{k}) v_2$.
For all value $i \geq 3$, $v_{i+1} = \frac{1}{k} \sum_{1 \leq j \leq i} v_j = \frac{1}{k}(v_i + \sum_{1 \leq j \leq i-1} v_j) = \frac{1}{k} v_i + v_i = (1 + \frac{1}{k}) v_i$.
Thus, by induction, we have $v_{i+1} = (1 + \frac{1}{k})^{i-1} v_2 = (1 + \frac{1}{k})^{i-1} \frac{c_1}{k}$. □

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